

Exact Solutions of Relativistic Two-Body Motion in Lineal Gravity

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Abstract

We develop the canonical formalism for a system of N bodies in lineal gravity and obtain exact solutions to the equations of motion for $N = 2$. The determining equation of the Hamiltonian is derived in the form of a transcendental equation, which leads to the exact Hamiltonian to infinite order of the gravitational coupling constant. In the equal mass case explicit expressions of the trajectories of the particles are given as the functions of the proper time, which show characteristic features of the motion depending on the strength of gravity (mass) and the magnitude and sign of the cosmological constant. As expected, we find that a positive cosmological constant has a repulsive effect on the motion, while a negative one has an attractive effect. However, some surprising features emerge that are absent for vanishing cosmological constant. For a certain range of the negative cosmological constant the motion shows a double maximum behavior as a combined result of an induced momentum-dependent cosmological potential and the gravitational attraction between the particles. For a positive cosmo-

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logical constant, not only bounded motions but also unbounded ones are realized. The change of the metric along the movement of the particles is also exactly derived.

1 INTRODUCTION

Solving the motion of a system of N particles interacting through their mutual gravitational forces has long been one of the important themes in physics [1]. Though an exact solution is known in the $N = 2$ case in Newtonian theory, in the context of the general theory of relativity the motion of the N bodies cannot be solved exactly due to dissipation of energy in the form of gravitational radiation, even when $N = 2$. Hence analysis of a two body system in general relativity (e.g. binary pulsars) necessarily involves resorting to approximation methods such as a post-Newtonian expansion [2, 3].

However in the past decade lower dimensional versions of general relativity (both in (1+1) and (2+1) dimensions) have been extensively investigated from both classical and quantum perspectives. Here the reduced dimensionality yields an absence of gravitational radiation. Although this desirable physical feature is lost in such models (at least in the vacuum), most (if not all) of the remaining conceptual features of relativistic gravity are retained. Hence their mathematical simplicity offers the hope of obtaining a deep understanding of the nature of gravitation in a wide variety of physical situations. It is with this motivation that we consider the N -body problem in lower dimensional gravity.

Specifically, we consider the gravitational N -body problem in two spacetime dimensions. Such lineal theories of gravity have found widespread use in other problems in physics. The simplest such theory (sometimes referred to as Jackiw-Teitelboim (JT) theory [4]) sets the Ricci scalar equal to a constant, with other matter fields evolving in this constant-curvature two-dimensional spacetime. Another such theory (sometimes referred to as $R = T$ theory) sets the Ricci scalar equal to the trace of the stress-energy of the prescribed matter fields and sources – in this manner, matter governs the evolution of spacetime curvature which reciprocally governs the evolution of matter [5]. This theory has a consistent Newtonian limit [5] (a problematic issue for a generic (1 + 1)-dimensional gravity theory [6]), and reduces to JT theory if the stress-energy is that of a cosmological constant.

The N -body problem, then, can be formulated in relativistic gravity by taking the matter action to be that of N point-particles minimally coupled to gravity. In previous work we developed the canonical formalism for this action in $R = T$ lineal gravity [7] and derived the exact Hamiltonian for $N = 2$ as a solution to a transcendental equation which is valid to infinite order in the gravitational coupling constant [8]. In the slow motion, weak field limit this Hamiltonian coincides with that of Newtonian gravity in (1 + 1) dimensions, and in the limit where all bodies are massless, spacetime is flat.

More recently we have extended this case to include a cosmological constant Λ , so that in the limit where all bodies are massless, spacetime has constant curvature (ie the JT theory is obtained), and when Λ vanishes the situation described in the previous paragraph is recovered [9]. For $N = 2$, we derived an exact solution for the Hamiltonian as a function of the proper separation and the centre-of-inertia momentum of the bodies. In the equal mass case an exact solution to the equations of motion for the proper separation of the two point masses as a function of their mutual proper time was also obtained. The trajectories showed characteristic structures depending on the values of a cosmological constant Λ . The purpose of this paper is to more fully describe these results and to expand upon them. Specifically, we generalize our previous formalism with $\Lambda = 0$ [7] to a system of N particles in (1+1) dimensional gravity with cosmological constant. When $N = 2$ we obtain exact solutions for

the motion of two bodies of unequal (and equal) mass.

Since the Einstein action is a topological invariant in (1+1) dimensions, we must incorporate a scalar (dilaton) field into the action [10]. By a canonical reduction of the action, the Hamiltonian is defined as a spatial integral of the second derivative of the dilaton field, which is a function of the canonical variables of the particles (coordinates and momenta) and is determined from the constraint equations. For a system of two particles an equation which determines the Hamiltonian in terms of the remaining degrees of freedom of the system is derived from the matching conditions of the solution to the constraint equations. We refer to this equation as the determining equation; it is a transcendental equation which allows one to determine the Hamiltonian in terms of the centre of inertia momentum and proper separation of the bodies. The canonical equations of motion are derived from the Hamiltonian. For the equal mass case they can be transformed so that the separation and momentum are given by differential equations in terms of the proper time. In this latter form they can be solved exactly in terms of hyperbolic and/or trigonometric functions.

Corresponding to the values of the magnitudes (and signs) of the energy and other parameters (e.g. gravitational coupling constant, masses, cosmological constant) several different types of motion are expected in the 2 body case. Broadly speaking, the two particles could remain either bounded or unbounded, or else balanced between these two conditions. We shall analyze these various states of motion, and discuss the transitions which occur between them. We find several surprising situations, including the onset of mutual repulsion for a range of values of negative Λ and the masses, and the diverging separation of the two bodies at finite proper time for a range of values of positive Λ .

We shall also consider the unequal mass case. In this situation the proper time is no longer the same for the two particles, and so a description of the motion requires a more careful analysis. We find that we are able to obtain phase space trajectories from the determining equation. We also can obtain explicit solutions for the proper separation in terms of a transformed time coordinate which reduces to the mutual proper time in the case of equal mass.

In Sec.II we describe the canonical reduction of the theory and define the Hamiltonian for the N -body system. In Sec.III we solve the constraint equations for the two-body case and get the determining equation of the Hamiltonian, from which the canonical equations of motion are explicitly derived. We investigate the motion for $\Lambda = 0$ in Sec.IV, by using the exact solutions to the canonical equations. The motion of equal masses for $\Lambda \neq 0$ are analyzed in Sec.V where a general discussion on the structure of the determining equation, the plots of phase space trajectories, the analysis of the explicit solutions in terms of the proper time are developed. We treat the unequal mass case in Sec.VI. Sec.VII contains concluding remarks and directions for further work. The solution of the metric tensor, a test particle approximation in the small mass limit of one of the particles and the causal relationships between particles in unbounded motion are given in Appendices.

2 CANONICALLY REDUCED HAMILTONIAN of N PARTICLES

The action integral for the gravitational field coupled to N point masses is

$$I = \int dx^2 \left[\frac{1}{2\kappa} \sqrt{-g} g^{\mu\nu} \left\{ \Psi R_{\mu\nu} + \frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi + \frac{1}{2} g_{\mu\nu} \Lambda \right\} + \sum_a \int d\tau_a \left\{ -m_a \left(-g_{\mu\nu}(x) \frac{dz_a^\mu}{d\tau_a} \frac{dz_a^\nu}{d\tau_a} \right)^{1/2} \right\} \delta^2(x - z_a(\tau_a)) \right], \quad (1)$$

where Ψ is the dilaton field, Λ is the cosmological constant, $g_{\mu\nu}$ and g are the metric and its determinant, R is the Ricci scalar, and τ_a is the proper time of a -th particle, respectively, with $\kappa = 8\pi G/c^4$. The symbol ∇_μ denotes the covariant derivative associated with $g_{\mu\nu}$.

The field equations derived from the action (1) are

$$R - g^{\mu\nu} \nabla_\mu \nabla_\nu \Psi = 0, \quad (2)$$

$$\frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi - \frac{1}{4} g_{\mu\nu} \nabla^\lambda \Psi \nabla_\lambda \Psi + g_{\mu\nu} \nabla^\lambda \nabla_\lambda \Psi - \nabla_\mu \nabla_\nu \Psi = \kappa T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \Lambda, \quad (3)$$

$$\frac{d}{d\tau_a} \left\{ g_{\mu\nu}(z_a) \frac{dz_a^\nu}{d\tau_a} \right\} - \frac{1}{2} g_{\nu\lambda,\mu}(z_a) \frac{dz_a^\nu}{d\tau_a} \frac{dz_a^\lambda}{d\tau_a} = 0, \quad (4)$$

where the stress-energy due to the point masses is

$$T_{\mu\nu} = \sum_a m_a \int d\tau_a \frac{1}{\sqrt{-g}} g_{\mu\sigma} g_{\nu\rho} \frac{dz_a^\sigma}{d\tau_a} \frac{dz_a^\rho}{d\tau_a} \delta^2(x - z_a(\tau_a)). \quad (5)$$

Eq.(3) guarantees the conservation of $T_{\mu\nu}$. Inserting the trace of Eq.(3) into Eq.(2) yields

$$R - \Lambda = \kappa T^\mu{}_\mu. \quad (6)$$

Eqs. (4) and (6) form a closed system of equations for the matter-gravity system. The evolution of the dilaton then follows from inserting the solutions to these equations into(2), and then solving for its motion, the traceless part of (3) being identities once the other equations are satisfied. Alternatively, one can solve the independent parts of equations (2), (3), and (4) for the metric, dilaton and matter degrees of freedom, which is the approach we shall take in this paper. If the masses of all particles are taken to be zero then the field equations reduce to those of constant curvature lineal gravity, or JT theory [4, 5].

Consider next the transformation of the action (1) to canonical form. We decompose the scalar curvature in terms of the extrinsic curvature K via

$$\sqrt{-g} R = -2\partial_0(\sqrt{\gamma} K) + 2\partial_1(\sqrt{\gamma} N^1 K - \gamma^{-1} \partial_1 N_0), \quad (7)$$

where $K = (2N_0\gamma)^{-1}(2\partial_1 N_1 - \gamma^{-1} N_1 \partial_1 \gamma - \partial_0 \gamma)$, and the metric is

$$ds^2 = -N_0^2 dt^2 + \gamma \left(dx + \frac{N_1}{\gamma} dt \right)^2, \quad (8)$$

so that $\gamma = g_{11}$, $N_0 = (-g^{00})^{-1/2}$ and $N_1 = g_{10}$, and then rewrite the particle action in first-order form. After some manipulation we find that the action (1) may be rewritten in the form

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a(t)) + \pi \dot{\gamma} + \Pi \dot{\Psi} + N_0 R^0 + N_1 R^1 \right\}, \quad (9)$$

where π and Π are conjugate momenta to γ and Ψ , respectively, and

$$R^0 = -\kappa \sqrt{\gamma} \pi^2 + 2\kappa \sqrt{\gamma} \pi \Pi + \frac{1}{4\kappa \sqrt{\gamma}} (\Psi')^2 - \frac{1}{\kappa} \left(\frac{\Psi'}{\sqrt{\gamma}} \right)' - \frac{\Lambda}{2\kappa} \sqrt{\gamma} - \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2} \delta(x - z_a(t)), \quad (10)$$

$$R^1 = \frac{\gamma'}{\gamma} \pi - \frac{1}{\gamma} \Pi \Psi' + 2\pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(t)), \quad (11)$$

with the symbols $(\dot{})$ and $()'$ denoting ∂_0 and ∂_1 , respectively.

The action (9) leads to the system of equations

$$\begin{aligned} \dot{\pi} + N_0 \left\{ \frac{3\kappa}{2} \sqrt{\gamma} \pi^2 - \frac{\kappa}{\sqrt{\gamma}} \pi \Pi + \frac{1}{8\kappa \sqrt{\gamma} \gamma} (\Psi')^2 - \frac{1}{4\sqrt{\gamma} \kappa} \right. \\ \left. - \sum_a \frac{p_a^2}{2\gamma^2 \sqrt{\frac{p_a^2}{\gamma} + m_a^2}} \delta(x - z_a(t)) \right\} \\ + N_1 \left\{ -\frac{1}{\gamma^2} \Pi \Psi' + \frac{\pi'}{\gamma} + \sum_a \frac{p_a}{\gamma^2} \delta(x - z_a(t)) \right\} + N_0' \frac{1}{2\kappa \sqrt{\gamma} \gamma} \Psi' + N_1' \frac{\pi}{\gamma} = 0, \quad (12) \end{aligned}$$

$$\dot{\gamma} - N_0 (2\kappa \sqrt{\gamma} \gamma \pi - 2\kappa \sqrt{\gamma} \Pi) + N_1 \frac{\gamma'}{\gamma} - 2N_1' = 0, \quad (13)$$

$$R^0 = 0, \quad (14)$$

$$R^1 = 0, \quad (15)$$

$$\dot{\Pi} + \partial_1 \left(-\frac{1}{\gamma} N_1 \Pi + \frac{1}{2\kappa \sqrt{\gamma}} N_0 \Psi' + \frac{1}{\kappa \sqrt{\gamma}} N_0' \right) = 0, \quad (16)$$

$$\dot{\Psi} + N_0 (2\kappa \sqrt{\gamma} \pi) - N_1 \left(\frac{1}{\gamma} \Psi' \right) = 0, \quad (17)$$

$$\begin{aligned} \dot{p}_a + \frac{\partial N_0}{\partial z_a} \sqrt{\frac{p_a^2}{\gamma} + m_a^2} - \frac{N_0}{2\sqrt{\frac{p_a^2}{\gamma} + m_a^2}} \frac{p_a^2}{\gamma^2} \frac{\partial \gamma}{\partial z_a} - \frac{\partial N_1}{\partial z_a} \frac{p_a}{\gamma} \\ + N_1 \frac{p_a}{\gamma^2} \frac{\partial \gamma}{\partial z_a} = 0, \quad (18) \end{aligned}$$

$$\dot{z}_a - N_0 \frac{\frac{p_a}{\gamma}}{\sqrt{\frac{p_a^2}{\gamma} + m_a^2}} + \frac{N_1}{\gamma} = 0. \quad (19)$$

In the equations (18) and (19), all metric components (N_0 , N_1 , γ) are evaluated at the point $x = z_a$ and

$$\frac{\partial f}{\partial z_a} \equiv \left. \frac{\partial f(x)}{\partial x} \right|_{x=z_a} .$$

This system of equations can be shown to be equivalent to the set of equations (2), (3) and (4).

Since N_0 and N_1 are Lagrange multipliers, equations (14) and (15) are constraints; specifically they are the energy and momentum constraints of the (1+1) dimensional gravitational system we consider. We may solve them for $(\Psi'/\sqrt{\gamma})'$ and π' in terms of the dynamical and gauge (*i.e.* co-ordinate) degrees of freedom, since these are the only linear terms in these constraints. We identify these coordinate degrees of freedom by writing the generator arising from the variation of the action at the boundaries in terms of $(\Psi'/\sqrt{\gamma})'$ and π' , and then finding which quantities serve to fix the frame of the physical space-time coordinates in a manner similar to the (3+1)-dimensional case.

Carrying out the same procedure as in the $\Lambda = 0$ case [8] we find that we can consistently choose the coordinate conditions

$$\gamma = 1 \quad \text{and} \quad \Pi = 0 . \quad (20)$$

Eliminating the constraints, the action reduces to

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \mathcal{H} \right\} , \quad (21)$$

and the reduced Hamiltonian for the N -body system is

$$H = \int dx \mathcal{H} = -\frac{1}{\kappa} \int dx \Delta \Psi , \quad (22)$$

where Ψ is a function of z_a and p_a , determined by solving the constraints which are under the coordinate conditions (20)

$$\Delta \Psi - \frac{1}{4}(\Psi')^2 + \kappa^2 \pi^2 - \frac{\Lambda}{2} + \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) = 0 , \quad (23)$$

$$2\pi' + \sum_a p_a \delta(x - z_a) = 0 . \quad (24)$$

The expression for the Hamiltonian (22) is analogous to the reduced Hamiltonian in (3+1) dimensional general relativity. In (1+1) dimensions it is determined by the dilaton field at spatial infinity. The consistency of this canonical reduction may be demonstrated in a manner analogous to that employed in the $\Lambda = 0$ case: namely the canonical equations of motion derived from the reduced Hamiltonian (22) are identical with the equations (18) and (19).

The methodology at this point is then as follows. First we must solve (23) and (24) for Ψ and π in terms of the p_a and the z_a , consistently matching solutions across the boundaries of the particles. Then we compute from (22) the Hamiltonian in terms of the independent momenta and coordinates of the particles. This expression is sufficient to obtain the phase-space trajectories for a given set of initial conditions. Finally we solve equations (12 – 19) to obtain a complete solution for the N body system. Throughout the remainder of this paper we shall consider only $N = 2$, *i.e.* 2-body dynamics.

3 SOLUTION TO THE CONSTRAINT EQUATIONS AND THE HAMILTONIAN FOR A SYSTEM OF TWO PARTICLES

The standard approach for investigating the dynamics of particles is to derive an explicit expression of the Hamiltonian, from which the equations of motion and the solution of trajectories are obtained. In this section we explain how to derive the Hamiltonian from the solution to the constraint equations (23) and (24) and get the explicit Hamiltonian for two particles in a spacetime with a cosmological constant.

We first express the equations (23) and (24) as

$$\Delta\Psi = \frac{1}{4}(\Psi')^2 - \kappa^2(\chi')^2 + \frac{1}{2}\Lambda - \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a), \quad (25)$$

$$\Delta\chi = -\frac{1}{2} \sum_a p_a \delta(x - z_a), \quad (26)$$

where we set $\chi' = \pi$. Rewriting (25) as

$$\left(1 + \frac{\Psi}{4}\right)\Delta\Psi = \frac{1}{8}\Delta(\Psi^2 - 4\kappa^2\chi^2) + \frac{1}{2}\Lambda + \kappa^2\chi\Delta\chi - \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a), \quad (27)$$

and using (26), we can rewrite (22) as

$$\begin{aligned} H &= \sum_a \frac{\sqrt{p_a^2 + m_a^2}}{1 + \frac{1}{4}\Psi(z_a)} + \frac{\kappa}{2} \sum_a \frac{p_a \chi(z_a)}{1 + \frac{1}{4}\Psi(z_a)} \\ &\quad - \frac{1}{8\kappa} \int dx \frac{1}{1 + \frac{1}{4}\Psi(x)} \Delta(\Psi^2 - 4\kappa^2\chi^2 + 2\Lambda x^2), \end{aligned} \quad (28)$$

which can also be obtained by inserting (25) and (26) into (22) and iterating by partial integration (assuming convergence). We shall refer to this formula later when we consider boundary conditions.

Defining ϕ by

$$\Psi = -4\log|\phi|, \quad (29)$$

the constraints (25) and (26) for a two-particle system become

$$\begin{aligned} \Delta\phi - \frac{1}{4} \left\{ \kappa^2(\chi')^2 - \frac{1}{2}\Lambda \right\} \phi &= \frac{\kappa}{4} \left\{ \sqrt{p_1^2 + m_1^2} \phi(z_1) \delta(x - z_1) \right. \\ &\quad \left. + \sqrt{p_2^2 + m_2^2} \phi(z_2) \delta(x - z_2) \right\}, \end{aligned} \quad (30)$$

$$\Delta\chi = -\frac{1}{2} \{p_1 \delta(x - z_1) + p_2 \delta(x - z_2)\}. \quad (31)$$

The general solution to (31) is

$$\chi = -\frac{1}{4} \{p_1 |x - z_1| + p_2 |x - z_2|\} - \epsilon X x + \epsilon C_\chi. \quad (32)$$

The factor ϵ ($\epsilon^2 = 1$) has been introduced in the constants X and C_χ so that the T-inversion (time reversal) properties of χ are explicitly manifest. By definition, ϵ changes sign under time reversal and so, therefore, does χ .

Consider first the case $z_2 < z_1$, for which we may divide space into three regions: $z_1 < x$ ((+) region), $z_2 < x < z_1$ ((0) region) and $x < z_2$ ((-) region). In each region χ' is constant:

$$\chi' = \begin{cases} -\epsilon X - \frac{1}{4}(p_1 + p_2) & (+) \text{ region,} \\ -\epsilon X + \frac{1}{4}(p_1 - p_2) & (0) \text{ region,} \\ -\epsilon X + \frac{1}{4}(p_1 + p_2) & (-) \text{ region.} \end{cases} \quad (33)$$

General solutions to the homogeneous equation $\Delta\phi - \frac{1}{4}\{\kappa^2(\chi')^2 - \frac{1}{2}\Lambda\}\phi = 0$ in each region are

$$\begin{cases} \phi_+(x) = A_+ e^{\frac{1}{2}K_+x} + B_+ e^{-\frac{1}{2}K_+x}, \\ \phi_0(x) = A_0 e^{\frac{1}{2}K_0x} + B_0 e^{-\frac{1}{2}K_0x}, \\ \phi_-(x) = A_- e^{\frac{1}{2}K_-x} + B_- e^{-\frac{1}{2}K_-x}, \end{cases} \quad (34)$$

where

$$\begin{cases} K_+ = \sqrt{\kappa^2 \left(X + \frac{\epsilon}{4}(p_1 + p_2)\right)^2 - \frac{1}{2}\Lambda} & (+) \text{ region,} \\ K_0 = \sqrt{\kappa^2 \left(X - \frac{\epsilon}{4}(p_1 - p_2)\right)^2 - \frac{1}{2}\Lambda} & (0) \text{ region,} \\ K_- = \sqrt{\kappa^2 \left(X - \frac{\epsilon}{4}(p_1 + p_2)\right)^2 - \frac{1}{2}\Lambda} & (-) \text{ region.} \end{cases} \quad (35)$$

For these solutions to be the actual solutions to Eq.(30) with delta function source terms, they must satisfy the following matching conditions at $x = z_1, z_2$:

$$\phi_+(z_1) = \phi_0(z_1) = \phi(z_1), \quad (36a)$$

$$\phi_-(z_2) = \phi_0(z_2) = \phi(z_2), \quad (36b)$$

$$\phi'_+(z_1) - \phi'_0(z_1) = \frac{\kappa}{4}\sqrt{p_1^2 + m_1^2}\phi(z_1), \quad (36c)$$

$$\phi'_0(z_2) - \phi'_-(z_2) = \frac{\kappa}{4}\sqrt{p_2^2 + m_2^2}\phi(z_2). \quad (36d)$$

The conditions (36a) and (36c) lead to

$$e^{\frac{1}{2}K_+z_1}A_+ + e^{-\frac{1}{2}K_+z_1}B_+ = e^{\frac{1}{2}K_0z_1}A_0 + e^{-\frac{1}{2}K_0z_1}B_0, \quad (37)$$

$$\begin{aligned} e^{\frac{1}{2}K_+z_1}A_+ - e^{-\frac{1}{2}K_+z_1}B_+ &= \frac{\kappa\sqrt{p_1^2 + m_1^2} + 2K_0}{2K_+}e^{\frac{1}{2}K_0z_1}A_0 \\ &+ \frac{\kappa\sqrt{p_1^2 + m_1^2} - 2K_0}{2K_+}e^{-\frac{1}{2}K_0z_1}B_0, \end{aligned} \quad (38)$$

yielding

$$A_+ = \frac{\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 + 2K_+}{4K_+}e^{\frac{1}{2}(K_0 - K_+)z_1}A_0$$

$$+ \frac{\kappa\sqrt{p_1^2 + m_1^2} - 2K_0 + 2K_+}{4K_+} e^{-\frac{1}{2}(K_0+K_+)z_1} B_0, \quad (39a)$$

$$B_+ = -\frac{\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+}{4K_+} e^{\frac{1}{2}(K_0+K_+)z_1} A_0 \\ - \frac{\kappa\sqrt{p_1^2 + m_1^2} - 2K_0 - 2K_+}{4K_+} e^{-\frac{1}{2}(K_0-K_+)z_1} B_0. \quad (39b)$$

Similarly from (36b) and (36d) we have

$$e^{\frac{1}{2}K_-z_2} A_- + e^{-\frac{1}{2}K_-z_2} B_- = e^{\frac{1}{2}K_0z_2} A_0 + e^{-\frac{1}{2}K_0z_2} B_0, \quad (40)$$

$$-e^{\frac{1}{2}K_-z_2} A_- + e^{-\frac{1}{2}K_-z_2} B_- = \frac{\kappa\sqrt{p_2^2 + m_2^2} - 2K_0}{2K_-} e^{\frac{1}{2}K_0z_2} A_0 \\ + \frac{\kappa\sqrt{p_2^2 + m_2^2} + 2K_0}{2K_-} e^{-\frac{1}{2}K_0z_2} B_0, \quad (41)$$

and then

$$A_- = -\frac{\kappa\sqrt{p_2^2 + m_2^2} - 2K_0 - 2K_-}{4K_-} e^{\frac{1}{2}(K_0-K_-)z_2} A_0 \\ - \frac{\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-}{4K_-} e^{-\frac{1}{2}(K_0+K_-)z_2} B_0, \quad (42a)$$

$$B_- = \frac{\kappa\sqrt{p_2^2 + m_2^2} - 2K_0 + 2K_-}{4K_-} e^{\frac{1}{2}(K_0+K_-)z_2} A_0 \\ + \frac{\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 + 2K_-}{4K_-} e^{-\frac{1}{2}(K_0-K_-)z_2} B_0. \quad (42b)$$

Since the magnitudes of both ϕ and χ increase with increasing $|x|$, it is necessary to impose a boundary condition which guarantees that the surface terms which arise in transforming the action vanish and simultaneously preserves the finiteness of the Hamiltonian. A consideration of (28) implies that we may choose the boundary condition

$$\Psi^2 - 4\kappa^2\chi^2 + 2\Lambda x^2 = C_{\pm}x \quad \text{for (+) and (-) regions} \quad (43)$$

with C_{\pm} being constants to be determined. This boundary condition means

$$A_- = B_+ = 0, \quad (44)$$

$$\{2K_+x + 4\log|A_+|\}^2 - 4\kappa^2 \left\{ - \left[\epsilon X + \frac{1}{4}(p_1 + p_2) \right] x + \epsilon C_{\chi} + \frac{1}{4}(p_1z_1 + p_2z_2) \right\}^2 \\ + 2\Lambda x^2 = C_+x, \quad (45)$$

$$\{2K_-x - 4\log|B_-|\}^2 - 4\kappa^2 \left\{ - \left[\epsilon X - \frac{1}{4}(p_1 + p_2) \right] x + \epsilon C_{\chi} - \frac{1}{4}(p_1z_1 + p_2z_2) \right\}^2 \\ + 2\Lambda x^2 = C_-x. \quad (46)$$

It may seem that instead of (44) we could have made the alternate choices $(A_+ = 0, B_- = 0)$, $(A_+ = 0, A_- = 0)$ or $(B_+ = 0, B_- = 0)$. However the definitions (35) imply that K_{\pm} are positive quantities, which in turn leads to a negative Hamiltonian for the choice $(A_+ = 0, B_- = 0)$. For the choices $(A_+ = 0, A_- = 0)$ and $(B_+ = 0, B_- = 0)$ the Hamiltonian identically vanishes.

The terms quadratic in x from (45) and (46) merely recapitulate the definitions of K_{\pm} . Equating terms linear in x yield the relations

$$16K_+ \log|A_+| + 8\kappa^2 \left[\epsilon X + \frac{1}{4}(p_1 + p_2) \right] \left[\epsilon C_{\chi} + \frac{1}{4}(p_1 z_1 + p_2 z_2) \right] = C_+ , \quad (47)$$

$$-16K_- \log|B_-| + 8\kappa^2 \left[\epsilon X - \frac{1}{4}(p_1 + p_2) \right] \left[\epsilon C_{\chi} - \frac{1}{4}(p_1 z_1 + p_2 z_2) \right] = C_- . \quad (48)$$

Equating the constant terms of (45) and (46) leads to

$$16 (\log|A_+|)^2 - 4\kappa^2 \left[\epsilon C_{\chi} + \frac{1}{4}(p_1 z_1 + p_2 z_2) \right]^2 = 0 , \quad (49)$$

$$16 (\log|B_-|)^2 - 4\kappa^2 \left[\epsilon C_{\chi} - \frac{1}{4}(p_1 z_1 + p_2 z_2) \right]^2 = 0 . \quad (50)$$

We choose the solutions

$$\log|A_+| = -\frac{\kappa}{2} \left[C_{\chi} + \frac{\epsilon}{4}(p_1 z_1 + p_2 z_2) \right] , \quad (51)$$

$$\log|B_-| = \frac{\kappa}{2} \left[C_{\chi} - \frac{\epsilon}{4}(p_1 z_1 + p_2 z_2) \right] .$$

Before proceeding, we add a remark to (51). In solving (49) and (50), there are actually four combinations (\mp, \pm) of sign choices for $\log|A_+|$ and $\log|B_-|$. However the choices $(+, +)$ and $(-, -)$ do not lead to any relations among the coefficients and gives us an unphysical Hamiltonian. The choices $(+, -)$ and $(-, +)$ lead to identical physical results once the signs of the momenta p_i and the coefficient C_{χ} are reversed.

The condition (44) leads to

$$\frac{A_0}{B_0} = -\frac{\kappa\sqrt{p_1^2 + m_1^2} - 2K_0 - 2K_+}{\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+} e^{-K_0 z_1} , \quad (52)$$

and

$$\frac{A_0}{B_0} = -\frac{\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-}{\kappa\sqrt{p_2^2 + m_2^2} - 2K_0 - 2K_-} e^{-K_0 z_2} . \quad (53)$$

From (52) and (53) we obtain

$$\begin{aligned} & \left(\kappa\sqrt{p_1^2 + m_1^2} - 2K_0 - 2K_+ \right) \left(\kappa\sqrt{p_2^2 + m_2^2} - 2K_0 - 2K_- \right) \\ & = \left(\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+ \right) \left(\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_- \right) e^{K_0(z_1 - z_2)} , \end{aligned} \quad (54)$$

which we shall refer to as the determining equation for X . The Ψ fields in (\pm) regions are

$$\Psi_+(x) = -4\log|A_+| - 2K_+x, \quad (55)$$

$$\Psi_-(x) = -4\log|B_-| + 2K_-x,$$

and the Hamiltonian is

$$\begin{aligned} H &= -\frac{1}{\kappa} \int dx \Delta \Psi = -\frac{1}{\kappa} [\Psi']_{-\infty}^{\infty} \\ &= \frac{2(K_+ + K_-)}{\kappa}. \end{aligned} \quad (56)$$

Once the solution for X is obtained from (54), the Hamiltonian is explicitly determined from (56) in terms of the degrees of freedom of the system (i.e. the coordinates and momenta of the particles).

From (39a), (42b), (52) and (53), A_+ and B_- are expressed in terms of A_0 as

$$\begin{aligned} A_+ &= \frac{4K_0}{2K_0 + 2K_+ - \kappa\sqrt{p_1^2 + m_1^2}} e^{\frac{1}{2}(K_0 - K_+)z_1} A_0, \\ B_- &= \frac{4K_0}{2K_0 - 2K_- + \kappa\sqrt{p_2^2 + m_2^2}} e^{\frac{1}{2}(K_0 + K_-)z_2} A_0, \end{aligned} \quad (57)$$

and from (51) and (57) the coefficients A_0 and C_χ (and hence A_+ , B_- and B_0) are also determined

$$C_\chi = \frac{1}{2\kappa} \log \frac{(2K_0 + 2K_+ - \kappa\sqrt{p_1^2 + m_1^2})(\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+)}{(2K_0 + 2K_- - \kappa\sqrt{p_2^2 + m_2^2})(\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-)} + \frac{1}{2\kappa} (K_+z_1 + K_-z_2), \quad (58)$$

$$\log|A_+| = \frac{1}{2} \log \frac{2K_0 + 2K_- - \kappa\sqrt{p_2^2 + m_2^2}}{\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+} - \frac{1}{4} (K_0 + K_+ + \frac{\kappa\epsilon}{2} p_1) z_1 + \frac{1}{4} (K_0 - K_- - \frac{\kappa\epsilon}{2} p_2) z_2, \quad (59)$$

$$\log|B_-| = \frac{1}{2} \log \frac{2K_0 + 2K_+ - \kappa\sqrt{p_1^2 + m_1^2}}{\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-} - \frac{1}{4} (K_0 - K_+ + \frac{\kappa\epsilon}{2} p_1) z_1 + \frac{1}{4} (K_0 + K_- - \frac{\kappa\epsilon}{2} p_2) z_2, \quad (60)$$

$$\begin{aligned} \log|A_0| &= \frac{1}{2} \log \frac{(2K_0 + 2K_+ - \kappa\sqrt{p_1^2 + m_1^2})(\kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-)}{(4K_0)^2} + \frac{1}{4} (K_+ - K_0 - \frac{\kappa\epsilon}{2} p_1) z_1 \\ &\quad - \frac{1}{4} (K_0 + K_- + \frac{\kappa\epsilon}{2} p_2) z_2, \end{aligned} \quad (61)$$

$$\begin{aligned} \log|B_0| &= \frac{1}{2} \log \frac{(2K_0 + 2K_- - \kappa\sqrt{p_2^2 + m_2^2})(\kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+)}{(4K_0)^2} + \frac{1}{4} (K_0 + K_+ - \frac{\kappa\epsilon}{2} p_1) z_1 \\ &\quad - \frac{1}{4} (K_- - K_0 + \frac{\kappa\epsilon}{2} p_2) z_2. \end{aligned} \quad (62)$$

The parameters C_{\pm} are determined from (47) and (48). From (52), (53) and (57) it is evident that an overall common sign of A_+ , B_- , A_0 and B_0 has no physical meaning, and so we can choose all these coefficients to be positive.

The previous expressions are somewhat cumbersome. We can express them more compactly by making use of the following notation:

$$\begin{aligned}
K_1 &\equiv 2K_0 + 2K_- - \kappa\sqrt{p_2^2 + m_2^2}, \\
K_2 &\equiv 2K_0 + 2K_+ - \kappa\sqrt{p_1^2 + m_1^2}, \\
K_{01} &\equiv K_0 - K_+ + \frac{\kappa\epsilon}{2}p_1, \\
K_{02} &\equiv K_0 - K_- - \frac{\kappa\epsilon}{2}p_2, \\
\mathcal{M}_1 &\equiv \kappa\sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+, \\
\mathcal{M}_2 &\equiv \kappa\sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-, \\
Y_+ &\equiv \kappa\left[X + \frac{\epsilon}{4}(p_1 + p_2)\right], \\
Y_0 &\equiv \kappa\left[X - \frac{\epsilon}{4}(p_1 - p_2)\right], \\
Y_- &\equiv \kappa\left[X - \frac{\epsilon}{4}(p_1 + p_2)\right].
\end{aligned} \tag{63}$$

The coefficients A_+ , B_- , A_0 and B_0 can then be rewritten as

$$\begin{aligned}
A_+ &= \left(\frac{K_1}{\mathcal{M}_1}\right)^{1/2} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) - \frac{1}{2}K_+z_1}, \\
B_- &= \left(\frac{K_2}{\mathcal{M}_2}\right)^{1/2} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) + \frac{1}{2}K_-z_2}, \\
A_0 &= \frac{(K_2\mathcal{M}_2)^{1/2}}{4K_0} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) - \frac{1}{2}K_0z_2}, \\
B_0 &= \frac{(K_1\mathcal{M}_1)^{1/2}}{4K_0} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) + \frac{1}{2}K_0z_1},
\end{aligned} \tag{64}$$

and the solution for ϕ is then

$$\begin{aligned}
\phi_+ &= \left(\frac{K_1}{\mathcal{M}_1}\right)^{\frac{1}{2}} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) + \frac{1}{2}K_+(x-z_1)}, \\
\phi_0 &= \frac{1}{4K_0} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2)} \left\{ (K_1\mathcal{M}_1)^{1/2} e^{-\frac{1}{2}K_0(x-z_1)} + (K_2\mathcal{M}_2)^{1/2} e^{\frac{1}{2}K_0(x-z_2)} \right\}, \\
\phi_- &= \left(\frac{K_2}{\mathcal{M}_2}\right)^{\frac{1}{2}} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) - \frac{1}{2}K_-(x-z_2)}.
\end{aligned} \tag{65}$$

Repeating the analysis for $z_1 < z_2$ yields a similar solution with $p_i \rightarrow -p_i$. Hence the full solution is obtained from the preceding expressions by replacing p_i and $z_1 - z_2$

by $\tilde{p}_i = p_i \operatorname{sgn}(z_1 - z_2)$ and $|z_1 - z_2|$, respectively. The determining equation (54) of the Hamiltonian is then expressed as

$$K_1 K_2 = \mathcal{M}_1 \mathcal{M}_2 e^{K_0 |z_1 - z_2|} . \quad (66)$$

or

$$\begin{aligned} & \left(4K_0^2 + [\kappa\sqrt{p_1^2 + m_1^2} - 2K_+] [\kappa\sqrt{p_2^2 + m_2^2} - 2K_-] \right) \tanh\left(\frac{1}{2}K_0 |z_1 - z_2|\right) \\ & = -2K_0 \left([\kappa\sqrt{p_1^2 + m_1^2} - 2K_+] + [\kappa\sqrt{p_2^2 + m_2^2} - 2K_-] \right) , \end{aligned} \quad (67)$$

where the momentum p_i is replaced by \tilde{p}_i .

For the expression (28) to have a definite meaning as the Hamiltonian, K_{\pm} should be real. This imposes the restriction $H^2 + 8\Lambda/\kappa^2 > 16(p_1 + p_2)^2$. However K_0 need not be real. If Λ takes a sufficiently large positive value K_0 will be imaginary and the above analysis must be repeated. In the (0) region the soliton to the ϕ equation (30) becomes

$$\phi_0(x) = A_s \sin\frac{1}{2}\tilde{K}_0 x + A_c \cos\frac{1}{2}\tilde{K}_0 x , \quad (68)$$

where

$$\begin{aligned} \tilde{K}_0 & = -iK_0 \\ & = \sqrt{\frac{1}{2}\Lambda - \kappa^2 \left(X - \frac{\epsilon}{4}(\tilde{p}_1 - \tilde{p}_2) \right)^2} . \end{aligned} \quad (69)$$

Under the same matching conditions (36a-36d) and the boundary condition (43) we get, instead of (67), a new determining equation for the Hamiltonian

$$\begin{aligned} & \left(4\tilde{K}_0^2 - [\kappa\sqrt{p_1^2 + m_1^2} - 2K_+] [\kappa\sqrt{p_2^2 + m_2^2} - 2K_-] \right) \tan\left(\frac{1}{2}\tilde{K}_0 |z_1 - z_2|\right) \\ & = 2\tilde{K}_0 \left([\kappa\sqrt{p_1^2 + m_1^2} - 2K_+] + [\kappa\sqrt{p_2^2 + m_2^2} - 2K_-] \right) , \end{aligned} \quad (70)$$

which is just the equation derived from (67) by formally replacing K_0 with $i\tilde{K}_0$. Similarly, the solution for ϕ for imaginary K_0 is also identical with that obtained from (65) by the same replacement. Hence equation (67) is valid for all values of K_0 , and may be regarded as a transcendental equation which determines H as a function of the independent coordinates and momenta of the system.

We have previously shown that in the case of zero cosmological constant the solution for H can be expressed in terms of the Lambert W function. In this more general case with $\Lambda \neq 0$ the solution for H from (67) cannot be expressed in terms of known functions. Rather we must regard H as being implicitly determined in terms of the coordinates and momenta via (67).

Finally, the components of the metric may be computed from the equations (12), (13), (16) and (17) under the coordinate conditions (20). The derivation and the explicit solutions of the metric are given in Appendix A.

The canonical equations for the 2-body system can be derived by differentiating the determining equation (66) with respect to the variables z_i and p_i . For the variables p_1 and z_1 this yields

$$\begin{aligned}\dot{p}_1 &= -\frac{\partial H}{\partial z_1} = -\frac{2}{\kappa} \left(\frac{\partial K_+}{\partial z_1} + \frac{\partial K_-}{\partial z_1} \right) = -2 \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{\partial X}{\partial z_1} \\ &= -\frac{2}{\kappa} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1 K_2}{J},\end{aligned}\quad (71)$$

$$\begin{aligned}\dot{z}_1 &= \frac{\partial H}{\partial p_1} = \frac{2}{\kappa} \left(\frac{\partial K_+}{\partial p_1} + \frac{\partial K_-}{\partial p_1} \right) \\ &= \frac{\epsilon}{2} \left(\frac{Y_+}{K_+} - \frac{Y_-}{K_-} \right) + 2 \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{\partial X}{\partial p_1} \\ &= \epsilon \frac{Y_+}{K_+} + \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1}{\mathcal{M}_1} \left\{ \frac{p_1}{\sqrt{p_1^2 + m_1^2}} - \epsilon \frac{Y_+}{K_+} \right\},\end{aligned}\quad (72)$$

where

$$\begin{aligned}J &= 2 \left\{ \left(\frac{Y_0}{K_0} + \frac{Y_+}{K_+} \right) K_1 + \left(\frac{Y_0}{K_0} + \frac{Y_-}{K_-} \right) K_2 \right\} \\ &\quad - 2 \left\{ \left(\frac{Y_0}{K_0} - \frac{Y_+}{K_+} \right) \frac{1}{\mathcal{M}_1} + \left(\frac{Y_0}{K_0} - \frac{Y_-}{K_-} \right) \frac{1}{\mathcal{M}_2} \right\} K_1 K_2 - \frac{Y_0}{K_0} K_1 K_2 (z_1 - z_2).\end{aligned}\quad (73)$$

Similarly, for particle 2 the equations are

$$\dot{p}_2 = \frac{2}{\kappa} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1 K_2}{J},\quad (74)$$

$$\dot{z}_2 = -\epsilon \frac{Y_-}{K_-} + \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_2}{\mathcal{M}_2} \left\{ \frac{p_2}{\sqrt{p_2^2 + m_2^2}} + \epsilon \frac{Y_-}{K_-} \right\}.\quad (75)$$

It is straightforward to show that these canonical equations guarantee the conservation of the Hamiltonian (i.e. $\dot{H} = 0$) and the total momentum $p_1 + p_2$ (i.e. $\dot{p}_1 + \dot{p}_2 = 0$).

Alternatively, the equations of motion (18) and (19) derived from the action (9) become

$$\dot{p}_a = -\frac{\partial N_0}{\partial z_a} \sqrt{p_a^2 + m_a^2} + \frac{\partial N_1}{\partial z_a} p_a,\quad (76)$$

$$\dot{z}_a = N_0 \frac{p_a}{\sqrt{p_a^2 + m_a^2}} - N_1,\quad (77)$$

under the coordinate conditions (20). Insertion of the solutions of the metric components given in the Appendix A into (76) and (77) reproduces the canonical equations of motion (71), (72), (74) and (75) when the partial derivatives at z_1, z_2 are defined by

$$\frac{\partial N_{0,1}}{\partial z_i} \equiv \frac{1}{2} \left\{ \frac{\partial N_{0,1}}{\partial x} \Big|_{x=z_i+0} + \frac{\partial N_{0,1}}{\partial x} \Big|_{x=z_i-0} \right\}.\quad (78)$$

Thus consistency between the geodesic equations and the canonical equations of motion is explicitly verified, while the formal proof of the consistency in the case of $\Lambda = 0$ [7] can be easily generalized to the $\Lambda \neq 0$ case.

4 EXACT SOLUTIONS OF THE TRAJECTORIES IN THE $\Lambda = 0$ CASE

In a previous paper [8] we showed that in the $\Lambda = 0$ case the determining equation (66) can be solved explicitly and the Hamiltonian for the equal mass is expressed in the center of inertia (C.I.) system $p_1 = -p_2 = p$ as

$$H = \sqrt{p^2 + m^2} + \epsilon p \operatorname{sgn}(r) - 8 \frac{W \left[-\frac{\kappa}{8} (|r| \sqrt{p^2 + m^2} - \epsilon p r) \exp \left(\frac{\kappa}{8} (|r| \sqrt{p^2 + m^2} - \epsilon p r) \right) \right]}{\kappa |r|}, \quad (79)$$

where $W(x)$ is the Lambert W function defined via

$$y \cdot e^y = x \quad \Rightarrow \quad y = W(x). \quad (80)$$

The $W(x)$ has two real branches W_0 and W_{-1} for real x [11].

The Hamiltonian (79) is exact to infinite order in the gravitational constant and for arbitrary values of m and p . We can view the whole structure of the theory from the weak field to the strong field limits. By setting $H = H_0$ we can draw a phase space trajectory in (r, p) space. This phase space trajectory should be, as a matter of course, obtainable directly from the solution $r(t), p(t)$ to the canonical equations by eliminating the time variable t . Indeed, this can be verified by numerically solving the equations

$$\dot{p} = - \frac{\frac{\kappa}{4} (H - 2\epsilon\tilde{p})(H - \epsilon\tilde{p} - \sqrt{p^2 + m^2})}{2 - \frac{\kappa r}{4} (H - \epsilon\tilde{p} - \sqrt{p^2 + m^2})} \operatorname{sgn}(r), \quad (81)$$

$$\dot{r} = 2\epsilon \left\{ 1 - \frac{H - 2\epsilon\tilde{p}}{2 - \frac{\kappa r}{4} (H - \epsilon\tilde{p} - \sqrt{p^2 + m^2})} \cdot \frac{1}{\sqrt{p^2 + m^2}} \right\} \operatorname{sgn}(r), \quad (82)$$

which are obtained in the case of $\Lambda = 0$ from (71), (72), (74) and (75). However for certain values of the parameters, superficial singularities appear in $r(t)$ and $p(t)$ due to the zero points of the denominator $\{2 - \kappa r (H - \epsilon\tilde{p} - \sqrt{p^2 + m^2})/4\}$. These singularities correspond to $W(x) = -1$ representing the transit point between two branches W_0 and W_{-1} . In a spacetime description the singularities are coordinate singularities and are a consequence of t being a coordinate time.

We can deal with this problem by describing the trajectories of the particles in terms of some invariant parameter. The natural candidate is the proper time τ_a of each particle. From the metric components given in Appendix A and the canonical equations (77), the proper time is

$$\begin{aligned} d\tau_a^2 &= dt^2 \left\{ N_0(z_a)^2 - (N_1(z_a) + \dot{z}_a)^2 \right\} \\ &= dt^2 N_0(z_a)^2 \frac{m_a^2}{p_a^2 + m_a^2} \quad (a = 1, 2). \end{aligned} \quad (83)$$

For the equal mass case it is common for both particles

$$d\tau = d\tau_1 = d\tau_2 = \frac{(H - 2\epsilon\tilde{p})m}{\left\{ 2 - \frac{\kappa r}{4} (H - 2\epsilon\tilde{p}) \right\} (\sqrt{p^2 + m^2} - \epsilon\tilde{p}) \sqrt{p^2 + m^2}} dt, \quad (84)$$

and the canonical equations (81) and (82) become

$$\frac{dp}{d\tau} = -\frac{\kappa}{4m}\sqrt{p^2 + m^2} \left\{ H(\sqrt{p^2 + m^2} - \epsilon\tilde{p}) - m^2 \right\} \text{sgn}(r), \quad (85)$$

$$\frac{dr}{d\tau} = \frac{2\epsilon}{m}(\sqrt{p^2 + m^2} - \epsilon\tilde{p}) \left\{ \frac{\left[2 - \frac{\kappa r}{4}(H - \epsilon\tilde{p} - \sqrt{p^2 + m^2}) \right] \sqrt{p^2 + m^2}}{H - 2\epsilon\tilde{p}} - 1 \right\} \text{sgn}(r). \quad (86)$$

Remarkably, the equations (85) and (86) have an exact solution. We can obtain it in the following way. First we solve Eq.(85) for $p(\tau)$. From this $r(\tau)$ can be extracted by directly solving (86) after substituting the solution for p or by solving (66) for r . This yields an exact expression for the proper separation r of the two bodies as a function of their mutual proper time. Note that since $\gamma = 1$, at a fixed time $dt = 0$ (and hence a fixed $d\tau = 0$), the separation $r = z_1 - z_2$ is the proper distance between the two particles.

In the $r > 0$ region the solution is

$$p(\tau) = \frac{\epsilon m}{2} \left(f_0(\tau) - \frac{1}{f_0(\tau)} \right), \quad (87)$$

$$r(\tau) = \frac{16}{\kappa \left\{ H - m \left(f_0 - \frac{1}{f_0} \right) \right\}} \tanh^{-1} \left[\frac{H - m \left(f_0 + \frac{1}{f_0} \right)}{H - m \left(f_0 - \frac{1}{f_0} \right)} \right], \quad (88)$$

with

$$f_0(\tau) = \frac{H}{m} \left[1 - \frac{\sqrt{p_0^2 + m^2} - \epsilon p_0 - \frac{m^2}{H}}{\sqrt{p_0^2 + m^2} - \epsilon p_0} e^{\frac{\epsilon \kappa m}{4}(\tau - \tau_0)} \right], \quad (89)$$

where p_0 is the initial momentum at $\tau = \tau_0$. In the $r < 0$ region the solution is

$$p(\tau) = -\frac{\epsilon m}{2} \left(\bar{f}_0(\tau) - \frac{1}{\bar{f}_0(\tau)} \right), \quad (90)$$

$$r(\tau) = \frac{-16}{\kappa \left\{ H - m \left(\bar{f}_0 - \frac{1}{\bar{f}_0} \right) \right\}} \tanh^{-1} \left[\frac{H - m \left(\bar{f}_0 + \frac{1}{\bar{f}_0} \right)}{H - m \left(\bar{f}_0 - \frac{1}{\bar{f}_0} \right)} \right], \quad (91)$$

with

$$\bar{f}_0(\tau) = \frac{H}{m} \left[1 - \frac{\sqrt{p_0^2 + m^2} + \epsilon p_0 - \frac{m^2}{H}}{\sqrt{p_0^2 + m^2} + \epsilon p_0} e^{\frac{\epsilon \kappa m}{4}(\tau - \tau_0)} \right]. \quad (92)$$

In Fig.1 and 2 we show the typical plots of $r(\tau)$ and phase space trajectories for various values of H_0 with $m = 0.5$. (In all plots in this paper we choose $\kappa = 1$.) In Fig.1 it is seen that as H_0 increases, not only the amplitude and the period of the bounded motion become large, but also the shape of $r(\tau)$ deforms from the parabolic (Newtonian) shape. This deformation is also present in the phase space trajectories shown in Fig.2. At higher energy the trajectories become more and more S shaped. This is due to the fact that the trajectory smoothly moves over the two branches W_0 and W_{-1} [8].

We see in Fig.3 the striking distinction between the separations of the particles in the non-relativistic and relativistic cases once the value of the energy becomes large relative to

the mass (here $H_0 = 25$ and $m = 0.5$). The maximal separation of the particles is much smaller than in its Newtonian counterpart (the dashed curve), and is achieved far more quickly. After maximal separation, the particles move toward each other at a slower velocity until they are very close together. At this point (less than 10% of their maximal separation), they accelerate toward the same point, after which the motion repeats with the particles interchanged.

Fig.1

The exact r vs τ curves in the case of $\Lambda = 0$ for $m = 0.5$ and four different values of H_0 .

Fig.2

Phase space trajectories corresponding to the $r(\tau)$ curves in Fig.1.

Fig.3

The exact r vs τ curve for $H_0 = 25$ with $m = 0.5$ and the Newtonian curve for the same H_0

5 EXACT SOLUTIONS OF THE TRAJECTORIES FOR THE EQUAL MASS IN THE $\Lambda \neq 0$ CASE

5.1 General Discussion

In this section we consider a system of two particles with equal mass for the $\Lambda \neq 0$ case. In the C.I. system, depending upon the sign of $(\sqrt{H^2 + 8\Lambda/\kappa^2} - 2\epsilon\tilde{p})^2 - 8\Lambda/\kappa^2$ the determining equations (67) and (70) become

$$(\mathcal{J}_\Lambda^2 + B^2) \tanh\left(\frac{\kappa}{8}\mathcal{J}_\Lambda |r|\right) = 2\mathcal{J}_\Lambda B, \quad (93)$$

and

$$(\tilde{\mathcal{J}}_\Lambda^2 - B^2) \tan\left(\frac{\kappa}{8}\tilde{\mathcal{J}}_\Lambda |r|\right) = -2\tilde{\mathcal{J}}_\Lambda B, \quad (94)$$

respectively, where

$$\begin{aligned}\mathcal{J}_\Lambda &= \sqrt{\left(\sqrt{H^2 + \frac{8\Lambda}{\kappa^2}} - 2\epsilon\tilde{p}\right)^2 - \frac{8\Lambda}{\kappa^2}}, \\ \tilde{\mathcal{J}}_\Lambda &= \sqrt{\frac{8\Lambda}{\kappa^2} - \left(\sqrt{H^2 + \frac{8\Lambda}{\kappa^2}} - 2\epsilon\tilde{p}\right)^2}, \\ B &= H - 2\sqrt{p^2 + m^2}.\end{aligned}\tag{95}$$

Equation (93) may be further divided into two types:

$$\tanh\left(\frac{\kappa}{16}\mathcal{J}_\Lambda|r|\right) = \frac{B}{\mathcal{J}_\Lambda} \quad (\text{tanh-type A}),\tag{96}$$

or

$$\tanh\left(\frac{\kappa}{16}\mathcal{J}_\Lambda|r|\right) = \frac{\mathcal{J}_\Lambda}{B} \quad (\text{tanh-type B}).\tag{97}$$

When $\Lambda = 0$ the tanh-type B equation is excluded because \mathcal{J}_Λ/B exceeds 1. When a cosmological constant is introduced, this equation has solutions in some restricted range of the parameters.

Likewise, eq.(94) may also be divided into

$$\tan\left(\frac{\kappa}{16}\tilde{\mathcal{J}}_\Lambda|r|\right) = -\frac{B}{\tilde{\mathcal{J}}_\Lambda} \quad (\text{tan-type A}),\tag{98}$$

or

$$\tan\left(\frac{\kappa}{16}\tilde{\mathcal{J}}_\Lambda|r|\right) = \frac{\tilde{\mathcal{J}}_\Lambda}{B} \quad (\text{tan-type B}).\tag{99}$$

For all of these four types of determining equations the canonical equations of motion are identical:

$$\dot{p} = -\frac{\kappa\mathcal{J}_\Lambda^2(\mathcal{J}_\Lambda^2 - B^2)}{16C} \text{sgn}(r),\tag{100}$$

$$\dot{r} = 2\epsilon\sqrt{1 + \frac{8\Lambda}{\kappa^2 H^2}} \left(1 - \frac{\mathcal{J}_\Lambda^2}{C}\right) \text{sgn}(r) + \frac{2\mathcal{J}_\Lambda^2}{C} \frac{p}{\sqrt{p^2 + m^2}}.\tag{101}$$

where

$$C = \frac{1}{\sqrt{1 + \frac{8\Lambda}{\kappa^2 H^2}}} \left\{ \sqrt{1 + \frac{8\Lambda}{\kappa^2 H^2}} \mathcal{J}_\Lambda^2 - \left(\sqrt{H^2 + \frac{8\Lambda}{\kappa^2}} - 2\epsilon\tilde{p}\right) \left(B + \frac{\kappa}{16}(\mathcal{J}_\Lambda^2 - B^2) r\right) \right\}.\tag{102}$$

5.2 Phase-space Trajectories

For a given value of Λ the equations (93) or (94) describe the surface in (r, p, H) space of all allowed phase-space trajectories, from which the (r, p) trajectory is obtained by setting $H = H_0$.

Fig.4 shows phase-space plots for a small $H_0 = 2.2m$ and three different values of Λ under identical initial conditions. First we note that the trajectories of the relativistic motion (solid curve) are slightly distorted compared to the Newtonian motion (dashed curve). Second, a trajectory with $\Lambda > 0$ (dash-dot curve) is expanded, reflecting the repulsive effect of the cosmological constant, while a trajectory of $\Lambda < 0$ (dotted curve) shrinks, due to the additional attractive effect. The relativistic plots in Fig.4 correspond to the choice of $\epsilon = 1$. Plots for $\epsilon = -1$ (the time-reversed solutions) are obtained by reflection in the $p = 0$ axis. The phase space plots for a large $H_0 = 8m$ are shown in Fig.5. The trajectories for $\Lambda \geq 0$ become extremely S -shaped, while for $\Lambda < 0$ the trajectory is still a distorted oval due to the attractive effect of Λ . The effects of the cosmological constant ($\Lambda > 0$ repulsive; $\Lambda < 0$ attractive) are precisely analyzed in terms of the exact $r(\tau)$ plots in the next subsection.

Fig.4

Non-relativistic (Newtonian) and relativistic trajectories for $m = 0.5, H_0 = 1.1$ and three different values of Λ . The undistorted oval (which is really a pair of parabolas intersecting at the $r = 0$ axis) is the Newtonian trajectory.

Fig.5

Relativistic trajectories for $m = 0.5, H_0 = 4$ and three different values of Λ .

5.3 Explicit Solutions

The phase-space trajectories discussed in the previous subsection can be obtained from the solution to the canonical equations (100) and (101). For the equal mass case there is a common proper time for both particles

$$d\tau = d\tau_1 = d\tau_2 = \frac{m}{\sqrt{p^2 + m^2}} \frac{\mathcal{J}_\Lambda^2}{C} dt, \quad (103)$$

via which the canonical equations (100) and (101) may be expressed in the form

$$\frac{dp}{d\tau} = - \frac{\kappa \sqrt{p^2 + m^2} (\mathcal{J}_\Lambda^2 - B^2)}{16m} \operatorname{sgn}(r), \quad (104)$$

$$\frac{dr}{d\tau} = \frac{2\epsilon}{m} \left\{ \sqrt{1 + \frac{8\Lambda}{\kappa^2 H^2} \left(\frac{C}{\mathcal{J}_\Lambda^2} - 1 \right)} \sqrt{p^2 + m^2} + \epsilon \tilde{p} \right\} \operatorname{sgn}(r). \quad (105)$$

First we solve Eq.(104) for p and then (96)-(99) for r . In the $r > 0$ region Eq.(104) leads to

$$\begin{aligned} \int_{p_0}^p \frac{dp}{\left\{ \sqrt{p^2 + m^2} - \epsilon \sqrt{1 + \frac{8\Lambda}{\kappa^2 H^2}} p - \frac{m^2}{H} \right\} \sqrt{p^2 + m^2}} &= -\frac{\kappa H}{4m} \int_{\tau_0}^{\tau} d\tau \\ &= -\frac{\kappa H}{4m} (\tau - \tau_0), \end{aligned} \quad (106)$$

provided the condition

$$1 + \frac{8\Lambda}{\kappa^2 H^2} \geq 0 \quad (107)$$

is satisfied. Hence for $\Lambda < 0$ the motion is allowed as long as H satisfies

$$H \geq \sqrt{-\frac{8\Lambda}{\kappa^2}}. \quad (108)$$

We perform the integration of the LHS of (106) for three separate cases which depend on the value of Λ relative to m and H . The solution $p(\tau)$ is

$$p(\tau) = \frac{\epsilon m}{2} \left(f(\tau) - \frac{1}{f(\tau)} \right), \quad (109)$$

with

$$f(\tau) = \begin{cases} \frac{\frac{H}{m}(1+\sqrt{\gamma_H}) \left\{ 1 - \eta e^{\frac{\epsilon \kappa m}{4} \sqrt{\gamma_m}(\tau - \tau_0)} \right\}}{1 + \sqrt{\gamma_m} + (\sqrt{\gamma_m} - 1) \eta e^{\frac{\epsilon \kappa m}{4} \sqrt{\gamma_m}(\tau - \tau_0)}} & \gamma_m > 0, \\ \frac{1 + \sqrt{\gamma_H}}{\frac{m}{H} + \frac{\sigma}{m - \sigma \frac{\epsilon \kappa H}{8}}(\tau - \tau_0)} & \gamma_m = 0, \\ \frac{\frac{H}{m}(1 + \sqrt{\gamma_H})}{1 + \sqrt{-\gamma_m} \frac{\sigma + \frac{m^2}{H} \sqrt{-\gamma_m} \tan \left[\frac{\epsilon \kappa m}{8} \sqrt{-\gamma_m}(\tau - \tau_0) \right]}{\frac{m^2}{H} \sqrt{-\gamma_m} - \sigma \tan \left[\frac{\epsilon \kappa m}{8} \sqrt{-\gamma_m}(\tau - \tau_0) \right]}} & \gamma_m < 0, \end{cases} \quad (110)$$

where

$$\begin{aligned} \gamma_H &= 1 + \frac{8\Lambda}{\kappa^2 H^2}, & \gamma_m &= 1 + \frac{8\Lambda}{\kappa^2 m^2}, \\ \eta &= \frac{\sigma - \frac{m^2}{H} \sqrt{\gamma_m}}{\sigma + \frac{m^2}{H} \sqrt{\gamma_m}}, & \sigma &= (1 + \sqrt{\gamma_H}) (\sqrt{p_0^2 + m^2} - \epsilon p_0) - \frac{m^2}{H}, \end{aligned} \quad (111)$$

with p_0 being the initial momentum at $\tau = \tau_0$.

Similarly the solution in $r < 0$ region is

$$p(\tau) = -\frac{\epsilon m}{2} \left(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)} \right), \quad (112)$$

with

$$\bar{f}(\tau) = \begin{cases} \frac{\frac{H}{m}(1+\sqrt{\gamma_H}) \left\{ 1 - \bar{\eta} e^{\frac{\epsilon \kappa m}{4} \sqrt{\gamma_m}(\tau - \tau_0)} \right\}}{1 + \sqrt{\gamma_m} + (\sqrt{\gamma_m} - 1) \bar{\eta} e^{\frac{\epsilon \kappa m}{4} \sqrt{\gamma_m}(\tau - \tau_0)}} & \gamma_m > 0, \\ \frac{1 + \sqrt{\gamma_H}}{\frac{m}{H} + \frac{\bar{\sigma}}{m - \bar{\sigma} \frac{\epsilon \kappa H}{8}}(\tau - \tau_0)} & \gamma_m = 0, \\ \frac{\frac{H}{m}(1 + \sqrt{\gamma_H})}{1 + \sqrt{-\gamma_m} \frac{\bar{\sigma} + \frac{m^2}{H} \sqrt{-\gamma_m} \tan \left[\frac{\epsilon \kappa m}{8} \sqrt{-\gamma_m}(\tau - \tau_0) \right]}{\frac{m^2}{H} \sqrt{-\gamma_m} - \bar{\sigma} \tan \left[\frac{\epsilon \kappa m}{8} \sqrt{-\gamma_m}(\tau - \tau_0) \right]}} & \gamma_m < 0, \end{cases} \quad (113)$$

where

$$\bar{\sigma} = (1 + \sqrt{\gamma_H})(\sqrt{p_0^2 + m^2} + \epsilon p_0) - \frac{m^2}{H}, \quad \bar{\eta} = \frac{\bar{\sigma} - \frac{m^2}{H}\sqrt{\gamma_m}}{\bar{\sigma} + \frac{m^2}{H}\sqrt{\gamma_m}}. \quad (114)$$

The solution for $r(\tau)$ for each of the determining equations (96)- (99) is obtained as follows

tanh-type A:

$$r(\tau) = \begin{cases} \frac{16}{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2 - 8\Lambda}} \tanh^{-1} \left[\frac{\kappa(H-m|f(\tau) + \frac{1}{f(\tau)}|)}{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2 - 8\Lambda}} \right] & r > 0, \\ \frac{-16}{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2 - 8\Lambda}} \tanh^{-1} \left[\frac{\kappa(H-m|\bar{f}(\tau) + \frac{1}{\bar{f}(\tau)}|)}{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2 - 8\Lambda}} \right] & r < 0, \end{cases} \quad (115)$$

tanh-type B:

$$r(\tau) = \begin{cases} \frac{16}{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2 - 8\Lambda}} \tanh^{-1} \left[\frac{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2 - 8\Lambda}}{\kappa(H-m|f(\tau) + \frac{1}{f(\tau)}|)} \right] & r > 0, \\ \frac{-16}{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2 - 8\Lambda}} \tanh^{-1} \left[\frac{\sqrt{(\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2 - 8\Lambda}}{\kappa(H-m|\bar{f}(\tau) + \frac{1}{\bar{f}(\tau)}|)} \right] & r < 0, \end{cases} \quad (116)$$

tan-type A:

$$r(\tau) = \begin{cases} \frac{16}{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2}} \left(\tan^{-1} \left[\frac{\kappa(m|f(\tau) + \frac{1}{f(\tau)}| - H)}{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2}} \right] + n\pi \right) & r > 0, \\ \frac{-16}{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2}} \left(\tan^{-1} \left[\frac{\kappa(m|\bar{f}(\tau) + \frac{1}{\bar{f}(\tau)}| - H)}{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2}} \right] + n\pi \right) & r < 0, \end{cases} \quad (117)$$

tan-type B:

$$r(\tau) = \begin{cases} \frac{16}{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2}} \left(\tan^{-1} \left[\frac{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(f(\tau) - \frac{1}{f(\tau)}))^2}}{\kappa(H-m|f(\tau) + \frac{1}{f(\tau)}|)} \right] + n\pi \right) & r > 0, \\ \frac{-16}{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2}} \left(\tan^{-1} \left[\frac{\sqrt{8\Lambda - (\sqrt{\kappa^2 H^2 + 8\Lambda} - m\kappa(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)}))^2}}{\kappa(H-m|\bar{f}(\tau) + \frac{1}{\bar{f}(\tau)}|)} \right] + n\pi \right) & r < 0. \end{cases} \quad (118)$$

The exact $r(\tau)$ solutions corresponding to the phase space trajectories in Fig.4 are given by tanh-type A solution (115) and are plotted in Fig.6. The motions are bounded and

periodic. Comparison of three curves in Fig.6 indicates that a negative cosmological constant $\Lambda < 0$ acts effectively as an attractive force: for the same value of H_0 , the particles do not achieve as wide a proper separation, and the frequency of oscillation is more rapid. As well, a positive $\Lambda > 0$ acts as a repulsive force: the frequency of oscillation decreases and the particles achieve a wider proper separation.

Fig.6

The exact r vs τ curves corresponding to the phase space trajectories in Fig.4.

The period T for the bounded motion is obtained from tanh-type A solution with the condition $r = 0$ and $p = \pm p_0$:

$$T = \begin{cases} \frac{16}{\sqrt{\kappa^2 m^2 + 8\Lambda}} \tanh^{-1} \left(\frac{\sqrt{\kappa^2 m^2 + 8\Lambda} \sqrt{H^2 - 4m^2}}{\kappa H m} \right) & \gamma_m > 0, \\ \frac{16\sqrt{H^2 - 4m^2}}{\kappa H m} & \gamma_m = 0, \\ \frac{16}{\sqrt{-\kappa^2 m^2 - 8\Lambda}} \tan^{-1} \left(\frac{\sqrt{-\kappa^2 m^2 - 8\Lambda} \sqrt{H^2 - 4m^2}}{\kappa H m} \right) & \gamma_m < 0. \end{cases} \quad (119)$$

In figure 7 we plot $r(\tau)$ for fixed $\Lambda = -1.5$ and $H_0 = 16$ for several different values of m . Though the attractive effect of a negative Λ is common in all cases, a special (and rather surprising) situation arises. As the motion becomes more relativistic (i.e. m gets smaller) we find that a second maximum develops in the curve (see $m = 0.05$ curve). The description of the motion is as follows. The two particles start at $r = 0$ depart in opposite directions, reaching a maximum separation. They then go back toward one another for a certain period of proper time. However at some point they each reverse direction, reaching a second maximal separation. They then reverse direction again, finally returning to their starting point where the motion then repeats itself.

As the mass becomes very small, the second maximum prevails. This peculiar behavior [9] is due to a subtle interplay between the gravitational attraction, cosmological constant and relativistic motion of the particles. To our knowledge it has never been previously observed. The changes of the peaks are clearly grasped in the phase space trajectories in Fig.8. (In Fig.7 the first maximum of $m = 0.001$ curve could not be drawn due to plotting precision.)

Fig.7

*A sequence of equal mass curves for $\Lambda = -1.5$ and $H_0 = 16$.
Note the presence of the second maximum for $m = 0.05$.*

Fig.8

Change of peaks for $\Lambda = -1.5$ and $H_0 = 16$ as m gets smaller.

As the negative value of Λ approaches its lower bound of $-\kappa^2 H^2/8$, the form of the phase space trajectories transforms from an S -shaped curve to a double peaked one and then to a diamond shape. Figure 9 illustrates these characteristics for the case of $m = 0.5$ and $H_0 = 100$, in which a double peak structure appears for $\Lambda = -70$. For each trajectory in Fig.9 the corresponding $r(\tau)$ plot is shown in Fig.10.

Fig.9

Phase space trajectories for $m = 0.5, H_0 = 100$ and different values of negative Λ .

Fig.10

The r vs τ curves corresponding the phase space trajectories in Fig.9.

The double-peak structure shown in Figs.7-10 is a consequence of having a momentum-dependent potential. We can gain some insight into this behaviour by computing a perturbative Hamiltonian. The structure of the determining equations (93) and (94) suggests that we can carry out a 2-parameter expansion of the Hamiltonian in terms of κ and Λ/κ^2 . To the third order the result is

$$\begin{aligned}
 H &= H_0(p, r) + H_\Lambda(p, r) \\
 &= 2\sqrt{p^2 + m^2} + \frac{\kappa}{4}(\sqrt{p^2 + m^2} - \epsilon\tilde{p})^2 |r| + \frac{\kappa^2}{4^2}(\sqrt{p^2 + m^2} - \epsilon\tilde{p})^3 r^2 \\
 &\quad + \frac{7\kappa^3}{6 \times 4^3}(\sqrt{p^2 + m^2} - \epsilon\tilde{p})^4 |r|^3 - \frac{\Lambda}{2\kappa} \cdot \frac{\epsilon\tilde{p}}{\sqrt{p^2 + m^2}} |r| - \frac{\Lambda}{16} \cdot \frac{\epsilon\tilde{p} m^2}{p^2 + m^2} r^2 \\
 &\quad + \frac{\Lambda^2}{4\kappa^3} \cdot \frac{\epsilon\tilde{p}}{(p^2 + m^2)^{3/2}} |r| + \dots \quad .
 \end{aligned} \tag{120}$$

It is straightforward to show that the terms in H_Λ have the form

$$\frac{p^{s_1}}{(\sqrt{p^2 + m^2})^{s_2}} |r|^{s_3} \quad , \tag{121}$$

to arbitrary order in Λ/κ^2 , where the s_i are positive integers and $s_2 \geq s_1$. One of the characteristics common to all such terms is that they vanish as $p \rightarrow 0$, since in this case the determining equation of the Hamiltonian becomes

$$\tanh\left(\frac{\kappa}{16}H|r|\right) = \frac{H - 2m}{H}. \quad (122)$$

which is Λ -independent. Another important characteristic is that they have a single maximum at $p^2 = \frac{s_1 m^2}{s_2 - s_1}$ if $s_2 > s_1$ in the case of $\Lambda < 0$.

Consider the situation depicted in Figs.7-10, that of two particles initially at the origin, each having initial momentum p_0 , with $\Lambda < 0$. In this case $H_0 = 2\sqrt{p^2 + m^2}$, $H_\Lambda = 0$, and the particles initially move apart as though they were free. From (109) the momentum is a monotonically decreasing function of time. For p_0/m sufficiently small, the particles will execute a motion which is a perturbation of that described in section 4 since the momentum never becomes large enough to cross the maximum in H_Λ . However if p_0/m is sufficiently large, the terms in H_Λ will grow as p decreases, and the Λ -dependent part of the potential will continue to increase. The terms in H_0 will decrease as p decreases, even though r is increasing. Eventually a maximum in r is reached, after which both r and p are decreasing. In the generic case r will continue to decrease toward zero. However a second extremum will appear if the H_Λ terms get too small too rapidly before $p = 0$, which can happen for Λ within a certain range. Since $H = H_0 + H_\Lambda$ is a constant of the motion, the only way to preserve the constancy of H will be for r to increase again. Essentially the particles are repelled due to their kinetic energy within this range of Λ . Of course r cannot increase too much, because p continues to get small – eventually r must reach a 2nd maximum, and then turns around again until $p = 0$, after which the motion continues to $r = 0$ whereupon the particles interchange roles.

We can see this effect in a simple non-relativistic model with

$$\hat{H} = \hat{p}^2/m + Km \frac{\hat{p}}{1 + \hat{p}^2/m^2} |\hat{r}| = m \left(p^2 + K \frac{p}{1 + p^2} |r| \right), \quad (123)$$

where in the latter equation p and r have been rescaled in units of m . The potential has been chosen so that $s_2 = 1 + s_1 = 2$ in terms of (121) above. Since $\hat{H} = mh$ is a constant of the motion, we can write

$$|r| = \frac{(-p^2 + h)(1 + p^2)}{Kp}, \quad (124)$$

which has extrema at $p = p_\pm \equiv \frac{1}{6}\sqrt{-6 + 6h \pm 6\sqrt{1 - 14h + h^2}}$. For $p_0^2 = h < 7 + 4\sqrt{3}$, the extreme values of $|r|$ are not real, and so there will be no double peak structure in the phase-space trajectory. However for $p_0^2 = h > 7 + 4\sqrt{3}$ two extrema appear, and the particle gets a bounce. This motion is viewed qualitatively in a potential diagram shown in Fig.11. As the momentum decreases from an initial value p_0 to zero, the potential curve (which is linear in $|r|$) changes from A to $B \rightarrow \dots \rightarrow E \rightarrow F \rightarrow E \rightarrow \dots \rightarrow B \rightarrow A \rightarrow G \rightarrow I \rightarrow \dots$. Accordingly the particle moves from points 0 to 10 in numerical order.

The equations of motion for the Hamiltonian (123) are easily solved, yielding

$$p(t) = \sqrt{W(h e^{h-2Kt})}, \quad r(t) = \frac{\{h - W(h e^{h-2Kt})\} \{1 + W(h e^{h-2Kt})\}}{K \sqrt{W(h e^{h-2Kt})}}, \quad (125)$$

where $W(x)$ is the Lambert-W function (80). The function $p(t)$ is monotonically decreasing. Provided $h > 7 + 4\sqrt{3}$ the particle gets a bounce at $r = r(p_-)$ before moving out to infinity, as illustrated in Figs. 12 and 13 for the case of $h = (7 + 4\sqrt{3}) + 20$. In these figures the numbers on the curves denote the corresponding numbers in the potential diagram of Fig.11.

Fig.11

A schematic view of the motion with a large p_0 in a potential diagram.

Fig.12

*The $p(t)$ and $r(t)$ curves in the non-relativistic model for $h = (7 + 4\sqrt{3}) + 20$.
A bounce occurs at $t = t_- = 18.08$.*

Fig.13

The phase space trajectory in the non-relativistic model for $h = (7 + 4\sqrt{3}) + 20$.

The expansion (120) has similar features, except that there is an additional gravitational and cosmological attraction which prevents the separation from diverging. To order Λ the potential is a sum of two terms of the form in (121), with $s_2 = s_1 = 1$ in the first term and $s_2 = 1 + s_1 = 2$ in the second term. The latter provides the bounce effect described above, but is always overwhelmed by the first term for small $|r|$. However the Λ^2 term has $s_2 = 2 + s_1 = 3$ and is a pure bounce term. Hence there exists a range of Λ which can provide a bounce. Fig.14 shows the phase space trajectories in $r > 0$ region for $\Lambda = -0.3$, $H_0 = 5$ and two different masses ($m = 0.2, 0.7$), in which the dotted curves represent the motions to the first order of Λ in the Hamiltonian (120) and the solid curves do the motions corresponding to the second order of Λ . We see that to order Λ , as the motion becomes relativistic (m gets smaller) the trajectory simply expands and changes from S -shape to a diamond shape. When the Λ^2 term is included, the double peak structure (a solid curve with $m = 0.2$) appears.

Fig.14

*Phase space trajectories for the perturbative Hamiltonian
for $\Lambda = -0.3, H_0 = 5$ and $m = 0.2, 0.7$.*

This perturbative analysis indicates that the bounce effect is a result of the negative cosmological constant inducing a momentum dependent potential with positive coefficients. For $\Lambda > 0$, odd powers of Λ/κ^2 are strongly repulsive, and suppress the attractive effects of even powers of Λ/κ^2 , eliminating the double peak structure.

More generally, a positive cosmological constant acts effectively as a repulsive force. Figure 15 shows $r(\tau)$ plots for fixed $\Lambda = 1.5, H_0 = 16$ and several different values of m . The motion becomes unbounded between $m = 4.72$ and $m = 4.73$. The $r(\tau)$ plots in Fig.16 are for fixed $m = 0.5, H_0 = 16$ and different values of Λ , showing also the transition from bounded to unbounded motion. This transition occurs at $\mathcal{J}_\Lambda = 0$ and the critical value of Λ is given by $\Lambda_c = \frac{\kappa^2 m^4}{2(H^2 - 4m^2)}$.

As $\Lambda \rightarrow \Lambda_c$ the particles rapidly separate, remaining nearly stationary for an increasingly large period of proper time before coming together again. At $\Lambda = \Lambda_c$ this separation time becomes infinite, and for $\Lambda > \Lambda_c$, the separation diverges at finite τ .

Fig.15

A sequence of curves of equal mass for $\Lambda = 1.5, H_0 = 16$.

Fig.16

A sequence of curves near $\Lambda_c = 20.008333$ for $m = 7, H_0 = 16$.

Though all the above solutions are derived from tanh-type equations (96) and (97), for a positive cosmological constant there exist also a countably infinite set of unbounded motions specified by tan-type A, B equations (98) and (99). Then for $0 < \Lambda < \Lambda_c$, both bounded and unbounded motions are realized for a fixed value of H , as shown in Fig.17. In the unbounded motion two particles simply approach one another at some minimal value of $|r|$ and then reverse direction toward infinity. In the trajectories the dotted curves come from tan-type A solution (117) and the dashed curves do from tan-type B solution (118). As Λ approaches Λ_c , two bulges of the solid curve (tanh-type A) and the dotted curve (tan-type A: n=0) come close and contact. When Λ exceeds Λ_c two curves switch to the unbounded trajectories as shown in the solid curves in Fig.18. The particles cross one another before receding toward infinity. The upper solid curve represents the motion in which p approaches the asymptotic values $p_\pm \equiv \frac{1}{2\kappa}(\pm\sqrt{\kappa^2 H^2 + 8\Lambda} + \sqrt{8\Lambda})$ as $r \rightarrow \pm\infty$.

As noted previously, one peculiar feature of this motion is that the two particles diverges to infinite separation at finite proper time. The time τ_∞ for $r \rightarrow \infty$ is

$$\tau_\infty = \frac{4}{\kappa m \sqrt{\gamma_m}} \log \left(\frac{H(1 + \sqrt{\gamma_H}) - (p_+ + \sqrt{p_+^2 + m^2})(1 + \sqrt{\gamma_m})}{\eta [H(1 + \sqrt{\gamma_H}) - (p_+ + \sqrt{p_+^2 + m^2})(\sqrt{\gamma_m} - 1)]} \right). \quad (126)$$

The lower solid curve represents the motion in reversed direction. For $\Lambda > \Lambda_c$ only unbounded motions are realized.

Fig.17

*Phase space trajectories of the bouned and the unbounded motions
for $\Lambda = 1, m = 1$ and $H_0 = 2.1$.*

Fig.18

*Phase space trajectories of the unbounded motions
for $\Lambda = 1.5, m = 1$ and $H_0 = 2.1$.*

We discuss in Appendix C the causal relationship between the two particles in the unbounded case.

6 THE UNEQUAL MASS CASE

For the unequal masses the proper time (83) of each particle is

$$\begin{aligned} d\tau_1 &= dt \frac{16YK_0K_1m_1}{JKM_1\sqrt{p^2 + m_1^2}}, \\ d\tau_2 &= dt \frac{16YK_0K_2m_2}{JKM_2\sqrt{p^2 + m_2^2}}, \end{aligned} \quad (127)$$

where $K \equiv K_+ = K_-$ and $Y \equiv Y_+ = Y_-$. In this situation, choosing the time coordinate to be the proper time of one particle introduces an asymmetry into the description of the motion. Instead we seek a time variable which is symmetric with respect to $1 \leftrightarrow 2$ and reduces to the proper time (103) when $m_1 = m_2$. From (127) we choose

$$d\tilde{\tau} \equiv dt \frac{16YK_0}{JK} \left(\frac{K_1K_2m_1m_2}{M_1M_2\sqrt{p^2 + m_1^2}\sqrt{p^2 + m_2^2}} \right)^{1/2}. \quad (128)$$

In terms of this variable the canonical equations are expressed as

$$\frac{dp}{d\tilde{\tau}} = -\frac{1}{4\kappa} \left(\frac{K_1 K_2 M_1 M_2 \sqrt{p^2 + m_1^2} \sqrt{p^2 + m_2^2}}{m_1 m_2} \right)^{1/2}, \quad (129)$$

$$\frac{dz_i}{d\tilde{\tau}} = (-1)^{i+1} \left(\frac{M_1 M_2 \sqrt{p^2 + m_1^2} \sqrt{p^2 + m_2^2}}{K_1 K_2 m_1 m_2} \right)^{1/2} \left\{ \frac{\epsilon J}{16K_0} + \frac{K_i}{M_i} \left(\frac{p}{\sqrt{p^2 + m_i^2}} - \epsilon \frac{Y}{K} \right) \right\}, \quad (130)$$

$$\begin{aligned} \frac{dr}{d\tilde{\tau}} = & \left(\frac{M_1 M_2 \sqrt{p^2 + m_1^2} \sqrt{p^2 + m_2^2}}{K_1 K_2 m_1 m_2} \right)^{1/2} \\ & \times \left\{ \frac{\epsilon J}{8K_0} + \frac{K_1}{M_1} \left(\frac{p}{\sqrt{p^2 + m_1^2}} - \epsilon \frac{Y}{K} \right) + \frac{K_2}{M_2} \left(\frac{p}{\sqrt{p^2 + m_2^2}} - \epsilon \frac{Y}{K} \right) \right\}. \quad (131) \end{aligned}$$

Note that r still describes the proper distance between the particles at any fixed instant.

Unlike the equal mass case, the integration $\int dp (K_1 K_2 M_1 M_2 \sqrt{p^2 + m_1^2} \sqrt{p^2 + m_2^2})^{-1/2}$ can not be performed within the framework of elementary calculus. Hence we solve (129) numerically.

In the case of a negative cosmological constant the $r(\tau)$ plots in Fig.19 show the trajectories for various mass ratios m_1/m_2 in the fixed $\Lambda = -1, m_2 = 1$ and $H_0 = 10$. Compared with the equal mass case $m_1 = m_2 = 1$, as the mass ratio gets larger, the gravitational attraction is stronger and the proper distance between two particles as well as the period become shorter. When the mass ratio gets a small value than unity, the gravity becomes weak. However, for quite a small mass ratio a strong attractive effect of the cosmological constant prevails and the period changes to become shorter. At the same time the double peak structure (the second maximum) appears and finally the first maximum fades out. These characteristics are very clear in the unequal mass case.

Fig.19

*r(τ) plots for the different values of the mass ratio m_1/m_2
for $\Lambda = -1, m_2 = 1$ and $H_0 = 10$.*

For a positive cosmological constant the situation is simple. As shown in Fig.20, as the mass ratio becomes small, the particles separate with an increasingly larger period of bounded motion. This is due to a repulsive effect of the cosmological constant and beyond the critical value the motion becomes unbounded.

Fig.20

*r(τ) plots for the different values of the mass ratio m_1/m_2
for $\Lambda = 0.003$, $m_2 = 1$ and $H_0 = 10$.*

7 CONCLUSIONS

In general relativity the relationship between the motion of a set of N bodies and the structure of space-time is non-linear and quite complicated, even for $N = 2$. Expanding upon the solution presented in [9], we have obtained an exact solution to the 2-body problem in $(1 + 1)$ dimensions with a cosmological constant. To our knowledge this is the first non-perturbative relativistic curved-spacetime treatment of this problem, providing new avenues for investigation of one-dimensional self-gravitating systems.

We recapitulate the main results of our paper:

- (1) We formulated the canonical formalism for a system of N bodies in a lineal theory of gravity with a cosmological constant Λ . The system is described by a conservative Hamiltonian. The effect of Λ is incorporated into the potential.
- (2) For $N = 2$ the determining equation of the Hamiltonian is a transcendental equation derived from the matching conditions and appropriate boundary conditions at infinity. From these the canonical equations of motion may be derived. The metric components are also completely determined.
- (3) For the equal mass case we obtained explicitly the exact solutions to the canonical equations in terms of the mutual proper time of the particles. Using the solutions we analyzed the motion in both $r(\tau)$ plots and phase-space trajectories.
- (4) As expected, a positive cosmological constant yields a repulsive effect on the motion relative to their mutual gravitational attraction. For $0 < \Lambda < \Lambda_c$ both bounded and unbounded motions are realized, while for $\Lambda_c < \Lambda$ only the unbounded motions are allowed. As $\Lambda \rightarrow \Lambda_c$ the particles separate to an infinite proper distance in infinite proper time. For $\Lambda > \Lambda_c$ this infinite separation occurs in finite proper time.
- (5) A negative cosmological constant has an additional attractive effect, and the motion of the particles is bounded. However for a certain range of the parameters, a repulsive effect sets in, resulting the double-peaked structures of Figs.7-10. This effect is due to a subtle interplay between the momentum-dependent Λ potential and the gravitational attraction.
- (6) In the unequal mass case the same basic features also occur; indeed the double peak behavior shows up more clearly than in the equal mass case. Although eq. (129) cannot be integrated in terms of elementary functions, it is straightforward to numerically integrate. An exact solution in the small mass limit of the particle 1 was also obtained.

Several interesting features of the motion remain to be explored. The divergent separation of the bodies at finite proper time needs to be better understood. Another issue concerns the condition (108) which means that for a given value of $\Lambda = -|\Lambda|$ the motion is allowed for the total energy larger than $\sqrt{8|\Lambda|/\kappa^2}$. What is the physical meaning of this condition? It seems to suggest that as the attractive effect of $\Lambda < 0$ exceeds a critical value the two particle system is no longer stable and transforms into some other system (probably making a black hole). To formulate the canonical formalism to treat this problem is our next subject.

APPENDIX A: SOLUTION OF THE METRIC TENSOR

Under the coordinate conditions (20) the field equations (12), (13), (16) and (17) become

$$\begin{aligned} \dot{\pi} + N_0 \left\{ \frac{3\kappa}{2} \pi^2 + \frac{1}{8\kappa} (\Psi')^2 - \frac{1}{4} \left(\frac{\Lambda}{\kappa} \right) - \frac{p_1^2}{2\sqrt{p_1^2 + m_1^2}} \delta(x - z_1(t)) - \frac{p_2^2}{2\sqrt{p_2^2 + m_2^2}} \delta(x - z_2(t)) \right\} \\ + N_1 \{ \pi' + p_1 \delta(x - z_1(t)) + p_2 \delta(x - z_2(t)) \} + \frac{1}{2\kappa} N_0' \Psi' + N_1' \pi = 0, \end{aligned} \quad (132)$$

$$\kappa \pi N_0 + N_1' = 0, \quad (133)$$

$$\partial_1 \left(\frac{1}{2} N_0 \Psi' + N_0' \right) = 0, \quad (134)$$

$$\dot{\Psi} + 2\kappa N_0 \pi - N_1 \Psi' = 0. \quad (135)$$

The solution to (134) is

$$N_0 = A e^{-\frac{1}{2}\Psi} = A \phi^2 = \begin{cases} A \phi_+^2 & (+) \text{ region} \\ A \phi_0^2 & (0) \text{ region} \\ A \phi_-^2 & (-) \text{ region} \end{cases}, \quad (136)$$

A being an integration constant.

Eq.(133) is

$$N_1' = -\kappa A \chi' \phi^2. \quad (137)$$

The solution in each region is

$$\begin{cases} N_{1(+)} = \epsilon \left\{ A \frac{Y_+}{K_+} \phi_+^2 - D_+ \right\} & (+) \text{ region}, \\ N_{1(0)} = \epsilon \left\{ A \frac{Y_0}{K_0} \left[A_0^2 e^{K_0 x} - B_0^2 e^{-K_0 x} \right] + 2A Y_0 A_0 B_0 x + D_0 \right\} & (+) \text{ region}, \\ N_{1(-)} = -\epsilon \left\{ A \frac{Y_-}{K_-} \phi_-^2 - D_- \right\} & (+) \text{ region}, \end{cases} \quad (138)$$

where D_+ , D_- and D_0 are integration constants.

The matching conditions $N_{1(+)}(z_1) = N_{1(0)}(z_1)$ and $N_{1(-)}(z_2) = N_{1(0)}(z_2)$ lead to

$$A = \frac{8K_0(D_+ + D_-)}{J} e^{\frac{1}{2}(K_{01}z_1 - K_{02}z_2)}, \quad (139)$$

$$\begin{aligned} D_0 = & -\frac{D_+ - D_-}{2} + \frac{D_+ + D_-}{2J} \left\{ 2 \left[\left(\frac{Y_0}{K_0} + \frac{Y_+}{K_+} \right) K_1 - \left(\frac{Y_0}{K_0} + \frac{Y_-}{K_-} \right) K_2 \right] \right. \\ & \left. - 2 \left[\left(\frac{Y_0}{K_0} - \frac{Y_+}{K_+} \right) \frac{1}{\mathcal{M}_1} - \left(\frac{Y_0}{K_0} - \frac{Y_-}{K_-} \right) \frac{1}{\mathcal{M}_2} \right] K_1 K_2 - \frac{Y_0}{K_0} K_1 K_2 (z_1 + z_2) \right\} \end{aligned} \quad (140)$$

In deriving these relations the expressions (64) for $A_{+,0}$, $B_{-,0}$ and the determining equation (66) were used. As for the equation (132), first take the δ function at $x = z_1$:

$$\begin{aligned} & \{ \delta \text{ function part of LHS (132) at } x = z_1 \} \\ & = \frac{1}{2} p_1 \delta(x - z_1) \left\{ \dot{z}_1 - N_0(z_1) \frac{p_1}{\sqrt{p_1^2 + m_1^2}} + N_1(z_1) \right\} \\ & = \frac{1}{2} p_1 \delta(x - z_1) \epsilon \left(\frac{Y_+}{K_+} - D_+ \right), \end{aligned} \quad (141)$$

where $N_0(z_1), N_1(z_1)$ and the canonical equation were inserted. Then the integration constant D_+ should be

$$D_+ = \frac{Y_+}{K_+}, \quad (142)$$

and similiary

$$D_- = \frac{Y_-}{K_-}. \quad (143)$$

Now the metric tensor is completely determined:

$$\begin{aligned} N_{0(+)}(x) &= \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1}{\mathcal{M}_1} e^{K_+(x-z_1)}, \\ N_{0(0)}(x) &= \frac{1}{2K_0 J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \left[(K_1 \mathcal{M}_1)^{\frac{1}{2}} e^{-\frac{1}{2}K_0(x-z_1)} + (K_2 \mathcal{M}_2)^{\frac{1}{2}} e^{\frac{1}{2}K_0(x-z_2)} \right]^2, \\ N_{0(-)}(x) &= \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_2}{\mathcal{M}_2} e^{-K_-(x-z_2)}, \end{aligned} \quad (144)$$

$$\begin{aligned} N_{1(+)} &= \epsilon \frac{Y_+}{K_+} \left\{ \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1}{\mathcal{M}_1} e^{K_+(x-z_1)} - 1 \right\}, \\ N_{1(0)} &= \epsilon \left\{ \frac{Y_0}{2JK_0^2} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \left[K_2 M_2 e^{K_0(x-z_2)} - K_1 M_1 e^{-K_0(x-z_1)} \right. \right. \\ &\quad \left. \left. + 2K_0 (K_1 K_2 M_1 M_2)^{1/2} e^{\frac{1}{2}K_0(z_1-z_2)} x \right] + D_0 \right\}, \\ N_{1(-)} &= -\epsilon \frac{Y_-}{K_-} \left\{ \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_2}{\mathcal{M}_2} e^{-K_-(x-z_2)} - 1 \right\}. \end{aligned}$$

With this solution and the canonical equations, the field equation (132) can be proved to hold in a whole x space.

As we showed in the previous paper, to satisfy (135) the dilaton field Ψ needs an extra function $f(t)$, which has no effect on the dynamics of particles. After lengthy calculation Eq.(135) leads to

$$\begin{aligned} \dot{f}(t) &= -\frac{d}{dt}(K_{01}z_1 - K_{02}z_2) + \frac{2}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \left\{ 2K_0 K_1 \frac{p_1}{\sqrt{p_1^2 + m_1^2}} - 2K_0 K_2 \frac{p_2}{\sqrt{p_2^2 + m_2^2}} \right. \\ &\quad \left. + \epsilon \frac{Y_0}{K_0} \left(K_0 K_1 + K_0 K_2 - \frac{K_0 K_1 K_2}{\mathcal{M}_1} - \frac{K_0 K_1 K_2}{\mathcal{M}_2} \right) + 4\epsilon Y_0 K_0 \left(\frac{K_1}{\mathcal{M}_1} + \frac{K_2}{\mathcal{M}_2} \right) \right\}. \end{aligned} \quad (145)$$

Thus $f(t)$ is uniquely determined.

APPENDIX B: A TEST PARTICLE APPROXIMATION

For a single static source M the solution to the field equations (12) - (17) under the coordinate conditions (20) is

$$N_0 = e^{\frac{\kappa M}{4}|x|}, \quad N_1 = \epsilon \sqrt{1 + \frac{8\Lambda}{\kappa^2 M^2}} \frac{x}{|x|} \left[e^{\frac{\kappa M}{4}|x|} - 1 \right], \quad (146)$$

$$\pi = -\frac{\epsilon M}{4} \sqrt{1 + \frac{8\Lambda}{\kappa^2 M^2}}, \quad \Psi = -\frac{\kappa M}{2} |x| + \frac{\kappa \epsilon M}{2} \sqrt{1 + \frac{8\Lambda}{\kappa^2 M^2}} t.$$

From (18) and (19) the canonical equations for a test particle (mass μ) under the gravity of a static source are

$$\dot{p} = -\sqrt{p^2 + \mu^2} \frac{\partial N_0(z)}{\partial z} + p \frac{\partial N_1(z)}{\partial z}, \quad (147)$$

$$\dot{z} = \frac{p}{\sqrt{p^2 + \mu^2}} N_0(z) - N_1(z). \quad (148)$$

The Hamiltonian leading to these equations is

$$\begin{aligned} H &= M + \sqrt{p^2 + \mu^2} N_0(z) - p N_1(z) \\ &= M + \sqrt{p^2 + \mu^2} e^{\frac{\kappa M}{4}|z|} - \epsilon p \sqrt{1 + \frac{8\Lambda}{\kappa^2 M^2}} \frac{z}{|z|} \left[e^{\frac{\kappa M}{4}|z|} - 1 \right]. \end{aligned} \quad (149)$$

This Hamiltonian is also derived from the determining eq. (66) by setting

$$\begin{aligned} z_1 &= z, \quad m_1 = \mu, \quad p_1 = p, \quad \tilde{p}_1 = \tilde{p} = p \frac{z}{|z|}, \\ z_2 &= 0, \quad m_2 = M, \quad p_2 = 0, \end{aligned}$$

and retaining only the linear terms of $\sqrt{p^2 + \mu^2}$ and \tilde{p} .

In terms of the proper time of the test particle

$$d\tau^2 = dt^2 \left\{ N_0(z)^2 - (N_1(z) + \dot{z})^2 \right\} = dt^2 N_0^2 \frac{\mu^2}{p^2 + \mu^2}, \quad (150)$$

the canonical equations (147) and (148) are expressed as

$$\frac{dp}{d\tau} = -\frac{\kappa M}{4\mu} \sqrt{p^2 + \mu^2} \left\{ \sqrt{p^2 + \mu^2} \operatorname{sgn}(z) - \epsilon p \sqrt{1 + \frac{8\Lambda}{\kappa^2 M^2}} \right\}, \quad (151)$$

$$\frac{dz}{d\tau} = \frac{p}{\mu} - \epsilon \sqrt{1 + \frac{8\Lambda}{\kappa^2 M^2}} \operatorname{sgn}(z) \left(1 - e^{-\frac{\kappa M}{4}|z|} \right) \frac{\sqrt{p^2 + \mu^2}}{\mu}. \quad (152)$$

Eq.(151) can be integrated and in $z > 0$ region the solution $p(\tau)$ is

$$p(\tau) = \frac{\epsilon\mu}{2} \left(h(\tau) - \frac{1}{h(\tau)} \right), \quad (153)$$

with

$$h(\tau) = \begin{cases} \frac{(1+\sqrt{\gamma_M}) \left\{ 1 - \rho e^{\epsilon\sqrt{\frac{\Lambda}{2}}(\tau-\tau_0)} \right\}}{\sqrt{\gamma_M-1} \left\{ 1 + \rho e^{\epsilon\sqrt{\frac{\Lambda}{2}}(\tau-\tau_0)} \right\}} & \Lambda > 0, \\ \frac{\sqrt{p_0^2 + \mu^2 + \epsilon p_0}}{\mu} - \frac{\epsilon\kappa M}{4}(\tau - \tau_0) & \Lambda = 0, \\ \frac{1 - \frac{(1+\sqrt{\gamma_M})(\sqrt{p_0^2 + \mu^2 - \epsilon p_0})}{\mu\sqrt{1-\gamma_M}} \tan\left[\epsilon\sqrt{-\frac{\Lambda}{8}}(\tau-\tau_0)\right]}{\frac{\sqrt{p_0^2 + \mu^2 - \epsilon p_0}}{\mu} + \frac{\sqrt{1-\gamma_M}}{1+\sqrt{\gamma_M}} \tan\left[\epsilon\sqrt{-\frac{\Lambda}{8}}(\tau-\tau_0)\right]} & \Lambda < 0, \end{cases} \quad (154)$$

where

$$\gamma_M = 1 + \frac{8\Lambda}{\kappa^2 M^2}, \quad \rho = \frac{(1 + \sqrt{\gamma_M})(\sqrt{p_0^2 + \mu^2 - \epsilon p_0}) - \mu\sqrt{\gamma_M - 1}}{(1 + \sqrt{\gamma_M})(\sqrt{p_0^2 + \mu^2 - \epsilon p_0}) + \mu\sqrt{\gamma_M - 1}}, \quad (155)$$

with p_0 being the initial momentum at $\tau = \tau_0$.

In $z < 0$ region the solution is

$$p(\tau) = -\frac{\epsilon\mu}{2} \left(\bar{h}(\tau) - \frac{1}{\bar{h}(\tau)} \right), \quad (156)$$

with

$$\bar{h}(\tau) = \begin{cases} \frac{(1+\sqrt{\gamma_M}) \left\{ 1 - \bar{\rho} e^{\epsilon\sqrt{\frac{\Lambda}{2}}(\tau-\tau_0)} \right\}}{\sqrt{\gamma_M-1} \left\{ 1 + \bar{\rho} e^{\epsilon\sqrt{\frac{\Lambda}{2}}(\tau-\tau_0)} \right\}} & \Lambda > 0, \\ \frac{\sqrt{p_0^2 + \mu^2 - \epsilon p_0}}{\mu} - \frac{\epsilon\kappa M}{4}(\tau - \tau_0) & \Lambda = 0, \\ \frac{1 - \frac{(1+\sqrt{\gamma_M})(\sqrt{p_0^2 + \mu^2 + \epsilon p_0})}{\mu\sqrt{1-\gamma_M}} \tan\left[\epsilon\sqrt{-\frac{\Lambda}{8}}(\tau-\tau_0)\right]}{\frac{\sqrt{p_0^2 + \mu^2 + \epsilon p_0}}{\mu} + \frac{\sqrt{1-\gamma_M}}{1+\sqrt{\gamma_M}} \tan\left[\epsilon\sqrt{-\frac{\Lambda}{8}}(\tau-\tau_0)\right]} & \Lambda < 0, \end{cases} \quad (157)$$

where

$$\bar{\rho} = \frac{(1 + \sqrt{\gamma_M})(\sqrt{p_0^2 + \mu^2 + \epsilon p_0}) - \mu\sqrt{\gamma_M - 1}}{(1 + \sqrt{\gamma_M})(\sqrt{p_0^2 + \mu^2 + \epsilon p_0}) + \mu\sqrt{\gamma_M - 1}}. \quad (158)$$

When $p = p_0$ and $z = 0$ at $\tau = \tau_0$, the total energy is $H = H_0 = M + \sqrt{p_0^2 + \mu^2}$. The

solution for $z(\tau)$ is obtained from (149) and $p(\tau)$ as

$$z(\tau) = \begin{cases} \frac{4}{\kappa M} \log \frac{2\sqrt{p_0^2 + \mu^2} - (h - \frac{1}{h})\sqrt{\gamma_M}}{(h + \frac{1}{h}) - (h - \frac{1}{h})\sqrt{\gamma_M}} & z > 0, \\ -\frac{4}{\kappa M} \log \frac{2\sqrt{p_0^2 + \mu^2} - (\bar{h} - \frac{1}{\bar{h}})\sqrt{\gamma_M}}{(\bar{h} + \frac{1}{\bar{h}}) - (\bar{h} - \frac{1}{\bar{h}})\sqrt{\gamma_M}} & z < 0. \end{cases} \quad (159)$$

For the test particle solution the critical value of Λ is $\Lambda_c = \frac{\kappa^2 \mu^2 M^2}{8p_0^2}$.

Fig.21 and 22 show typical trajectories of the test particle $\mu = 0.1$ for $M = 10$ and $\Lambda = -10, 0, 0.4$ and 2 . The characteristics of these plots are common to those of the unequal mass case.

Fig.21

Phase space trajectories of a test particle for different values of Λ .

Fig.22

The r plots corresponding to the trajectories in Fig.21.

APPENDIX C: CAUSAL RELATIONSHIPS BETWEEN PARTICLES IN UNBOUNDED MOTION

We can explicitly verify that the particles lose causal contact with one another for $\tau > \tau_\infty$. Consider the unbounded motion of tan-type $A(n = 0)$ with $m = 1, H = 2.1$ and $\Lambda = 1.5$. The path $x(t)$ of light emitted from particle 2 at time T is governed by $d\tau = 0$, which reads

$$\left(\frac{dx}{dt}\right)^2 + 2N_1 \frac{dx}{dt} - (N_0^2 - N_1^2) = 0 \quad (160)$$

and so the equation of the light signal directed to particle 1 is

$$\frac{dx}{dt} = N_0(x(t), z_1(t), z_2(t), p(t)) - N_1(x(t), z_1(t), z_2(t), p(t)). \quad (161)$$

The light emitted in the opposite direction is described by

$$\frac{dx}{dt} = -N_0(x(t), z_1(t), z_2(t), p(t)) - N_1(x(t), z_1(t), z_2(t), p(t)) . \quad (162)$$

Numerically solving (161) and (162) yields the solutions shown in Figs.23 and 24, where the trajectories of light signals emitted from particle 2 at various times T are plotted. For small T ($T < 3$), the particles are in causal contact (a dotted curve in (+) direction in Fig.23), but for $T \approx 3$ the signal just barely catches up with particle 1, which is almost in light-like motion (a dashed curve in (+) direction in Fig.23). For $T = 4$ the world line $x(t)$ in the (+) direction is parallel to $z_1(t)$ at large t and in the (-) direction it goes nearly on the same trajectory with the particle 2. For large T ($T > 4.82$) the particles are out of causal contact with each other (Fig.24): a light ray sent from particle 2 toward particle 1 receives a strong repulsive effect and ultimately reverses direction, following behind particle 2. In Fig.24 the trajectories of the light signal emitted to (-) direction can not be discriminated from those of the particle 2.

Fig.23

The trajectories of light signals emitted at $T = 2, 3$ and 4.

Fig.24

The trajectories of light signals emitted at $T = 4.82, 5$ and 6.

A flat-space model of these effects can be constructed as follows. Consider the following expression for the 2-velocity

$$u^\mu = (f(\sigma\tau), \sqrt{f^2(\sigma\tau) - 1}) \quad (163)$$

where $f(\sigma\tau)$ is some function and

$$d\tau^2 = dt^2 - dx^2 \quad (164)$$

is the flat metric. We have $\frac{dt}{d\tau} = f$, $\frac{dx}{d\tau} = \sqrt{f^2 - 1}$ and so

$$\frac{dx}{dt} = \frac{\sqrt{f^2 - 1}}{f} . \quad (165)$$

The general expression for the acceleration of a particle with 2-velocity (163) is

$$a^\mu = \frac{du^\mu}{d\tau} = \sigma f' \left(1, \frac{f}{\sqrt{f^2(\sigma\tau) - 1}} \right) \quad (166)$$

where $f' = df(\tau)/d\tau$. We have $u \cdot u = 1$ and $a \cdot u = 0$ and

$$a \cdot a = \frac{(\sigma f')^2}{f^2(\sigma\tau) - 1} \quad (167)$$

for the magnitude of the acceleration. In general we have the following possibilities:

1) The function $f \rightarrow f_0$ where f_0 is finite at $\tau \rightarrow \infty$. In this case the particle never becomes lightlike.

2) The function $f \rightarrow \infty$ as $\tau \rightarrow \infty$. In this case the particle becomes lightlike, but it takes an infinite proper time (and coordinate time) for this to happen. The standard example is $f = \cosh(\sigma\tau)$, the constant acceleration example.

3) The function $f \rightarrow \infty$ as $\tau \rightarrow \tau_0$, where τ_0 is finite. In this case the particle becomes lightlike in a finite amount of proper time, but an infinite amount of coordinate time. An example would be $f = \sec(\sigma\tau)$. The acceleration is not constant, but increases as a function of proper time, diverging at $\tau = \tau_0$. This last situation is realized by our exact solutions (117–118) with $\Lambda > \Lambda_c$.

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