CERN-TH-99-151
ECM-UB-PF-99-12

Effective Transport Equations for non-Abelian Plasmas

Daniel F. Litim *,a and Cristina Manuel †,b

[a] *Departament ECM & IFAE, Facultat de Física*
Univ. de Barcelona, Diagonal 647, E-08028 Barcelona, Spain.

[b] *Theory Division, CERN, CH-1211 Geneva 23, Switzerland.*

Abstract

Starting from classical transport theory, we derive a set of covariant equations describing the dynamics of mean fields and their statistical fluctuations in a non-Abelian plasma in or out of equilibrium. A general procedure is detailed for integrating-out the fluctuations as to obtain the effective transport equations for the mean fields. In this manner, collision integrals for Boltzmann equations are obtained as correlators of fluctuations. The formalism is applied to a hot non-Abelian plasma close to equilibrium. We integrate-out explicitly the fluctuations with typical momenta of the Debye mass, and obtain the collision integral in a leading logarithmic approximation. We also identify a source for stochastic noise. The resulting dynamical equations are of the Boltzmann-Langevin type. While our approach is based on classical physics, we also give the necessary generalizations to study the quantum plasmas. Ultimately, the dynamical equations for soft and ultra-soft fields change only in the value for the Debye mass.

*E-Mail: Litim@ecm.ub.es

†E-Mail: Cristina.Manuel@cern.ch

I. INTRODUCTION

This article presents in full detail an approach to the dynamics of non-Abelian plasmas based on classical transport theory, the main results of which have been summarized in [1].

In the recent years, there has been an increasing interest in the dynamics of non-Abelian plasmas at very high temperatures or high densities. Due to the asymptotic freedom of quantum chromodynamics, one expects that quarks and gluons are no longer confined under such extreme conditions, but rather behave as free entities forming the so-called quark-gluon plasma. Within the next few years, a lot of efforts will be given to detect experimentally this new state of matter using heavy-ion colliders. Another domain of application concerns the physics of the early universe. If baryogenesis can finally be understood within an electroweak scenario, an understanding of the physics of the electroweak model in the high temperature regime where the spontaneous broken symmetry is restored, is essential for a computation of the rate of baryon number violation.

It is therefore mandatory to devise reliable theoretical tools for a quantitative description of non-Abelian plasmas both in or out of equilibrium. While some progress has been achieved in the recent years [2,3], we are still far away from having a satisfactory understanding of the relevant relaxation and transport processes in non-Abelian plasmas, in particular when it comes to out-of-equilibrium situations.

There are different approaches in the literature to study non-Abelian plasmas, ranging from thermal field theory to quantum transport equations or lattice studies. Even in the close-to-equilibrium plasma, and for small gauge coupling, the situation is complicated due to the non-perturbative character of long-wavelength excitations in the plasma. Most attempts to tackle this problem are based on a quantum field theoretical description of the non-Abelian interactions [4,5]. It has been conjectured that the plasma close to equilibrium allows for a description in terms of soft classical fields, as the occupation number for the soft excitations are large. Surprisingly, a classical transport theory approach, as developed in [6], has never been exploited in full detail for the non-Abelian case. The opposite holds true for Coulomb plasmas, where all the essential transport phenomena have been studied longly using techniques developed within (semi-)classical kinetic theory [7], while a quantum field theoretical approach has been undertaken only recently.

Our approach aims at filling this gap in the literature of classical non-Abelian plasmas. Here, we follow the philosophy of Klimontovitch [7], and our equations can be seen as the generalization of classical kinetic theory for Abelian plasmas to non-Abelian ones. Our essential contribution is considering the non-Abelian colour charges as dynamical variables and introducing the concept of ensemble average to the non-Abelian kinetic equations. Equally important is the consistent treatment of the intrinsic non-linearities of non-Abelian gauge interactions. The observation that Klimontovitch's procedure leads to the Balescu-Lenard collision integral for Coulomb plasmas has motivated earlier derivations of similar (semi-)classical kinetic equations for non-Abelian plasmas [8–10]. However, these implementations are not fully consistent, and have never been worked out in all generality.

The starting point for a classical transport theory of non-Abelian plasmas is considering an ensemble of classical point particles carrying a non-Abelian charge. They interact through self-consistent fields, that is, the fields generated by the particles themselves. The microscopic dynamics is governed by the classical equations of motion given by the Wong equations [11]. When the number of particles is large, one has to abandon a microscopic description of the system in favour of a macroscopical one based on an ensemble average of all the microscopic quantities. This leads naturally to a description in terms of averaged quantities, and their statistical fluctuations. By averaging the microscopic dynamical equations, we obtain effective transport equations for mean quantities. These contain the collision integrals of the macroscopic Boltzmann equation, which appear in this formulation as statistical correlators of fluctuating quantities. By subtracting the exact microscopic equations from the mean ones, we obtain the dynamical equations for the fluctuations themselves. In principle, these two set of equations should be enough to consider all the transport phenomena in the plasma.

This method is then applied in full detail to a thermal non-Abelian plasma close to equilibrium, which allows to employ some approximations. For a small plasma parameter, the two-particle correlators are small and can be neglected. In the case of small fluctuations, the dynamical equations simplify considerably. These conditions are always met for a small gauge coupling parameter, which, for simplicity, will be assumed throughout. Our approximate equations are the leading order ones in a consistent expansion in the gauge coupling. However, we shall also see that the condition for a kinetic description to be valid could also be met for large gauge couplings. After taking statistical averages, we are able to explicitly integrate-out the fluctuations with momenta about the Debye mass. This gives the collision integrals which appear in the transport equations for the mean fields. In addition to the dissipative processes in the plasma described by the collision term, we are able to deduce the stochastic source which prevents the system from abandoning equilibrium. This is an important result, because it allows to prove explicitly that the fluctuation-dissipation theorem holds, when switching from a microscopic to a macroscopic description of the system. These findings for classical plasmas can be generalized to the case of quantum ones. The resulting dynamical equation match perfectly the effective theory for the ultra-soft modes as found by other approaches [12–14].

The lesson to be learned is thus two-fold: There exists a fully self-contained formalism to study classical non-Abelian plasmas in the first place, which opens in particular a door for applications to out-of-equilibrium situations. Second, this approach is -technically speaking- much easier than approaches based on the full QFT. Some of the intrinsic complications of a quantum field theoretical description (like gauge-fixing, ghost degrees of freedom) can be avoided, and in the close to equilibrium plasma, the same effective dynamical equations are obtained.

The paper is organized as follows. We begin with a review of the microscopic picture, based on the classical equations of motions for coloured point particles (sect. II). Changing to a macroscopic description needs the introduction of a statistical average, which also allows the computation of correlators of fluctuations (sect. III). This procedure is then applied to the fields and the distribution function as to obtain dynamical equations for their mean values and their fluctuations. The dynamical equations are given in its most general form. Possible approximation schemes are detailed, and the interpretation of statistical correlators in terms of collision integrals is given (sect. IV). The consistency of the procedure with the requirements of gauge invariance is shown for the complete set of equations, and some approximations to them (sect. V). In order to apply the formalism to a plasma close to equilibrium, we discuss first the relevant physical scales for both classical and quantum plasmas (sect. VI). This is followed by a fully detailed derivation of the mean field dynamical equation for classical plasmas, which includes the integrating-out of the fluctuations with momenta about the Debye mass, and the computation of the collision integral and the related noise variable in a leading logarithmic approximation. Explicit expressions for the non-local ultra-soft current and the colour conductivity are given as well (sect. VII). We argue that these results can be translated to the case of quantum plasmas and detail the necessary changes. Some comments on related work are added as well (sect. VIII). Finally, we present our conclusions (sect. IX), deferring to the appendices some technical details regarding the Darboux variables for SU(N) colour charges (appendix A), and the derivation of a useful algebraic identity (appendix B).

II. MICROSCOPIC EQUATIONS FOR NON-ABELIAN CHARGED PARTICLES

Let us consider a system of particles carrying a colour charge Q^a, where the colour index runs from $a = 1$ to $N^2 - 1$ for a SU(N) gauge group. Within a microscopic description, the trajectories in phase space are known exactly. The trajectories $\hat{x}(\tau), \hat{p}(\tau)$ and $\hat{Q}(\tau)$ for every particle are solutions of their classical equations of motions, the Wong equations [11]

$$m\frac{d\hat{x}^\mu}{d\tau} = \hat{p}^\mu \ , \tag{2.1a}$$

$$m\frac{d\hat{p}^\mu}{d\tau} = g\hat{Q}^a F_a^{\mu\nu}\hat{p}_\nu \ , \tag{2.1b}$$

$$m\frac{d\hat{Q}^a}{d\tau} = -gf^{abc}\hat{p}^\mu A_\mu^b \hat{Q}^c \ . \tag{2.1c}$$

Here, A_μ denotes the microscopic gauge field. The corresponding microscopic field strength $F^a_{\mu\nu}$ and the energy momentum tensor of the gauge fields $\Theta^{\mu\nu}$ are given by

$$F^a_{\mu\nu}[A] = \partial_\mu A^a_\mu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \ , \tag{2.2}$$

$$\Theta^{\mu\nu}[A] = \tfrac{1}{4}g^{\mu\nu}F^a_{\rho\sigma}F_a^{\rho\sigma} + F_a^{\mu\rho}F^{a\ \nu}_{\rho} \tag{2.3}$$

and f^{abc} are the structure constants of SU(N). We set $c = k_B = \hbar = 1$ and work in natural units, unless otherwise indicated. Note that the non-Abelian charges are also subject to dynamical evolution. Equation (2.1c) can be rewritten as $D_\tau Q = 0$, where $D_\tau = \frac{d\hat{x}^\mu}{d\tau}D_\mu$ is the covariant derivative along the world line, and $D^{ac}_\mu[A] = \partial_\mu\delta^{ac} + gf^{abc}A^b_\mu$ the covariant derivative in the adjoint representation. With Q_a and $F^a_{\mu\nu}$ transforming in the adjoint representation, the Wong equations can be shown to be invariant under gauge transformations. The equation (2.1c) ensures the conservation under dynamical evolution of the set of $N - 1$ Casimir of the $SU(N)$ group.[‡]

[‡]For $SU(2)$, it is easy to verify explicitly the conservation of the quadratic Casimir Q_aQ_a. For $SU(3)$, both the quadratic and cubic Casimir $d_{abc}Q_aQ_bQ_c$, where d_{abc} are the symmetric structure

The colour current associated to each particle can be constructed once the solutions of the Wong equations are known. For every single particle it reads

$$j_a^\mu(x) = g \int d\tau \frac{d\hat{x}^\mu}{d\tau} \, \hat{Q}_a(\tau) \; \delta^{(4)}(x - \hat{x}(\tau)) \; . \tag{2.4}$$

Employing the Wong equations (2.1) we find that j^μ is covariantly conserved, $D_\mu j^\mu = 0$ [11]. Similarly, the energy momentum tensor associated to a single particle is given by [11]

$$t^{\mu\nu}(x) = \int d\tau \, \frac{d\hat{x}^\mu}{d\tau} \, \hat{p}^\nu(\tau) \, \delta^{(4)}(x - \hat{x}(\tau)) \; , \tag{2.5}$$

It is convenient to describe the ensemble of particles introducing a phase space density which depends on the whole set of coordinates x^μ, p^μ and Q_a. We define the function

$$n(x,p,Q) = \sum_i \int d\tau \, \delta^{(4)}(x - \hat{x}_i(\tau)) \; \delta^{(4)}(p - \hat{p}_i(\tau)) \; \delta^{(N^2-1)}(Q - \hat{Q}_i(\tau)) \; , \tag{2.6}$$

where the index i labels the particles. This distribution function is constructed in such a way that the colour current

$$J_a^\mu(x) = g \int d^4p \, d^{(N^2-1)}Q \, \frac{p^\mu}{m} Q_a \, n(x,p,Q) \tag{2.7}$$

coincides with the sum over all currents associated to the individual particles, $J_a^\mu = \sum_i j_a^\mu$, and is covariantly conserved, $D_\mu J^\mu = 0$. It is convenient to make the following changes in the choice of the distribution function. We will define a new function $f(x,p,Q)$ such that the physical constraints like the on-mass shell condition, positive energy and conservation of the group Casimirs are factored out into the phase space measure. We introduce the momentum measure[§]

$$dP = d^4p \, 2\theta(p_0)\delta(p^2 - m^2) \; , \tag{2.8}$$

and the measure for the colour charges

$$dQ = d^3Q \, c_R \delta(Q_a Q_a - q_2) \; , \tag{2.9}$$

in the case of SU(2). For $SU(3)$ the measure is

$$dQ = d^8Q \, c_R \delta(Q_a Q_a - q_2)\delta(d_{abc}Q^a Q^b Q^c - q_3) \; . \tag{2.10}$$

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constants of the group, are conserved under the dynamical evolution. The last conservation can be checked using (2.1c) and a Jacobi-like identity which involves the symmetric d_{abc} and antisymmetric f_{abc} constants.

[§]Note that in [1], we used a slightly different normalization of the measure. Here, the measure has an additional factor of $(2\pi)^3$.

For $SU(N)$, $N-1$ δ-functions ensuring the conservation of the set of $N-1$ Casimirs have to be introduced into the measure dQ. We have also introduced the representation-dependent normalization constant c_R into the measure dQ, which is fixed by requiring $\int dQ = 1$ (see appendix A). The constant C_2 is defined as

$$\int dQ Q_a Q_b = C_2 \delta_{ab} \ , \tag{2.11}$$

and depends on the group representation of the particles.

With these conventions the colour current (2.7) reads now

$$J_a^\mu(x) = g \int dP dQ \, p^\mu \, Q_a \, f(x,p,Q) \ , \tag{2.12}$$

while the energy momentum tensor associated to the particles obtains as

$$t^{\mu\nu}(x) = \int dP dQ \ p^\mu p^\nu \ f(x,p,Q) \ . \tag{2.13}$$

We will now come to the dynamical equation of the microscopic distribution functions $n(x,p,Q)$ and $f(x,p,Q)$, which will serve as the starting point for the subsequent formalism. The dynamical equation for $n(x,p,Q)$ is the same as for $f(x,p,Q)$. This is so because the physical constraints which we have factored out as to obtain $f(x,p,Q)$ are not affected by the Wong equations. Employing (2.1), we find

$$p^\mu \left(\frac{\partial}{\partial x^\mu} - g f^{abc} A_\mu^b Q^c \frac{\partial}{\partial Q^a} - g Q_a F_{\mu\nu}^a \frac{\partial}{\partial p_\nu} \right) f(x,p,Q) = 0 \ , \tag{2.14a}$$

which can be checked explicitly by direct inspection of (2.6) into (2.14a) (see appendix A of [5]). In a self-consistent picture this equation is completed with the Yang-Mills equation,

$$(D_\mu F^{\mu\nu})_a(x) = J_a^\nu(x) \ , \tag{2.14b}$$

and the current being given by (2.12). It is worth noticing that (2.14) is *exact* in the sense that no approximations have been made so far. The effects of collisions are included inasmuch as the Wong equations do account for them, although (2.14a) looks formally like a *collisionless* Boltzmann equation.

For the energy momentum tensor of the gauge fields we find

$$\partial_\mu \Theta^{\mu\nu} = -F_a^{\nu\mu} J_\mu^a \ . \tag{2.15}$$

On the other hand, using (2.14a) and the definition (2.13) we find that

$$\partial_\mu t^{\mu\nu} = \ F_a^{\nu\mu} J_\mu^a \ . \tag{2.16}$$

which establishes that the total energy momentum tensor is conserved, $\partial_\mu(\Theta^{\mu\nu} + t^{\mu\nu}) = 0$.

To finish the review of the microscopic description of the system, let us recall the gauge symmetry properties of the set of equations (2.14) (see [5] for more details). From the definition of $f(x,p,Q)$ it follows that it transforms as a scalar under a (finite) gauge transformation, $f'(x,p,Q') = f(x,p,Q)$. This implies the gauge covariance of (2.14b) because the current (2.7) transforms like the vector Q_a in the adjoint. The non-trivial dependence of f on the non-Abelian colour charges implies that the partial derivative $\partial_\mu f$ does not yet transform as a scalar, but rather its covariant derivative $D_\mu f$, which is given by

$$D_\mu[A]f(x,p,Q) \equiv [\partial_\mu - gf^{abc}Q_c A_{\mu,b}\partial_a^Q]f(x,p,Q) \ . \tag{2.17}$$

Notice that (2.17) combines the first two terms of (2.14a). (Here, and in the sequel we use the shorthand notation $\partial_\mu \equiv \partial/\partial x^\mu$, $\partial_\mu^p \equiv \partial/\partial p^\mu$ and $\partial_a^Q \equiv \partial/\partial Q^a$.) The invariance of the third term in (2.14a) follows from the trivial observation that $Q_a F^a_{\mu\nu}$ is invariant under gauge transformations. This establishes the gauge invariance of (2.14a).

III. STATISTICAL AVERAGES

If the system under study contains a large number of particles it is impossible to follow their individual trajectories. One has then to switch to a statistical description of the system.

In this section, we describe in detail the statistical average to be used in the sequel. As we are studying classical point particles in phase space, the appropriate statistical average corresponds to the Gibbs ensemble average for classical systems. We will review the main features of this procedure, defined in phase space. Let us remark that this derivation is completely general, valid for any classical system, and does not require equilibrium situations.

We will introduce two basic functions. The first one is the phase space density function $\mathcal{N}$ which gives, after integration over a phase space volume element, the number of particles contained in that volume. In a microscopic description it is just a deterministic quantity, and a function of the time t, the vectors $\mathbf{x}$ and $\mathbf{p}$, and the set of canonical (Darboux) variables $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$ associated to the colour charges Q (see appendix A). For $SU(N)$, there are $N(N-1)/2$ pairs of canonical variables, which we denote as $\boldsymbol{\phi} = (\phi_1, \dots, \phi_{N(N-1)/2})$ and $\boldsymbol{\pi} = (\pi_1, \dots, \pi_{N(N-1)/2})$. The microscopic phase space density is given by

$$\mathcal{N}(\mathbf{x},\mathbf{p},\boldsymbol{\phi},\boldsymbol{\pi}) = \sum_i \delta^{(3)}(\mathbf{x}-\hat{\mathbf{x}}_i(t))\ \delta^{(3)}(\mathbf{p}-\hat{\mathbf{p}}_i(t))\ \delta(\boldsymbol{\phi}-\hat{\boldsymbol{\phi}}_i(t))\ \delta(\boldsymbol{\pi}-\hat{\boldsymbol{\pi}}_i(t)) \ , \tag{3.1}$$

where the sum runs over all particles of the system, and $(\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_i, \hat{\boldsymbol{\phi}}_i, \hat{\boldsymbol{\pi}}_i)$ refers to the trajectory of the i-th particle in phase space. Then $\mathcal{N} d\mathbf{x}\, d\mathbf{p}\, d\boldsymbol{\phi} d\boldsymbol{\pi}$ gives the number of particles at time t in an infinitesimal volume element of phase space around the point $z = (\mathbf{x},\mathbf{p},\boldsymbol{\phi},\boldsymbol{\pi})$

Let us now define the distribution function $\mathcal{F}$ of the microstates of a system of L identical classical particles. Due to Liouville's theorem, $d\mathcal{F}/dt = 0$. Thus, it can be normalized as

$$\int dz_1 dz_2 \dots dz_L\, \mathcal{F}(z_1, z_2, \dots, z_L, t) = 1 \ . \tag{3.2}$$

Here z_i denotes all the phase space variables associated to the particle i. For simplicity we have considered that there is only one species of particles in the system, although the generalization to several species of particles is rather straightforward.

The statistical average of any function $\mathcal{G}$ defined in phase space is given by

$$\langle \mathcal{G} \rangle = \int dz_1 dz_2, \dots dz_L \, \mathcal{G}(z_1, z_2, \dots, z_L) \, \mathcal{F}(z_1, z_2, \dots, z_L, t) \ . \tag{3.3}$$

The one-particle distribution function is obtained from $\mathcal{F}$ as

$$f_1(z_1, t) = V \int dz_2 \dots dz_L \mathcal{F}(z_1, z_2, \dots, z_L, t) \ , \tag{3.4}$$

where V denotes the phase space volume. Correspondingly, the two-particle distribution function is

$$f_2(z_1, \dots, z_k, t) = V^2 \int dz_3 \dots dz_L \, \mathcal{F}(z_1, z_2, \dots, z_L, t) \ , \tag{3.5}$$

and similarly for the k-particle correlators. A complete knowledge of $\mathcal{F}$ would allow us to obtain all the set of $(f_1, f_2, \dots, f_L)$ functions; this is, however, not necessary for our present purposes.

Notice that we have allowed for an explicit dependence on the time t of the function $\mathcal{F}$, as this would be typically the case in out of equilibrium situations. We will drop this t dependence from now on to simplify the formulas.

Using the above definition one can obtain the mean value of the microscopic phase space density. Microscopically $\mathcal{N}(z) = \sum_{i=1}^{L} \delta(z - \hat{z}_i)$, where $\hat{z}_i$ describes the trajectory in phase space of the particle i. The statistical average of this function is

$$\langle \mathcal{N}(z) \rangle = \int dz_1 dz_2, \dots, dz_L \mathcal{F}(z_1, z_2, \dots, z_L) \sum_{i=1}^{L} \delta(z - \hat{z}_i) = \frac{L}{V} f_1(z) \ . \tag{3.6}$$

The second moment $\langle \mathcal{N}(z) \mathcal{N}(z') \rangle$ can equally be computed, and it is not difficult to see that gives

$$\langle \mathcal{N}(z) \mathcal{N}(z') \rangle = \frac{L}{V} \delta(z - z') f_1(z) + \frac{L(L-1)}{V^2} f_2(z, z') \ . \tag{3.7}$$

Let us now define a deviation of the phase space density from its mean value

$$\delta \mathcal{N}(z) \equiv \mathcal{N}(z) - \langle \mathcal{N}(z) \rangle \ . \tag{3.8}$$

By definition $\langle \delta \mathcal{N}(z) \rangle = 0$, although the second moment of this statistical fluctuation does not vanish in general, since

$$\langle \delta \mathcal{N}(z) \delta \mathcal{N}(z') \rangle = \langle \mathcal{N}(z) \mathcal{N}(z') \rangle - \langle \mathcal{N}(z) \rangle \langle \mathcal{N}(z') \rangle \ . \tag{3.9}$$

If the number of particles is large $L \gg 1$ then one then finds

$$\langle \delta \mathcal{N}(z) \delta \mathcal{N}(z') \rangle = \left(\frac{L}{V} \right) \delta(z - z') f_1(z) + \left(\frac{L}{V} \right)^2 g_2(z, z') \ , \tag{3.10}$$

where the function $g_2(z, z') = f_2(z, z') - f_1(z) f_1(z')$ measures the two-particle correlations in the system. For a completely uncorrelated system $g_2 = 0$.

Notice that the above statistical averages are well defined in the thermodynamic limit, $L, V \to \infty$ but L/V remaining constant. Higher order moments and higher order correlators can as well be defined.

We finally point out that the the function $\mathcal{N}(\mathbf{x}, \mathbf{p}, \boldsymbol{\phi}, \boldsymbol{\pi})$ agrees with the microscopic function $f(x, p, Q)$ introduced earlier, except for a representation-dependent normalization constant c_R as introduced in sect. II. We will also swallow the density factors L/V into the mean functions $\bar{f}$. Those small changes in the normalizations allow to simplify slightly the notations of the equations.

IV. AVERAGING THE MICROSCOPIC EQUATIONS

A. The mean fields and the fluctuations

In this section we perform the step from a microscopic to a macroscopic formulation of the problem. Using the prescription explained in sect. III, we take statistical averages of the microscopic equations (2.14). This implies that the distribution function $f(x, p, Q)$, which in the microscopic picture is a deterministic quantity, has now a probabilistic nature and can be considered as a random function, given by its mean value and statistical (random) fluctuation about it. Let us define the quantities

$$f(x, p, Q) = \bar{f}(x, p, Q) + \delta f(x, p, Q) \ , \tag{4.1a}$$

$$J^\mu_a(x) = \bar{J}^\mu_a(x) + \delta J^\mu_a(x) \ , \tag{4.1b}$$

$$A^a_\mu(x) = \bar{A}^a_\mu(x) + a^a_\mu(x) \ , \tag{4.1c}$$

where the quantities carrying a bar denote the mean values, e.g. $\bar{f} = \langle f \rangle$, $\bar{J} = \langle J \rangle$ and $\bar{A} = \langle A \rangle$, while the mean value of the statistical fluctuations vanish by definition, $\langle \delta f \rangle = 0$, $\langle \delta J \rangle = 0$ and $\langle a \rangle = 0$. This separation into mean fields and statistical, random fluctuations corresponds effectively to a split into low frequency (long wavelength) modes associated to the mean fields, and high frequency (short wavelength) modes associated to the fluctuations.** We also split the field strength tensor as

$$F^a_{\mu\nu} = \bar{F}^a_{\mu\nu} + f^a_{\mu\nu} \ , \tag{4.2a}$$

$$\bar{F}^a_{\mu\nu} = F^a_{\mu\nu}[\bar{A}] \ , \tag{4.2b}$$

$$f^a_{\mu\nu} = (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a + g f^{abc} a^b_\mu a^c_\nu \ , \tag{4.2c}$$

using also $\bar{D}_\mu \equiv D_\mu[\bar{A}]$. The term $f^a_{\mu\nu}$ contains terms linear and quadratic in the fluctuations. Note that the statistical average of the field strength $\langle F^a_{\mu\nu} \rangle$ is given by $\langle F^a_{\mu\nu} \rangle = \bar{F}^a_{\mu\nu} + g f^{abc} \langle a^b_\mu a^c_\nu \rangle$, due to quadratic terms contained in $f^a_{\mu\nu}$.

In the same light, we split the energy momentum tensor of the gauge fields into the part from the mean fields and the fluctuations, according to

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**In the close-to-equilibrium plasma (sect. VII), we identify the relevant momentum scales explicitly.

$$\Theta^{\mu\nu} = \bar{\Theta}^{\mu\nu} + \theta^{\mu\nu} \tag{4.3a}$$

$$\bar{\Theta}^{\mu\nu} = \tfrac{1}{4}g^{\mu\nu}\bar{F}^a_{\rho\sigma}\bar{F}^{\rho\sigma}_a + \bar{F}^{\mu\rho}_a\bar{F}^{a\ \nu}_{\rho} \ , \tag{4.3b}$$

$$\theta^{\mu\nu} = \tfrac{1}{2}g^{\mu\nu}\bar{F}^a_{\rho\sigma}f^{\rho\sigma}_a + \bar{F}^{\mu\rho}_a f^a_{\rho\nu} + \bar{F}^{\nu\rho}_a f^a_{\rho\mu} + \tfrac{1}{4}g^{\mu\nu}f_{\rho\sigma,a}f^{\rho\sigma,a} + f^{\mu\rho}_a f^{a\ \nu}_{\rho} \ , \tag{4.3c}$$

The term $\theta^{\mu\nu}$ contains the fluctuations up to quartic order. Due to the non-linear character of the theory, we find that the ensemble average of the energy momentum tensor reads $\langle\Theta^{\mu\nu}\rangle = \bar{\Theta}^{\mu\nu} + \langle\theta^{\mu\nu}\rangle$.

B. Dynamical equations for the mean fields and the fluctuations

We perform now the step from the microscopic to the macroscopic Boltzmann equation, taking the statistical average of (2.14). This yields the dynamical equation for the mean values,

$$p^\mu\left(\bar{D}_\mu - gQ_a\bar{F}^a_{\mu\nu}\partial^\nu_p\right)\bar{f} = \langle\eta\rangle + \langle\xi\rangle \ . \tag{4.4a}$$

We have made use of the covariant derivative of f as introduced in (2.17). The macroscopic Yang-Mills equations are obtained as

$$\bar{D}_\mu\bar{F}^{\mu\nu} + \langle J^\nu_{\rm fluc}\rangle = \bar{J}^\nu \ . \tag{4.4b}$$

In (4.4), we collected all terms quadratic or cubic in the fluctuations into the functions $\eta(x,p,Q), \xi(x,p,Q)$ and $J_{\rm fluc}(x)$. They read explicitly

$$\eta(x,p,Q) \equiv gQ_a\,p^\mu\partial^\nu_p f^a_{\mu\nu}\,\delta f(x,p,Q) \ , \tag{4.5a}$$

$$\xi(x,p,Q) \equiv gp^\mu f^{abc}Q^c\left(\partial^Q_a a^b_\mu\ \delta f(x,p,Q)\ + ga^a_\mu a^b_\nu\partial^\nu_p\bar{f}(x,p,Q)\right), \tag{4.5b}$$

$$J^{a,\nu}_{\rm fluc}(x) \equiv g\left\{f^{dbc}\bar{D}^\mu_{ad}a_{b,\mu}a^\nu_c + f^{abc}a_{b,\mu}\left((\bar{D}^\mu a^\nu - \bar{D}^\nu a^\mu)_c + gf^{cde}a^\mu_d a^\nu_e\right)\right\}(x). \tag{4.5c}$$

The corresponding equations for the fluctuations are obtained by subtracting (4.4) from (2.14). The result is

$$p^\mu\left(\bar{D}_\mu - gQ_a\bar{F}^a_{\mu\nu}\partial^\nu_p\right)\delta f = gQ_a(\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a p^\mu\partial^p_\nu\bar{f} \ ,$$

$$+gp^\mu a_{b,\mu}f^{abc}Q_c\partial^Q_a\bar{f} + \eta + \xi - \langle\eta+\xi\rangle \tag{4.6a}$$

$$\left(\bar{D}^2a^\mu - \bar{D}^\mu(\bar{D}_\nu a^\nu)\right)^a + 2gf^{abc}\bar{F}^{\mu\nu}_b a_{c,\nu} + J^{a,\mu}_{\rm fluc} - \langle J^{a,\mu}_{\rm fluc}\rangle = \delta J^{a,\mu} \ . \tag{4.6b}$$

The above set of dynamical equations is enough to describe all transport phenomena in the plasma.

While the dynamics of the mean fields (4.4) depends on correlators quadratic and cubic in the fluctuations, the dynamical equations for the fluctuations (4.6) do in addition depend on higher order terms (up to cubic order) in the fluctuations themselves. The dynamical equations for the higher order correlation functions are contained in (4.6). To see this, consider for example the dynamical equation for the correlators $\langle\delta f\,\delta f\rangle$. After multiplying (4.6a) with δf and taking the statistical average, we obtain

$$p^\mu \left(\bar{D}_\mu - gQ_a \bar{F}^a_{\mu\nu}\partial^\nu_p\right) \langle \delta f \, \delta f\rangle = gQ_a p^\mu \partial^p_\nu \bar{f} \ \left\langle (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a \delta f\right\rangle$$
$$+gp^\mu f^{abc} Q_c \partial^Q_a \bar{f} \ \langle a_{b,\mu}\, \delta f\rangle + \langle (\eta+\xi)\, \delta f\rangle \ . \qquad (4.7)$$

In the same way, we find for $\langle \delta f \, \delta f \, \delta f\rangle$ the dynamical equation

$$p^\mu \left(\bar{D}_\mu - gQ_a \bar{F}^a_{\mu\nu}\partial^\nu_p\right) \langle \delta f \, \delta f \, \delta f\rangle = gQ_a p^\mu \partial^p_\nu \bar{f} \ \left\langle (\bar{D}_\mu a_\nu - \bar{D}_\nu a_\mu)^a \, \delta f \, \delta f\right\rangle - \langle \eta+\xi\rangle \langle \delta f \, \delta f\rangle$$
$$+gp^\mu f^{abc} Q_c \partial^Q_a \bar{f} \ \langle a_{b,\mu}\, \delta f \, \delta f\rangle + \langle (\eta+\xi)\, \delta f \, \delta f\rangle \ , \qquad (4.8)$$

and similarly for higher order correlators. Typically, the dynamical equations for correlators of n fluctuations will couple to correlators ranging from the order $(n-1)$ up to order $(n+2)$ in the fluctuations. From cubic order onwards, the back-coupling contains terms non-linear in the correlation functions.††

The dynamical equation for the energy momentum tensor of the gauge fields obtains from the average of (2.15). The corresponding one for the particles is found after integrating (2.14a) over $dPdQ\, p^\mu$. They read

$$\partial_\nu \bar{\Theta}^{\mu\nu} + \partial_\nu \langle \theta^{\mu\nu}\rangle = -\bar{F}^{\mu\nu}_a \bar{J}_{\nu a} - \langle f^{\mu\nu}_a \delta J_{\nu a}\rangle - \langle f^{\mu\nu}_a\rangle \bar{J}^a_\nu \ , \qquad (4.9)$$
$$\partial_\nu \bar{t}^{\mu\nu} = \bar{F}^{\mu\nu}_a \bar{J}_{\nu a} + \langle f^{\mu\nu}_a \delta J_{\nu a}\rangle + \langle f^{\mu\nu}_a\rangle \bar{J}^a_\nu \ , \qquad (4.10)$$

such that the total energy momentum tensor is conserved.

Finally, the condition for the microscopic current conservation translates, after averaging, into two equations, one for the mean fields, and another one for the fluctuation fields. From $\langle D_\mu J^\mu\rangle = 0$ we obtain

$$(\bar{D}_\mu \bar{J}^\mu)_a + g f_{abc} \langle a^b_\mu \delta J^{c,\mu}\rangle = 0 \ . \qquad (4.11)$$

For the fluctuation current, we learn from $D_\mu J^\mu - \langle D_\mu J^\mu\rangle = 0$ that

$$(\bar{D}_\mu \delta J^\mu)_a + g f_{abc} \left(a^b_\mu \bar{J}^\mu_c + a^b_\mu \delta J^\mu_c - \langle a^b_\mu \delta J^\mu_c\rangle\right) = 0 \ . \qquad (4.12)$$

In sect. V B, it is shown that (4.11) and (4.12) are consistent with the corresponding equations as obtained from the Yang-Mills equations.

C. Second moment approximation and small coupling expansion

The dynamical equations, as derived and presented here, are exact. No approximations have been performed. In order to solve them, it will be essential to apply some approximations, or to find a reasonable truncation for the hierarchy of dynamical equations for correlator functions. Here, we will indicate two approximation schemes, the second moment

††The resulting hierarchy of dynamical equations for the correlators is very similar to the BBGKY hierarchy. While the BBGKY hierarchy links the dynamical equations for n-particle distribution functions with each other, here, we rather have a hierarchy for the correlator functions.

approximation and the small coupling expansion. Although they have a distinct origin in the first place, we will see below (sect. V) that they are intimately linked due to the requirements of gauge invariance.

The so-called *second moment approximation* (sometimes also referred to as the *polarization approximation* [7]) is employed in order to simplify the dynamics of the fluctuations. In consists in equating

$$\eta = \langle \eta \rangle \ , \qquad \xi = \langle \xi \rangle \ , \qquad J_{\text{fluc}} = \langle J_{\text{fluc}} \rangle \ . \tag{4.13}$$

The essence of this approximation is that the dynamical equations for the correlators becomes homogeneous. It is easy to see that (4.7) or (4.8) depend only on quadratic or cubic correlators, respectively, once (4.13) is imposed. This approximation allows to cut the infinite hierarchy of equations down to a closed system of differential equations for both mean quantities and statistical fluctuations. The mean fields couple to quadratic correlators, and all correlators of degree n couple amongst each others. This turns the dynamical equation for the fluctuations (4.6) into a differential equation linear in the fluctuations. This approximation is viable if the fluctuations and the two-particle correlators are small (see also sect. VI).

A seemingly different approximations concerns the non-Abelian sector of the theory. We shall perform a systematic perturbative expansion in powers of the gauge coupling g, keeping only the leading order terms. This can be done, because the differential operator appearing in the Boltzmann equation (2.14a) or (4.4a) admits such an expansion. In a small coupling expansion, the term $gQ_a\bar{F}^a_{\mu\nu}\partial^\nu_p$ is suppressed by a power in g as compared to the leading order term $p^\mu\bar{D}_\mu$. Notice, that expanding the g appearing within $p^\mu\bar{D}_\mu$ of (2.17) is not allowed as it will break gauge invariance. In this spirit, we expand as well

$$\bar{f} = \bar{f}^{(0)} + g\,\bar{f}^{(1)} + g^2\,\bar{f}^{(2)} + \dots \tag{4.14}$$

and similarly for δf. This is at the basis for a systematic organization of the dynamical equations in powers of g. To leading order, this concerns in particular the cubic correlators in $\langle \eta \rangle$ and $\langle J_{\text{fluc}} \rangle$, which are suppressed by a power of g as compared to the quadratic ones.

In principle, after these approximations are done, it should be possible to express the correlators of fluctuations appearing in (4.4) through known functions. This requires finding a solution of the fluctuation dynamics first.

D. Collision integrals

Let us comment on the interpretation of $\langle \eta \rangle$ and $\langle \xi \rangle$ as collision integrals of the macroscopic Boltzmann equation. The functions $\langle \eta \rangle$ and $\langle \xi \rangle$ appear only after the splitting (4.1) has been performed. This introduces new terms in the corresponding Boltzmann equation (4.4) for the mean fields, which are interpreted as effective collision integrals for the macroscopic transport equation. In this formalism, the collision integrals appear naturally as correlators of statistical fluctuations. The fluctuations in the gauge fields cause random changes in the motion of the particles, while random changes in the distribution function of the particles induce changes in the gauge fields. This is having the same effect as collisions, and yields a precise recipe as to obtain collision integrals from the microscopic theory.

In this light, the second moment approximation (4.13) can be interpreted as neglecting the back-coupling of collisions to the dynamics of the fluctuations. Also, neglecting cubic correlators appearing in $\langle\eta\rangle$ or $\langle J_{\text{fluc}}\rangle$ in favour of quadratic ones, to leading order in an expansion in the gauge coupling and for small fluctuations, is interpreted as neglecting the three-particle collisions in favour of two-particle collisions.

In order to find explicitly the corresponding collision integrals for the non-Abelian plasma, one has to solve first the dynamical equations for the fluctuations in the background of the mean fields. This step is interpreted as integrating-out the fluctuations. In general, this is a difficult task, in particular due to the non-linear terms present in (4.6). As argued above, this will only be possible when some approximations have been performed.

In the Abelian limit, (4.4) and (4.6) reduce to the known set of kinetic equations for Abelian plasmas [7]. In this limit, only the collision integral $\langle\eta\rangle$ survives. Here it is known that $\langle\eta\rangle$ can be expressed explicitly as the Balescu-Lenard collision integral, after solving the dynamical equations for the fluctuations and computing the correlators involved [7]. This proofs in a rigorous way the correspondence between fluctuations and collisions in an Abelian plasma.

V. CONSISTENCY WITH GAUGE SYMMETRY

In this section we shall discuss the consistency of the present approach with the requirements of the non-Abelian gauge symmetry. This discussion will concern the consistency of the general set of equations. The question of consistent approximations will be raised as well. In this section, we shall for convenience switch to a matrix notation, using the conventions $A \equiv A^a t_a$, $Q \equiv Q^a t_a$ etc., as well as $[t_a, t_b] = f_{abc} t^c$ and Tr $t_a t_b = -\frac{1}{2}\delta_{ab}$.

A. Gauge covariance of the macroscopic equations

As a consequence of the Wong equations being gauge invariant, we already established in sect. II that the microscopic set of equations (2.14) transforms covariantly under (finite) gauge transformations

$$gA'_\mu = U(x)(\partial_\mu + gA_\mu)U^{-1}(x)\ , \quad U(x) = \exp -g\epsilon^a(x)t_a\ , \tag{5.1}$$

with parameter $\epsilon^a(x)$. We also have $f'(x,p,Q') = f(x,p,Q)$, and

$$Q' = U(x)\, Q\, U^{-1}(x)\ , \quad \partial'_Q = U^{-1}(x)\, \partial_Q\, U(x)\ , \quad F'_{\mu\nu} = U(x)\, F_{\mu\nu}\, U^{-1}(x)\ . \tag{5.2}$$

The question raises as to which extend this symmetry is conserved under the statistical average, performed when switching to a macroscopic description. This concerns in particular the subsequent split of the gauge field into a mean (or background) field, and a fluctuation field

$$A_\mu = \bar{A}_\mu + a_\mu\ . \tag{5.3}$$

This separation is very similar to what is done in the background field method (BFM) [15]. Two symmetries are left after the splitting is performed, the *background gauge symmetry*,

$$g\bar{A}'_\mu = U(x)(\partial_\mu + g\bar{A}_\mu)U^{-1}(x) \ , \quad a'_\mu = U(x)\, a_\mu \, U^{-1}(x) \ , \tag{5.4}$$

and the *fluctuation gauge symmetry*,

$$g\bar{A}'_\mu = 0 \ , \quad ga'_\mu = U(x) \left(\partial_\mu + g(\bar{A}_\mu + a_\mu)\right) U^{-1}(x) \ . \tag{5.5}$$

Under the background gauge symmetry, the fluctuation field transforms covariantly (as a vector in the adjoint). In the first step, we will split (2.14a) according to (5.3). It follows trivially, that the resulting equation is invariant under both (5.4) and (5.5), if both $\bar{f}$ and δf transform as f, that is, as a scalar.

The next step involves the statistical average. We require that the statistical average of the fluctuation vanishes, $\langle a\rangle = 0$. This constraint is fully compatible with the background gauge symmetry, as $\langle a\rangle = 0$ is invariant under (5.4). Any inhomogeneous transformation law for a, and in particular (5.5), can no longer be a symmetry of the macroscopic equations as the constraint $\langle a\rangle = 0$ is not invariant.[‡‡]

We rewrite now the transport equations in matrix convention. We have

$$p^\mu \left(\bar{D}_\mu + 2g \ \mathrm{Tr}\,(Q\bar{F}_{\mu\nu})\, \partial_p^\nu\right) \bar{f} = \langle\eta\rangle + \langle\xi\rangle \ . \tag{5.6a}$$

$$\left[\bar{D}_\mu, \bar{F}^{\mu\nu}\right] + \langle J^\nu_{\text{fluc}}\rangle \ = \bar{J}^\nu \ . \tag{5.6b}$$

with

$$\eta(x,p,Q) = -2g \ \mathrm{Tr}\,(Q\, f_{\mu\nu})\, p^\mu \partial_p^\nu \delta f(x,p,Q) \ , \tag{5.7a}$$

$$\xi(x,p,Q) = -2gp^\mu \ \mathrm{Tr}\,([Q,\partial^Q]\, a_\mu)\, \delta f(x,p,Q) \ - 2g^2 \ \mathrm{Tr}\,([a_\mu, a_\nu]Q)\, p^\mu \partial_p^\nu \bar{f}(x,p,Q), \tag{5.7b}$$

$$J^\nu_{\text{fluc}}(x) = g\left[\bar{D}^\mu, [a_\mu, a^\nu]\right] + \left[a_\mu, [\bar{D}^\mu, a^\nu] - [\bar{D}^\nu, a^\mu]\right] + g^2\, [a_\mu, [a^\mu, a^\nu]] \, . \tag{5.7c}$$

and

$$p^\mu \left(\bar{D}_\mu - g \ \mathrm{Tr}\,(Q\bar{F}_{\mu\nu})\, \partial_p^\nu\right) \delta f \ = -2\, g\mathrm{Tr}\,(Q[\bar{D}_\mu, a_\nu] - Q[\bar{D}_\nu, a_\mu])\, p^\mu \partial^p_\nu \bar{f}$$
$$-2g\, p^\mu \ \mathrm{Tr}\,([Q,\partial^Q]a_\mu)\, \bar{f} + \eta + \xi - \langle \eta + \xi\rangle \ , \tag{5.8a}$$

$$\left[\bar{D}_\nu, [\bar{D}^\nu, a^\mu]\right] - \left[\bar{D}^\mu, [\bar{D}_\nu, a^\nu]\right] \ + 2g[\bar{F}^{\mu\nu}, a_\nu] + J^\mu_{\text{fluc}} - \langle J^\mu_{\text{fluc}}\rangle = \delta J^\mu \ . \tag{5.8b}$$

It is now straightforward to realize that (5.6) – (5.8) transform covariantly under (5.4). It suffices to employ the cyclicity of the trace, and to note that a_μ and background covariant derivatives of it transform covariantly.

[‡‡]This is similar to what happens in the BFM, where the fluctuation gauge symmetry can no longer be seen once the expectation value of the fluctuation field is set to zero. However, the symmetry (5.5) will be observed in both (4.4a) and (4.6a), as long as the terms linear in $\langle a\rangle$ are retained.

B. Consistent current conservation

In (4.11) and (4.12), we have given the equations which imply the covariant current conservation of the mean and the fluctuation current. However, this information is contained both in the transport and in the Yang-Mills equation. It remains to be shown that these equations are self-consistent.

We start with the mean current $\bar{J}$. Performing $g \int dPdQ\, Q$ of the transport equation (5.6a), we find

$$0 = [\bar{D}_\mu, \bar{J}^\mu] + g\langle [a_\mu, \delta J^\mu] \rangle \ . \tag{5.9}$$

This is (4.11). Here, we used that

$$\int dP \ \eta(x,p,Q) = 0 \ , \tag{5.10a}$$

$$\int dP \ p^\mu F_{\mu\nu} \partial_p^\nu f(x,p,Q) = 0 \ , \tag{5.10b}$$

$$\int dPdQ\, Q \ \xi(x,p,Q) = -[a_\mu, \delta J^\mu] \ , \tag{5.10c}$$

$$g \int dPdQ\, Q \ p^\mu \bar{D}_\mu \bar{f}(x,p,Q) = [\bar{D}_\mu, \bar{J}^\mu] \ . \tag{5.10d}$$

Taking the background covariant derivative of (5.6b), we find

$$0 = [\bar{D}_\mu, \bar{J}^\mu] - [\bar{D}_\mu, \langle J^\mu_{\rm fluc} \rangle] \ , \tag{5.11}$$

which has to be consistent with (5.9). Thus, combining these two equations we end up with the consistency condition

$$0 = [\bar{D}_\mu, \langle J^\mu_{\rm fluc} \rangle] + g\langle [a_\mu, \delta J^\mu] \rangle \ . \tag{5.12}$$

In the appendix B, we established the identity

$$0 = [\bar{D}_\mu, J^\mu_{\rm fluc}] + g[a_\mu, \delta J^\mu] + g[a_\mu, \langle J^\mu_{\rm fluc} \rangle] \ , \tag{5.13}$$

which follows using the explicit expressions for δJ^μ from (5.8b) and for $J_{\rm fluc}$ from (5.7c). Taking the average of (5.13) reduces it to (5.12), and establishes the self-consistent conservation of the mean current.

The analogous consistency equation for the fluctuation current obtains from (5.8a) after performing $g \int dPdQ\, Q$, and reads

$$0 = [\bar{D}_\mu, \delta J^\mu] + g[a_\mu, \delta J^\mu] + g[a_\mu, \bar{J}^\mu] - g\langle [a_\mu, \delta J^\mu] \rangle \ . \tag{5.14}$$

This is (4.12). Here, in addition to (5.10), we made use of

$$2\, g \int dPdQ\, Q \ {\rm Tr}\left([Q, \partial^Q]\, a_\mu \right) \bar{f}(x,p,Q) = g[a_\mu, \bar{J}^\mu] \ . \tag{5.15}$$

The background covariant derivative of (5.8b) is given as

$$0 = [\bar{D}_\mu, \delta J^\mu] + g\left[a_\nu, [\bar{D}_\mu, \bar{F}^{\mu\nu}]\right] - [\bar{D}_\mu, J^\mu_{\text{fluc}}] + [\bar{D}_\mu, \langle J^\mu_{\text{fluc}}\rangle] \; . \tag{5.16}$$

Subtracting these equations yields the consistency condition

$$0 = [\bar{D}_\mu, J^\mu_{\text{fluc}}] + g[a_\mu, \delta J^\mu] - [\bar{D}_\mu, \langle J^\mu_{\text{fluc}}\rangle] - g\langle [a_\mu, \delta J^\mu]\rangle + g[a_\mu, \bar{J}^\mu] - g\left[a_\nu, [\bar{D}_\mu, \bar{F}^{\mu\nu}]\right] \; . \tag{5.17}$$

Using (5.6b), (5.12) and (5.13) we confirm (5.17) explicitly. This establishes the self-consistent conservation of the fluctuation current.

C. Gauge-consistent approximations

We close this section with a comment on the gauge consistency of *approximate* solutions. The consistent current conservation can no longer be taken for granted when it comes to finding approximate solutions of the equations. On the other hand, finding an explicit solution will require some type of approximations to be performed. The relevant question in this context is to know which approximations will be consistent with gauge invariance.

Consistency with gauge invariance requires that approximations have to be consistent with the background gauge symmetry. From the general discussion above we can already conclude that dropping any of the explicitly written terms in (4.4) – (4.6) is consistent with the background gauge symmetry (5.4). This holds in particular for the polarization or second moment approximation (4.13).

Consistency of the second moment approximation (4.13) with covariant current conservation turns out to be more restrictive. Employing $J_{\text{fluc}} = \langle J_{\text{fluc}}\rangle$ implies that (5.12) is only satisfied, if in addition

$$0 = \left[\bar{D}_\nu, \langle [a_\mu, [a^\mu, a^\nu]]\rangle\right] \tag{5.18}$$

holds true. This is in accordance with neglecting cubic correlators for the collision integrals.

Similarly, the consistent conservation of the fluctuation current implies the consistency condition (5.17), and holds if

$$0 = [a_\mu, \langle J^\mu_{\text{fluc}}\rangle] \tag{5.19}$$

vanishes. It is interesting to note that the consistent current conservation relates the second moment approximation with the neglection of correlators of gauge field fluctuations. We conclude, that (4.13) with (5.18) and (5.19) form a gauge-consistent set of approximations.

VI. PHYSICAL SCALES: CLASSICAL VERSUS QUANTUM PLASMAS

In the remaining part of the paper, we study the classical and quantum non-Abelian plasma close to equilibrium. Prior to this, we shall present a discussion of the relevant physical scales of both relativistic classical and quantum plasmas, close to equilibrium. We will restore here the fundamental constants $\hbar, c$ and k_B in the formulas.

To discuss the relevant physical scales in the classical non-Abelian plasma, it will turn out convenient to discuss first the simpler Abelian case, which has been considered in detail in the literature [7]. At equilibrium the classical distribution function is given by the relativistic Maxwell distribution,

$$\bar{f}^{\rm eq}(p_0) = A e^{-p_0/k_B T} \ , \tag{6.1}$$

where A is a dimensionful constant which is fixed once the mean density of particles in the system is known. For massless particles $p_0 = pc$, and if no further internal degrees of freedom are present, the mean density is given by $\bar{N} = 8\pi A (k_B T/c)^3$. The inter-particle distance is then $\bar{r} \sim \bar{N}^{-1/3}$. As we are considering a classical plasma, we are assuming $\bar{r} \gg \lambda_{\rm dB}$, where $\lambda_{\rm dB}$ is the de Broglie wave length, $\lambda_{\rm dB} \sim \hbar/p$, with p some typical momenta associated to the particles, thus $p \sim k_B T/c$. The previous inequality implies therefore $A \ll 1/\hbar^3$, which is the condition under which quantum statistical effects can be neglected.

There is another typical scale in a plasma close to equilibrium, which is the Debye length r_D. The Debye length is the distance over which the screening effects of the electric fields in the plasma are felt. For an electromagnetic plasma, the Debye length squared is given by [7]

$$r_D^2 = k_B T/4\pi \bar{N} e^2 \ . \tag{6.2}$$

Notice that the electric charge contained in the above formula is a dimensionful parameter: it is just the electric charge of the point particles of the system.

In the classical case, and in the absence of the fundamental constant $\hbar$, the only dimensionless quantity that can be constructed from the basic scales of the problem is the *plasma parameter* ϵ. The plasma parameter is defined as the ratio [7]

$$\epsilon = \bar{r}^3/r_D^3 \ . \tag{6.3}$$

The quantity $1/\epsilon$ gives the number of particles contained in a sphere of radius r_D. If $\epsilon \ll 1$ this implies that a large number of particles are in that sphere, and thus a large number of particles are interacting in this volume, and the collective character of their interactions in the plasma can not be neglected. For the kinetic description to make sense, ϵ has to be small [7]. This does not require, in general, that the interactions have to be weak and treated perturbatively.

Let us now consider the non-Abelian plasma. The inter-particle distance is defined as in the previous case. The main differences with respect to the Abelian case concerns the Debye length, defined as the distance over which the screening effects of the non-Abelian electric fields in the plasma are noticed. It reads

$$r_D^2 = k_B T/4\pi \bar{N} g^2 C_2 \ , \tag{6.4}$$

where C_2, defined in (2.11), is a dimensionful quantity, carrying the same dimensions as the electric charge squared in (6.2). The coupling constant g is a dimensionless parameter. In the non-Abelian plasma one can also construct the plasma parameter, defined as in (6.3).

It is interesting to note that there are two natural dimensionless parameters in the non-Abelian plasma: ϵ and g. The condition for the plasma parameter being small translates into

$$\left(\frac{4\pi C_2}{k_B T}\right)^{3/2} \bar{N}^{1/2} g^3 \ll 1 \ , \tag{6.5}$$

which is certainly satisfied for small gauge coupling constant $g \ll 1$. But it can also be fulfilled for a rarefied plasma. Thus, one may have a small plasma parameter *without* having a small gauge coupling constant. This is an interesting observation, since the inequalities $\epsilon \ll 1$ and $g \ll 1$ have different physical meanings. A small gauge coupling constant allows for treating the non-Abelian interaction perturbatively, while $\epsilon \ll 1$ just means having a collective field description of the physics occurring in the plasma. In principle, these two situations are different. If we knew how to treat the non-Abelian interactions *exactly*, we could also have a kinetic description of the classical non-Abelian plasmas without requiring $g \ll 1$.

Now we consider the quantum non-Abelian plasma, and consider the quantum counterparts of all the above quantities, as derived from quantum field theory. For a quantum plasma at equilibrium the one particle distribution function is

$$\bar{f}^{\rm eq}_{\rm B}(p_0) = \frac{1}{e^{p_0/k_B T} - 1} \ , \qquad \bar{f}^{\rm eq}_{\rm F}(p_0) = \frac{1}{e^{p_0/k_B T} + 1} \ , \tag{6.6}$$

where the subscript B/F refers to the bosonic/fermionic statistics. For a plasma of massless particles the mean density is $\bar{N} \sim (k_B T/\hbar c)^3$. The inter-particle distance $\bar{r} \sim \bar{N}^{-1/3}$ becomes of the same order that the de Broglie wavelength, which is why quantum statistics effects can not be neglected in this case.

The value of the Debye mass is obtained from quantum field theory. It depends on the specific quantum statistics of the particles and their representation of $SU(N)$. From the quantum Debye mass one can deduce the value of the Debye length, which is of order

$$r_D^2 \sim \frac{1}{g^2} \left(\frac{\hbar c}{k_B T}\right)^2 \ . \tag{6.7}$$

It is not difficult to check that the plasma parameter, defined as in (6.3), becomes proportional to g^3. Thus ϵ is small *if and only if* $g \ll 1$. This is so, because in a quantum field theoretical formulation one does not have the freedom to fix the mean density $\bar{N}$ in an arbitrary way, as in the classical case. This explains why the kinetic description of a quantum non-Abelian plasma is deeply linked to the small gauge coupling regime of the theory.

VII. THE CLASSICAL PLASMA CLOSE TO EQUILIBRIUM

In this section we put the method to work for a hot non-Abelian plasma close to equilibrium. A prerequisite for a kinetic description to be viable is a small plasma parameter $\epsilon \ll 1$. We shall ensure this by imposing a small gauge coupling constant $g \ll 1$. Then, all further approximations as detailed in the sequel can be seen as a systematic expansion in powers of the gauge coupling.

A. Non-Abelian Vlasov equations

We begin with the set of mean field equations (4.4) and neglect the effect of statistical fluctuations entirely, $\delta f \equiv 0$. In that case, (4.4) becomes the non-Abelian Vlasov equations [6]

$$p^\mu \left(\bar{D}_\mu - g\, Q_a \bar{F}^a_{\mu\nu}\, \partial^\nu_p\right) \bar{f} = 0 \ , \tag{7.1a}$$

$$\bar{D}_\mu \bar{F}^{\mu\nu} = \bar{J}^\nu \ , \tag{7.1b}$$

where the colour current is given by

$$\bar{J}^\mu_a(x) = g \sum_{\substack{\text{helicities}\\ \text{species}}} \int dP dQ\, Q_a p^\mu \bar{f}(x,p,Q) \ , \tag{7.1c}$$

We will omit the species and helicity indices on the distribution functions, and in the sequel, we will also omit the above sum, in order to keep the notation as simple as possible. We will solve (7.1a) perturbatively, as it admits a consistent expansion in powers of g. Close to equilibrium, we expand the distribution function as in (4.14) up to leading order in the coupling constant

$$\bar{f}(x,p,Q) = \bar{f}^{\rm eq}(p_0) + g\bar{f}^{(1)}(x,p,Q) \ . \tag{7.2}$$

In the strictly classical approach, the relativistic Maxwell distribution (6.1) at equilibrium is used. Here, we consider only massless particles, or massive particles with $m \ll T$, such that the masses can be neglected in a first approximation. We will consider also internal degrees of freedom, two helicities associated to every particle.

It is convenient to re-write the equations in terms of current densities. Consider the current densities

$$J^\rho_{a_1\cdots a_n}(x,p) = g\ p^\rho \int dQ\, Q_{a_1} \cdots Q_{a_n} f(x,p,Q), \tag{7.3a}$$

$$\mathcal{J}^\rho_{a_1\cdots a_n}(x,v) = \int d\tilde{P}\, J^\rho_{a_1\cdots a_n}(x,p) \ . \tag{7.3b}$$

Here, $v^\mu = (1, \mathbf{v})$ with $\mathbf{v}^2 = 1$. The measure $d\tilde{P}$ integrates over the radial components. It is related to (2.8) by $dP = d\tilde{P} d\Omega/4\pi$, and reads

$$d\tilde{P} = 4\pi\, dp_0\, d|\mathbf{p}|\, |\mathbf{p}|^2\, 2\Theta(p_0)\, \delta(p^2) \tag{7.4}$$

for massless particles. The colour current is obtained performing the remaining angle integration $J(x) = \int \frac{d\Omega}{4\pi} \mathcal{J}(x,v)$. From now on we will omit the arguments of the current density $\mathcal{J}$, unless necessary to avoid confusion.

We now insert (7.2) into (7.1) and expand in powers of g. The leading order term $p \cdot D\bar{f}^{\rm eq.}(p_0)$ vanishes. After multiplying (4.4a) by $gQ_a p^\rho/p_0$, summing over two helicities, and integrating over $d\tilde{P}dQ$, we obtain for the mean current density at order g

$$v^\mu \bar{D}_\mu \bar{\mathcal{J}}^\rho + m_D^2 v^\rho v^\mu \bar{F}_{\mu 0} = 0 \ , \tag{7.5a}$$

$$\bar{D}_\mu \bar{F}^{\mu\nu} = \bar{J}^\nu \ , \tag{7.5b}$$

with the Debye mass

$$m_D^2 = -2g^2 C_2 \int d\tilde{P} d\bar{f}^{\rm eq}(p)/dp \ . \tag{7.6}$$

The solution to (7.5a) is now constructed with the knowledge of the retarded Green's function

$$iv^\mu \bar{D}_\mu \, G_{ret}(x, y; v) = \delta^{(4)}(x - y) \ . \tag{7.7}$$

It reads

$$G_{ret}(x, y; v)_{ab} = -i\theta(x_0 - y_0)\delta^{(3)} \left(\mathbf{x} - \mathbf{y} - \mathbf{v}(x_0 - y_0)\right) \bar{U}_{ab}(x, y) \ , \tag{7.8}$$

where $\bar{U}_{ab}$ is the parallel transporter obeying $v^\mu \bar{D}^x_\mu \bar{U}_{ab}(x, y)|_{y=x-vt} = 0$, and $\bar{U}_{ab}(x, x) = \delta_{ab}$. One finds

$$\bar{J}^\mu_a(x) = -m_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int_0^\infty d\tau \ \bar{U}_{ab}(x, x - v\tau) v^\mu v^\nu \bar{F}_{\nu 0, b}(x - v\tau) \ . \tag{7.9}$$

The above colour current agrees with the hard thermal loop (HTL) colour current [4,5], except for the value of the Debye mass.

From (7.5a) it is easy to estimate the typical momentum scale of the mean fields. If the effects of statistical fluctuations are neglected (and as we will see, this is equivalent to neglecting collisions), the typical momentum scales associated to the mean current and the mean field strength are of the order of the Debye mass m_D. We will refer to those scales as soft scales. The momentum scales with momenta $\ll m_D$ will be referred to as ultra-soft from now on.

B. Leading order dynamics for mean fields and fluctuations

We now allow for small statistical fluctuations $\delta f(x, p, Q)$ around (7.2), writing

$$f(x, p, Q) = \bar{f}^{\rm eq.}(p_0) + g\bar{f}^{(1)}(x, p, Q) + \delta f(x, p, Q) \tag{7.10}$$

and re-write the approximations to (4.4) and (4.6) in terms of current densities and their fluctuations. Note that the fluctuations $\delta f(x, p, Q)$ in the close to equilibrium case are already of the order of g. This observation is important for the consistent approximation in powers of the gauge coupling. As a consequence, the term $g\bar{f}^{(1)}$ in (7.10) will now account for the ultra-soft modes for momenta $\ll m_D$. Integrating-out the fluctuations results in an effective theory for the latter.

As before, we obtain the dynamical equation for the mean current density at leading order in g, after multiplying (4.4a) by $gQ_a p^\rho/p_0$, summing over two helicities, and integrating over $d\tilde{P}dQ$. The result is

$$v^\mu \bar{D}_\mu \bar{\mathcal{J}}^\rho + m_D^2 v^\rho v^\mu \bar{F}_{\mu 0} = \langle \eta^\rho \rangle + \langle \xi^\rho \rangle \ , \tag{7.11a}$$

$$\bar{D}_\mu \bar{F}^{\mu\nu} + \langle J^\nu_{\rm fluc} \rangle = \bar{J}^\nu \ . \tag{7.11b}$$

In a systematic expansion in g, we have to neglect cubic correlator terms as compared to quadratic ones, as they are suppressed explicitly by an additional power in g. Therefore, we find to leading order

$$\eta_a^\rho = -g \int \frac{d\tilde{P}}{p_0} \left((\bar{D}_\mu a^\rho - \bar{D}^\rho a_\mu)^b \, \delta J^\mu_{ab}(x,p) - \frac{p^\rho}{p_0} (\bar{D}_\mu a_0 - \bar{D}_0 a_\mu)^b \, \delta J^\mu_{ab}(x,p) \right) \ , \tag{7.12a}$$

$$\xi_a^\rho = -g f_{abc} v^\mu \, a_\mu^b \, \delta \mathcal{J}^{c,\rho} \ , \tag{7.12b}$$

$$J^{\rho,a}_{\rm fluc} = g f^{dbc} \Big(\bar{D}^{ad}_\mu a^\mu_b a^\rho_c + \delta^{ad} a^b_\mu \Big(\bar{D}^\mu a^\rho - \bar{D}^\rho a^\mu \Big)^c \Big) \ . \tag{7.12c}$$

The same philosophy is applied to the dynamical equations for the fluctuations. To leading order in g, the result reads

$$\Big(v^\mu \bar{D}_\mu \, \delta \mathcal{J}^\rho \Big)_a = -m_D^2 v^\rho v^\mu \left(\bar{D}_\mu a_0 - \bar{D}_0 a_\mu \right)_a - g f_{abc} v^\mu a^b_\mu \bar{\mathcal{J}}^{c,\rho} \ , \tag{7.13a}$$

$$\Big(v^\mu \bar{\partial}_\mu \delta_{ac} \delta_{bd} + g \bar{A}^m_\mu \left(f_{amc} \, \delta_{bd} + f_{bmd} \delta_{ac} \right) \Big) \, \delta \mathcal{J}^\rho_{cd} = g v^\mu a^m_\mu \left(f_{mac} \, \delta_{bd} + f_{mbd} \delta_{ac} \right) \bar{\mathcal{J}}^\rho_{cd}, \tag{7.13b}$$

$$\Big(\bar{D}^2 a^\mu - \bar{D}^\mu (\bar{D} a) \Big)_a + 2 g f_{abc} \bar{F}^{\mu\nu}_b a_{c,\nu} = \delta J^\mu_a \ . \tag{7.13c}$$

The typical momentum scale associated to the fluctuations can be estimated from (7.13). We find that it is of the order of the Debye mass $\sim m_D$, that is, of the same order as the mean fields in (7.5). This confirms explicitly the discussion made above. The typical momentum scales associated to the mean fields in (7.11) is therefore $\ll m_D$.

C. Integrating-out the fluctuations

We solve the equations for the fluctuations (7.13) with an initial boundary condition for δf, and $a_\mu(t=0) = 0$. Exact solutions to (7.13a) and (7.13b) can be obtained.

Let us start by solving the homogeneous differential equation

$$v^\mu \bar{D}_\mu \, \delta \mathcal{J}^\rho = 0 \ , \tag{7.14}$$

with the initial condition $\delta \mathcal{J}^\mu_a(t=0, \mathbf{x}, v)$. It is not difficult to check, by direct inspection, that the solution to the homogeneous problem is

$$\delta \mathcal{J}^\rho_a(x,v) = \bar{U}_{ab}(x, x - vt) \, \delta \mathcal{J}^\rho_b(t=0, \mathbf{x} - \mathbf{v}t, v) \ . \tag{7.15}$$

The solution of (7.13a) is now constructed using the retarded Green's function (7.8). For $x_0 \equiv t \geq 0$ the complete solution can be expressed as

$$\delta \mathcal{J}^\rho_a(x,v) = -\int_0^\infty d\tau \, \bar{U}_{ab}(x, x_\tau) \Big(\, m_D^2 v^\rho v^\mu \left(\bar{D}_\mu a_0 - \bar{D}_0 a_\mu \right)^b (x_\tau) + g f_{bdc} v^\mu a^d_\mu(x_\tau) \bar{\mathcal{J}}^\rho_c(x_\tau, v) \Big)$$
$$+ \bar{U}_{ab}(x, x_t) \, \delta \mathcal{J}^\rho_b(x_t, v) \, . \tag{7.16}$$

We have introduced $x_\tau \equiv x - v\tau$, and thus $x_t = (0, \mathbf{x} - \mathbf{v}t)$. Since $a_\mu(t=0) = 0$, one can check that the above current obeys the correct initial condition.

The equation (7.13b) can be solved in a similar way. The solution is

$$
\begin{aligned}
\delta\mathcal{J}^\rho_{ab}(x,v) &= \bar{U}_{am}(x,x_t)\bar{U}_{bn}(x,x_t)\,\delta\mathcal{J}^\rho_{mn}(x_t,v) \\
&\quad -g\int_0^\infty d\tau\,\bar{U}_{am}(x,x_\tau)\bar{U}_{bn}(x,x_\tau)\,(f_{mpc}\delta_{nd}+f_{npd}\delta_{mc})\,v^\mu a^p_\mu(x_\tau)\bar{\mathcal{J}}^\rho_{cd}(x_\tau,v)\ .
\end{aligned} \quad (7.17)
$$

Now we seek for solutions to the equation (7.13c) with the colour current of the fluctuation as found above. However, notice that this equation is non-local in a_μ, which makes it difficult to find exact solutions. Nevertheless, one can solve the equation in an iterative way, by making a double expansion in both $g\bar{A}$ and $g\bar{\mathcal{J}}$. This is possible since the parallel transporter $\bar{U}$ admits an expansion in $g\bar{A}$, so that the current $\delta\mathcal{J}^\rho$ can be expressed as a power series in $g\bar{A}$

$$
\delta\mathcal{J}^\rho = \delta\mathcal{J}^{\rho(0)} + \delta\mathcal{J}^{\rho(1)} + \delta\mathcal{J}^{\rho(2)} + \cdots\ , \quad (7.18)
$$

and thus (7.13c) can be solved for every order in $g\bar{A}$. To lowest order in $g\bar{A}$, using $\bar{U}_{ab} = \delta_{ab} + \mathcal{O}(g\bar{A})$, equation (7.13c) becomes

$$
\partial^\mu\left(\partial_\mu a^{(0)}_{\nu,a} - \partial_\nu a^{(0)}_{\mu,a}\right) = \delta J^{(0)}_{\nu,a}\ . \quad (7.19)
$$

Using the one-sided Fourier transform [7] and (7.16) to find

$$
\begin{aligned}
\delta J^{\mu\,(0)}_{a+}(k) &= \Pi^{\mu\nu}_{ab}(k)a^{(0)}_{\nu,b}(k) - gf_{abc}\int\frac{d\Omega_{\mathbf{v}}}{4\pi}\frac{1}{-i\ k\cdot v}\int\frac{d^4q}{(2\pi)^4}v^\rho a^{b(0)}_\rho(q)\ \bar{\mathcal{J}}^{\mu,c}(k-q,v) \\
&\quad + \int\frac{d\Omega_{\mathbf{v}}}{4\pi}\frac{\delta\mathcal{J}^\mu_a(t=0,\mathbf{k},v)}{-i\ k\cdot v}\ ,
\end{aligned} \quad (7.20)
$$

where $\Pi^{\mu\nu}_{ab}(k)$ is the polarization tensor in the plasma, which reads

$$
\Pi^{\mu\nu}_{ab}(k) = \delta_{ab}m_D^2\left(-g^{\mu0}g^{\nu0} + k_0\int\frac{d\Omega_{\mathbf{v}}}{4\pi}\frac{v^\mu v^\nu}{k\cdot v}\right)\ , \quad (7.21)
$$

and agrees with the HTL polarization tensor of QCD [4,5], except in the value of the Debye mass. Retarded boundary conditions are assumed above, with the prescription $k_0 \to k_0 + i0^+$.

We solve (7.19) iteratively in momentum space for a_μ as an infinite power series in $g\bar{\mathcal{J}}$,

$$
a^{(0)}_\mu = a^{(0,0)}_\mu + a^{(0,1)}_\mu + a^{(0,2)}_\mu + \ldots \quad (7.22)
$$

where the second index counts the powers of the background current $g\bar{\mathcal{J}}$. Notice that in this type of Abelianized approximation, the equation (7.19) has a (perturbative) Abelian gauge symmetry associated to the fluctuation a_μ. This symmetry is only broken by the term proportional to $\bar{\mathcal{J}}$ in the current. It is an exact symmetry for the term $a^{(0,0)}_\mu$ in the above expansion. We will use this perturbative gauge symmetry in order to simplify the computations, and finally check that the results of the approximate collision integrals do not depend on the choice of the fluctuation gauge.

Using the one-sided Fourier transform, we find the following results for the longitudinal fields, in the gauge $\mathbf{k}\cdot\mathbf{a}^{(0,0)} = 0$,

$$a^{(0,0)}_{0,a\,+}(k) = \frac{1}{\mathbf{k}^2 - \Pi_L} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{\delta \mathcal{J}_{0,a}(t=0,\mathbf{k},v)}{-i\ k\cdot v}\ , \tag{7.23a}$$

$$a^{(0,1)}_{0,a\,+}(k) = \frac{-g f_{abc}}{\mathbf{k}^2 - \Pi_L} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{1}{-i\ k\cdot v} \int \frac{d^4q}{(2\pi)^4} v^\mu a^{b(0,0)}_\mu(q)\ \bar{\mathcal{J}}^c_0(k-q,v)\ , \tag{7.23b}$$

while we find

$$a^{T(0,0)}_{i,a\,+}(k) = \frac{1}{-k^2 + \Pi_T} \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{\delta \mathcal{J}^T_{i,a}(t=0,\mathbf{k},v)}{-i\ k\cdot v}\ , \tag{7.24a}$$

$$a^{T(0,1)}_{i,a\,+}(k) = \frac{-g f_{abc}}{-k^2 + \Pi_T} P^T_{ij}(\mathbf{k}) \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{1}{-i\ k\cdot v} \int \frac{d^4q}{(2\pi)^4} v^\mu a^{b(0,0)}_\mu(q)\ \bar{\mathcal{J}}^c_j(k-q,v)\ , \tag{7.24b}$$

for the transverse fields.[§§] The functions $\Pi_{L/T}(k)$ are the longitudinal/transverse polarization tensor of the plasma, $P^T_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/\mathbf{k}^2$ the transverse projector, and $a^T_i \equiv P^T_{ij} a_j$.

In the approximation $g \ll 1$, it will be enough to consider the solution of leading (zeroth) order in $g\bar{A}$, and the zeroth and first order in $g\bar{\mathcal{J}}$. The remaining terms are subleading in the leading logarithmic approximation. However, notice that we have all the tools necessary to compute the complete (perturbative) series. If we could solve equation (7.13c) exactly, it would not be necessary to use this perturbative expansion.

D. The statistical correlator of fluctuations

With the explicit expressions obtained in (7.20), (7.23) and (7.24), we can express all fluctuations in terms of initial conditions $\delta\mathcal{J}^\mu_a(t=0,\mathbf{x},v)$ and the mean fields.

In order to compute the correlator of initial conditions, we will make use of the result obtained in sect. III. For each species of particles or internal degree of freedom, the statistical average over initial conditions can be expressed as

$$\langle \delta f(t=0,\mathbf{x},p,Q)\ \delta f(t=0,\mathbf{x}',p',Q')\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(Q-Q')\bar{f}(x,p,Q) + \tilde{g}_2(\mathbf{x},p,Q;\mathbf{x}',p',Q')\ , \tag{7.25}$$

where the function $\tilde{g}_2$ obtains from the two-particle correlator, and

$$\delta(Q-Q') = \frac{1}{c_R}\delta(\boldsymbol{\phi}-\boldsymbol{\phi}')\ \delta(\boldsymbol{\pi}-\boldsymbol{\pi})\ , \tag{7.26}$$

and $\boldsymbol{\phi}$, $\boldsymbol{\pi}$ are the Darboux variables associated to the colour charges Q_a. The appearance of the factor $1/c_R$ in the above expression is due to the change of normalization factors associated to the functions $\mathcal{N}$ and f, as we mentioned at the end of sect. III. The above statistical average is all we need to evaluate the collision integrals below.

[§§]In [1], we used a more condensed notation. There, the functions $a^{(0,0)}_{i,+}$ and $a^{(0,1)}_{i,+}$ have been denoted $a^{(0)}_{i,+}$ and $a^{(1)}_{i,+}$.

From (7.25) one deduces the statistical average over colour current densities $\delta\mathcal{J}$. We expand the momentum $\delta-$function in polar coordinates

$$\delta^{(3)}(\mathbf{p}-\mathbf{p}') = \frac{1}{p^2}\,\delta(p-p')\,\delta^{(2)}(\Omega_{\mathbf{v}}-\Omega_{\mathbf{v}'}) \ , \tag{7.27}$$

where $\Omega_{\mathbf{v}}$ represents the angular variables associated to the vector $\mathbf{v}=\mathbf{p}/|\mathbf{p}|$. After simple integrations we arrive at

$$\begin{aligned}\Big\langle \delta\mathcal{J}^a_\mu(t=0,\mathbf{x},v)\;\delta\mathcal{J}^b_\nu(t=0,\mathbf{x}',v')\Big\rangle &= 2g^2 B_C\, C_2\,\delta^{ab}\,v_\mu v'_\nu\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta^{(2)}(\Omega_{\mathbf{v}}-\Omega_{\mathbf{v}'})\\ &\quad +\tilde{g}^{ab}_{2,\mu\nu}(\mathbf{x},v;\mathbf{x}',v') \ ,\end{aligned} \tag{7.28}$$

where $v^\mu=(1,\mathbf{v})$, and

$$B_C = 16\pi^2\int_0^\infty dp\,p^2\,\bar{f}^{eq}(p) \ . \tag{7.29}$$

The function $\tilde{g}^{ab}_{2,\mu\nu}$ is obtained from the two-particle correlation function $\tilde{g}_2$. Notice that we have neglected the piece $g\bar{f}^{(1)}$ above, as this is subleading in an expansion in g.

Since we know the dynamical evolution of all fluctuations we can also deduce the dynamical evolution of the correlators of fluctuations, with the initial condition (7.25). This corresponds to solving (4.7) in the present approximation. It is convenient to proceed as follows [7]. We separate the colour current (7.16) into a source part and an induced part,

$$\delta\mathcal{J}^\mu = \delta\mathcal{J}^\mu_{sou}+\delta\mathcal{J}^\mu_{ind} \ . \tag{7.30}$$

The induced piece is the part of the current which contains the dependence on a_μ, and thus takes the polarization effects of the plasma into account. The source piece is the part of the current which depends only on the initial condition. This splitting will be useful since ultimately all the relevant correlators can be expressed in terms of correlators of $\delta\mathcal{J}^\mu_{sou}$.

From the explicit solution (7.16) and the average (7.28) we then find, at leading order in g and neglecting the non-local term in (7.28)

$$\begin{aligned}&\Big\langle \delta\mathcal{J}^a_{\mu,sou}(x,v)\;\delta\mathcal{J}^b_{\nu,sou}(x',v')\Big\rangle\\ &= 2g^2 B_C\, C_2 v_\mu v'_\nu \delta^{(3)}(\mathbf{x}-\mathbf{x}'-\mathbf{v}(t-t'))\delta^{(2)}(\Omega_{\mathbf{v}}-\Omega_{\mathbf{v}'})\bar{U}^{ac}(x,x-vt)\bar{U}^{bc}(x',x'-v't') \ .\end{aligned} \tag{7.31}$$

Here, and from now on, we neglect the non-local piece $\tilde{g}^{ab}_{2,\mu\nu}$. It can be shown [7] that they give contributions which decrease rapidly with time, so that for asymptotic large times, they vanish.

Expanding the parallel transporter $\bar{U}$, and switching to momentum space we find the spectral density for the zeroth order in $g\bar{A}$

$$\Big\langle \delta\mathcal{J}^a_\mu\;\delta\mathcal{J}^b_\nu\Big\rangle^{sou\,(0)}_{k,v,v'} = 2g^2 B_C\, C_2\delta^{ab} v_\mu v'_\nu \delta^{(2)}(\Omega_{\mathbf{v}}-\Omega_{\mathbf{v}'})(2\pi)\delta(k\cdot v) \ . \tag{7.32}$$

As an illustrative example, let us compute the correlator of two transverse fields a. Using (7.24a) and (7.32) one arrives at

$$\left\langle a_{i,a}^{T(0,0)}(k)a_{j,b}^{T(0,0)}(q)\right\rangle = g^2 B_C\, C_2 \delta^{ab}(2\pi)^4\delta^{(4)}(k+q)\frac{P_{ik}^T(\mathbf{k})P_{jl}^T(\mathbf{k})}{|-k^2+\Pi_T|^2}\int\frac{d\Omega_{\mathbf{v}}}{4\pi}v_k v_l\,\delta(k\cdot v)\ . \quad (7.33)$$

Since the imaginary part of the polarization tensor, which describes Landau damping, can be expressed as [5]

$$\mathrm{Im}\,\Pi_{ab}^{\mu\nu}(k) = -\delta_{ab}m_D^2\pi k_0\int\frac{d\Omega_{\mathbf{v}}}{4\pi}v^\mu v^\nu\,\delta(k\cdot v)\ , \quad (7.34)$$

the statistical correlator can finally be written as

$$\left\langle a_{i,a}^{T(0,0)}(k)a_{j,b}^{T(0,0)}(q)\right\rangle = \frac{4\pi T}{k_0}\frac{\mathrm{Im}\,\Pi_{ij,T}^{ab}(k)}{|-k^2+\Pi_T|^2}(2\pi)^3\delta^{(4)}(k+q)\ . \quad (7.35)$$

Here, we have used the relation

$$2g^2C_2B_C = 4\pi T m_D^2\ . \quad (7.36)$$

Equation (7.35) is a form of the fluctuation dissipation theorem (FDT), which links the dissipative processes occurring in the plasma with statistical fluctuations.

E. The collision integral

We are now ready to compute at leading order in g the collision integrals appearing on the r.h.s. of (7.11a). We shall combine the expansions introduced earlier to expand the collision integrals in powers of $\bar{\mathcal{J}}$ (while retaining only the zeroth order in $g\bar{A}$),

$$\langle\xi\rangle = \langle\xi^{(0)}\rangle + \langle\xi^{(1)}\rangle + \langle\xi^{(2)}\rangle + \dots\ , \quad (7.37)$$

and similarly for $\langle\eta\rangle$ and $\langle J_{\rm fluc}\rangle$. We find that the induced current $\langle J_{\rm fluc}^{(0)}\rangle$ vanishes, as do the fluctuation integrals $\langle\eta^{(0)}\rangle$ and $\langle\xi^{(0)}\rangle$. The vanishing of $\langle J_{\rm fluc}^{(0)}\rangle$ is deduced trivially from the fact that $\langle a_a^{(0,0)}a_b^{(0,0)}\rangle \sim \delta_{ab}$, while this correlator always appears contracted with the antisymmetric constants f_{abc} in $J_{\rm fluc}$. To check that $\langle\eta^{(0)}\rangle = 0$, one needs the statistical correlator $\langle\delta J_a^\mu\delta J_{ab}^\rho\rangle$, which is proportional to $\sum_a d_{aab} = 0$ for $SU(N)$. The vanishing of $\langle\eta^{(0)}\rangle$ is consistent with the fact that in the Abelian limit the counterpart of $\langle\eta\rangle$ vanishes at equilibrium [7]. Finally, $\langle\xi^{(0)}\rangle = 0$ due to a contraction of f_{abc} with a correlator symmetric in the colour indices.

In the same spirit we evaluate the terms in the collision integrals containing one $\bar{\mathcal{J}}$ field and no background gauge $\bar{A}$ fields. Consider

$$\left\langle\xi_{\rho,a}^{(1)}\right\rangle = gf_{abc}v^\mu\left(-\left\langle a_{\mu,b}^{(0,1)}(x)\,\delta\mathcal{J}_{\rho,c}^{(0)}(x,v)\right\rangle + gf_{cde}v^\nu\int_0^\infty d\tau\,\bar{\mathcal{J}}_{\rho,e}(x_\tau,v)\left\langle a_{\mu,b}^{(0,0)}(x)\ a_{\nu,d}^{(0,0)}(x_\tau)\right\rangle\right). \quad (7.38)$$

Using the values for a_μ and $\delta\mathcal{J}^{(0)}$ as found earlier, we obtain in momentum space

$$\left\langle\xi_{\rho,a}^{(1)}(k,v)\right\rangle \approx -g^4C_2NB_Cv^\rho\int\frac{d\Omega_{\mathbf{v}'}}{4\pi}C(\mathbf{v},\mathbf{v}')\left(\bar{\mathcal{J}}_a^0(k,v)-\bar{\mathcal{J}}_a^0(k,v')\right)\ , \quad (7.39)$$

with

$$C(\mathbf{v},\mathbf{v}') = \int \frac{d^4q}{(2\pi)^4} \left| \frac{v_i P^T_{ij}(q) v'_j}{-q^2 + \Pi_T} \right|^2 (2\pi)\delta(q\cdot v)(2\pi)\delta(q\cdot v') \ . \tag{7.40}$$

Here, the symbol $\approx$ means that only the leading terms have been retained. To arrive at the above expression we have used the $SU(N)$ relation $f_{abc}f_{abd} = N\delta_{cd}$. Within the momentum integral, we have neglected in the momenta of the mean fields, k, in front of the momenta of the fluctuations, q. As we discussed above, the momenta associated to the background fields is much smaller that that associated to the fluctuations. Notice that we have only written the part arising from the transverse fields a, as the one associated to the longitudinal modes is subleading. This is easy to see once one realizes that the above integral is logarithmic divergent in the infrared (IR) region, while the longitudinal contribution is finite. At this point, we can also note that the collision integral computed this way is independent on the (perturbative) Abelian gauge used to solve equation (7.19). This is so because the collision integral computed this way can always be expressed in terms of the imaginary parts of the polarization tensors (7.34) in the plasma, which are known to be gauge-independent.

The integral (7.40) has also been obtained in [13], on the basis of a phenomenological derivation of the Boltzmann collision integral for a quantum plasma. The only difference consists in the value of the Debye mass appearing in the polarization tensor.

In any case, the transverse polarization tensor Π_T vanishes at $q_0 = 0$, and the dynamical screening is not enough to make (7.40) finite. An IR cutoff must be introduced by hand in order to evaluate the integral. With a cutoff of order gm_D we thus find at logarithmic accuracy

$$C(\mathbf{v},\mathbf{v}') \approx \frac{2}{\pi^2 m_D^2} \ln(1/g) \frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}} \tag{7.41}$$

Using also the relation (7.36) we finally arrive at the collision integral to leading logarithmic accuracy,

$$\left\langle \xi^{(1)}_{\rho,a}(x,v) \right\rangle = -\frac{g^2}{4\pi} NT \ln(1/g)\, v_\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \mathcal{I}(v,v') \bar{\mathcal{J}}^0_a(x,v'), \tag{7.42}$$

$$\mathcal{I}(\mathbf{v},\mathbf{v}') = \delta^{(2)}(\mathbf{v}-\mathbf{v}') - \mathcal{K}(\mathbf{v},\mathbf{v}') \ , \qquad \mathcal{K}(\mathbf{v},\mathbf{v}') = \frac{4}{\pi} \frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}} \ , \tag{7.43}$$

where we have introduced $\delta^{(2)}(\mathbf{v}-\mathbf{v}') \equiv 4\pi\delta^{(2)}(\Omega_{\mathbf{v}} - \Omega'_{\mathbf{v}})$, $\int \frac{d\Omega_{\mathbf{v}}}{4\pi}\delta^{(2)}(\mathbf{v}-\mathbf{v}') = 1$.

We can verify explicitly that the leading logarithmic solution is consistent with gauge invariance. This should be so, as the approximations employed have been shown in sect. V C on general grounds to be consistent with gauge invariance. Evaluating the correlator in (4.11) in the leading logarithmic approximation yields

$$g f_{abc} \left\langle a^b_\mu(x) \delta J^\mu_c(x) \right\rangle = -\frac{g^2}{4\pi} NT \ln(1/g) \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \frac{d\Omega_{\mathbf{v}'}}{4\pi} \mathcal{I}(\mathbf{v},\mathbf{v}') \bar{\mathcal{J}}^0_a(x,v') \ , \tag{7.44}$$

which vanishes, because

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi}\, \mathcal{I}(\mathbf{v},\mathbf{v}') = 0 \ . \tag{7.45}$$

We thus establish that $\bar{D}_\mu \bar{J}^\mu = 0$, in accordance with (7.11b) in the present approximation.

F. The source for stochastic noise

The collision integral obtained above describes a dissipative process in the plasma, so in principle, it could trigger the system to abandon equilibrium. Whenever dissipative processes are encountered, it is important to identify as well the stochastic source related to it. This is the essence of the fluctuation-dissipation theorem (FDT). Phenomenologically, this is well known, and sometimes used the other way around: imposing the FDT allows to add by hand a source for stochastic noise with the strength of its self-correlator fixed by the dissipative processes.

In the present formalism, we are able to identify directly the source for stochastic noise which prevents the system from abandoning equilibrium. This proves, that the FDT does hold (analogous considerations have been presented in [12]). The relevant noise term is given by the contributions from the transversal gauge fields in $\xi^{(0)}$. While its average vanishes, $\langle \xi^{(0)} \rangle = 0$, its correlator

$$\Big\langle \xi^{\rho(0)}_a(x,v)\, \xi^{\sigma(0)}_b(y,v') \Big\rangle = g^2 f_{apc} f_{bde} v^\mu v^\nu \left\langle a^p_\mu(x)\, \delta\mathcal{J}^{\rho,c}_{sou}(x,v)\, a^d_\mu(y)\, \delta\mathcal{J}^{\sigma,e}_{sou}(y,v') \right\rangle^{(0)} \tag{7.46}$$

does not. In order to evaluate this correlator we switch to Fourier space. Within the second moment approximation we expand the correlator $\langle \delta f \delta f \delta f \delta f \rangle$ into products of second order correlators $\langle \delta f \delta f \rangle \langle \delta f \delta f \rangle$ and find

$$\begin{aligned} \Big\langle \xi^{\rho(0)}_a(k,v)\, \xi^{\sigma(0)}_b(p,v') \Big\rangle = {} & g^2 f_{apc} f_{bde} v^\mu v^\nu \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4r}{(2\pi)^4} \\ & \Big\{ \Big\langle a^{(0,0)}_{\mu p}(q)\, a^{(0,0)}_{\nu d}(r) \Big\rangle \Big\langle \delta\mathcal{J}^{(0)\rho,c}_{sou}(k-q,v)\, \delta\mathcal{J}^{(0)\sigma,e}_{sou}(p-r,v') \Big\rangle \\ & + \Big\langle a^{(0,0)}_{\mu p}(q)\, \delta\mathcal{J}^{(0)\sigma,e}_{sou}(p-r,v') \Big\rangle \Big\langle \delta\mathcal{J}^{(0)\rho,c}_{sou}(k-q,v)\, a^{(0,0)}_{\nu d}(r) \Big\rangle \Big\} \end{aligned} \tag{7.47}$$

In the leading logarithmic approximation (that is, retaining only the transverse modes) one finally arrives at

$$\Big\langle \xi^{i(0)}_a(x,v) \xi^{j(0)}_b(y,v') \Big\rangle = \frac{g^6 N C_2^2 B_C^2}{(2\pi)^3 m_D^2} \ln(1/g) v^i v'^j \mathcal{I}(\mathbf{v},\mathbf{v}') \delta_{ab} \delta^{(4)}(x-y) \ . \tag{7.48}$$

After integrating over the angular variables, and using (7.36), we obtain

$$\Big\langle \xi^{i(0)}_a(x) \xi^{j(0)}_b(y) \Big\rangle = 2T m_D^2 \frac{g^2 N T \ln(1/g)}{12\pi} \delta_{ab} \delta^{ij} \delta^{(4)}(x-y) \ . \tag{7.49}$$

which identifies $\xi^{(0)}(x)$ as a source of white noise.

G. Mean Field Equations and Non-Abelian Ohm's law

We have managed to obtain the following set of mean field equations, after integrating-out the statistical fluctuations (from now on, we drop the bar to denote the mean fields)

$$v^\mu D_\mu \mathcal{J}^\rho(x,v) + m_D^2 v^\rho v^\mu F_{\mu 0}(x) = -\gamma\ v^\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi}\, \mathcal{I}(\mathbf{v},\mathbf{v}')\mathcal{J}^0(x,v') + \zeta^\rho(x,v)\,, \tag{7.50a}$$

$$D_\mu F^{\mu\nu} = J^\nu\,. \tag{7.50b}$$

Here, we denote by $\zeta(x,v)$ the stochastic noise term identified in the preceding section, with $\zeta^0 = 0$, $\langle \zeta^i \rangle = 0$ and $\langle \zeta^i \zeta^j \rangle$ given by the r.h.s. of (7.48). We also introduced

$$\gamma = \frac{g^2}{4\pi} N T \ln{(1/g)}\ \ , \tag{7.51}$$

which will be identified as (twice) the damping rate for the ultra-soft currents. We shall refer to (7.50a) as a Boltzmann-Langevin equation, as it accounts for particle interactions via a collision integral as well as for the stochastic character of the underlying fluctuations.

The mean field dynamical equation is an integro-differential equation for the current density $\mathcal{J}^\rho$. Using (7.43), we rewrite the Boltzmann-Langevin equation (7.50a) as

$$\begin{aligned}(v^\mu D_\mu + 2\gamma)\mathcal{J}^\rho(x,v) = &-m_D^2 v^\rho v^\mu F_{\mu 0}(x) + \zeta^\rho(x,v)\\ &+\gamma\ v^\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi}\, \mathcal{K}(\mathbf{v},\mathbf{v}')\ \mathcal{J}^0(x,v')\ .\end{aligned} \tag{7.52}$$

We first seek for the retarded Green's function of the differential operator

$$i\,(v^\mu D_\mu + \gamma)\ G_{ret}(x,y;v) = \delta^{(4)}(x-y)\ , \tag{7.53}$$

which reads, for $t = x_0 - y_0$

$$G_{ret}(x,y;v)_{ab} = -i\theta(t)\delta^{(3)}\left(\mathbf{x}-\mathbf{y}-\mathbf{v}t\right)\exp(-\gamma t)\, U_{ab}(x,y)\ . \tag{7.54}$$

The problem in finding a solution to (7.52) is that the last term in term (7.52) is a function of the current density itself. A solution for (7.52) is obtained if we apply an iterative procedure. Disregarding the last term (7.52) in a first step, we obtain

$$\mathcal{J}^{\rho,a}_{(0)}(x,v) = \int_0^\infty d\tau \exp(-\gamma\tau) U^{ab}(x,x-v\tau)\left\{-m_D^2 v^\rho v^j F_{j0,b}(x-v\tau) + \zeta^{\rho,a}(x-v\tau,v)\right\} \tag{7.55}$$

Then we re-insert the 0-component of $\mathcal{J}^\rho_{(0)}$ into the r.h.s. of (7.52) to obtain a new solution $\mathcal{J}^\rho_{(1)}$, and so forth. This procedure induces corrections to (7.55). These additional terms have however a simple $\mathbf{v}$-dependence, namely proportional to

$$v^\rho\ \mathcal{K}(\mathbf{v},\mathbf{v}')\ . \tag{7.56}$$

This implies that the terms induced during the iteration, that is, from the last term in (7.52), will not contribute to the current $J^i(x)$, because

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi} \, \mathbf{v} \, \mathcal{K}(\mathbf{v}, \mathbf{v}') = 0 \; . \tag{7.57}$$

We thus conclude that the non-local solution for the spatial component of the colour current is obtained from (7.55) as

$$J^i_a(x) = -m_D^2 \int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int_0^\infty d\tau \left\{ \exp(-\gamma\tau) U_{ab}(x, x - v\tau) v^i v^j F_{j0,b}(x - v\tau) \right\} + \nu^i_a(x) \; , \tag{7.58a}$$

where ν^i_a is a non-local function of $\zeta(x, v)$

$$\nu^i_a(x) = -\int \frac{d\Omega_{\mathbf{v}}}{4\pi} \int_0^\infty d\tau \exp(-\gamma\tau) U_{ab}(x, x - v\tau) \zeta^i_b(x - v\tau, v) \; . \tag{7.58b}$$

The 0-component of the colour current can be obtained as indicated above, only that the contributions induced by the last term in (7.52) survive after angle averaging. Alternatively, one can make use of the covariant current conservation $D_\mu J^\mu = 0$, which follows from (7.50a) after angle averaging the $\rho = 0$ component. Thus, we finally end up with J^0 expressed as a function of J^i and A_μ.

It is worth noticing that, apart from the presence of the stochastic source ν in (7.58a), the ultra-soft colour current we have found has the same functional dependence on F_{i0} and on U than the soft colour current (7.9); there is, however, an additional damping factor $e^{-\gamma\tau}$ in the integrand.

Let us stress that (7.58a) defines the non-local version of the non-Abelian Ohm's law. It can be used to define the colour conductivity tensor

$$\sigma^{ij}_{ab}(x, y) = \frac{\delta J^i_a(x)}{\delta E^b_j(y)} \; . \tag{7.59}$$

In momentum space, the colour conductivity tensor obtains as an infinite power series in the vector gauge field A. The leading order term reads

$$\sigma^{ij}_{ab}(k) = \delta_{ab} m_D^2 \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \frac{v^i v^j}{-i(k \cdot v) + \gamma} \; . \tag{7.60}$$

We shall now consider the local limit of the above equations. Consider the mean field current (7.58a). The terms contributing to this current are exponentially suppressed for times τ much larger than the characteristic time scale $1/\gamma$. On the other hand, the fields occurring in the integrand vary typically very slowly, that is on time scales $\ll 1/m_{\rm D}$. Thus, the leading contribution consists in approximating

$$U_{ab}(x, x - v\tau) \approx U_{ab}(x, x) = \delta_{ab} \; , \quad F_{j0}(x - v\tau) \approx F_{j0}(x) \; . \tag{7.61}$$

In this case the remaining integration can be performed. The local limit of (7.58) is

$$J^i_a(x) = \sigma E^i_a + \nu^i_a \ , \qquad \sigma = \frac{4\pi m_D^2}{3Ng^2 T \ln(1/g)} \tag{7.62a}$$

while the noise term becomes

$$\nu^i_a(x) = \frac{1}{\gamma}\int \frac{d\Omega_{\mathbf{v}}}{4\pi}\zeta^i_a(x,v) \ , \qquad \left\langle \nu^i_a(x)\, \nu^j_b(y)\right\rangle = 2\,T\,\sigma\,\delta^{ij}\delta_{ab}\,\delta^{(4)}(x-y) \tag{7.62b}$$

Within this last approximation, the noise term appearing in the Yang-Mills equation becomes white noise. The fluctuation-dissipation theorem is fulfilled because the strength of the noise-noise correlator (7.62b) is precisely given by the dissipative term of (7.62a). This is the simplest form of the FDT. The colour conductivity (7.60) becomes

$$\sigma^{ij}_{ab}(k \to 0) = \sigma\,\delta^{ij}\,\delta_{ab} \tag{7.63}$$

in accordance with (7.62a). The colour conductivity in the local limit has been discussed by several authors in the literature [13,16].

The complete set of gauge field equations in the local limit are thus

$$D_\mu F^{\mu i} = \sigma E^i + \nu^i \ , \tag{7.64a}$$

$$D_i E^i = J^0 \tag{7.64b}$$

$$D_0 J^0 = -\sigma J^0 - D_i \nu^i \ . \tag{7.64c}$$

It is worth pointing out that already in the leading logarithmic approximation the noise term appearing in the Yang Mills equation (7.58b) is not white, except in the local limit (7.62b). The noise in the Boltzmann-Langevin equation, on the other hand, is white (see (7.49)), when averaged over the directions of $\mathbf{v}$.

For numerical computations, which can in principle take into account the non-localities of the problem, it might be more convenient to work with the two set of equations (7.50), rather than with a non-local stochastic gauge field equation.

VIII. THE QUANTUM PLASMA CLOSE TO EQUILIBRIUM

A. The quantum plasma from transport theory

Up to now we have made an entirely classical derivation of a Boltzmann equation with collision integrals and stochastic sources, and we have finally derived the mean gauge field equations. The basic ingredients for such a derivation were the classical equations of motion and the classical statistical averages introduced in sect. III. The following natural step is quantizing the whole procedure in order to obtain quantum Boltzmann equations and the corresponding mean gauge field equations.

In order to quantize this formulation one has to abandon the concept of classical trajectories, and introduce commutators for all the canonical conjugate pairs of variables. One should also take into account that quantum particles are indistinguishable. The natural formulation of the quantum problem is then given in terms of Wigner functions and density matrices. We will not present a rigorous discussion of the quantum counterpart of our formulation, but rather present a minimal set of changes in our equations which allow to consider also quantum plasmas close to equilibrium. We leave for a future project a much more rigorous discussion based on first principles of the quantum formulation of the problem.

Our starting observation is that even in quantum plasmas the physics occurring at soft and ultra-soft scales can be encoded into classical or semiclassical equations. The reason for this is that the occupation number for soft modes close to equilibrium is very large, suggesting that a description in terms of classical equations might also be valid to describe the physics of the soft and ultra-soft scales in the quantum plasmas.

Therefore, in order to consider a quantum plasma, we shall need to make several changes. The first step consists in expanding the mean distribution function around the appropriate quantum statistical distribution function. For a plasma close to equilibrium, these are given by (6.6). It is also common to change the phase space measure d^3xd^3p to the standard quantum normalization $d^3xd^3p/(2\pi\hbar)^3$ (although we will keep on working in the units $\hbar = 1$)***. If fluctuations are neglected, this is all that has to be changed. This suffices to change the Debye mass appearing in the Vlasov equation to its quantum value, and thus to reproduce fully the HTL effective theory in the leading order in g [5].

When fluctuations are considered as well, it is equally important to modify also the classical correlator (7.25) to the corresponding quantum statistical one. For bosons, and for every internal degree of freedom, one has

$$\begin{aligned}\langle \delta f_{\mathbf{x},p,Q}\, \delta f_{\mathbf{x}',p',Q'}\rangle &= \delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(Q-Q')\bar{f}_{\rm B}(1+\bar{f}_{\rm B}) \\ &\quad + \tilde{g}_2^{\rm B}(\mathbf{x},p,Q;\mathbf{x}',p',Q')\ ,\end{aligned} \tag{8.1}$$

while the corresponding correlator for fermions is

$$\begin{aligned}\langle \delta f_{\mathbf{x},p,Q}\, \delta f_{\mathbf{x}',p',Q'}\rangle &= \delta^{(3)}(\mathbf{x}-\mathbf{x}')\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta(Q-Q')\bar{f}_{\rm F}(1-\bar{f}_{\rm F}) \\ &\quad + \tilde{g}_2^{\rm F}(\mathbf{x},p,Q;\mathbf{x}',p',Q')\ .\end{aligned} \tag{8.2}$$

The functions $\tilde{g}_2^{\rm B/F}$ are related to the bosonic/fermionic two-particle correlation function. The above relations should be derived from first principles in a similar way as our equation (7.25). In the limit $\bar{f}_{\rm B/F} \ll 1$ they reduce to the correct classical value. It has also to be pointed out that the correlators (8.1) and (8.2) have been derived for the case of both an ideal gas of bosons and ideal gas of fermions close to equilibrium, matching the change described above. This can be taken as the correct answer in the case that the non-Abelian interactions can be treated perturbatively.

With the above in mind, we can now describe the minimal set of changes to our computations of sect. VII which allows to treat the quark-gluon plasma close to equilibrium. We will consider gluons in the adjoint representation with $C_2 = N$, and N_F quarks and N_F antiquarks in the fundamental representation, with $C_2 = 1/2$. All particles carry two helicities. We will neglect the masses of the quarks, $m \ll T$.

***As done in the standard textbooks the factor $(2\pi\hbar)^3$ is introduced into the measure, although some authors introduce it into the distribution function. We follow the first option.

The value of the the quantum Debye mass squared becomes

$$m_D^2 = -\frac{g^2}{\pi^2} \int_0^\infty dp p^2 \left(N \frac{d\bar{f}^{\rm eq}_{\rm B}}{dp} + N_F \frac{d\bar{f}^{\rm eq}_{\rm F}}{dp} \right) \ . \tag{8.3}$$

Evaluating explicitly the integral, one finds $m_D^2 = g^2T^2(2N + N_F)/6$.

The correlator of colour currents densities are then modified, according to the changes mentioned above. Since we now consider different species of particles in (7.28) C_2B_C is replaced by a sum over different species of particles. For the quark gluon plasma, all our equations of sect. VII remain valid if we replace C_2B_C by

$$\sum_{\rm species} C_2B_C = \frac{2N}{\pi} \int_0^\infty dp p^2 \bar{f}^{\rm eq}_{\rm B}(1 + \bar{f}^{\rm eq}_{\rm B}) + \frac{2N_F}{\pi} \int_0^\infty dp p^2 \bar{f}^{\rm eq}_{\rm F}(1 - \bar{f}^{\rm eq}_{\rm F}) \ . \tag{8.4}$$

It is curious that the relation (7.36) for the quantum values of the above quantities remains unchanged, that is,

$$2g^2 \sum_{\rm species} C_2B_C = 4\pi T m_D^2 \tag{8.5}$$

holds true also in the quark-gluon plasma. Since this combination appears in front of all our collision integrals, we find a universal value for the coefficient of (7.42), γ for both the classical and quantum plasmas. This is always the case in the leading log approximation, if the IR cutoff used is of order gm_D, where m_D would correspond to the classical or quantum Debye mass, respectively. The value $\gamma/2$ can be identified with the damping rate of a hard transverse gluon [17].

With these observations, one does not need to repeat the computations that we performed in sect. VII. In particular, the final mean field equations of sect. VII G only change in the value of the Debye mass.

B. Comparison to related work

Let us briefly comment on some related work. A similar philosophy to ours has already been followed by Selikhov [8]. He used the semiclassical limit of quantum transport equations for the Wigner functions associated to gluons and quarks, which reduce to our starting classical transport equations. He used a procedure of splitting both the Wigner functions and vector gauge fields into mean values and statistical fluctuations. A key point is how the statistical correlator of fluctuations in a quantum framework can be derived. Selikhov relied on the same type of statistical correlator as derived in (7.28). However, it should be stressed that this statistical correlator is only correct in the pure classical framework, for classical statistics. This can not reproduce the correct prefactors of the quantum collision integral. Also, the FDT is not satisfied in this case. Instead, the correct correlators are given by (8.1) and (8.2). Also, the colour current he found is not covariantly conserved. This is so because the non-local term in the collision integral (7.42), proportional to $\mathcal{K}(\mathbf{v}, \mathbf{v}')$, has been neglected.

The first to derive the mean field equations (7.50) and the related noise correlator for the quantum plasma was Bödeker [12]. His approach uses the local version of the HTL action as starting point, and profits from the observation that the soft field modes behave classically. This allows the definition of a statistical average. Although close in spirit, this method seems technically quite different.

Arnold, Son and Yaffe [13] then realized that Bödeker's effective theory has a physical interpretation in terms of kinetic equations, deriving the relevant collision term of the Boltzmann equation on phenomenological grounds.

Very recently, the quantum collision integral has been derived within a quantum field theoretical setting by Blaizot and Iancu [14]. The derivation relies on a gauge covariant derivative expansion of the quantum field equations. While conceptually very different to our approach [1], the approximations used in [14] are very similar to the ones we performed. This is maybe not too surprising after all, as both approaches are based on a consistent expansion in powers of the gauge coupling constant. It seems only that the concept of statistical fluctuations has not been introduced in [14], which may be a reason for why the source of stochastic noise, necessary for a correct macroscopic description of the plasma, has not yet been identified. The stochastic noise can probably be derived by considering the effects of higher order correlation functions [18].

Based on our approach [1], the quantum collision integral for the transport equation has been obtained as well by Valle [19]. He started from the HTL effective theory and found the correct coefficient after imposing a fluctuation-dissipation relation. The noise term in his final equations is however missing, which again would entail that the system abandons equilibrium.

Finally, it is interesting that the collision integral can be interpreted in terms of Feynman diagrams [12,14]. Bödeker also made a diagrammatic derivation of his effective theory [12]. This is a much more lengthy and cumbersome task, and shows, on the other hand, the very efficiency of a kinetic approach, as it corresponds to a re-organisation of the perturbative series.

IX. DISCUSSION

We have presented a self-consistent approach to study classical non-Abelian plasmas. Let us summarize here again our starting assumptions to derive the effective transport equations. A system of point particles carrying non-Abelian charges is considered. Their microscopic equations of motions are the Wong equations. In order to describe an ensemble of these particles, we introduced an ensemble average, which takes also the colour charges as dynamical variables into account. This yields finally a set of transport equations for both mean quantities and statistical fluctuations, and gives a recipe to obtain explicitly the collision integrals for macroscopic transport equations. This approach is consistent with gauge invariance, and admits systematic approximations. Most particularly, it does not rely on close-to-equilibrium situations. These techniques, applied since long to Abelian plasmas, have never been fully exploited for the non-Abelian case. Our approach is aimed at closing this gap in the literature of non-Abelian plasmas.

We applied this method to non-Abelian plasmas close to thermal equilibrium. A sufficiently small gauge coupling parameter is at the basis for a systematic expansion of the dynamical equations. Neglecting fluctuations yields to leading order the known non-local expression for the soft current in terms of the soft gauge fields (HTL approximation). Integrating-out, in addition, the fluctuations to leading order, that is with momenta about m_D, results in a Boltzmann-Langevin equation for the ultra-soft modes,

$$v^\mu D_\mu \mathcal{J}^\rho(x,v) = -m_D^2 v^\rho v^\mu F_{\mu 0}(x) - \gamma\, v^\rho \int \frac{d\Omega_{\mathbf{v}'}}{4\pi} \mathcal{I}(\mathbf{v},\mathbf{v}')\mathcal{J}^0(x,v') + \zeta^\rho(x,v) \, .$$

It contains a collision term and a related noise term, with $\gamma = g^2 NT \ln{(1/g)}/4\pi$, while the stochastic source ζ^ρ with $\zeta^0 \equiv 0$ obeys

$$\left\langle \zeta^i_a(x,v)\, \zeta^j_b(y,v') \right\rangle = 2\,\gamma\, T\, m_D^2\, v^i\, v'^j\, \mathcal{I}(\mathbf{v},\mathbf{v}')\, \delta_{ab}\, \delta^{(4)}(x-y) \ .$$

Solving the Boltzmann-Langevin equation (see sect. VII G), one obtains the Yang-Mills equation for the ultra-soft fields

$$D_\mu F^{\mu\nu} = \int \frac{d\Omega_{\mathbf{v}}}{4\pi}\, \mathcal{J}^\nu(x,v) \ .$$

Surprisingly, the dynamical equations are the same for both classical and quantum plasmas, the only difference being the value for the Debye mass. This conclusion relies also on the use of an infrared cut-off of order gm_D, where m_D is the classical or quantum Debye mass. For the quantum plasmas, our result agrees with the quantum collision integrals found in the literature using different methods [12–14]. The main effect of the fluctuations with momenta about the Debye mass is the introduction of a damping term and a source of stochastic noise into the above expression. Note also that the damping coefficient γ is the same for classical and quantum plasmas.

Our work establishes a link even beyond the one-loop level between the classical transport theory approach as presented here, and a full quantum field theoretical treatment. It would be most desirable if this connection could be further substantiated. This should also yield a quantitative criterion for the applicability of the -technically speaking- much simpler approach based on the classical point particle picture.

Let us finally emphasize that the same IR problems, which are due to the unscreened magnetic modes, appear both for classical and quantum plasmas. This suggests that the solution for these IR divergences might also be the same in the two cases. Therefore, it seems profitable to seek for a solution to this problem in the much simpler framework of classical transport theory, rather than in a quantum field theoretical approach.

APPENDIX A: DARBOUX VARIABLES

The statistical averages defined in sect. III have to be performed in phase space. The colour charges Q_a are not real phase space variables [5]. It is possible to define the set of Darboux variables associated to the Q_a charges.

For $SU(2)$ we define the new set of variables ϕ, π, J by the transformation [5]

$$Q_1 = \cos\phi\sqrt{J^2 - \pi^2} \qquad Q_2 = \sin\phi\sqrt{J^2 - \pi^2} \qquad Q_3 = \pi \tag{A1}$$

where π is bounded by $-J \leq \pi \leq J$. The variables ϕ, π form a canonically conjugate pair, while J is fixed by the value of the quadratic Casimir, which is constant under the dynamical evolution. One can define Poisson brackets with these canonical variables, under which the colour charges form a representation of $SU(2)$, $\{Q_a, Q_b\}_{PB} = f_{abc}Q_c$. With the above change of variables, one can easily fix the value of the representation normalization constant c_R introduced in (2.9). From the condition $\int dQ = 1$ one finds $c_R = 1/4\pi\sqrt{q_2}$. From the condition $\int dQ Q_a Q_b = C_2\delta_{ab}$ one gets $q_2 = 3C_2$. This then entirely fixes the value of c_R as a function of C_2.

The Darboux variables associated to $SU(3)$ where defined in [5], and will not be discussed explicitly here.

We should also comment that in the pure classical framework, C_2 carries the same dimensions of $(\hbar c)^2$. After quantization, the quadratic Casimirs should take quantized values proportional to $\hbar^2$. The Poisson brackets then have to be replaced by commutators, as well.

APPENDIX B: CONSISTENT CURRENT CONSERVATION

In this appendix we verify explicitly the identity

$$0 = [\bar{D}_\mu, J^\mu_{\rm fluc}] + g[a_\mu, \delta J^\mu] + g[a_\mu, \langle J^\mu_{\rm fluc}\rangle] \ , \tag{B1}$$

which is at the basis for the proof of the consistent current conservation of both the mean field and the fluctuation current in sect. V B. The following check is entirely algebraic, and it will make use of symmetry arguments like the antisymmetry of the commutator and the tensors $\bar{F}_{\mu\nu}, f_{\mu\nu}$, and of the cyclic identity $[t_a,[t_b,t_c]] + [t_b,[t_c,t_a]] + [t_c,[t_a,t_b]] = 0$. The identity $[\bar{D}_\mu, \bar{D}_\nu] = g\bar{F}_{\mu\nu}$ is employed as well. To simplify the computation, we will seperate the fluctuation part of the field strength (4.2c) into the term linear and quadratic in a, according to

$$f_{\mu\nu} = f_{1,\mu\nu} + f_{2,\mu\nu} \ , \qquad f_{1,\mu\nu} = [\bar{D}_\mu, a_\nu] - [\bar{D}_\nu, a_\mu] \ , \qquad f_{2,\mu\nu} = g[a_\mu, a_\nu] \ . \tag{B2}$$

Recall furthermore, using (5.7c) and (5.8b), that

$$J^\mu_{\rm fluc} = [\bar{D}_\nu, f_2^{\nu\mu}] + g[a_\nu, f_1^{\nu\mu} + f_2^{\nu\mu}] \tag{B3}$$
$$\delta J^\mu = [\bar{D}_\nu, f_1^{\nu\mu}] + g[a_\nu, \bar{F}^{\nu\mu}] + J^\mu_{\rm fluc} - \langle J^\mu_{\rm fluc}\rangle \tag{B4}$$

are functions of the fluctuation field a. The first term of (B1) reads, after inserting $J_{\rm fluc}$ from (B3),

$$[\bar{D}_\mu, J^\mu_{\rm fluc}] = [\bar{D}_\nu, [\bar{D}_\mu, f_2^{\mu\nu}]] + g[\bar{D}_\nu, [a_\mu, f_1^{\mu\nu}]] + g[\bar{D}_\nu, [a_\mu, f_2^{\mu\nu}]] \ . \tag{B5}$$

Using δJ from (B4), it follows for the second term of (B1)

$$g[a_\mu, \delta J^\mu] = g^2[a_\nu, [a_\mu, \bar{F}^{\mu\nu}]] + g[a_\nu, [\bar{D}_\mu, f_1^{\mu\nu}]] + g\,[a_\nu, J^\nu_{\rm fluc}] - g\,[a_\nu, \langle J^\nu_{\rm fluc}\rangle] \tag{B6}$$

The last term of (B6) will be canceled by the last term in (B1). We show now that the first three terms of (B5) and (B6) do cancel one by one. The first term in (B5) can be re-written as

$$[\bar{D}_\nu, [\bar{D}_\mu, f_2^{\mu\nu}]] = [[\bar{D}_\nu, \bar{D}_\mu], f_2^{\mu\nu}] - [\bar{D}_\nu, [\bar{D}_\mu, f_2^{\mu\nu}]] = \tfrac{1}{2} g [\bar{F}_{\nu\mu}, f_2^{\mu\nu}] \tag{B7}$$

Similarly, the first term of (B6) yields

$$g^2[a_\nu, [a_\mu, \bar{F}^{\mu\nu}]] = -g^2[\bar{F}^{\mu\nu}, [a_\nu, a_\mu]] - g^2[a_\nu, [a_\mu, \bar{F}^{\mu\nu}]] = -\tfrac{1}{2} g [\bar{F}_{\mu\nu}, f_2^{\nu\mu}] \ . \tag{B8}$$

For the second term in (B5) we have

$$g[\bar{D}_\nu, [a_\mu, f_1^{\mu\nu}]] = g[a_\mu, [\bar{D}_\nu, f_1^{\mu\nu}]] + g[[\bar{D}_\nu, a_\mu], f_1^{\mu\nu}] = -g[a_\mu, [\bar{D}_\nu, f_1^{\nu\mu}]] \ , \tag{B9}$$

which equals (minus) the second term of (B6). Finally, consider the third term of (B6),

$$\begin{aligned} g\,[a_\nu, J^\nu_{\text{fluc}}] &= g^2[a_\nu, [\bar{D}_\mu, [a^\mu, a^\nu]]] + g^2[a_\nu, [a_\mu, f^{\mu\nu}]] \\ &= \tfrac{1}{2} g [f_2^{\mu\nu}, f_{1,\mu\nu}] - g[\bar{D}_\mu, [f_2^{\mu\nu}, a_\nu]] - \tfrac{1}{2} g [f_2^{\mu\nu}, f_{1,\mu\nu}] \\ &= -g[\bar{D}_\mu, [a_\nu, f_2^{\nu\mu}]] \end{aligned} \tag{B10}$$

which equals (minus) the third term of (B5). This establishes (B1).

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