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RESONANCE FREE LATTICES FOR A.G. MACHINES

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Abstract

A part of an alternating gradient circular machine composed of a number of identical cells N_c will not contribute to the excitation of most of the non-linear resonances to first order in multipole strength, if the two phase advances per cell take the values $2\pi k_1/N_c$ and $2\pi k_2/N_c$. k_1 and k_2 are any integers and $k_1 \neq k_2$. This property is demonstrated here. Its application to synchrotron light sources and colliders is discussed.

1 INTRODUCTION

The problem of non-linear resonances has been considered from the beginning in the design of AG machines [1]. The first synchrotrons had a high periodicity to avoid systematic resonances and this principle is still used widely. For a collider this is not possible because of the small number of insertions. In this case, the choice of a proper phase advance per cell in the arcs makes it possible to avoid most of the systematic non-linear resonances, thanks to the cancellation of their driving term per super-period.

It is shown here how to design these parts of a machine which are constituted of identical cells, so that they do not contribute to most of the non-linear resonances. Firstly some results of the resonance theory are recalled. The resonance driving term is computed for an ensemble of identical cells and the condition for resonance cancellation is derived. Eventually some applications are given.

2 DRIVING TERM OF A NON-LINEAR RESONANCE

The theory of non-linear resonances is quite old [2]. Apart from the pioneer Moser [1] many authors contributed in the accelerator field. The formulae used here are extracted from [3]. More modern approaches based on normal form analysis lead to similar results (see for instance [4]).

We consider here only the response in amplitude of an harmonic oscillator driven by the non-linear field associated with the unperturbed linear oscillation, **to first order in multipole strength.** The change of frequency with amplitude is not taken into account.

On a resonance of order n defined by $n_x.Q_x + n_y.Q_y = integer$, with $|n_x| + |n_y| = n$, the driving term, which originates from the Fourier transform of the non-linear field created by the excursion of the linear motion in the multipole b_n, is proportional to the circumferential integral [3] :

$$\mathcal{I} = \left| \int_0^C b_n \beta_x^{\frac{n_x}{2}} \beta_y^{\frac{n_y}{2}} e^{i(n_x\mu_x + n_y\mu_y)} ds \right| \qquad (1)$$

The resonance order n is the same as the multipole index for the main resonance. It is equal to $n-2m$, where m is an integer, for the sub-resonances of order $n-2m$ associated with the same multipole. These sub-resonances have a similar driving term.

The integral $\mathcal{I}$ can be easily calculated for a sequence of identical cells. Indeed the optics functions β_x and β_y have identical values at homologous places inside each cell and the phase advances in the $(p+1)^{th}$ cell are given by : $\mu_{x,p+1}(s) = \mu_{x,0}(s) + p\mu_{x,c}$ and $\mu_{y,p+1}(s) = \mu_{y,0}(s) + p\mu_{y,c}$, if the longitudinal coordinate s has its origin at homologous places inside each cell. $\mu_{x,c}$ and $\mu_{y,c}$ are the phase advances per cell in both planes, i.e. constants. Thus, making a straightforward change of variable : $s \rightarrow s + p \times Lc$, where Lc is the cell length, the contribution of the $(p+1)^{th}$ cell to the integral $\mathcal{I}$ (1) defined by equation (1) can then be written as the product of an integral, which is the same for all cells, and a phase term depending only on the index p and the phase advances per cell :

$$e^{ip(n_x\mu_{x,c} + n_y\mu_{y,c})} \int_{cell} \beta_x^{\frac{n_x}{2}} \beta_y^{\frac{n_y}{2}} e^{i(n_x\mu_{x,0} + n_y\mu_{y,0})} ds. \qquad (2)$$

The integral $\mathcal{I}$ is obtained eventually from the modulus of the sum of all the terms given by (2) associated with the different cells. They all contain the same integral which can be factorised and does not need to be evaluated for our purpose. The modulus of the sum of the phase terms can be referred to as the amplification factor of the resonance $\{n_x, n_y\}$ since it tells us by how much the driving term associated with a single cell has to be multiplied to obtain the driving term associated with the whole structure.

3 RESONANCE CANCELLATION

The driving term on resonance associated with the ensemble of N_c cells vanishes if the resonance amplification factor defined in the preceding section is zero, i.e. :

$$\left| \sum_{p=0}^{N_c-1} e^{ip(n_x\mu_{x,c} + n_y\mu_{y,c})} \right| = \sqrt{\frac{1-\cos[N_c(n_x\mu_{x,c} + n_y\mu_{y,c})]}{1-\cos(n_x\mu_{x,c} + n_y\mu_{y,c})}} = 0 \qquad (3)$$

This is achieved if :

$$N_c(n_x\mu_{x,c} + n_y\mu_{y,c}) = 2k\pi \quad (4)$$

provided the denominator of equation (3) is non zero, i.e. :

$$n_x\mu_{x,c} + n_y\mu_{y,c} \neq 2k'\pi \quad (5)$$

k and k' are any integers. In what follows, the letter k with or without indices will mean "any integer". The resonance amplification factor has a value oscillating between zero and about one for almost all values of the variable $n_x\mu_{x,c}+ n_y\mu_{y,c}$ except those satisfying the equality in 5. In the latter case it is equal to N_c. Starting from this value, it decreases to zero when the variable is equal to $2\pi/N_c$ and takes the value $N_c/2$ for a value of $2\sqrt{3}/N_c$. This gives the range of the variable for which the resonance amplification factor takes large values.

Setting now the phase advances to the values :

$$\mu_{x,c}/2\pi = k_1/N_c, \quad \mu_{y,c}/2\pi = k_2/N_c, \quad (6)$$

we see that the condition for the cancellation of resonances (4) is satisfied. Indeed, introducing these values into equation (4), we obtain :

$$N_c(n_x 2\pi k_1/N_c + n_y 2\pi k_2/N_c) = 2k\pi$$

which can be simplified into :

$$n_x k_1 + n_y k_2 = k$$

For any integer values of n_x, k_1, n_y, k_2, the value of the expression $n_x k_1 + n_y k_2$ is always an integer. This means that this equation is satisfied for any value of k_1 and k_2. This results in the following important property :

" A part of a circular machine containing N_c identical cells will not contribute to the excitation of any non-linear resonance, except those defined by $n_x\mu_{x,c} + n_y\mu_{y,c} = 2k_3\pi$, if the phase advances per cell satisfy the two conditions :

$N_c\mu_{x,c} = 2k_1\pi$ **(cancellation of one-D horizontal non-linear resonances)**

$N_c\mu_{y,c} = 2k_2\pi$ **(cancellation of one-D vertical non-linear resonances)**

k_1 k_2 **and** k_3 **being any integers."**

The usefulness of this property lies in the fact that, for a given resonance order, there are much more resonances cancelled than excited. For certain orders all resonances are cancelled. Numerical examples are given below.

It is interesting to point out a useful by-product of this property. The linear coupling resonance defined by $n_x = 1, n_y = -1$ is cancelled provided $\mu_{x,c} - \mu_{y,c} \neq 2k'\pi$. Taking $k_2 = k_1 \pm 1$, it is sure that this condition is satisfied for FODO cells. Thus, for an ensemble of FODO cells, the linear coupling resonance is always cancelled when the above property holds. The demonstration of the cancellation of the linear coupling has been done in LEP [5]. The solution retained was $\mu_{y,c} = 60°$ and $\mu_{x,c} = 71.5°$. The value of $60°$ of the vertical phase was needed for the non-linear chromaticity correction, it does not fulfil the above condition. The number of cells per arc of 31 imposes a value of the horizontal phase advance of $60°$ + $k \times 11.613°$, the value retained for $\mu_{x,c}$ corresponds to $k = 1$.

4 NUMEROLOGY

We want to find good values of the three parameters : N_c, k_1, k_2, i.e. values which satisfy equation 5 for the largest number of resonances. In order to solve this equation, $\mu_{x,c}$ and $\mu_{y,c}$ are replaced by their values given by equation (6). This leads to the Diophantine equation :

$$n_x k_1 + n_y k_2 \neq k' N_c \quad (7)$$

For instance for machines with strong systematic multipoles of low-order, it is extremely interesting to seek the smallest number of cells and the phase advances for which there are no second nor third order resonances. Thus equation (7) has to be solved for n_x and $n_y \in [1, 2, 3]$. The easiest procedure is in fact to list the resonances satisfying the equality in this equation after having fixed N_c, k_1 and k_2 in order to decide whether these numbers are acceptable.

At first it is clear that N_c must not be equal to one, two or three. As the addition of a multiple of N_c on either side of equation (7) do dot modify it, it is sufficient to examine values smaller than N_c. As equation 7 is symmetrical in k_1 and k_2 it is sufficient to examine the cases $k_1 > k_2$. To suppress the linear coupling, it is necessary that $k_1 \neq k_2$. To suppress the third order resonances, k_1 must be different from $2k_2$.

n_x	0	1	2	3	4	5	6
n_y	5	1	2	-2	-1	0	2
res. order	5	2	4	5	5	5	8

Table 1: Resonances satisfying the equality in equation 7 associated with k_1=3, k_2=2 and N_c=5. The resonance order is equal to $|n_x| + |n_y|$, it has been limited to 10.

For $N_c = 4$ there are only two couples $\{k_1, k_2\}$ to consider : {3,1} and {3,2}. Both have systematic second order but no third orders. The fourth order are all systematic, which prevents the use of systematic octupole excitation to adjust the anharmonicity.

For N_c=5, there are five solutions with always at least one resonance of order smaller than 3 excited. There are two solutions without third order resonance {3,2} and {4,1}. The list of systematic resonances associated with {3,2} is given in table 1. This case is interesting for a machine with magnets containing no systematic do-decapole component both erect and skew.

The first value of N_c for which there is a solution without first second and third order is 8. The search was stopped here as the objective of this study was to find a solution for LHC with N_c=25. Before examining it, we consider the case of purely periodic machines for which it is worth clarifying the concept of systematic resonances.

5 PURE SUPER-PERIODIC MACHINES

This is the case of almost all accelerators or synchrotron light sources. For a number of super-periods equal to N_c, the resonance amplification factor can be written in terms of the tunes Q_x and Q_y :

$$\sqrt{\frac{1-\cos 2\pi(n_x Q_x + n_y Q_y)}{1-\cos[2\pi(n_x Q_x + n_y Q_y)/N_c]}} \quad (8)$$

It is clear that this expression is zero only on resonance. Away from the resonance, the factor becomes of the order of unity. Consequently a large value of N_c is interesting to gain more freedom to find tunes which maximise the denominator. Note that, if the anharmonicity can bring a tune on resonance which zeroes the expression (8) at a certain amplitude, the resonance will not appear in the phase-space plot. However the plots will be distorted because the resonance has a non zero effect except for a single value of the tune.

As an example the synchrotron light source ESRF was designed with 16 super-periods and the design tunes were Qx=32.2, Qy=11.2. For the values of Qx=32 and Qy=11, the amplification factor associated with the third order resonances is exactly zero. For the actual tune values, it is 1.54 for the resonance 3Qx and 1.09 for Qx+2Qy. This is why a sextupole arrangement with more than two families had to be found [6] to cancel these resonances. This situation will be encountered in any purely super-periodic machine.

6 MACHINES WITH INSERTIONS

This is **the** case to fully exploit the results of section 3. Insertions with an arbitrary phase advance make it possible to adjust the fractional part of the tunes independently of the phase advance of the arc cells.

A nice example is that of the SPEAR 3 upgrade project. It has a racetrack layout, i.e. two arcs jointed with straight sections, with 9 cells per arc and matching cells in the straight sections [7]. For N_c=9, two couples $\{k_1, k_2\}$ make the lattice free from all third order resonances : {5,4} and {7,2}. The second order sum coupling resonance is systematic for both. In a study of this lattice by tracking trajectories and systematic tune scan [7], two sets of phase advances per cell giving the largest dynamic aperture were found. The first set is $\mu_{x,c} = 0.79 * 2\pi$ and $\mu_{y,c} = 0.25 * 2\pi$, i.e. close to the values corresponding to the couple {7,2} : $\mu_{x,c} = \frac{7}{9}2\pi$ and $\mu_{y,c} = \frac{2}{9}2\pi$. The second set is $\mu_{x,c} = 0.78 * 2\pi$ and $\mu_{y,c} = 0.42 * 2\pi$ again close to the couple {7,4} which has only a skew third order systematic resonance.

For a superconducting storage ring, like LHC at CERN, there are systematic octupole components in the dipoles and fourth order resonances have to be avoided. LHC is composed of eight arcs with 23 FODO cells and one dispersion suppressor at each end. The latter break the periodicity of the cell's arrangement. Assimilating a dispersion suppressor with one cell, we have to deal with 25 cells per arc. Because of the design gradient of the arc quadrupoles and the restricted aperture, the possible couples $\{k_1, k_2\}$ are {7,6} and {6,5}. Trajectories have been tracked with systematic a4, b4 and b5 producing separately a relative field error of $0.5.10^{-4}$ at 17mm from the dipole axis, for the couple {7,6}. The dynamic aperture is increased by more than 50% for a model arc made from 25 cells and insertions, compared with the nominal lattice with a phase advance per cell close to 90°. For the actual LHC with the dispersion suppressors, there is less improvement but the lattice is much less sensitive to systematic expected octupole errors [8]. This opens the possibility of powering the octupole spool pieces correctors which have the same periodicity as the cells, without killing the dynamic aperture, to adjust the anharmonicity for Landau damping.

7 CONCLUSION

The analysis done in this paper shows that paying attention to the phase advance per cell is extremely beneficial to storage rings with strong focusing or large multipole errors and no strong non-linear chromaticity due to low-β insertions.

It is shown that a high superperiodicity is not the best ingredient to avoid non-linear resonances. It is better to design a machine with a low superperiodicity and arc cells with a proper phase advance so that the effect of the arc non-linearities is minimised. This is one of the strategies followed for LHC in order to minimise the effect of systematic multipole components which have different values in each of the eight arcs. It has been shown that such a lattice is rather insensitive to a substantial increase of the systematic per arc octupole component. It is also insensitive to a possible dangerous systematic per arc skew sextupole component.

8 REFERENCES

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