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Low Scale Unification, Newton's Law and Extra Dimensions.

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Abstract

Motivated by recent work on low energy unification, in this short note we derive corrections on Newton's inverse square law due to the existence of extra decompactified dimensions. In the four-dimensional macroscopic limit we find that the corrections are of Yukawa type. Inside the compactified space of n-extra dimensions the sub-leading term is proportional to the $(n+1)$- power of the distance over the compactification radius ratio. Some physical implications of these modifications are briefly discussed.

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One of the most tantalizing mysteries in modern unified theories is the magnitude of the unification scale. A well known result in the weakly coupled heterotic string theory is that the string scale, is of the order of the Planck mass M_P [1]. Recent developments have revealed the possibility that the string scale can be arbitrarily low in Type I and Type IIB theories [2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

According to a recently proposed scenario, the hierarchy problem may be solved[3] assuming the existence of extra spatial dimensions at low energies[12]. In this picture, strong gravitational effects –which could not be described accurately by Newton's law– may appear at short distances of the order of the compactification scale of the extra dimensions. If so, gravitons may propagate freely inside the space of extra dimensions, while all ordinary particles would leave in the four dimensional world. Experimental searches for possible deviations from Newton's inverse square law imply that such effects should be limited below the sub-millimeter range[13]. We note that this scenario can find a realization in the context of D–branes[14, 15]. Matter fields may live in a 9 or 3–brane, while gravitons can live in a larger dimensional bulk.

Deviations from the gravitational law have been intensively studied also in the past. In [16] the theoretical aspects of a gravitationally repulsive term in supergravity theories were investigated, while in [17] string loop corrections which affect gravitational couplings were considered.

In this letter we examine corrections to the gravitational force which are of particular importance in the case of experimental searches in the vicinity of the compactification radii. In the presence of n compact spatial dimensions of radii $R_{1,\ldots n}$, the fundamental scale M_X of the theory for very short or very large distances can be estimated using the Gauss law. The approximate forms of the gravitational potential in two limiting cases in the presence of n-compactified extra dimensions are given as follows[3]. Assuming for simplicity that all compactification radii are the same $R_i = R$, inside the volume of the extra dimensions i.e.

when $r \ll R$, the Gauss low gives

$$V(r) \sim \frac{1}{M_{P_{n+4}}^{n+2}} \frac{1}{r^{n+1}}, \quad r \ll R \tag{1}$$

where $M_{P_{n+4}}^{n+2}$ is the Planck mass in $n+4$–dimensional space and is identified with the fundamental scale M_X. For large distances compared to the mean compactification radius of the extra dimensions i.e., when $r \gg R$, the n-dimensional compactified volume confines the gravitational flux and as a result the approximate potential is given by

$$V(r) \sim \frac{1}{M_{P_{n+4}}^{n+2}} \frac{1}{R^n r}, \quad r \gg R \tag{2}$$

The latter should be identical to the known $4-d$ gravitational potential

$$V(r) = \frac{1}{M_{P_4}^2} \frac{1}{r} \tag{3}$$

The comparison of the last two formulae for distances far beyond the compactification scale $M_C \sim \frac{1}{R}$ gives an approximate relation between the latter and the Planck mass in 4 and $n+4$ dimensions

$$\frac{1}{M_C} \sim R \sim \frac{1}{M_{P_{n+4}}} \left(\frac{M_{P_4}}{M_{P_{n+4}}} \right)^{2/n} \tag{4}$$

For distances comparable to the compactification scale corrections are expected to modify the above formulae. In what follows, we will present some analytic results for the case of $n=1$ and $n=2$ compactified dimensions. We will see that some important modifications of the above formulae will show up in both cases. In particular, inside the compactification circle, i.e., $r < R$, the first sub-dominant term will be shown to have a power dependence on the ratio r/R, while at large distances the potential has a Yukawa type correction, proportional to the form $e^{-r/R}/r$.

We will solve the Laplace equation in $(n+3)$ spatial dimensions where n of them are compactified on a torus with radius R. Assume the coordinates $x_{1,2,3}$ for the 3–dimensional ordinary space and x_i^c, $i = 1, \cdots n$ for the compactified ones. Defining the angles $\theta_{1,2,\ldots n}$ for the compactified dimensions with $\theta_i \in [0, 2\pi]$, we write them as $x_i^c = R\theta_i$ where we assumed for simplicity one common radius R. The Laplace equation may be written as follows

$$\vec{\nabla}^2\Phi = -\delta^3(\vec{x}-\vec{y})\frac{1}{R^n}\delta^n(\vec{\theta}-\vec{\theta}_0) \tag{5}$$

where the δ–functions on the right-hand side (RHS) are given as usually by

$$\begin{aligned} \delta^3(\vec{x}-\vec{y}) &= \frac{1}{(2\pi)^3}\int d^3k e^{\imath\vec{k}\cdot(\vec{x}-\vec{y})} \\ \delta^n(\vec{x}^c-\vec{x}_0^c) &= \frac{1}{(2\pi R)^n}\sum_{\vec{m}} e^{\imath\vec{m}\cdot(\vec{\theta}^c-\vec{\theta}_0^c)} \end{aligned}$$

and the sums extend form $-\infty$ to ∞ for all indices $m_{1,2\ldots n}$. Using the Fourier transform, one finds

$$\Phi(r,q) = \frac{1}{(2\pi)^{n+3}}\frac{1}{R^n}\sum_{\vec{m}}\int d^3k\left\{e^{\imath\vec{k}\cdot\vec{r}+\imath\vec{m}\cdot\vec{q}}\int_0^\infty d\,s e^{-s[k^2+(\frac{\vec{m}}{R})^2]}\right\} \tag{6}$$

where for simplicity we have denoted $\vec{r}=\vec{x}-\vec{y}$ and $\vec{q}=\vec{\theta}-\vec{\theta}_0$. In the integrand of (6), the summation is taken over the infinite tower of KK–excitations in all the additional space dimensions, $\vec{m}=(m_1,\ldots m_n)$. It is easy now to perform the integration with respect to $\vec{k}$. The result is

$$\Phi(r,q) = \frac{1}{(4\pi)^{3/2}}\frac{1}{(2\pi R)^n}\int_0^\infty d\,s s^{-3/2}e^{-\frac{r^2}{4s}}\sum_{\vec{m}} e^{\imath\vec{m}\cdot\vec{q}-s(\frac{\vec{m}}{R})^2} \tag{7}$$

In the above summations, $\vec{m}^2 = m_1^2+\cdots+m_n^2$ and $\vec{m}\cdot\vec{q} = m_1q_1+\cdots m_nq_n$ is the inner product over the $n-$dimensional compactified space. The above result can also be written in terms of a product of *theta* functions as follows

$$\Phi(r,q) = \frac{1}{(4\pi)^{3/2}}\frac{1}{(2\pi R)^n}\int_0^\infty d\,s s^{-3/2}e^{-\frac{r^2}{4s}}\prod_{j=1}^n\theta_3\left(\frac{q_j}{2\pi},\imath\frac{s}{\pi R^2}\right) \tag{8}$$

where,

$$\theta_3(\nu,\tau) = \sum_{n=-\infty}^{\infty} p^{\frac{n^2}{2}}z^n \tag{9}$$

with $p=e^{2\imath\pi\tau}$ and $z=e^{2\imath\pi\nu}$. Performing the integral we obtain

$$\Phi(r,q) = \frac{1}{4\pi}\frac{1}{(2\pi R)^n}\frac{1}{r}\left\{1+2\sum_{\vec{m}}^{\infty}e^{-|\vec{m}|\frac{r}{R}}\cos(\vec{m}\cdot\vec{q})\right\} \tag{10}$$

In the particular case of one extra dimension, $n = 1$, we may obtain an exact result of the above integral. We first perform the integration in (7) to obtain

$$\Phi(r,q) \;=\; \frac{1}{8\pi^2}\frac{1}{R}\frac{1}{r}\left\{1 + 2\sum_{m=1}^{\infty} e^{-m\frac{r}{R}}\cos(mq)\right\} \tag{11}$$

Performing the sum in this formula one gets the final expression for $n = 1$. Suppressing an overall numerical factor, we have the following form for the potential

$$V_{n=1}(r) \;\propto\; \frac{1}{M_{P_5}^3}\left(1 + 2\frac{e^{\frac{r}{R}}\cos q - 1}{e^{2\frac{r}{R}} - 2e^{\frac{r}{R}}\cos q + 1}\right)\frac{1}{Rr} \tag{12}$$

The dependence on the distance r in this formula is exact and valid for any value of r. For fixed r, its maximum value is obtained when $q = 0$, while for fixed q the maxima are along the path determined by the equation $r = R\log(1 \pm \sin q)$. The resulting potential as a function of r and $q = \theta - \theta_0$ is plotted in figure 1.

In order to compare with the approximate formulae of the potential given in the introduction, we wish now to take the limit $q = 0$ in (12) which gives

$$V \;=\; \frac{1}{M_{P_5}^3}\frac{e^{\frac{r}{R}} + 1}{e^{\frac{r}{R}} - 1}\frac{1}{Rr} \tag{13}$$

The formula for $r \ll R$ becomes

$$V_{n=1} \sim \frac{1}{M_{P_5}^3}\frac{2}{r^2}\left(1 + \frac{1}{12}\frac{r^2}{R^2}\right) \tag{14}$$

This formula which is valid for small r, differs by a factor of 2 compared to the approximation (1). For $r \gg R$ we obtain an exponential correction of the form

$$\begin{aligned} V_{n=1} \;&\sim\; \frac{1}{M_{P_5}^3}\frac{1}{Rr}\left(1 + 2e^{-\frac{r}{R}}\right) \\ &\sim\; \frac{M_C}{M_{P_5}^3}\frac{1}{r}\left(1 + 2e^{-M_C r}\right) \end{aligned} \tag{15}$$

which is a Yukawa type correction valid for large distances compared to the compactification radius. The approximation used, gives us the chance to compare directly the above formula with the usual parametrization of the long-range forces of gravitational strength in the literature[18, 19]

$$V(r) \propto \frac{1}{r}(1 + \alpha e^{-\frac{r}{\lambda}}). \tag{16}$$

Comparing the two formulae, we have a definite prediction for the strength α of Yukawa type gravitational corrections in the case of one extra compact dimension which is $\alpha = 2$. Using the α - λ plot of[13] which gives the experimentally determined region, we conclude that the allowed radius has an upper bound of the order $\lambda \equiv R \sim 1mm$.

Next, let us return to the approximate formulae in (1, 3) which can be written in a single expression as,

$$V \sim \frac{1}{M_{P_5}^3} \left(\frac{1}{r^2} \theta(r - R) + \frac{1}{rR} \theta(R - r) \right). \tag{17}$$

The formula (17) is plotted in figure 2 versus the exact expression (13). The plot shows that the two expressions coincide only for $r \gg R$. For distances $r \sim R$ and $r < R$ there exist significant deviations which might lead to interesting corrections in calculating various effects in physical processes.

For more than one compact dimensions ($n > 1$), we will work out approximated forms of the potential. As already stated, the approximations are straightforward in the case where the radii of the extra compactified dimensions are either very big or enormously small compared to the distance that the potential is estimated, being those obtained from the Gauss' law in the 'spherically' symmetric case. At relatively large distances, $r > R$, we may also keep the first two terms of the series expansion in (7), to obtain the result

$$\Phi(r, R) \quad = \quad \frac{1}{4\pi} \frac{1}{r} \frac{1}{(2\pi R)^n} \left(1 + 2ne^{-r/R} \right) \tag{18}$$

which is a straightforward generalization of the approximation (15) for arbitrary n. The other interesting case, which may have particular importance for the experimental verification of strongly coupled gravity at the TeV scale, is when the distance is comparable with the compactification radius.

When the experimental measurement is taken in distances smaller than the compactification radius of the extra dimensions $r \leq R$, the behavior of the infinite sum is not manifest since an infinite number of terms may contribute. Then, the most effective tool to extract the asymptotic behaviour of the potential in the transition region where R becomes effectively large, is the Jacobi's transformation of theta functions[20]

$$1 + 2\sum_{m=1}^{\infty} e^{-m^2\ell^2}\cos(2\pi m\ell z) \;=\; \frac{\sqrt{\pi}}{\ell}e^{-\pi^2 z^2}\left(1 + 2\sum_{m=1}^{\infty} e^{-m^2\pi^2/\ell^2}\cosh\frac{2\pi^2 mz}{\ell}\right). \quad (19)$$

Substitution of the above formula in (7) gives

$$\begin{aligned}\Phi(r,q) \;&=\; \frac{1}{(2\sqrt{\pi})^{n+3}}\int_0^{\infty} d\,s s^{-\frac{n+3}{2}} e^{-r^2/4s} \\ &\times \prod_{j=1}^{n} e^{-q_j^2 R^2/4s}\left(1 + 2\sum_{m_j}^{\infty} e^{-(m_j\pi R)^2/s}\cosh\frac{m_j q_j \pi R^2}{s}\right)\end{aligned} \quad (20)$$

Now, for $R > r$ the exponentials in the sum converge rapidly and a certain number of terms in the product may give a good approximation.

We are interested in the case of two extra dimensions. Taking the case of zero angles, i.e. $q_j = \theta_j - \theta_{0j} = 0$ for all j's and $n = 2$ we can split (20) into three integrals which can be evaluated. The results are,

$$I_0 \;=\; \frac{1}{8\pi^2}\frac{1}{r^3} \quad (21)$$

$$I_1 \;=\; \frac{4}{8\pi^2}\sum_k \frac{1}{(r^2 + 4\pi^2 k^2 R^2)^{3/2}} \quad (22)$$

$$I_2 \;=\; \frac{4}{8\pi^2}\sum_{k=1}\sum_{\ell=1}\frac{1}{(r^2 + 4\pi^2(k^2+\ell^2)R^2)^{3/2}} \quad (23)$$

We note that the number 4 multiplying the corrections is the product of the factor 2 in front of the sum in the integral (20) times the number of dimensions $n = 2$. Defining the parameter $\rho = r/(2\pi R)$, for $\rho < 1$ we may expand to obtain

$$I_1 \;\approx\; \frac{1}{8\pi^2}\frac{4}{(2\pi R)^3}\left(\zeta(3) - \frac{3}{2}\zeta(5)\rho^2\right) \quad (24)$$

$$I_2 \;\approx\; \frac{1}{8\pi^2}\frac{4}{(2\pi R)^3}\left(\zeta_2(3) - \frac{3}{2}\zeta_2(5)\rho^2\right) \quad (25)$$

where in the above expressions $\zeta(\ell)$ is the Riemann zeta function and we have introduced the notation $\zeta_2(\ell) = \sum_{k,m}(k^2+m^2)^{-\ell}$. Thus, we obtain an approximation for the corrections

$$V(r) \sim \frac{1}{r^3} + 4\frac{2.24}{(2\pi R)^3} \tag{26}$$

An estimation of the correction terms may also be given in the limiting case $\rho \to 1$. Putting $\rho = 1$ and performing the sums we obtain

$$V(r) \sim \frac{1}{r^3} + 4\frac{1.32}{(2\pi R)^3} \tag{27}$$

The general result for $n = 2$ can be written as a double convergent sum as follows

$$\Phi_{n=2} = \frac{1}{8\pi^2}\frac{1}{r^3}\left[1 + 4\rho^3 \sum_{k=0}^{\infty}\sum_{l=1}^{\infty} \frac{1}{(\rho^2 + (k^2 + l^2))^{\frac{3}{2}}}\right] \tag{28}$$

The double sum takes also into account the degeneracy of a particular KK-contribution. Clearly, the sum of the two integers $k^2 + l^2 = N^2$ which appears in the denominator is related to the degeneracy.

The generalization of the above result to higher dimensions is straightforward. Here, we have restricted in examining the corrections to the Newtonian gravity due to the possible existence of $n = 1$ or $n = 2$ extra space-time dimensions. We have succeeded to obtain useful exact forms of the potential for the case $n = 1$. In the cases where $n > 1$ the long range corrections can be approximated by a Yukawa type interaction, and the potential is written

$$V_{r>R} \sim \frac{1}{R^n r}(1 + 2ne^{-r/R})$$

where r is the distance and R a common compactification radius of the n extra dimensions. When the compact radius is effectively large, the modifications are expressed as powers of the ratio r/R,

$$V_{r<R} \sim \frac{1}{r^{n+1}}\left(1 + 2nc_n\left(\frac{r}{R}\right)^{n+1}\right)$$

where c_n is a calculable coefficient which for cases $n = 1, 2$ is given by the expressions (14) and (28) in this work.

As this work was being written, we received [21] where a similar analysis is presented.

References

[1] J. Polchinski, *String Theory*, **V 2**, p. 343, Cambridge U.P., 1998.

[2] E. Witten, Nucl. Phys. **B 471** (1996) 135; [hep-th/9602070]

[3] Nima Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett.**B 429** (1998) 263-272, [hep-ph/9803315]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, Gia Dvali, Phys. Lett. **B 436** (1998) 257-263, [hep-ph/9804398] .

[4] J. Lykken, Nucl. Phys.**B 543** (1999) 105-121, [hep-th/9806080]

[5] K. Dienes, E. Dudas and T. Gherghetta, Phys. Lett. **B 436** (1998)55; hep-ph/9806292.

[6] C. Bachas, hep-ph/9807415;
I. Antoniadis and C. Bachas, Phys. Lett. **B 450**(1999) 83-91, [hep-th/9812093].

[7] G. Shiu and S.-H.H. Tye, Phys. Rev. **1998** 106007;
Z. Kakushadze and S.H. Tye, hep-ph/9809147.

[8] C.P. Burges, L.E. Ibañez and F. Quevedo, hep-ph/9810535.

[9] I. Antoniadis and B. Pioline, hep-th/9902055.

[10] K. Benakli, hep-ph/9809582.

[11] G.K. Leontaris and N.D. Tracas, Phys. Lett. **B** (1999) to appear, [hep-ph/9902368.]

[12] I. Antoniadis, Phys. Lett. **B 246** (1990) 377.

[13] G.C. Long, H. Chang, J. Price, Nucl. Phys. **B 539** (1999) 23, [hep-ph/9805217].

[14] J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724;

[15] J. Polchinski, S. Chaudhuri and C.V. Johnson, hep-th/9602052;
J. Polchinski, hep-th/9611050.

[16] J. Scherk, Phys. Lett. **B 88** (1979) 265;
K. Zachos, Phys. Lett. **B 76** (1978) 329.

[17] T. Taylor, G. Veneziano, Phys. Lett. **B 213** (1988) 450.

[18] E. Fischbach and C. Talmadge, Proc. XXXI, Moriond, Les Arcs, Jan. 1996, eds R. Ansari et al.

[19] A. Cook, Cont. Phys. **282** (1987) 159.

[20] M.J. Lighthill, *"Introduction to Fourier analysis and generalised functions."*, Cambridge U.P., 1958.

[21] A. Kehagias and C. Sfetsos, `hep-ph/9905417`.

FIGURE CAPTIONS

FIGURE 1: The potential $V(r)$ for one extra dimension as a function of the distance-compactification ratio r/R and the angle $q = \theta - \theta_0$.

FIGURE 2: Comparison of the exact (upper curve) and approximate forms (lower curve) in the $n = 1$ potential.

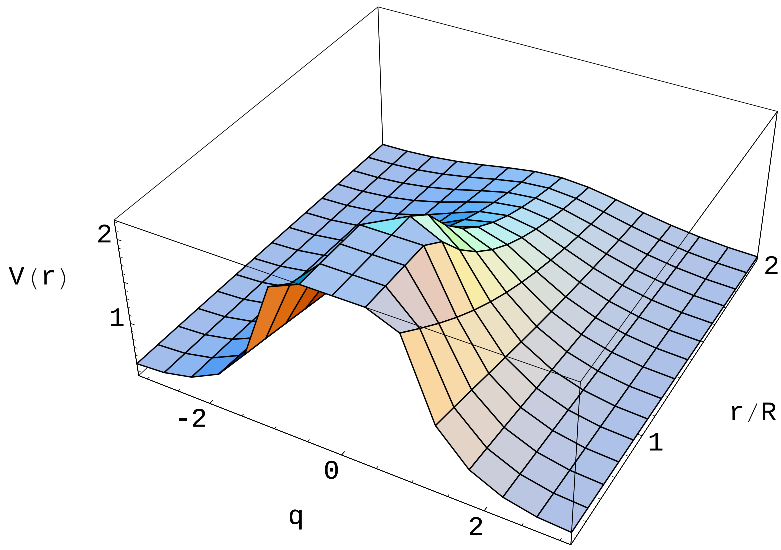

FIGURE 1

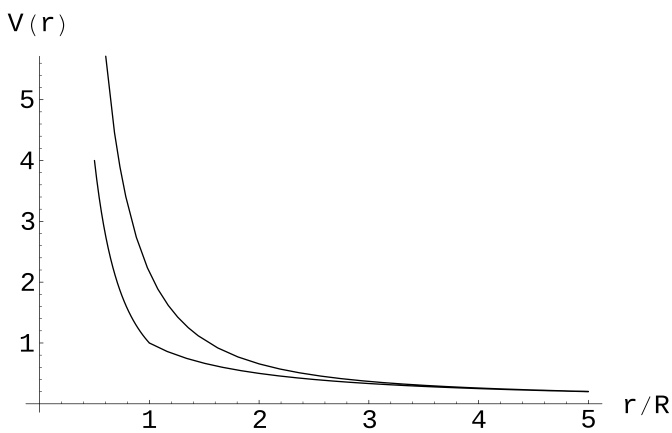

FIGURE 2.