CERN–TH/99–141
OUTP–99–24P
hep–th/9905132

Gauge Coupling Instability and Dynamical Mass Generation in $\mathcal{N}=1$ Supersymmetric QED$_3$

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Using superfield Dyson–Schwinger equations, we compute the infrared dynamics of the semi–amputated full vertex, corresponding to the effective running gauge coupling, in N–flavour $\mathcal{N}=1$ supersymmetric QED$_3$. It is shown that the presence of a supersymmetry–preserving mass for the matter multiplet stabilizes the infrared gauge coupling against oscillations present in the massless case, and we therefore infer that the massive vacuum is thus selected at the level of the (quantum) effective action. We further demonstrate that such a mass can indeed be generated dynamically in a self–consistent way by appealing to the superfield Dyson–Schwinger gap equation for the full matter propagator.

I. INTRODUCTION.

Three dimensional U(1) gauge field theories, whether as toy models of four dimensional physics or as interesting physical theories in their own right, have attracted considerable attention [1–8]. In particular, they have been shown to exhibit a rich and non–trivial infrared structure, exhibiting dynamical mass generation and critical behaviour; the study of these phenomena requires non–perturbative techniques. The question of how the incorporation of supersymmetry affects the infrared physics of these theories has also been addressed [1,9,10] and it has been found that when $\mathcal{N}=1$ mass can be generated dynamically without breaking the supersymmetry, while mass generation in the $\mathcal{N}=2$ model is forbidden. For the $\mathcal{N}=1$ model an important question remains unanswered: whether the massive or massless solutions are selected by the theory. In non–supersymmetric theories this question can be addressed by appealing to the effective potential and energetic favourability [1,2]; in supersymmetric theories the effective potential vanishes and this argument fails. It has been suggested [1]

that the issue can be resolved in supersymmetric theories at the level of the (quantum) effective action (which does not vanish) and the Ward identities arising from supersymmetry. The Ward identities have been found to play a crucial rôle in this selection in models which have additional constraints [11, 12], but do not yield any extra information for the unconstrained U(1) gauge field theory [10]. In this paper we propose a resolution of this problem for the basic N–flavour $\mathcal{N}=1$ U(1) theory at the level of the effective action: by considering an appropriate set of Dyson–Schwinger equations we show that a mass can be self–consistently generated without breaking supersymmetry, and that this mass stabilizes the infrared effective gauge coupling which in the massless case oscillates without definite sign.

We adopt a superfield formalism throughout this paper, and work with a large number N of matter flavours. We consider the superfield Dyson–Schwinger equation for the semi–amputated full three–point vertex [13–15] which corresponds to the effective (running) gauge coupling [16], and compute the infrared dynamics of this vertex in the presence of vanishing and non–vanishing masses for the matter multiplet. In the massless case the dynamics are described by a non–linear differential equation of Emden–Fowler type [17, 18], which admits only oscillatory solutions in the infrared; we interpret this as indicating instabilities in the (quantum) effective action. Repeating the computation with the insertion (by hand) of a finite mass for the matter multiplet we find that these instabilities disappear, and the solution shows the existence of a non–trivial infrared fixed point for the running coupling. We then demonstrate that a mass for the matter superfield can be dynamically generated in a self–consistent way in this approach by appealing to the Dyson–Schwinger equation for the full matter propagator cast in terms of the semi–amputated full vertex. In agreement with reference [10] we find no evidence for a critical flavour number, above which dynamical mass generation does not occur.

In reference [10] the gap equation for the full matter propagator was studied incorporating a full vertex which by construction satisfied the U(1) Ward identity; instead, in this work, the form of the full vertex in the deep infrared will be determined self–consistently from the Dyson–Schwinger equation. The truncated form of the Dyson–Schwinger equation is not manifestly gauge invariant, i.e. it does not respect the Ward identity for general incoming momenta and there is a residual explicit dependence on the covariant gauge fixing parameter. However in the physical, on–shell limit of vanishing incoming momenta we find that the Ward identity is satisfied and that the dependence on the gauge fixing parameter drops out. Since in the present work we are interested only in obtaining and analysing the structure of the vertex in the deep infrared, the lack of gauge invariance away from vanishing incoming momentum will not adversely affect our conclusions. The superfield formalism we adopt keeps supersymmetry manifest and thereby radically simplifies the system of equations one would obtain for the full vertices in a component computation. The disadvantages of the superfield formalism [19, 20] lie in the propagation of gauge artifacts in the connexion superfield which result in spurious infrared divergences. By working in a general (covariant) gauge we will have enough flexibility to remove these divergences by an appropriate gauge choice.

This paper is organized as follows: in section II we construct the action functional for $\mathcal{N}=1$ supersymmetric QED_3 in superfield formalism and give the form of the dressed propagators for the matter and connexion superfields. In section III we introduce the superfield

semi–amputated vertex and construct its Dyson–Schwinger equation; we then solve the Dyson–Schwinger equation in the presence of vanishing and non–vanishing masses for the matter superfield and interpret the results. In section IV we discuss the infrared properties of the matter and connexion superfield propagators, in association with approximations made in section III; we show in section V that when the previously computed vertex is incorporated in the gap equation for the matter propagator, a mass is dynamically generated self–consistently. Finally we present our conclusions in section VI.

II. THE ACTION.

We consider a model with $\mathcal{N} = 1$ supersymmetry, N matter flavours and local U(1) gauge invariance. The required action functional then comprises three parts: the gauge invariant classical field strength term for the (spinor) connexion Γ_α, a (Lorentz) gauge fixing term, and a locally U(1) invariant kinetic term for the matter superfields Φ and Φ^*:

$$S = S_g^{\rm class} + S_g^{\rm GF} + S_m; \tag{2.1}$$

$$\begin{aligned} S_g^{\rm class} &= \int d^3x\, d^2\theta\ \Gamma_\alpha \left(-\frac{1}{8} D^\eta D^\alpha D^\beta D_\eta\right) \Gamma_\beta, \\ S_g^{\rm GF} &= \int d^3x\, d^2\theta\ \Gamma_\alpha \left(\frac{1}{4\xi} D^\alpha D^2 D^\beta\right) \Gamma_\beta, \\ S_m &= \int d^3x\, d^2\theta\ \left(-\frac{1}{2}\right) \left[\nabla^\alpha \Phi\right]^* \left[\nabla_\alpha \Phi\right]. \end{aligned} \tag{2.2}$$

We have included in the matter part an implicit sum over N flavours, which do not interact with each other directly, but interact with the same connexion superfield. The U(1) covariant derivative ∇^α is given by

$$\nabla^\alpha \doteq D^\alpha - ie\Gamma^\alpha,$$

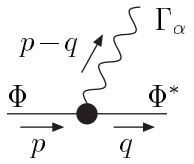
where e is the (dimensionful) gauge coupling. We work within large–N, in which the quantity

$$\frac{e^2 N}{4} = \alpha$$

is kept fixed (but large) as N becomes large. From the action (2.1) it is easy to derive the dressed propagators for the matter and connexion superfields:

$$\begin{aligned} \Delta(p;12) &= \frac{i}{A(p)} \frac{D^2(p) - M(p)}{p^2 + M^2(p)} \delta^2(12), \\ \Delta_{\alpha\beta}(p;12) &= -i \frac{1}{p^4} \frac{1}{B(p)} \left[(1+\xi)\, p_{\alpha\beta}\, D^2 - (1-\xi)\, C_{\alpha\beta}\, p^2\right] \delta^2(12). \end{aligned} \tag{2.3}$$

The matter superfield contains a possible dynamically generated mass function $M(p)$ and the scalar functions A and B parameterize the dressing of the matter and connexion superfields respectively. The unknown functions A, B, and M can at least in principle be determined from the appropriate Dyson–Schwinger equations; we return to these in sections IV and V. The dressed three–point vertex derived from the action above is shown in figure 1.



$$: \; -\tfrac{e}{2}\, G(p, p-q, q)\, C^{\alpha\beta}\, D_\beta(q)$$

FIG. 1. Structure and momentum assignments of the dressed three–point vertex; solid lines represent matter superfields and wavy lines the connexion superfield.

In this paper we use a convenient spinor index notation for three–vectors, in which they are represented as symmetric second rank spinors; spinor indices are raised and lowered by the antisymmetric metric C. We collect here some useful identities which will be used in what follows [21] ($\{\partial_\mu, \theta^\nu\} = \delta_\mu^\nu$):

$$\begin{aligned}
&A^\alpha = C^{\alpha\beta} A_\beta, \\
&A_\beta = A^\alpha C_{\alpha\beta} \\
&A^2 \doteq \frac{1}{2} A^\alpha A_\alpha, \\
&C_{\mu\nu} C^{\alpha\beta} = \delta^\alpha_{[\mu}\, \delta^\beta_{\nu]}, \\
&p_{\mu\nu} q^{\nu\alpha} = \delta^\alpha_\mu\, p \cdot q, \\
&D^\mu(q) = \partial^\mu + \theta^\nu q_{\mu\nu}, \\
&D^2(q) = \partial^2 + \theta^\mu q_{\mu\nu} \partial^\nu + q^2 \theta^2, \\
&D^\mu(q) D^2(q) = q^{\mu\lambda} D_\lambda(q), \\
&D^\mu(q) D^\nu(q) = q^{\mu\nu} + C^{\nu\mu} D^2(q).
\end{aligned} \tag{2.4}$$

In the next section we construct and analyse the truncated Dyson–Schwinger equation for the full three–point vertex shown in figure 1 with and without a mass for the matter multiplet and demonstrate that oscillations present in the massless case disappear when a mass for the matter multiplet is included.

III. DYSON–SCHWINGER EQUATION FOR THE VERTEX.

Throughout this paper we use a truncation to leading order in $1/N$; the truncated Dyson–Schwinger equation for the full vertex is shown schematically in figure 2.

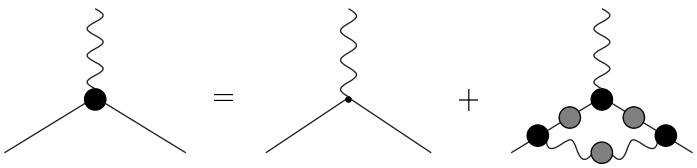


FIG. 2. Schematic form of the Dyson–Schwinger equation for the full vertex. Solid lines represent matter superfields, and wavy lines the connexion superfield. Blobs indicate full non–perturbative quantities.

The graph on the left hand side of figure 2 is written as follows

$$\left(-\frac{e}{2}G(p,p-q,q)\right)\int d^2\theta\ \Phi(-p,\theta)\Gamma_\alpha(p-q,\theta)\,D^\alpha(q)\Phi^*(q,\theta), \tag{3.1}$$

where we have assumed that we can perform the factorization into a scalar function G multiplying the superspace structure shown above; the first graph on the right of figure 2 is of the same form but with $G \mapsto 1$. This factorization is similar in effect to the approach taken in non–supersymmetric QED$_3$ in which the vertex is assumed to be a scalar function multiplying the usual Dirac matrix. This approximation is computationally convenient, but does not satisfy the Ward identity except at vanishing incoming momentum, where the full vertex becomes on–shell. Given this limitation we will only be interested in the behaviour of the theory in this regime; as we will show, even with this restriction we can obtain interesting information about the infrared fixed point structure. As we will see in section V, the present approach gives rise to the same qualitative picture as that obtained in reference [10], namely that a mass can be generated dynamically for the matter multiplet without breaking supersymmetry.

Following references [13–15] we define the semi–amputated full non–perturbative vertex $\hat{G}$ as

$$\hat{G}(p_1,p_2,p_3) \doteq Z(p_1,p_2,p_3)\,G(p_1,p_2,p_3), \tag{3.2}$$

where Z is defined in terms of the functions A and B appearing in the dressed propagators (2.3) as

$$Z(p_1,p_2,p_3) \doteq A^{-1/2}(p_1)\,B^{-1/2}(p_2)\,A^{-1/2}(p_3) \geqslant 0. \tag{3.3}$$

The quantity $e\hat{G}$ is the appropriate and natural generalization of the running charge in super–renormalizable gauge field theories [16]. This definition is the same as the generalization of the running charge in non–supersymmetric QED$_3$ [15]. As we will show, this definition also simplifies the structure of the integral equation for the vertex function considerably.

A. Vertex With Vanishing Mass.

The one–loop graph on the right of figure 2 reads as follows:

$$\begin{aligned}-\frac{e}{2}\left(-i\frac{e^2}{4}\right)\int \bar{d}^3k\,d^2\theta_1 d^2\theta_2 d^2\theta_3\,G^3\ &\left[\frac{D^\mu(p+k)D^2(p+k)}{A(p+k)\,(p+k)^2}\delta^2(12)\right]\times\\ &\times\left[\frac{D^\alpha(q+k)D^2(q+k)}{A(q+k)\,(q+k)^2}\delta^2(23)\right]\left[\frac{(1+\xi)k_{\mu\nu}-(1-\xi)C_{\mu\nu}k^2}{B(k)\,k^4}\delta^2(31)\right]\times\\ &\times\Phi(-p,\theta_1)\,\Gamma_\alpha(p-q,\theta_2)\,D^\nu(q)\Phi^*(q,\theta_3),\end{aligned} \tag{3.4}$$

where

$$G^3 = G(p,-k,p+k)\,G(p+k,p-q,q+k)\,G(q+k,k,q). \tag{3.5}$$

Following [15] we consider the external connexion superfield momentum to be vanishingly small, leaving one external scale p in the problem; computing the superspace part of equation (3.4) using the identities (2.4) we obtain an integral equation for the full vertex G. Multiplying this equation through by $Z(p)$ we obtain the desired integral equation for the semi–amputated vertex $\hat{G}$ from the Dyson–Schwinger equation in figure 2:

$$\hat{G}(p) = Z(p) + \frac{1}{4}e^2 \int \bar{d}^3k \; \hat{G}^3(k) \, \frac{(1+\xi)k \cdot (p+k) + (1-\xi)k^2}{(p+k)^2 \, k^4}. \tag{3.6}$$

We have made the approximation that $\hat{G}^3$ is the cube of one scalar function dependent on the scale k only [15]: this is justified by the self–consistency of our results. Note that in the absence of the inhomogeneous term Z on the right hand side of equation (3.6) one obtains a integral equation involving only $\hat{G}$, which can in principle be solved. In the following we will drop this inhomogeneous term, returning in section IV to a discussion of why this may be done with safety.

The angular integration is easy to perform, leaving the following equation, which we have recast in the dimensionless variables $x \doteq p/\alpha$, $y \doteq k/\alpha$:

$$\hat{G}(x) = \frac{1}{4\pi^2 N} \int dy \; \frac{\hat{G}^3}{y^2} \left\{ (1+\xi) \left(1 + \frac{y^2 - x^2}{2xy} \ln \left| \frac{y+x}{y-x} \right| \right) + (1-\xi) \frac{y}{x} \ln \left| \frac{y+x}{y-x} \right| \right\}. \tag{3.7}$$

Since we are interested in the deep infrared we consider the limit $x \ll 1$, expand the logarithms in the above equation to second order and obtain

$$x^2 \hat{G} \simeq \frac{(2-\xi)}{3\pi^2 N} \int_0^x dy \; \hat{G}^3(y) + \frac{x^2}{\pi^2 N} \int_x^\infty dy \; \frac{\hat{G}^3(y)}{y^2} + \frac{x^4 \xi}{3\pi^2 N} \int_x^\infty dy \; \frac{\hat{G}^3(y)}{y^4}. \tag{3.8}$$

We drop the last term in the limit $x \ll 1$ and after appropriate differentiations with respect to x we arrive the equivalent differential equation:

$$x^3 \hat{G}'' + \left[3x^2 + \frac{x(1+\xi)}{\pi^2 N} \hat{G}^2 \right] \hat{G}' + \frac{5-\xi}{3\pi^2 N} \hat{G}^3 = 0. \tag{3.9}$$

In the gauge $\xi = -1$ and changing variables $x \mapsto w = x^{-2}$ the above equation can be recast as a differential equation of Emden–Fowler type [17, 18]:

$$\frac{d^2}{dw^2} \hat{G} + \frac{1}{2\pi^2 N} w^{-3/2} \hat{G}^3 = 0; \tag{3.10}$$

in the limit $w \to \infty$ it has been shown [18] that the only (real and non–divergent) solutions of this equation are oscillatory. We interpret this as indicating that in the absence of a mass for the matter multiplet the gauge coupling is subject to instabilities which render it unphysical. In the next subsection we study the effects of a mass introduced by hand and show that these instabilities are removed, and that the coupling is driven to an non–trivial infrared fixed point.

B. Vertex With Non–Vanishing Mass.

In this subsection we repeat the computation above retaining the effects of a constant mass $M(p) \simeq M(0) = M \neq 0$ in the propagators (2.3). Here we will put in the mass by hand, in section V we will show that the semi–amputated full vertex we consider admits the dynamical generation of a mass. The one–loop graph on the right of figure 2 now reads

$$
\begin{aligned}
-\frac{e}{2}\left(-i\frac{e^2}{4}\right)\int \vec{d}^3k\, d^2\theta_1 d^2\theta_2 d^2\theta_3\; G^3\; \left[\frac{D^\mu(p+k)\,(D^2(p+k)-M)}{A(p+k)\;((p+k)^2+M^2)}\delta^2(12)\right]\times \\
\times\left[\frac{D^\alpha(q+k)\,(D^2(q+k)-M)}{A(q+k)\;((q+k)^2+M^2)}\delta^2(23)\right]\left[\frac{(1+\xi)k_{\mu\nu}-(1-\xi)C_{\mu\nu}k^2}{B(k)\,k^4}\delta^2(31)\right]\times \\
\times\Phi(-p,\theta_1)\,\Gamma_\alpha(p-q,\theta_2)\,D^\nu(q)\Phi^*(q,\theta_3), \qquad (3.11)
\end{aligned}
$$

where as before G^3 is given by equation (3.5). Again the superspace parts of this equation can be evaluated using the identities (2.4); in contrast to the case of non–supersymmetric QED$_3$, where the inclusion of a mass for the fermions significantly alters the structure of the integral equation, here the mass terms only appear in the denominators of the matter propagators, for the terms linear in M in the numerator cancel. In this respect, a mass for the matter superfield behaves like a trivial infrared regulator, similar to the effect of a photon mass in the non–supersymmetric model [15]. On considering the external connexion momentum vanishingly small and multiplying through by $Z(p)$ as before, the integral equation to be compared with (3.6) reads

$$
\hat{G}(p) = Z(p) + \frac{1}{4}e^2\int \vec{d}^3k\;\hat{G}^3(k)\,\frac{(1+\xi)k\cdot(p+k)+(1-\xi)k^2}{((p+k)^2+M^2)\;k^4}. \qquad (3.12)
$$

As before, it is easy to perform the angular integration, the result of which reads (in dimensionless variables $x \doteq p/\alpha$, $y \doteq k/\alpha$ and $m \doteq M/\alpha$)

$$
\begin{aligned}
\hat{G}(x) = \frac{1}{4\pi^2 N}\int dy\;\frac{\hat{G}^3}{y^2}\Bigg\{(1+\xi)\left(1+\frac{y^2-x^2-m^2}{4xy}\ln\left[\frac{(y+x)^2+m^2}{(y-x)^2+m^2}\right]\right) \\
+(1-\xi)\frac{y}{2x}\ln\left[\frac{(y+x)^2+m^2}{(y-x)^2+m^2}\right]\Bigg\}. \qquad (3.13)
\end{aligned}
$$

We have again dropped the inhomogeneous term Z (see section IV); considering the deep infrared limit $x \ll 1$, and now also $x \ll m$, we can expand the logarithms above to second order to obtain the approximate form:

$$
\begin{aligned}
\hat{G} \simeq \frac{(3-\xi)}{4\pi^2 N}\frac{1}{m^2}\int_0^x dy\;\hat{G}^3(y) - \frac{(1+\xi)}{4\pi^2 N}\frac{x^2}{m^2}\int_0^x dy\;\frac{\hat{G}^3(y)}{y^2} + \frac{1}{\pi^2 N}\int_x^\infty dy\;\frac{\hat{G}^3(y)}{y^2+m^2} \\
-\frac{(1+\xi)}{4\pi^2 N}x^2\int_x^\infty dy\;\frac{\hat{G}^3(y)}{y^2\,(y^2+m^2)}. \qquad (3.14)
\end{aligned}
$$

Differentiation with respect to x yields the integral–differential equation

$$\hat{G}'(x) = \frac{(3-\xi)}{4\pi^2 N}\frac{\hat{G}^3(x)}{m^2} - \frac{1}{\pi^2 N}\frac{\hat{G}^3(x)}{x^2+m^2} - \frac{(1+\xi)}{4\pi^2 N}\frac{\hat{G}^3(x)}{m^2} + \frac{(1+\xi)}{4\pi^2 N}\frac{\hat{G}^3(x)}{x^2+m^2}$$
$$-\frac{(1+\xi)}{2\pi^2 N}\frac{x}{m^2}\int_0^x dy\ \frac{\hat{G}^3(y)}{y^2} - \frac{(1+\xi)}{2\pi^2 N}x\int_x^\infty dy\ \frac{\hat{G}^3(y)}{y^2\,(y^2+x^2)}. \tag{3.15}$$

By the convenient choice of gauge $\xi = -1$ this can be reduced to a first order non–linear differential equation, which can be integrated with ease to yield

$$\hat{G}(x) = \frac{1}{\left[c + \frac{2}{\pi^2 N m^2}\left(m\ \arctan\left(\frac{x}{m}\right) - x\right)\right]^{1/2}}, \tag{3.16}$$

where c is an integration constant to be determined from the boundary condition obtained from the original integral equation. In the limit $x \to 0$ the renormalization group β function vanishes:

$$\lim_{x\to 0} x\frac{d\hat{G}(x)}{dx} = \lim_{x\to 0}\frac{1}{\pi^2 N}\frac{\hat{G}^3(x)\,x^3}{m^2\,(x^2+m^2)} \to 0, \tag{3.17}$$

and hence there is a non–trivial (N–independent) fixed point at $x = 0$ given by

$$\hat{G}(0) = c^{-1/2}. \tag{3.18}$$

Returning to the integral equation (3.12) and taking the limit $x \to 0$ we can investigate the constraints on the integration constant c:

$$\hat{G}(0) = \frac{1}{\pi^2 N}\int_0^\infty dy\ \frac{\hat{G}^3(y)}{y^2+m^2}. \tag{3.19}$$

Since the right hand side of this equation is manifestly positive definite, the trivial solution $\hat{G}(0) = 1/\sqrt{c} = 0$ is ruled out, unless $\hat{G}$ is trivially zero. Noting that in the small x limit $\hat{G}$ differs from its fixed point value by a quantity of order $\mathcal{O}(x^3)$, we can crudely approximate the integral in (3.19) as follows

$$\hat{G}(0) = \frac{1}{\pi^2 N}\int_0^m dy\ \frac{\hat{G}^3(0)}{y^2+m^2} + \frac{1}{\pi^2 N}\int_m^\infty dy\ \frac{1}{y^2+m^2}, \tag{3.20}$$

where we have set $\hat{G}$ in the second integral to its ultraviolet asymptote of unity: both of these approximations are underestimates, and therefore after performing the integrations we have

$$4\pi m\hat{G}(0) > \frac{1}{N}\left(1 + \hat{G}^3(0)\right). \tag{3.21}$$

In principle for a given m this leads to a (small) critical coupling, below which there is no mass generation. Note that even on restoring the inhomogeneous term Z (see section IV) the inequality is only modified to

$$4\pi m\left(\hat{G}(0) - Z(0)\right) > \frac{1}{N}\left(1 + \hat{G}^3(0)\right). \tag{3.22}$$

Since at the level of the original integral equation (3.12) in the gauge $\xi = -1$ it is obvious that $\hat{G} \geqslant Z$, we obtain again a (small) critical coupling. Note that in the above analysis there is no evidence for a critical flavour number.

To conclude this section we should check our assertion that the value of $\hat{G}$ at vanishing x is indeed ξ–independent. To accomplish this we return to the integral equation (3.14) and retain the general gauge dependence. The equation can then readily be converted to a differential equation:

$$x\hat{G}'' + \left[-1 - \frac{3}{2m^2\pi^2 N}(1-\xi)x\hat{G}^2 + \frac{3}{4\pi^2 N}(3-\xi)\frac{x}{x^2+m^2}\hat{G}^2\right]\hat{G}' + \frac{1}{\pi^2 N}\left[\frac{1}{m^2} - \frac{(5+\xi)}{4}\frac{1}{x^2+m^2} - \frac{2x^2}{(x^2+m^2)^2}\right]\hat{G}^3 = 0. \tag{3.23}$$

Considering the limit $x \to 0$ this reduces to a first order differential equation (in which the term $x\hat{G}''$ has been dropped as subleading in x)

$$\hat{G}' + \frac{(1+\xi)}{4\pi^2 m^2}\hat{G}^3 = 0$$

which can be integrated easily to give for small x

$$\hat{G}(x) \simeq \frac{1}{\left(c + \frac{(1+\xi)}{4\pi^2 N m^2}x\right)^{1/2}}, \tag{3.24}$$

demonstrating that the term in $\hat{G}''$ is indeed subleading and also that all the ξ dependence vanishes at $x = 0$, so information about the fixed point is ξ independent as expected from the on–shell nature of $\hat{G}(0)$.

We have shown in this section that in the presence of a mass for the matter multiplet the full vertex is stabilized and driven to a non–trivial infrared fixed point. In the above analysis the mass for the matter multiplet has been included by hand; however we will show in section V that a mass can be dynamically generated self–consistently by coupling the vertex equation to the corresponding Dyson–Schwinger equation for the matter propagator. In the next section we turn to a discussion of the inhomogeneous term Z and why it can be safely omitted.

IV. THE INHOMOGENEOUS TERM Z AND THE FUNCTIONS A AND B.

In this section we analyse the Dyson–Schwinger equations for the functions A and B with which we have respectively dressed the matter and connexion superfield propagators. First we consider the Dyson–Schwinger equation for the matter propagator, shown schematically in figure 3.

The difference between the graphs on the left of figure 3 is easily computed to be

$$-i(A(p)-1)\int d^2\theta\ \Phi(-p,\theta)D^2(p)\Phi^*(p,\theta) - i\frac{M(p)}{A(p)}\int d^2\theta\ \Phi(-p,\theta)\Phi^*(p,\theta). \tag{4.1}$$

We adopt the one–loop dressed approximation to the Dyson–Schwinger equation in which both vertices in figure 3 are fully dressed and all two–particle irreducible corrections are dropped [22]. In doing so the resulting integral equation can be cast in terms of $\hat{G}$ alone.

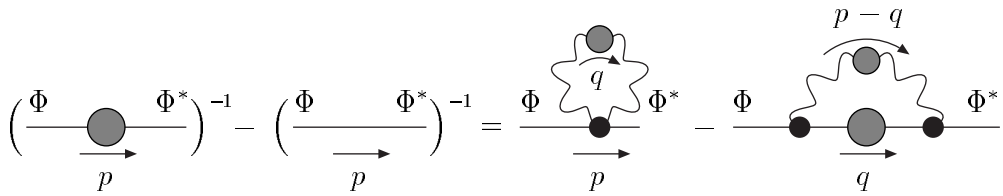


FIG. 3. Schematic form of the Dyson–Schwinger equation for the full matter propagator. Solid lines represent matter superfield propagators, and wavy lines gauge superfield propagators; blobs indicate full non–perturbative quantities.

The graphs on the right hand side of figure 3 can be manipulated to obtain functions multiplying the two superspace structures

$$\int d^2\theta\ \Phi(-p,\theta)\, D^2(p)\, \Phi^*(p,\theta),$$
$$\int d^2\theta\ \Phi(-p,\theta)\, \Phi^*(p,\theta); \tag{4.2}$$

comparison with equation (4.1) shows that the function multiplying the first of these structures is to be identified with the contribution to the wavefunction renormalization $A(p)$ and the function which multiplies the second corresponds to the self energy $M(p)/A(p)$, to which we will return in the next section. The first "seagull" graph on the right hand side gives only an irrelevant p–independent contribution to the wavefunction renormalization while the last graph on the right hand side of the figure contributes both to the wavefunction renormalization and the self energy. Evaluating the superspace parts of the last graph leads to the following integral equation for $A(p)$

$$A(p) = 1 + (1+\xi)A(p)\frac{ie^2}{2}\int \bar{d}^3k \frac{\hat{G}^2(k)}{k^2 + M^2(k)}\frac{k\cdot(k-p)}{(k-p)^4}. \tag{4.3}$$

In the gauge $\xi = -1$ which we adopt throughout this paper, this reduces to unity, leaving a constant $A(p)$. Note that this is in line with the result of reference [10], where the wavefunction renormalization was computed to be

$$A(p) = \left(\frac{p}{\alpha}\right)^{2(1+\xi)/N\pi^2}. \tag{4.4}$$

We turn now to the connexion superfield and the function $B(p)$. In standard large–N treatments [9,10] the function B includes the effects of massless matter loops resummed to leading order in $1/N$, giving

$$B(p) \sim 1 + \alpha/p.$$

Therefore in the deep infrared we find

$$Z(x) = A^{-1}(x)\, B^{-1/2}(x) \sim x^{1/2} \tag{4.5}$$

and the inhomogeneous term goes to zero and can be safely dropped compared with a non–vanishing $\hat{G}(0)$ in a large–N framework.

For completeness it is instructive to consider the function B outside this large–N resummation, and consider the Dyson–Schwinger equation shown schematically in figure 4. Again we adopt the one–loop dressed approximation, which enables us to recast the equation in terms of $\hat{G}$ alone.

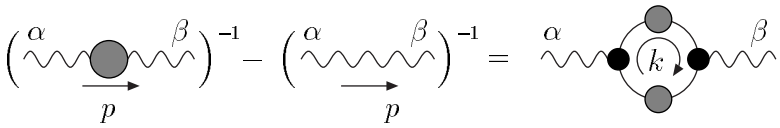


FIG. 4. Schematic form of the Dyson–Schwinger equation for the full connexion propagator. Solid lines represent matter superfields and wavy lines represent connexion superfields; blobs indicate full non–perturbative quantities.

The difference between the graphs on the left of figure 4 is easily computed (in the gauge $\xi = -1$) to be

$$(B(p) - 1)\, \frac{1}{2} C^{\alpha\beta} p^2 \int d^2\theta\; \Gamma_\alpha(-p,\theta)\, \Gamma_\beta(p,\theta). \tag{4.6}$$

In computing the graph on the right of figure 4 the superspace structure in equation (4.6) naturally appears, and the resulting integral equation for B is (in which we have already performed the simple angular integration)

$$p^3\, (B(p) - 1) = \frac{e^2 N}{16\pi^2} B(p) \int dk\; k \hat{G}^2(k)\, \ln\left(\frac{(k+p)^2 + M^2}{(k-p)^2 + M^2} \right). \tag{4.7}$$

Converting to dimensionless variables, considering constant M and expanding the logarithm as usual we can convert to an equivalent differential equation in which we have approximated $\hat{G}$ by its constant value at $x = 0$, noting from equation (3.16) that in doing so we have only dropped terms of order $\mathcal{O}(x^3)$:

$$\frac{d}{dx}\left(x^2 B^{-1}(x) \right) = 2x - \frac{\hat{G}^2(0)}{\pi^2} \frac{x^2}{m^2} + \frac{\hat{G}^2(0)}{\pi^2} \frac{x^2}{x^2 + m^2}. \tag{4.8}$$

This can be integrated easily with result

$$B^{-1}(x) = 1 - \frac{1}{3} \frac{\hat{G}^2(0)}{\pi^2} \frac{x}{m^2} + \frac{\hat{G}^2(0)}{\pi^2} \frac{1}{x} - \frac{\hat{G}^2(0)}{\pi^2} \frac{m}{x^2} \arctan\left(\frac{x}{m} \right); \tag{4.9}$$

for finite solutions the constant of integration must vanish. In the limit of small x there are cancellations to order $\mathcal{O}(x^3)$ and B reduces to

$$B(x) \simeq \frac{1}{1 + \mathcal{O}(x^3)}. \tag{4.10}$$

From this analysis it is clear that even when the connexion propagator is not resummed to leading order in $1/N$, the inhomogeneous term Z reduces to a finite constant in the infrared and can still be dropped with safety, justifying fully our assumption in section III.

V. MASS.

In this section we demonstrate that a mass for the matter multiplet can be generated dynamically when we feed the solution for the full vertex (3.16) into the Dyson–Schwinger equation for the matter propagator. The latter is shown schematically in figure 3, and now we evaluate the terms in the graph on the right hand side which multiply the second of the superspace structures in equation (4.2). Evaluating the superspace integrals leaves the following integral equation for the mass function

$$M(p) = \frac{ie^2}{2} \int \!\!\!\!\!\!- \, d^3k \frac{\hat{G}^2(k)\, M(k)}{(k^2 + M^2(k))\ (k-p)^4} \left[(1+\xi) k \cdot (k-p) + (1-\xi)(k-p)^2 \right]. \tag{5.1}$$

Choosing again the gauge $\xi = -1$, we see that it removes the spurious infrared divergences and simplifies the equation considerably. After performing the angular integration and expanding the resulting logarithms, we obtain in dimensionless variables

$$m(x) = \frac{2}{\pi^2 N x^2} \int_0^x dy\ \frac{y^2\, m(y)\, \hat{G}^2(y)}{y^2 + m^2(y)} + \frac{2}{\pi^2 N} \int_x^\infty dy\ \frac{m(y)\, \hat{G}^2(y)}{y^2 + m^2(y)}. \tag{5.2}$$

Differentiating with respect to x this can be converted to an equivalent differential equation,

$$x m'' + 3m' + \frac{4}{\pi^2 N} \frac{m\, \hat{G}^2}{x^2 + m^2} = 0. \tag{5.3}$$

In the deep infrared $x \ll m$, and approximating $\hat{G}(x)$ by $\hat{G}(0)$, noting again that the corrections to this approximation are of order $\mathcal{O}(x^3)$ (see equation (3.16)); equation (5.3) then admits the following approximate solution:

$$m(x) \simeq m(0) \left(1 - ax - \mathcal{O}(x^2)\right) \qquad a > 0 \tag{5.4}$$

which exhibits a constant dynamical mass $m(0)$ and has the correct (decreasing) behaviour away from $x = 0$; note that this solution is similar to the small x expansion of the mass found using the same method for non–supersymmetric QED$_3$ in reference [15]. The solution above is to be compared with the case when the vertex is chosen to satisfy the U(1) Ward identity for general momenta [10], where the solution for the mass behaves as

$$m(x) = m(0) e^{-x^2} \simeq m(0) \left(1 - x^2 + \mathcal{O}(x^4)\right).$$

The semi–amputated vertex considered here only satisfies the Ward identity at $x = 0$ where the vertex is on–shell, and therefore the small–x dependence may differ from the more complete solution, though the qualitative (decreasing) behaviour is retained.

VI. CONCLUSION.

In this paper we have proposed that the dynamical generation of a mass in $\mathcal{N} = 1$ supersymmetric QED$_3$ is selected by the dynamics over the massless alternative, for it stabilizes the running gauge coupling against oscillations. In particular the picture which has

emerged from our analysis is that in the absence of a mass for the matter multiplet, the full vertex oscillates in the infrared, leading to instability in the effective action. When a finite dynamical mass is added, the infrared physics is stabilized and there exists a non–trivial infrared fixed point. We have demonstrated that the incorporation of our full vertex solution into the Dyson–Schwinger equation for the matter propagator leads self–consistently to the dynamical generation of a mass.

Supersymmetry has been kept manifest throughout by adopting a superfield formalism which has the additional advantage of simplifying considerably the system of equations and Ward identities which would have to be solved in a component calculation. By working in a general gauge we have retained enough freedom to remove infrared divergences which arise from the propagation of spurious degrees of freedom in the connexion superfield [19, 20], and we have been able to simplify the analysis significantly. We have employed a large–N framework consistently throughout this paper; this in combination with the concept of a semi–amputated full vertex allowed us to decouple the Dyson–Schwinger equations so that the functions A and B used to dress the full propagators do not appear in the analysis of the vertex.

There is the interesting possibility that the techniques employed in this paper and in reference [15] could be extended to models in higher dimensions and with non–Abelian gauge groups.

ACKNOWLEDGEMENTS.

A.C.–S. would like to thank the CERN Theory Division for hospitality, and gratefully acknowledges financial support from P.P.A.R.C. (U.K.) (studentship number 96314661), which made his visit to CERN possible. N.E.M. is partially supported by P.P.A.R.C. (U.K.) under an Advanced Fellowship, and J.P. is funded by a Marie Curie Fellowship (TMR–ERBFMBICT 972024).

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[1] R. Pisarski, Physical Review **D 29**, 2423 (1984).
[2] T. Appelquist, M. Bowick, D. Karabali, and L. Wijewardhana, Physical Review **D 33**, 3704 (1986).
[3] M. Pennington and D. Walsh, Physics Letters **B 253**, 245 (1991).
[4] K. Kondo and H. Nakatani, Progress of Theoretical Physics **87**, 193 (1992).
[5] N. Dorey and N.E. Mavromatos, Nuclear Physics **B 386**, 614 (1992).
[6] P. Maris, Physical Review **D 54**, 4049 (1996).
[7] I. Aitchison and N.E. Mavromatos, Physical Review **B 53**, 9321 (1996).
[8] I. Aitchison, G. Amelino–Camelia, M. Klein–Kreisler, N.E. Mavromatos, and D. McNeill, Physical Review **B 56**, 2836 (1997).
[9] M. Koopmans and J. Steringa, Physics Letters **B 226**, 309 (1989).
[10] A. Campbell–Smith and N.E. Mavromatos, (1999), hep–th/9904173.

[11] O. Alvarez, Physical Review **D 17**, 1123 (1978).
[12] G. Diamandis, B. Georgalas, and N.E. Mavromatos, Modern Physics Letters **A 13**, 387 (1998).
[13] J. Papavassiliou and J. Cornwall, Physical Review **D 44**, 1285 (1991).
[14] J. Cornwall and D. Morris, Physical Review **D 52**, 6074 (1995).
[15] N.E. Mavromatos and J. Papavassiliou, (1999), hep–th/9904046.
[16] J. Cornwall, Physica **A 158**, 97 (1989).
[17] E. Kamke, *Differentialgleichungen, Lösungsmethoden und Lösungen* (Akedemische Verlagsgesellschaft, Geest & Portig K.–G., Leipzig, 1959).
[18] Emden, *Gaskugeln*; R.H. Fowler, Quarterly Journal **45**, 289 (1914); Quarterly Journal Oxford, **2** 259 (1931).
[19] T. Clark, O. Piguet, and K. Sibold, Annals of Physics **109**, 418 (1977).
[20] T. Clark and S. Love, Nuclear Physics **B 310**, 371 (1988).
[21] S. Gates, M. Grisaru, M. Roček, and W. Siegel, *Superspace: One Thousand And One Lessons In Supersymmetry*, Frontiers in Physics Vol. 58 (Benjamin/Cummings, Massachusetts, 1983).
[22] J. Cornwall, R. Jackiw, and E. Tomboulis, Physical Review **D 10**, 2428 (1974).