CERN - European Laboratory for Particle Physics

CLIC Note 399

BEAM LOADING VOLTAGE PROFILE OF AN ACCELERATING SECTION WITH A LINEARLY VARYING GROUP VELOCITY

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Abstract

The CLIC Tapered Damped accelerating Structure (TDS) has a 5.4% detuning of the lowest dipole mode. The geometrical variations that produce this detuning range also fix the fundamental mode's group velocity variation - very nearly linear with 0.108c (c is the speed of light) at the structure input to 0.054c at the output. In addition R'/Q also varies approximately linearly, from 22.3 kΩ/m at the input to 30 kΩ/m at the output. These variations result in a structure that is neither constant impedance nor constant gradient so the widely used relationships between structure length, input and average accelerating gradient are not applicable. In order to simplify the process of optimizing accelerator parameters an analytic expression for the voltage profile in a structure with a linearly varying group velocity has been derived. A more accurate numerical solution that includes the variation in R'/Q is also presented.

Geneva, Switzerland
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1. INTRODUCTION

The CLIC Tapered Damped accelerating Structure (TDS) [1] has a 5.4% detuning of the lowest dipole mode. The geometrical variations that produce this detuning range also fix the fundamental mode's group velocity variation - very nearly linear with 0.108c (c is the speed of light) at the structure input to 0.054c at the output. In addition R'/Q also varies approximately linearly, from 22.3 kΩ/m at the input to 30 kΩ/m at the output. These variations result in a structure that is neither constant impedance nor constant gradient so the widely used relationships between structure length, input and average accelerating gradient are not applicable. In order to simplify the process of optimizing accelerator parameters an analytic expression for the voltage profile in a structure with a linearly varying group velocity has been derived. A more accurate numerical solution that includes the variation in R'/Q is also presented.

2. SETTING UP THE DIFFERENTIAL EQUATION FOR POWER FLOW AND POWER LOSS

The basis for the derivation of the longitudinal electric field as a function of distance along the structure, $E(z)$, is the assumption that power flow along the structure, $P(z)$, is constant except for power dissipated in the walls of the structure and power either given to or taken from the beam. Wall losses are proportional to the power flowing and inversely proportional to the quality factor Q,

$$\frac{dP}{dz} = -\frac{\omega}{Q}\frac{1}{v_g}P \tag{2.1}$$

The power extracted (or left) by the beam depends on the local field strength. A charge dq produces a field change,

$$\begin{aligned} dE &= -2k'dq \\ &= -2\left(\frac{\omega}{4}\frac{R'}{Q}\right)Idt \end{aligned} \tag{2.2}$$

Power is related to the field through,

$$P = v_g \frac{E^2}{\omega \frac{R'}{Q}} \tag{2.3}$$

Differentiating,

$$\begin{aligned} dP &= v_g \frac{2}{\omega \frac{R'}{Q}} EdE \\ &= v_g \frac{2}{\omega \frac{R'}{Q}} \sqrt{\omega \frac{R'}{Q}\frac{P}{v_g}}\left(-\frac{\omega}{2Q}R'Idt\right) \\ &= -\sqrt{\omega v_g \frac{R'}{Q}} IP^{1/2}dt \end{aligned} \tag{2.4}$$

Changing over to a variation in space,

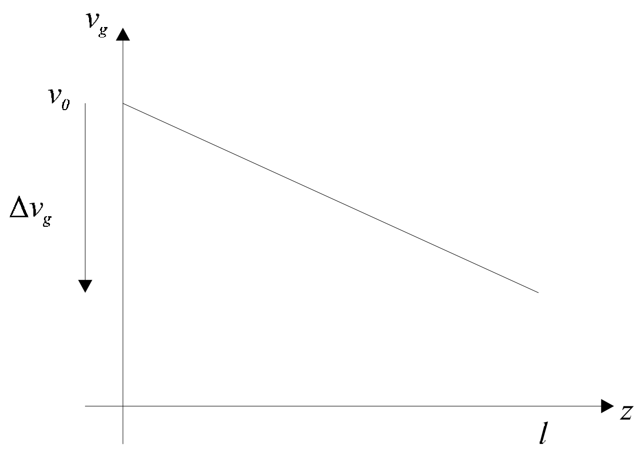
$$\begin{aligned}\frac{dP}{dz}&=\frac{dP}{dt}\frac{dt}{dz}\\&=\frac{1}{v_g}\frac{dP}{dt}\\&=-\sqrt{\frac{\omega}{v_g}\frac{R'}{Q}}IP^{1/2}\end{aligned} \tag{2.5}$$

The resulting differential equation for power as a function of distance along the structure, including both wall losses and interaction with the beam, is,

$$\frac{dP}{dz}=-\frac{\omega}{Q}\frac{1}{v_g}P-\sqrt{\frac{\omega}{v_g}\frac{R'}{Q}}IP^{1/2} \tag{2.6}$$

3. SOLVING FOR A LINEAR VARIATION IN GROUP VELOCITY (R'/Q CONSTANT)

Equation 2.6 can be solved analytically for a linearly varying group velocity with all other terms constant.



$$v_g(z)=v_0-\Delta v_g\frac{z}{l} \tag{3.1}$$

Equation 2.6 can be solved more easily by switching from power to energy density,

$$P=v_gW \tag{3.2}$$

$$\begin{aligned}\frac{dP}{dz}&=v_g\frac{dW}{dz}+W\frac{dv_g}{dz}\\&=v_g\frac{dW}{dz}-\frac{\Delta v_g}{l}\end{aligned} \tag{3.3}$$

$$v_g\frac{dW}{dz}=\left(-\frac{\omega}{Q}+\frac{\Delta v_g}{l}\right)W-\sqrt{\frac{\omega R'}{Q}}IW^{1/2} \tag{3.4}$$

This differential equation is separable,

$$\frac{dW}{\left(-\frac{\omega}{Q}+\frac{\Delta v_g}{l}\right)W-\sqrt{\frac{\omega R'}{Q}}IW^{1/2}}=\frac{dz}{v_0-\Delta v_g\frac{z}{l}} \qquad 3.5$$

Integrating (C is the constant of integration),

$$\frac{2}{\left(-\frac{\omega}{Q}+\frac{\Delta v_g}{l}\right)}\ln\left(\left(-\frac{\omega}{Q}+\frac{\Delta v_g}{l}\right)W^{1/2}-\sqrt{\frac{\omega Q}{R'}}I\right)=-\frac{l}{\Delta v_g}\ln\left(1-\frac{\Delta v_g}{v_0}\frac{z}{l}\right)+C \qquad 3.6$$

$$W^{1/2}=\frac{1}{\left(-\frac{\omega}{Q}+\frac{\Delta v_g}{l}\right)}\left[C\left(1-\frac{\Delta v_g}{v_0}\frac{z}{l}\right)^{\frac{1}{2}\left(\frac{\omega}{Q}\frac{l}{\Delta v_g}-1\right)}+\sqrt{\frac{\omega R'}{Q}}I\right] \qquad 3.7$$

Substituting ,

$$W=\frac{E^2}{\omega\frac{R'}{Q}} \qquad 3.8$$

into equation 3.7 then setting the boundary condition on field at the input of the structure, *E(0)*,

$$\begin{aligned}E(z)&=C\left(1-\frac{\Delta v_g}{v_0}\frac{z}{l}\right)^{\frac{1}{2}\left(\frac{\omega}{Q}\frac{l}{\Delta v_g}-1\right)}-\frac{IR'}{1-\frac{Q}{\omega}\frac{\Delta v_g}{l}}\\&=\left(E(0)+\frac{IR'}{1-\frac{Q}{\omega}\frac{\Delta v_g}{l}}\right)\left(1-\frac{\Delta v_g}{v_0}\frac{z}{l}\right)^{\frac{1}{2}\left(\frac{\omega}{Q}\frac{l}{\Delta v_g}-1\right)}-\frac{IR'}{1-\frac{Q}{\omega}\frac{\Delta v_g}{l}}\end{aligned} \qquad 3.9$$

4. THREE SPECIAL CASES

Solutions of equation 2.6 are now found for two special cases. The first is the case of constant group velocity (constant impedance), no wall losses (only beam loading) and no RF fed into the structure. This case is singular so requires a return to the initial differential equation (2.6) but with the damping term removed,

$$\frac{dP}{dz} = -\sqrt{\frac{\omega}{v_g}\frac{R'}{Q}} I P^{1/2}$$

$$\frac{dP}{P^{1/2}} = -\sqrt{\frac{\omega}{v_g}\frac{R'}{Q}} I dz \qquad 4.1$$

$$P = \frac{1}{4}\frac{\omega R'}{Q}\frac{I^2 z^2}{v_g}$$

In terms of field,

$$E = \sqrt{\omega \frac{R'}{Q}\frac{P}{v_g}}$$

$$= \frac{1}{2}\frac{\omega R'}{Q}\frac{Iz}{v_g} \qquad 4.2$$

$$= 2kq$$

The charge q is the current multiplied by the RF filling (travelling) time to the position z.

The second special case is constant impedance, with losses and with beam. This derivation also requires a return to equation 2.6 and a change of variable from P to W,

$$v_g \frac{dW}{dz} = -\frac{\omega}{Q} W - \sqrt{\frac{\omega R'}{Q}} I W^{1/2} \qquad 4.3$$

$$\frac{dW}{\frac{\omega}{Q} W + \sqrt{\frac{\omega R'}{Q}} I W^{1/2}} = -\frac{dz}{v_g} \qquad 4.4$$

Integrating,

$$2\frac{Q}{\omega}\ln\left(\frac{\omega}{Q}W^{1/2} + \sqrt{\frac{\omega R'}{Q}} I\right) = -\frac{z}{v_g} + C$$

$$\frac{\omega}{Q}W^{1/2} + \sqrt{\frac{\omega R'}{Q}} I = C e^{-\frac{\omega z}{2Qv_g}}$$

$$\frac{\omega}{Q}W^{1/2} = C e^{-\frac{\omega z}{2Qv_g}} - \sqrt{\frac{\omega R'}{Q}} I \qquad 4.5$$

$$W^{1/2} = \frac{\omega}{Q}\left[C e^{-\frac{\omega z}{2Qv_g}} - \sqrt{\frac{\omega R'}{Q}} I\right]$$

Now substituting for electric field,

$$E(z) = \sqrt{\frac{\omega R'}{Q}} W^{1/2}$$

$$= \sqrt{\frac{\omega R'}{Q}} C e^{-\frac{\omega z}{2Qv_g}} - IR' \qquad 4.6$$

$$= \left(E(0) + R'I\right) e^{-\frac{\omega z}{2Qv_g}} - IR'$$

In the limit of a very long structure the field just becomes -*IR'*.

Finally the case of no current and a linearly varying group velocity is considered. It is sufficient to take the general solution to the differential equation, 3.9, and substitute *I*=0,

$$E(z) = E(0)\left(1 - \frac{\Delta v_g}{v_0}\frac{z}{l}\right)^{\frac{1}{2}\left(\frac{\omega}{Q}\frac{l}{\Delta v_g} - 1\right)} \qquad 4.7$$

The condition for a constant gradient structure is that the exponent equals 0,

$$\frac{\omega}{Q}\frac{l}{\Delta v_g} = 1 \qquad 4.8$$

5. INPUT FIELD AS A FUNCTION OF AVERAGE ACCELERATING GRADIENT

The relationship between input and average accelerating gradient is found by integrating equation 3.9 over the length of the structure,

$$\int_0^l E(z)dz = \left(E(0) + \frac{IR'}{1 - \frac{Q}{\omega}\frac{\Delta v_g}{l}}\right)\frac{v_0 l}{\Delta v_g}\frac{1}{\frac{1}{2}\left(\frac{\omega l}{Q\Delta v_g} + 1\right)}\left(1 - \left(1 - \frac{\Delta v_g}{v_0}\right)^{\frac{1}{2}\left(\frac{\omega}{Q}\frac{l}{\Delta v_g} + 1\right)}\right) - \frac{IR' l}{1 - \frac{Q}{\omega}\frac{\Delta v_g}{l}} \qquad 5.1$$

Defining,

$$\langle E \rangle = \frac{1}{l}\int_0^l E(z)dz \qquad 5.2$$

Now solving for input gradient as a function of the average,

$$E(0) = \frac{\langle E \rangle + \frac{IR'}{1 - \frac{Q}{\omega}\frac{\Delta v_g}{l}}}{\frac{v_0}{\Delta v_g}\frac{2}{\left(\frac{\omega l}{Q\Delta v_g} + 1\right)}\left(1 - \left(1 - \frac{\Delta v_g}{v_0}\right)^{\frac{1}{2}\left(\frac{\omega}{Q}\frac{l}{\Delta v_g} + 1\right)}\right)} - \frac{IR'}{1 - \frac{Q}{\omega}\frac{\Delta v_g}{l}} \qquad 5.3$$

6. FILL TIME

The fill time, t_f, of the structure is given by the integral,

$$t_f = \int_0^l \frac{dz}{v_0 - \Delta v_g \frac{z}{l}} = \frac{l}{\Delta v_g} \ln\left(1 - \frac{\Delta v_g}{v_0}\right) \tag{6.1}$$

7. THE TDS AS AN EXAMPLE

The TDS fundamental mode parameters are summarized in Table 1.

Table 1: TDS RF parameters

	v_g [mm/sec]	Q	R'/Q [kΩ/m]
input	$3.24\text{x}10^{10}$	3910	22.3
output	$1.62\text{x}10^{10}$	3708	30.11
average	$2.38\text{x}10^{10}$	3875	26.0

The beam loading voltage profile for nominal CLIC parameters is shown in figure 1. The parameters are: an average loaded gradient of 150 MV/m, the v_g range given in the table, average Q and R'/Q values from the table, a structure length of 0.5 m (150 cells), a bunch charge of .64 nC, and a bunch spacing of .667 nsec.

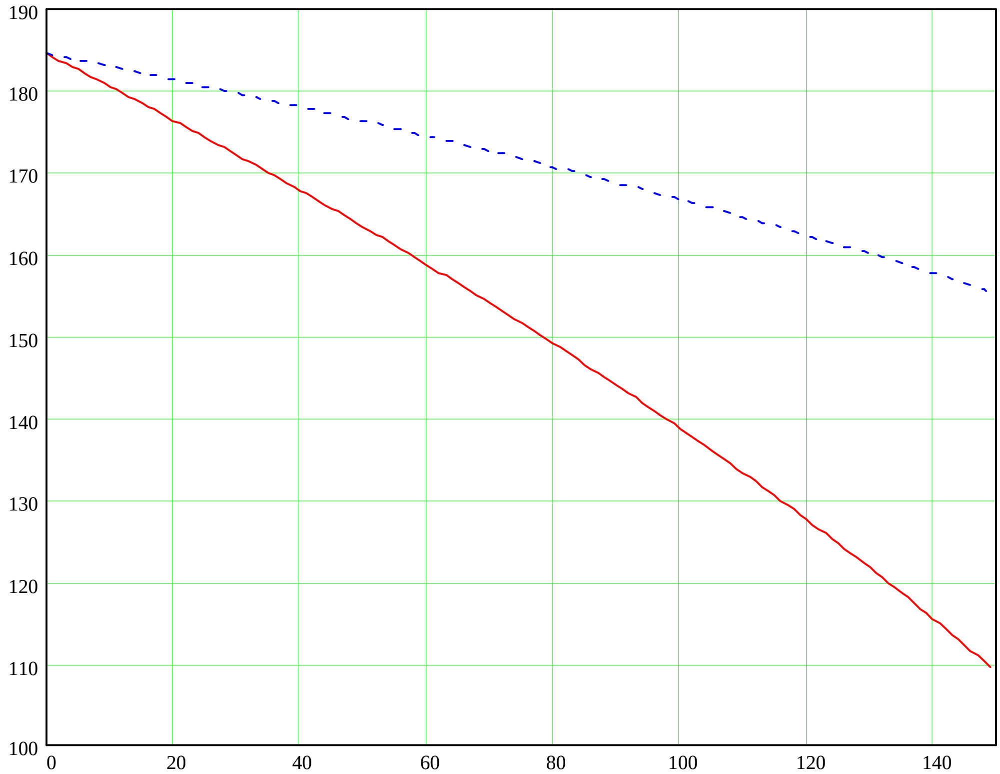


Figure 1: Analytic solution of the voltage profile in the TDS assuming a linear variation in v_g and constant Q and R'/Q. The average loaded gradient is 150 MV/m. The solid curve is the voltage profile with beam and the dashed curve is without beam.

8. SOLVING FOR A LINEAR VARIATION IN GROUP VELOCITY AND R'/Q

In order to improve the accuracy of the field profile calculation, the variation in R'/Q that is present in the TDS design must be taken into account. Equation 2.6 expressed in field (rather than power) is,

$$\frac{d}{dz}\left(\frac{E^2(z)v_g(z)}{R'/Q(z)}\right)+\frac{E^2(z)\omega}{R'/Q(z)Q}+E(z)I\omega=0 \qquad 8.1$$

This equation has been solved numerically for the case of a linear variation of R'/Q along the structure. The end values of the fit to R'/Q are those listed in table 1. The field profiles with and without beam are shown in figure 2. The input power required to produce a loaded accelerating gradient of 150 MV/m is consequently 229 MW. Including the varying R'/Q results in an input gradient that is about 10% lower than with R'/Q constant. The unloaded gradient is almost constant even though the TDS is not 'constant gradient' as defined by the condition given in equation 4.8.

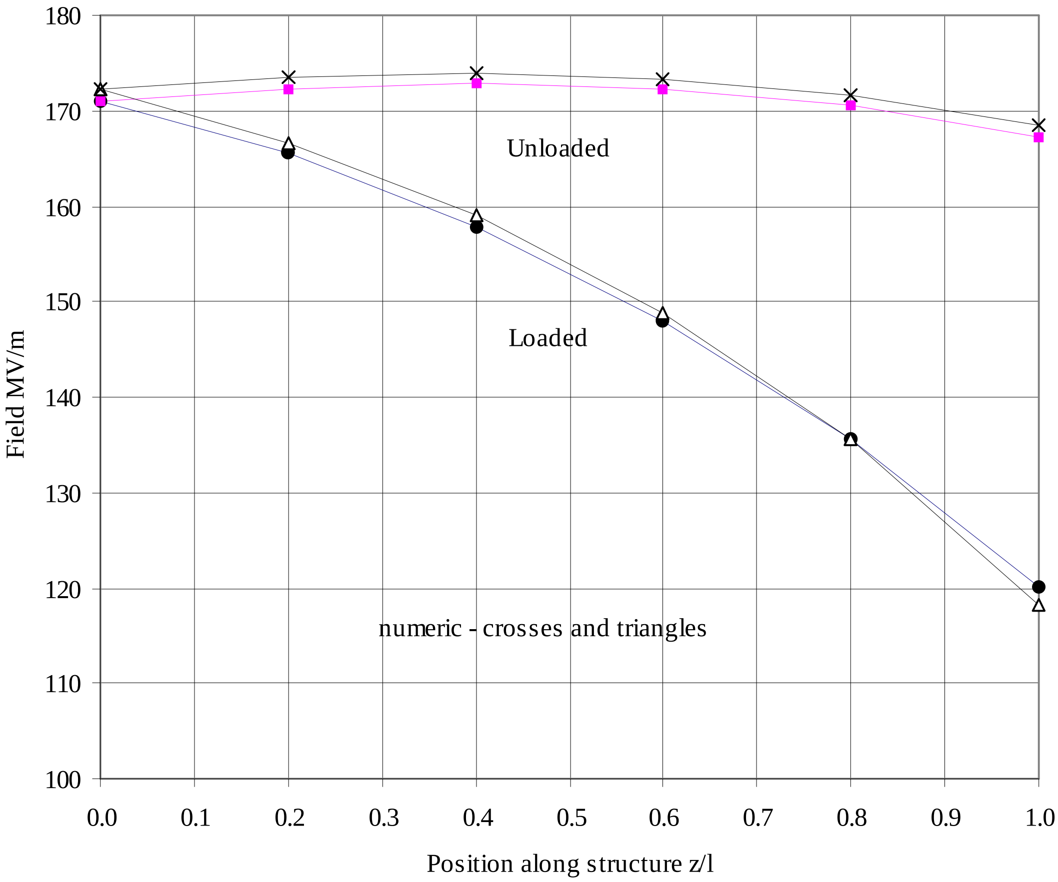


Figure 2: Comparison between numerical and analytic solutions of the voltage profile in the TDS assuming linear variations in v_g and R'/Q, and and average gradient of 150 MV/m.

An analytic approximation to the solution of equation 8.2 with a linear variation in v_g and R'/Q is given by [2],

$$E(z) = \sqrt{\frac{R'/Q(0) + \Delta R'/Q \frac{z}{l}}{v_0 - \Delta v_g \frac{z}{l}}} \left(1 - \frac{\Delta v_g}{v_0} \frac{z}{l}\right)^{\frac{\omega l}{2Q\Delta v_g}} * \left[E_0 \sqrt{\frac{v_0}{R'/Q(0)}} - \frac{l}{2} I\omega \frac{z}{l} \left(1 - \frac{\Delta v_g}{2v_0} \frac{z}{l}\right)^{-\frac{\omega l}{2Q\Delta v_g}} \sqrt{\frac{R'/Q(0) + \Delta R'/Q \frac{z}{2l}}{v_0 - \Delta v_g \frac{z}{2l}}} \right] \quad 8.2$$

This equation applied to the TDS is plotted in figure 2.

REFERENCES

[1] M. Dehler, I. Wilson, W. Wuensch, "A Tapered Damped Accelerating Structure for CLIC", LINAC98.
[2] G. Guignard, J. Hagel, "Analytical Solution of the Differential Equation for the Electric Field in a Loaded RF structure", CLIC note 386.