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On the twisted chiral potential in 2d and the analogue of rigid special geometry for 4-folds

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ABSTRACT: We discuss how to obtain an $N = (2, 2)$ supersymmetric SU(3) gauge theory in two dimensions via geometric engineering from a Calabi-Yau 4-fold and compute its non-perturbative twisted chiral potential $\widetilde{W}(\Sigma)$. The relevant compact part of the 4-fold geometry consists of two intersecting $\mathbb{P}^1$'s fibered over $\mathbb{P}^2$. The rigid limit of the local mirror of this geometry is a complex surface that generalizes the Seiberg-Witten curve and on which there exist two holomorphic 2-forms. These stem from the same meromorphic 2-form as derivatives w.r.t. the two moduli, respectively. The middle periods of this meromorphic form give directly the twisted chiral potential. The explicit computation of these and of the four-point Yukawa couplings allows for a non-trivial test of the analogue of rigid special geometry for a 4-fold with several moduli.

KEYWORDS: Field Theories in Lower Dimensions, Nonperturbative Effects, Supersymmetry and Duality, Supersymmetric Effective Theories.

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1. Introduction

The embedding of supersymmetric gauge theories into the framework of string theory and the subsequent application of symmetries and dualities of the latter has shown to be an extremely fruitful approach to the study of non-perturbative properties of gauge systems. One way to proceed is to exploit the fact that type IIA string theory gives rise to non-abelian gauge symmetries when compactified on certain singular Calabi-Yau manifolds. These are naturally $K3$-fibrations and a gauge theory of ADE-type arises when the $K3$ fiber develops a corresponding singularity. "Geometric engineering" [1] furthermore exploits the fact that in the field theory limit the relevant part of the compactification geometry is the local singularity structure. This allows us, for the purpose of extracting field theoretical properties, to just model manifolds that exhibit the correct configuration of exceptional divisors ($\mathbb{P}^1$'s that occur in resolving the singularities of the $K3$ fiber and whose intersection matrix equals the negative of the Cartan matrix of the respective gauge group) fibered over the appropriate base.

The literature on geometric engineering of $N = 2$ supersymmetric gauge theories in four dimensions is already vast. In this paper we follow [2] in applying this method to Calabi-Yau 4-folds, which leads to $N = (2, 2)$ supersymmetric gauge theories in two dimensions. The specific example we will investigate is an SU(3) gauge theory. This implies that the local compactification geometry on the type IIA side contains two intersecting $\mathbb{P}^1$'s fibered over a common compact complex two dimensional base, which we take to be a $\mathbb{P}^2$. The rigid limit of the local mirror to this geometry is

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a complex surface, which plays the rôle of the Seiberg-Witten curve. In analogy to the 3-fold case, this surface is no longer a Calabi-Yau manifold as it was in the SU(2) example of [2]. Instead, it has two holomorphic 2-forms that stem from the same meromorphic 2-form as the derivatives w.r.t. the two moduli, respectively. A major novelty of 4-folds, as compared to 3-folds, is that 4-forms are no longer dual to 2-forms but represent an independent part of the cohomology. In particular the primitive subspace of $H^{2,2}_{\bar{\partial}}(X)$, i.e. the one generated by forms in $H^{1,1}_{\bar{\partial}}(X)$, will play a predominant rôle. It is related to the occurrence of 4-fluxes ν_a, which as well as part of the intersection form $\eta^{(2)}$ on this subspace appear in the twisted chiral potential $\widetilde{W}$. It is given by

$$\widetilde{W} = \nu \cdot \eta^{(2)} \cdot \sigma_{D1} , \tag{1.1}$$

where σ_{D1} are the middle periods of the meromorphic 2-form on the rigid surface.

We will explicitly compute these periods and the four-point Yukawa couplings and then perform a non-trivial test of the generalization of rigid special geometry to 4-folds with several moduli. Recall that for 3-folds, X_3, special geometry implies that the prepotential F of the complex structure moduli space is given in terms of the periods Z^i, F_j of the holomorphic 3-form w.r.t. a symplectic basis of $H_3(X_3, \mathbb{Z})$ as $F = Z^i F_i(Z)/2$. The three-point Yukawa couplings are then given by $C_{(3)} = \partial^3 F$, where derivatives are w.r.t. the special projective coordinates Z^i. We will verify that for 4-folds the analogous structure in the rigid limit is given by $\widetilde{W}$ as in (1.1) and by

$$C_{(4)} \sim (\partial^2 \sigma_{D1}) \cdot \eta^{(2)} \cdot (\partial^2 \sigma_{D1}) .$$

We will make this more precise in the following.

The computation of the four-point Yukawa couplings is done in a global model of the compact Calabi-Yau 4-fold, a realization of which is given by the resolution of the Fermat hypersurface of degree 36 in the weighted projective space $\mathbb{P}^5(18, 12, 3, 1, 1, 1)$, which according to common convention we will call $X_{36}(18, 12, 3, 1, 1, 1)$. This manifold does not only represent a $K3$ fibration over $\mathbb{P}^2$, but the $K3$ is itself a fibration of an algebraic 2-torus over $\mathbb{P}^1$. Such elliptically fibered 4-folds are interesting in themselves as they give rise to phenomenologically more interesting $N = 1$ supersymmetric gauge theories in four dimensions when used as compactification manifolds for F-theory [3].

2. The holomorphic Fayet-Iliopoulos potentials

It is well known that for generic points in the moduli space spanned by the gauge multiplets the non-abelian gauge group of $N = 2$ supersymmetric gauge theories is broken down to the maximal abelian torus. Hence we will have to consider an effective gauge theory with $N = (2, 2)$ supersymmetry and abelian gauge group $\mathrm{U}(1)^k$ in two dimensions, where in our specific example $k = 2$. The superfields appearing in such theories comprise k real vector superfields V_a, $a = 1, \ldots, k$, with component expansion

$$V_a = -\sqrt{2}\Big(\theta^-\bar\theta^- v_{\bar z,a} + \theta^+\bar\theta^+ v_{z,a} - \theta^-\bar\theta^+\sigma_a - \theta^+\bar\theta^-\bar\sigma_a\Big) + \\ + i\Big(\theta^2\bar\theta^{\dot\alpha}\bar\lambda_{\dot\alpha,a} - \bar\theta^2\theta^\alpha\lambda_{\alpha,a}\Big) + \frac{1}{2}\theta^2\bar\theta^2 D_a\,,$$

r chiral superfields Φ_i, $i = 1, \ldots, r$, obeying $\bar D_+\Phi_i = \bar D_-\Phi_i = 0$, with component expansion

$$\Phi_i = \phi_i + \sqrt{2}\Big(\theta^+\psi_{+,i} + \theta^-\psi_{-,i}\Big) + \theta^2 F_i + \cdots\,,$$

where $\cdots$ are total derivative terms, as well as their complex conjugates $\bar\Phi_i$. As a novelty of two dimensions there are in addition twisted chiral superfields Σ_a, $a = 1, \ldots, k$, that satisfy $\bar D_+\Sigma_a = D_-\Sigma_a = 0$. Their component expansion reads

$$\begin{aligned}\Sigma_a &= \frac{1}{\sqrt{2}}\bar D_+ D_- V_a \\ &= \sigma_a - i\sqrt{2}\Big(\theta^+\bar\lambda_{+,a} + \bar\theta^-\lambda_{-,a}\Big) + \sqrt{2}\theta^+\bar\theta^-\Big(D_a - if_a\Big) + \cdots\end{aligned} \tag{2.1}$$

The most general Lagrangian involving these superfields consists of a generalized Kähler potential $K(\Sigma, \bar\Sigma, \Phi, \bar\Phi)$ as well as holomorphic chiral and twisted chiral potentials, $W(\Phi)$ and $\widetilde W(\Sigma)$.

For generic points in the moduli space the chiral matter fields will be massive, so that in the infrared, after having them integrated out, we are left with only the twisted chiral fields as the light degrees of freedom and an effective action involving $K(\Sigma, \bar\Sigma)$ and $\widetilde W(\Sigma)$. Taking the scaling dimension of Σ to equal 1, the Kähler potential has to be multiplied by the squared inverse of a dimensionful gauge coupling and therefore becomes irrelevant in the infrared. The twisted chiral potential on the other hand generalizes the Fayet-Iliopoulos term

$$\frac{i}{2\sqrt{2}}\int d\theta^+ d\bar\theta^-\ \widetilde W(\Sigma)\Big|_{\theta^- = \bar\theta^+ = 0} + \text{c.c.} = \sum_{a=1}^{k}\left(-\xi_a(\sigma)D_a + \frac{\theta_a(\sigma)}{2\pi}f_a\right) \tag{2.2}$$

and gives rise to effective, field dependent, complex FI couplings

$$\tau_a(\sigma) \equiv i\xi_a(\sigma) + \frac{\theta_a(\sigma)}{2\pi} = \frac{\partial\widetilde W(\Sigma)}{\partial\Sigma_a}\bigg|_{\theta=\bar\theta=0}\,, \qquad a = 1, \ldots, k\,. \tag{2.3}$$

These dimensionless couplings are known [4, 5] to receive logarithmic perturbative corrections to one-loop order

$$\tau_a(\sigma) = \tau_{a,0} - \frac{1}{2\pi i}\sum_{i=1}^{r} Q_i^a \log\left(\frac{\sqrt{2}}{\mu}\sum_{b=1}^{k} Q_i^b\sigma_b\right) + \cdots\,, \tag{2.4}$$

where μ is the RG scale and Q_i^a is the charge of the i^{th} massive chiral matter field under the a^{th} U(1) factor. In addition we expect non-perturbative corrections.

A further difference from four dimensional gauge theories is the appearance of a non-trivial scalar potential

$$V(\sigma) \sim \sum_{a=1}^{k} |\tau_a(\sigma)|^2 \,. \tag{2.5}$$

This potential makes the vacuum energy depend on the theta-angle [6, 4] and implies that supersymmetry is broken unless $\tau_a(\sigma) = 0$ for all $a = 1, \ldots, k$.

3. The twisted chiral potential via string theory

In compactifications of type IIA string theory on a Calabi-Yau 4-fold X, the twisted chiral superfields Σ_a of the resulting two dimensional gauge theory are in one-to-one correspondence with Kähler classes in $H^{1,1}_{\bar\partial}(X)$, while the chiral matter fields Φ_i correspond to the complex structure moduli belonging to $H^{3,1}_{\bar\partial}(X)$. The twisted chiral potential $\widetilde{W}(\Sigma)$ is holomorphic in the Σ's and is thus a holomorphic section of a line bundle over the moduli space of Kähler deformations of X. Since the type II dilaton resides in a different multiplet, $\widetilde{W}(\Sigma)$ does not receive any quantum space-time corrections and we can restrict ourselves to string tree level. Nevertheless $\widetilde{W}(\Sigma)$ suffers from non-perturbative corrections due to embeddings of the worldsheet ($\mathbb{P}^1$, since we are in string tree-level) into X, i.e. from worldsheet instantons. These lead to quantum corrections of the Kähler moduli space of X that show up in $\widetilde{W}(\Sigma)$. The way to compute these corrections is to use mirror symmetry, which allows us to consider instead type IIA string theory on the mirror manifold X^*, while the Kähler moduli space of X is mapped to the complex structure moduli space of X^* (isomorphic to $H^{3,1}_{\bar\partial}(X^*)$) and vice versa. Here the twisted chiral potential is a holomorphic section of a line bundle over the moduli space of complex structure deformations of X^*; as such, it receives no quantum corrections at all.

As was done in [2], we can identify the tree-level correlator we have to compute in order to obtain $\widetilde{W}(\Sigma)$ by considering the following tree-level Chern-Simons term in ten dimensional type IIA string theory

$$\mathcal{L}_{\text{CS}} = B \wedge F_4 \wedge F_4 \,,$$

where F_4 is the field strength of the RR 3-form field of type IIA theory. We expand the above forms in topological bases $\{\mathcal{O}^{(i)}_{a_i}\}$ of $H^{i,i}_{\bar\partial}(X)$ as

$$B = \sum_{a=1}^{h^{1,1}} \sigma_a \mathcal{O}^{(1)}_a \,, \qquad F_4 = \sum_{b=1}^{\tilde h^{2,2}} \nu_b \mathcal{O}^{(2)}_b + \sum_{a=1}^{h^{1,1}} F_a \wedge \mathcal{O}^{(1)}_a \,, \tag{3.1}$$

where F_a is the field strength of the twisted chiral superfield, related to the component f_a in the expansion (2.1) of Σ_a by $F_a = f_a d^2x$, with d^2x the volume form on the complement of X. Actually the expansion in the space $H^{2,2}_{\bar\partial}(X)$ is restricted to its primitive subspace, whose dimension we denote by $\tilde h^{2,2}$. Novel to 4-folds as compared to 3-folds is the rôle played by elements $\mathcal{O}_b^{(2)}$ and their corresponding coefficients ν_b. Denoting the dual cycles in $H_4(X,\mathbb{Z})$ by $\{\gamma_b\}$, we have

$$\int_{\gamma_b} F_4 = \nu_b\,, \tag{3.2}$$

and it is known [7] that at the quantum level these 4-fluxes have to be integers (or possibly half-integers if $p_1/4$ is not an integral class). Using the above expansion we obtain

$$\langle \mathcal{L}_{\rm CS}\rangle_{\rm 10d} \sim \sigma_a n_b \nu_c \langle \mathcal{O}_a^{(1)}\mathcal{O}_b^{(1)}\mathcal{O}_c^{(2)}\rangle_X\,, \tag{3.3}$$

where $n_b = 1/(2\pi)\int F_b$ is the first Chern class of the $b^{\rm th}$ U(1) bundle, i.e. the instanton number of the $b^{\rm th}$ gauge factor. Note that the correlator on the right hand side is a Yukawa coupling in a topological sigma model, known as the A model [8], which is obtained by twisting the superconformal sigma model on the worldsheet with target space X. The algebra of observables of this model is identified with the quantum deformation of the classical intersection algebra on $\mathcal{A} = \oplus_{p=0}^d H^p(X,\wedge^p T^*X)$, where d is the complex dimension of X. Another twist leads to the B model, whose algebra of observables is the algebra on $\mathcal{B} = \oplus_{p=0}^d H^p(X,\wedge^p TX) \cong \oplus_{p=0}^d H^p(X,\wedge^{d-p}T^*X)$. Mirror symmetry relates the A and B models to each other on a pair of mirror manifolds . It was shown in [9, 10, 11] how to compute the Frobenius subalgebra of $\mathcal{A}$ corresponding to the primitive part of the vertical cohomology of X around the large radius point, which is entirely determined in terms of the two- and three-point functions. For the case of several moduli, the authors of [10, 11] found in particular that the couplings $C_{a,b,c}^{(1,1,d-2)} : H_{\bar\partial}^{1,1}(X)\times H_{\bar\partial}^{1,1}(X)\times H_{\bar\partial}^{d-2,d-2}(X)\to\mathbb{C}$ are given by the period integrals of the holomorphic d-form on the mirror X^* as

$$C_{a,b,c}^{(1,1,d-2)} = \partial_{t_a}\partial_{t_b}\Pi_d^{(2)}\eta_{dc}^{(2)}\,.$$

Here t_a are the periods linear in logarithms[1] and $\Pi_d^{(2)}$ are the quadratic ones, both in a gauge such that the unique series solution for the periods around the large complex structure point is equal to 1. On the other hand, $\eta_{dc}^{(2)}$ are purely topological two-point functions. They define a metric on the primitive subspace of $H_{\bar\partial}^{2,2}(X)$ by the cup product pairing on a fixed topological basis $\{\mathcal{O}_a^{(2)}\}$ of this subspace as

$$\eta_{ab}^{(2)} = \langle \mathcal{O}_a^{(2)},\mathcal{O}_b^{(2)}\rangle = \int_X \mathcal{O}_a^{(2)}\wedge\mathcal{O}_b^{(2)}\,. \tag{3.4}$$

Via mirror symmetry it coincides with the analogous cup product pairing between B model observables on the mirror X^*, which are related to the above basis elements $\mathcal{O}_a^{(2)}$ by the mirror map. Hence $\eta^{(2)}$ is a symmetric, invertible $\tilde h^{2,2}\times\tilde h^{2,2}$ matrix with integer entries. We denote its inverse by $\eta_{(2)}$.

[1]These are the flat coordinates on the complex structure moduli space of X^*, i.e. the ones in which the Gauss-Manin connection on the bundle $\mathcal{B}\cong H^d(X^*,\mathbb{C})$ over this moduli space is flat. Under the mirror map they become the flat coordinates on the moduli space of the complexified Kähler structure of X.

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Our situation differs from that above in that we are interested in the point of the string moduli space where an enhanced gauge symmetry arises, which is not the large radius point. Furthermore we will take the field theory limit in which gravitational and stringy modes decouple. In this limit the mirror X^* turns into a complex surface $W_{\rm rig}$, which we will call the local rigid surface. The attribute local refers to the fact that it is determined by the local singularity structure of X, i.e. the fibration of exceptional divisors of the resolved $K3$ over the base $\mathbb{P}^2$. In particular the number of twisted chiral superfields that do not decouple in this limit is given by the number k of such exceptional divisors. Moreover, as was established in [12] for 3-folds, there exists a map $f : H_4(X^*, \mathbb{Z}) \to H_2(W_{\rm rig}, \mathbb{Z})$, such that the subset of periods of the holomorphic 4-form $\Omega_{(4,0)}$ on X^* that form a closed monodromy problem by themselves is given in the rigid limit by periods of a meromorphic 2-form λ on $W_{\rm rig}$ as $\int_\gamma \Omega_{(4,0)}\Big|_{\rm rigid} = \int_{f(\gamma)} \lambda$. These periods give the scalar components of the twisted chiral superfields not decoupling in the rigid limit and of their magnetic duals. In [2] it turned out that in the resulting $\mathrm{U}(1)^k$ gauge theory the rôle of the flat coordinates t_a is played by the k series solutions σ_a of the periods of λ and that the $\Pi_d^{(2)}$ are replaced by the k logarithmic solutions $\sigma_{D1,d}$. The latter arise as periods over cycles that are the image under the map f of 4-cycles in X^*, which are dual to observables of the B model on X^* that correspond, via the mirror map, to elements of the primitive subspace of $H^{2,2}_{\bar\partial}(X)$. Identifying the field theory limit of the exponential of (3.3) with $\sim \exp(2\pi i \tau_b(\sigma) n_b)$ we are hence led to

$$\partial_{\sigma_a}\partial_{\sigma_b}\widetilde{W}(\sigma) \sim \sum_c \nu_c \,\langle \mathcal{O}_a^{(1)}\mathcal{O}_b^{(1)}\mathcal{O}_c^{(2)}\rangle_X\Big|_{\rm rigid} \sim \sum_{c,d} \nu_c \partial_{\sigma_a}\partial_{\sigma_b}\sigma_{D1,d}(\sigma)\,\eta^{(2)}_{dc} \qquad (3.5)$$

or (modulo an additive constant and linear terms in σ)

$$\widetilde{W}(\sigma) \sim \sum_{c,d} \nu_c\,\eta^{(2)}_{cd}\,\sigma_{D1,d}(\sigma)\,. \qquad (3.6)$$

The indices $a, b, c, d = 1, \ldots, k$ correspond to the k indices in $\{1, \ldots, h^{1,1}(X)\}$ and $\{1, \ldots, \tilde h^{2,2}(X)\}$, respectively, which are related to periods surviving the field theory limit. Furthermore we kept the symbol $\eta^{(2)}$ for the resulting non-degenerate $k \times k$ submatrix of (3.4).

In view of (2.3) and (3.6) the Fayet-Iliopoulos couplings are given (modulo a possible additive constant) by

$$\tau_a(\sigma) \sim \sum_{c,d=1}^{k} \nu_c\,\eta^{(2)}_{cd}\,\frac{\partial \sigma_{D1,d}}{\partial \sigma_a} \equiv \sum_{c,d=1}^{k} \nu_c\,\eta^{(2)}_{cd}\,\hat\tau_{ad}(\sigma)\,, \qquad a = 1, \ldots, k\,. \qquad (3.7)$$

The objects $\hat\tau_{ad} = \partial_{\sigma_a}\sigma_{D1,d}$ are very reminiscent of the gauge couplings $\tau_{ij} = \partial_{a_i} a_{D,j}$ of the $N = 2$ supersymmetric $\mathrm{SU}(k+1)$ gauge theory in four dimensions. In that case the rigid limit of the mirror Calabi-Yau 3-fold is a genus k Riemann surface S, whose

k holomorphic 1-forms ω_i stem from a meromorphic 1-form $\lambda_{\rm SW}$ as derivatives w.r.t. the k moduli of the Coulomb branch. Its $2k$ independent 1-cycles can be chosen to form a symplectic basis $\{\alpha_j, \beta_j\}_{j=1}^k$ of $H_1(S,\mathbb{Z})$ such that the gauge couplings are given in terms of the $k \times k$ parts $A_{ij} = \int_{\alpha_j} \omega_i = \partial_{u_{i+1}} \int_{\alpha_j} \lambda_{\rm SW} = \partial_{u_{i+1}} a_j$ and $B_{ij} = \int_{\beta_j} \omega_i = \partial_{u_{i+1}} \int_{\beta_j} \lambda_{\rm SW} = \partial_{u_{i+1}} a_{D,j}$ of the period matrix as $\tau_{ij} = (A^{-1}B)_{ij}$.

In complete analogy, the local rigid surface $W_{\rm rig}$, which arises from the Calabi-Yau 4-fold we consider, is furnished with k holomorphic 2-forms ω_a stemming from a meromorphic 2-form λ as derivatives w.r.t. the k moduli. However, for even complex dimensional manifolds the vanishing of $\int \Omega \wedge \Omega$ implies quadratic algebraic dependences between the periods of the holomorphic form Ω. We therefore expect to find $3k$ independent 2-cycles $\{\alpha_j, \beta_j, \gamma_j\}_{j=1}^k$ in $H_2(W_{\rm rig}, \mathbb{Z})$, such that the αs intersect with γs and the βs only among themselves. Their $k \times 3k$ period matrix then comprises the $k \times k$ parts

$$\begin{aligned}
A_{ab} &= \int_{\alpha_b} \omega_a = \partial_{u_{a+1}} \int_{\alpha_b} \lambda = \partial_{u_{a+1}} \sigma_b \, , \\
B_{ab} &= \int_{\beta_b} \omega_a = \partial_{u_{a+1}} \int_{\beta_b} \lambda = \partial_{u_{a+1}} \sigma_{D1,b} \, , \\
C_{ab} &= \int_{\gamma_b} \omega_a = \partial_{u_{a+1}} \int_{\gamma_b} \lambda = \partial_{u_{a+1}} \sigma_{D2,b} \, ,
\end{aligned} \tag{3.8}$$

such that the couplings $\hat{\tau}_{ab}$, as defined in (3.7), are given by $\hat{\tau}_{ab} = (A^{-1}B)_{ab}$. Whereas the σ_b are series in the moduli, $\sigma_{D1,b}$ and $\sigma_{D2,b}$ are logarithmic and double logarithmic, respectively, and this is precisely what accounts for the logarithmic one-loop correction in (2.4).

In the following we will compute the periods σ_a, $\sigma_{D1,a}$ and $\sigma_{D2,a}$ using geometric engineering and in a subsequent subsection 3.2 the four-point Yukawa couplings. This will allow us to exhibit the analogue of rigid special geometry for 4-folds.

3.1 Geometric engineering

The relevant part of the 4-fold geometry is most efficiently described with the help of toric geometry. Doing so, the Mori vectors describing the two intersecting $\mathbb{P}^1$'s fibered over a base $\mathbb{P}^2$ (together with the canonical line bundle that will be irrelevant for the purpose of this section) are

$$\begin{aligned}
l^{(1)} &= (1, -2, 1, 0, 0, 0, 0) \, , \\
l^{(2)} &= (0, 1, -2, 1, 0, 0, 0) \, , \\
l^{(3)} &= (-3, 0, 0, 0, 1, 1, 1) \, .
\end{aligned} \tag{3.9}$$

The local mirror $X^*_{\rm loc}$ is then the complex surface given by

$$X^*_{\rm loc} = \left\{ 0 = P(y) = \sum_{i=1}^{7} a_i y_i \right\} \, , \tag{3.10}$$

where the variables $\{y_i\}$ are projective and subject to the constraints $1 = \prod_{i=1}^{7} y_i^{l_i^{(j)}}$, $j = 1, 2, 3$. A solution to these is given by

$$(y_1, \ldots, y_7) = \left(s^3, s^2 t, st^2, t^3, s^2 z, s^2 w, \frac{s^5}{zw}\right)$$

with $[s, t, z, w] \in \mathbb{P}^3$, such that P is a homogeneous Laurent polynomial of degree 3

$$P = a_1 s^3 + a_2 s^2 t + a_3 st^2 + a_4 t^3 + a_5 s^2 z + a_6 s^2 w + a_7 \frac{s^5}{zw} . \tag{3.11}$$

In terms of the algebraic coordinates a, b, c on the moduli space of complex structure deformations of X^*_{loc}, where

$$a = \frac{a_1 a_3}{a_2^2} , \qquad b = \frac{a_2 a_4}{a_3^2} , \qquad c = \frac{a_5 a_6 a_7}{a_1^3} , \tag{3.12}$$

the discriminant of X^*_{loc} reads

$$\tilde{\Delta} = abc\Delta = abc((\Delta_{\mathrm{cl}})^3 + 27a^3 c\, q(a, b, c)) , \tag{3.13}$$

with $q(a, b, c)$ a polynomial and

$$\Delta_{\mathrm{cl}} = -1 + 4a + 4b - 18ab + 27a^2 b^2 .$$

This discriminant is itself singular at the point $(a, b, c) = (1/3, 1/3, 0)$ and expanding around this singularity as

$$a = \frac{1}{3} - \left(\frac{1}{3}\right)^{2/3} \epsilon^2 u , \qquad b = \frac{1}{3} - \left(\frac{1}{3}\right)^{2/3} \epsilon^2 u + 3\epsilon^3 v , \qquad c = \epsilon^9 \Lambda^9 , \tag{3.14}$$

we find to lowest order in ϵ

$$\frac{1}{\epsilon^{18}} \Delta = -(4u^3 - 27v^2)^3 - 162\Lambda^9 v(4u^3 + 9v^2) + 27\Lambda^{18} + \mathcal{O}(\epsilon) . \tag{3.15}$$

Note that whereas for 3-folds the classical discriminant splits quadratically, for 4-folds we have a cubic splitting [2]. In order to find the correct variables on the field theory moduli space, we blow up the singular point $(a, b, c) = (1/3, 1/3, 0)$ until we get divisors with only normal crossings [13]. For one particular choice of coordinate patch this leads to the following variables

$$\begin{aligned} z_1 &= b - a = 3\epsilon^3 v , \\ z_2 &= \frac{(a - \frac{1}{3})^3}{(b - a)^2} = -\frac{1}{81} \frac{u^3}{v^2} , \\ z_3 &= \frac{c}{(b - a)^3} = \frac{1}{27} \frac{\Lambda^9}{v^3} . \end{aligned} \tag{3.16}$$

The sections $\sigma, \sigma_{D1}, \sigma_{D2}$ obey a Picard-Fuchs system of regular singular differential equations. In terms of the algebraic coordinates (a, b, c) this system takes the form

$$\begin{aligned} \mathcal{L}_1 &= (\theta_a - 3\theta_c)(\theta_a - 2\theta_b) - a(-2\theta_a + \theta_b)(-2\theta_a + \theta_b - 1)\,, \\ \mathcal{L}_2 &= \theta_b(-2\theta_a + \theta_b) - b(\theta_a - 2\theta_b)(\theta_a - 2\theta_b - 1)\,, \\ \mathcal{L}_3 &= \theta_c^3 - c(\theta_a - 3\theta_c)(\theta_a - 3\theta_c - 1)(\theta_a - 3\theta_c - 2)\,, \end{aligned} \tag{3.17}$$

where $\theta_a = a\partial_a$, etc. After transforming to the variables z_1, z_2, z_3, rescaling the periods $\pi_{\text{old}} = \epsilon\pi_{\text{new}} = \sqrt{\alpha'}\pi_{\text{new}}$, where $\epsilon \sim z_1^{1/3} z_3^{1/9}$, and taking the field theory limit $\epsilon \to 0$, we are left with two independent differential operators

$$\begin{aligned} \mathcal{D}_1 &= \frac{1}{27}\theta_2\left(\theta_2 - \frac{1}{3}\right) + z_2\left(\frac{2}{3}\theta_2 + \theta_3\right)\left(\frac{2}{3}\theta_2 + \theta_3 + \frac{1}{3}\right)\,, \\ \mathcal{D}_2 &= -\left(\theta_3 + \frac{1}{9}\right)^3 + z_3\left(\frac{2}{3}\theta_2 + \theta_3\right)\left(\frac{2}{3}\theta_2 + \theta_3 + \frac{1}{3}\right)\left(\frac{2}{3}\theta_2 + \theta_3 + \frac{2}{3}\right)\,, \end{aligned} \tag{3.18}$$

where $\theta_i = z_i\partial_{z_i}$. Solutions to these are easily found, using the Frobenius method, by making the ansatz

$$\tilde{\sigma}(s, t; z_2, z_3) = \sum_{n,m\geq 0} c(n, m; s, t) z_2^{n+s} z_3^{m+t}\,. \tag{3.19}$$

This determines the indices to be

$$(s, t) \in \left\{\left(0, -\frac{1}{9}\right), \left(\frac{1}{3}, -\frac{1}{9}\right)\right\} \tag{3.20}$$

and, remembering that we chose the twisted chiral fields to have scaling dimension 1, we fix the first coefficient to be $c(0, 0; s, t) = \Lambda$. Recursion relations then imply that the general coefficient is given by

$$\begin{aligned} c(n, m; s, t) = \Lambda &\left\{\prod_{i=1}^{n}\left[(-27)\left(\frac{2}{3}(i - 1 + s) + t\right)\left(\frac{2}{3}(i - 1 + s) + t + \frac{1}{3}\right)\right]\right\} \times \\ &\times \frac{\left(\frac{2}{3}(n + s) + t\right)_m \left(\frac{2}{3}(n + s) + t + \frac{1}{3}\right)_m \left(\frac{2}{3}(n + s) + t + \frac{2}{3}\right)_m}{\left(s + 1\right)_n \left(s + \frac{2}{3}\right)_n \left[\left(t + \frac{10}{9}\right)_m\right]^3}\,, \end{aligned} \tag{3.21}$$

where we have used the Pochhammer symbol $(a)_m = \Gamma(a + m)/\Gamma(a) = \prod_{i=0}^{m-1}(a + i)$. Since the derivatives w.r.t. s and t commute with $\mathcal{D}_1$ and $\mathcal{D}_2$ and since the first two derivatives w.r.t. t of the indicial equations vanish at our given pairs of indices, we find the following set of solutions in a neighbourhood of $(z_2, z_3) \sim (0, 0)$: two series solutions

$$\tilde{\sigma}_i(z_2, z_3) = \tilde{\sigma}(s, t; z_2, z_3)\big|_{(s,t)_i}\,, \tag{3.22}$$

two logarithmic solutions

$$\begin{aligned} \tilde{\sigma}_{D1,i}(z_2, z_3) &= \partial_t\, \tilde{\sigma}(s, t; z_2, z_3)\big|_{(s,t)_i} \\ &= \log(z_3)\tilde{\sigma}_i(z_2, z_3) + \sum_{n,m\geq 0}\Big(\partial_t c(n, m; s, t)\Big)\, z_2^{n+s} z_3^{m+t}\Big|_{(s,t)_i} \end{aligned} \tag{3.23}$$

and two double-logarithmic solutions

$$\tilde{\sigma}_{D2,i}(z_2,z_3) = \partial_t^2\tilde{\sigma}(s,t;z_2,z_3)\Big|_{(s,t)_i}\,. \tag{3.24}$$

The expansions for the series solutions read

$$\begin{aligned}
\tilde{\sigma}_1(z_2,z_3) &= \frac{\Lambda}{z_3^{1/9}}\Big(1+z_2-\frac{10}{729}z_3-4z_2^2+\frac{440}{729}z_2z_3-\frac{1540}{531441}z_3^2+\frac{77}{3}z_2^3- \\
&\qquad -\frac{10472}{729}z_2^2z_3+\frac{261800}{531441}z_2z_3^2-\frac{12042800}{10460353203}z_3^3+\cdots\Big) \\
\tilde{\sigma}_2(z_2,z_3) &= \frac{\Lambda z_2^{1/3}}{z_3^{1/9}}\Big(1-z_2+\frac{28}{729}z_3+5z_2^2-\frac{910}{729}z_2z_3+\frac{7280}{531441}z_3^2-\frac{104}{3}z_2^3+ \\
&\qquad +\frac{19760}{729}z_2^2z_3-\frac{760760}{531441}z_2z_3^2+\frac{76076000}{10460353203}z_3^3+\cdots\Big)
\end{aligned} \tag{3.25}$$

and for the logarithmic solutions as functions of the series solutions

$$\begin{aligned}
\tilde{\sigma}_{D1,1}(\tilde{\sigma}_1,\tilde{\sigma}_2) &= 9\,\tilde{\sigma}_1\log\left(\frac{\Lambda}{\tilde{\sigma}_1}\right)+\frac{27\tilde{\sigma}_2^9}{8\tilde{\sigma}_1^8}-\frac{27\tilde{\sigma}_2^6}{10\tilde{\sigma}_1^5}+\frac{9\tilde{\sigma}_2^3}{2\tilde{\sigma}_1^2}+\frac{55\Lambda^9\tilde{\sigma}_2^3}{27\tilde{\sigma}_1^{11}}- \\
&\qquad -\frac{11\Lambda^9}{243\tilde{\sigma}_1^8}-\frac{2443\Lambda^{18}}{354294\tilde{\sigma}_1^{17}}+\cdots \\
\tilde{\sigma}_{D1,2}(\tilde{\sigma}_1,\tilde{\sigma}_2) &= 9\,\tilde{\sigma}_2\log\left(\frac{\Lambda}{\tilde{\sigma}_1}\right)-\frac{27\tilde{\sigma}_2^{10}}{10\tilde{\sigma}_1^9}+\frac{27\tilde{\sigma}_2^7}{14\tilde{\sigma}_1^6}-\frac{9\tilde{\sigma}_2^4}{4\tilde{\sigma}_1^3}-\frac{5\Lambda^9\tilde{\sigma}_2^4}{\tilde{\sigma}_1^{12}}+ \\
&\qquad +\frac{59\Lambda^9\tilde{\sigma}_2}{243\tilde{\sigma}_1^9}+\frac{12157\Lambda^{18}\tilde{\sigma}_2}{177147\tilde{\sigma}_1^{18}}+\cdots\,.
\end{aligned} \tag{3.26}$$

These sections have a geometric interpretation as period integrals of a meromorphic 2-form λ over 2-cycles in the rigid surface, which arises from the local mirror in the field theory limit and which generalizes the Seiberg-Witten curve arising from 3-folds. The physical set of periods σ_b, $\sigma_{D1,b}$, $\sigma_{D2,b}$ is one that corresponds to an integral basis of 2-cycles with Weyl-invariant intersection form such that the series solutions satisfy the Casimir relations

$$\begin{aligned}
\sigma_1^2(u,v)+\sigma_2^2(u,v)-\sigma_1(u,v)\sigma_2(u,v) &= u+\cdots\,, \\
\sigma_1(u,v)\sigma_2(u,v)[\sigma_1(u,v)-\sigma_2(u,v)] &= v+\cdots\,,
\end{aligned} \tag{3.27}$$

where $\cdots$ indicate corrections that vanish in the classical limit (which in our coordinate patch (3.16) is at $v\to\infty$). Using the notation $\sigma_{(0,b)}\equiv\sigma_b$, $\sigma_{(1,b)}\equiv\sigma_{D1,b}$ and $\sigma_{(2,b)}\equiv\sigma_{D2,b}$, such a set of periods is obtained from the above solutions of the Picard-Fuchs equations by the linear transformation

$$\sigma_{(a,b)} = c_a\sum_{c=1}^{2}M_{bc}\tilde{\sigma}_{(a,c)}\,,\qquad a=0,1,2 \tag{3.28}$$

with

$$M = \begin{pmatrix} -(1-i\sqrt{3})/(2\,3^{1/3}) & -1 \\ -1/(3^{1/3}) & (-1)^{2/3} \end{pmatrix} \tag{3.29}$$

and $c_0 = 1$, $c_1 = i/6\pi$ and $c_2 = -1/36\pi^2$. The monodromy around $v \sim \infty$ for constant u then acts on the period vector $(\sigma_1, \sigma_2, \sigma_{D1,1}, \sigma_{D1,2}, \sigma_{D2,1}, \sigma_{D2,2})^t$ as matrix multiplication by[2]

$$M_v^{(\infty)} = \begin{pmatrix} N & 0 & 0 \\ N & N & 0 \\ N & 2N & N \end{pmatrix} \qquad \text{where} \qquad N = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.30}$$

Note that N^3 is the identity matrix.

For our specific example we can easily write down the equation for the local rigid surface and the meromorphic 2-form λ. After shifting the t-variable such that the quadratic term in t disappears from (3.11), and using (3.12) and (3.14), the local rigid surface takes the form

$$W_{\text{rig}} = \left\{ z + w + \frac{\Lambda^9}{zw} + P_{A_2}(x; u, v) = 0 \right\},$$

with $P_{A_2}(x; u, v) = x^3 - ux - v$ the simple singularity of type A_2. Equivalently we can take the polynomial form

$$f_{\text{rig}} = z^2 w + zw^2 + \Lambda^9 + zw P_{A_2}(x; u, v)\,. \tag{3.31}$$

As for the Seiberg-Witten curve, this rigid surface is no longer a Calabi-Yau space. Rather, we expect to find two holomorphic 2-forms that are derivatives of λ w.r.t. u and v, respectively. There are curves in the field theory moduli space (spanned by u and v) above which $\{f_{\text{rig}}(w, x, z) = 0\}$ is a singular space. Away from such subvarieties, i.e. where f_{rig} defines a smooth surface, we can use the Poincaré residue map to construct a holomorphic 2-form. Indeed, whenever we have $\partial_w f_{\text{rig}}|_{\{f_{\text{rig}}=0\}} \equiv [-4\Lambda^9 z + z^2(z + P_{A_2})^2]^{1/2} \neq 0$ a suitable meromorphic 2-form is given by

$$\lambda = -\frac{1}{z} \log\left[2z(z + P_{A_2}) + 2\sqrt{-4\Lambda^9 z + z^2(z + P_{A_2})^2} \right] dz \wedge dx\,, \tag{3.32}$$

such that

$$\omega_1 = \partial_v \lambda = \frac{dz \wedge dx}{\sqrt{-4\Lambda^9 z + z^2(z + P_{A_2})^2}} = \frac{dz \wedge dx}{\partial_w f_{\text{rig}}}$$

[2]Actually the blocks below the diagonal could be altered by a change of homology basis, which adds series solutions to logarithmic ones and series and logarithmic solutions to double logarithmic ones and would still meet our requirements. It is however important to notice that such a change of basis leaves the relations (3.5) and (3.38) invariant and reflects the indeterminacy mentioned above (3.6).

and

$$\omega_2 = \partial_u \lambda = \frac{x\, dz \wedge dx}{\sqrt{-4\Lambda^9 z + z^2(z + P_{A_2})^2}} = \frac{x\, dz \wedge dx}{\partial_w f_{\mathrm{rig}}}$$

are two holomorphic 2-forms, as we had expected. In the case $\partial_w f_{\mathrm{rig}}|_{\{f_{\mathrm{rig}}=0\}} = 0$, but $\partial_z f_{\mathrm{rig}}|_{\{f_{\mathrm{rig}}=0\}} \equiv [-4\Lambda^9 w + w^2(w + P_{A_2})^2]^{1/2} \neq 0$, we just have to trade z for w in the above forms.

3.2 The Yukawa couplings and rigid special geometry

In (3.5) we had found that the second derivatives of the twisted chiral potential are proportional to linear combinations of three-point Yukawa couplings of the A model with target space the Calabi-Yau 4-fold X. These couplings in turn were identified with second derivatives of the logarithmic solutions to the period integrals of a meromorphic 2-form over 2-cycles in the rigid surface. In the previous subsection we used geometric engineering and local mirror symmetry to compute these periods. The purpose of this subsection is to compute the four-point Yukawa couplings directly from the Picard-Fuchs system. This allows us to exhibit the structure that generalizes rigid special geometry to 4-folds in an example with two moduli.

The 4-fold we used above is a non-compact toric Calabi-Yau manifold, more precisely the total space of a canonical line bundle. This non-compactness did not bother us as long as we were only interested in properties of the rigid local mirror since, for these, only the compact base space of the bundle was relevant. On the other hand, since the ring structure on the cohomology of a non-compact manifold is not well defined, we need to use a global model of the compact Calabi-Yau 4-fold X in order to compute the Yukawa couplings (taking the field theory limit not until the end). As already alluded to in the introduction, such a model is furnished, for example, by the Fermat hypersurface of degree 36 in the resolution of the weighted projective space $\mathbb{P}^5(18, 12, 3, 1, 1, 1)$, which we called $X_{36}(18, 12, 3, 1, 1, 1)$. This space is a fibration over $\mathbb{P}^2$ with fiber a $K3$ surface given as $X_{12}(6, 4, 1, 1)$ that is itself a fibration over $\mathbb{P}^1$ with fiber this time a 2-torus $X_6(3, 2, 1)$. Note that this is the same geometry as that of $X_{24}(12, 8, 2, 1, 1)$ [14, 15], except that the base $\mathbb{P}^1$ of that $K3$-fibration has been traded for a $\mathbb{P}^2$ base. Since it is the family of $K3$-fibers that determines the gauge symmetries, most of the analysis is similar to the 3-fold case [16].

The weighted projective space $\mathbb{P}^5(18, 12, 3, 1, 1, 1)$ is a toric variety whose fan comprises the one-dimensional cones

$$\begin{aligned}
\nu_1 &= (1, 0, 0, 0, 0)\,, & \nu_2 &= (0, 1, 0, 0, 0)\,, \\
\nu_3 &= (0, 0, 1, 0, 0)\,, & \nu_4 &= (0, 0, 0, 1, 0)\,, \\
\nu_5 &= (0, 0, 0, 0, 1)\,, & \nu_6 &= (-18, -12, -3, -1, -1)\,.
\end{aligned}$$

They represent vertices of a reflexive polyhedron Δ, which contains in addition the origin ν_0 as the only interior point, as well as the two integral vertices $\nu_7 = (\nu_4+\nu_5+\nu_6)/3$ and $\nu_8 = (3\nu_3+\nu_4+\nu_5+\nu_6)/6$. The latter represent exceptional divisors that are introduced in the process of resolving the quotient singularities of the weighted projective space. The enlarged vertices, $\bar{\nu}_i = (1,\nu_i) \in \mathbb{Z}^6$, satisfy three independent relations $0 = \sum_{i=0}^{8} l_i^{(k)} \bar{\nu}_i$, $k=1,2,3$, which define the Mori vectors

$$\begin{aligned} l^{(1)} &= (-6,3,2,0,0,0,0,0,1)\,, \\ l^{(2)} &= (0,0,0,0,1,1,1,-3,0)\,, \\ l^{(3)} &= (0,0,0,1,0,0,0,1,-2)\,. \end{aligned} \tag{3.33}$$

Applying standard techniques [16, 17, 18, 19] it follows that the mirror X^* is again a hypersurface in the same weighted projective space $\mathbb{P}^5(18,12,3,1,1,1)$, given by

$$\begin{aligned} p = {} & a_0 z_1 z_2 z_3 z_4 z_5 z_6 + a_1 z_1^2 + a_2 z_2^3 + a_3 z_3^{12} + a_4 z_4^{36} + a_5 z_5^{36} + a_6 z_6^{36} \\ & + a_7 (z_4 z_5 z_6)^{12} + a_8 (z_3 z_4 z_5 z_6)^6\,. \end{aligned} \tag{3.34}$$

In terms of the algebraic coordinates $a = a_1^3 a_2^2 a_8 / a_0^6$, $b = a_4 a_5 a_6 / a_7^3$, $c = a_3 a_7 / a_8^2$ on the moduli space of complex structure deformations of X^* the Picard-Fuchs system for the periods $\pi_i(a,b,c)$ of the holomorphic 4-form on X^* reads

$$\begin{aligned} \mathcal{D}_1 &= \theta_a(\theta_a - 2\theta_c) - 12a(6\theta_a+1)(6\theta_a+5)\,, \\ \mathcal{D}_2 &= \theta_b^3 - b(-3\theta_b+\theta_c)(-3\theta_b+\theta_c-1)(-3\theta_b+\theta_c-2)\,, \\ \mathcal{D}_3 &= \theta_c(-3\theta_b+\theta_c) - c(\theta_a - 2\theta_c)(\theta_a - 2\theta_c - 1)\,. \end{aligned} \tag{3.35}$$

From the analysis of the $X_{24}(12,8,2,1,1)$ model in [14] we can furthermore infer that the point of SU(3) gauge symmetry enhancement is located in the string moduli space in rescaled variables $(x,y,z) = (432a, 27b, 4c)$ at $(x,y,z) = (\infty, 0, 1)$. Expanding around this point as

$$x = \frac{1}{2\epsilon^3 u^{3/2}}\,, \qquad y = 3\sqrt{3}\,\epsilon^9 \Lambda^9\,, \qquad z = 1 - 2\epsilon^3 u^{3/2} - 3\sqrt{3}\,\epsilon^3 v\,, \tag{3.36}$$

we indeed find the discriminant of (3.35) to coincide with (3.15) at lowest order and up to an irrelevant factor.

Next, we compute the four-point Yukawa couplings following [19]. This means that we first compute the four-point functions of the B model on X^* directly from the Picard-Fuchs system (3.35), transform them to the field theory variables and to the correct gauge and take the rigid limit. Whereas in [19] this gauge was chosen such that the unique fundamental period around the large complex structure point was scaled to 1, in our situation it is fixed by the requirement that the Picard-Fuchs system (3.35), when transformed to this gauge, reduces in the field theory limit to the rigid system (3.18). The rigid limit of the A model four-point Yukawa couplings

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is then obtained[3] by going to the flat coordinates σ_i. As has already been mentioned, the four-point couplings are entirely determined by the two- and three-point functions as

$$C_{\sigma_i\sigma_j\sigma_k\sigma_l} = \frac{1}{4!} \sum_{\pi\in S_4} \sum_{e,f=1}^{\tilde{h}^{2,2}} C^{(1,1,2)}_{\sigma_{\pi(i)},\sigma_{\pi(j)},e}\ \eta^{ef}_{(2)}\ C^{(1,1,2)}_{\sigma_{\pi(k)},\sigma_{\pi(l)},f}\,. \tag{3.37}$$

Our identification of the A model couplings $C^{(1,1,2)}_{i,j,k}\Big|_{\mathrm{rigid}} \sim \partial_{\sigma_i}\partial_{\sigma_j}\sigma_{D1,h}\eta^{(2)}_{hk}$ hence leads to the following relations

$$\begin{aligned} C_{\sigma_i\sigma_j\sigma_k\sigma_l}\Big|_{\mathrm{rigid}} &= \frac{\mathrm{const}}{4!} \sum_{\pi\in S_4} \sum_{e,f,g,h} \left(\partial_{\sigma_{\pi(i)}}\partial_{\sigma_{\pi(j)}}\sigma_{D1,g}\ \eta^{(2)}_{ge}\right) \eta^{ef}_{(2)} \left(\partial_{\sigma_{\pi(k)}}\partial_{\sigma_{\pi(l)}}\sigma_{D1,h}\ \eta^{(2)}_{hf}\right) \\ &= \frac{\mathrm{const}}{4!} \sum_{\pi\in S_4} \sum_{e,f} \left(\partial_{\sigma_{\pi(i)}}\partial_{\sigma_{\pi(j)}}\sigma_{D1,e}\right) \eta^{(2)}_{ef} \left(\partial_{\sigma_{\pi(k)}}\partial_{\sigma_{\pi(l)}}\sigma_{D1,f}\right)\,. \end{aligned} \tag{3.38}$$

Here $\eta^{(2)}$ is the non-degenerate submatrix of the intersection form (3.4) on the primitive subspace of $H^{2,2}_{\bar\partial}(X)$ corresponding to periods that survive the rigid limit. It is a symmetric, invertible 2×2 matrix with integer coefficients, whose inverse we denote by $\eta_{(2)}$. The constant of proportionality appearing in (3.38) is of course the same for all couplings. Having calculated the four-point functions and the periods, we can indeed match the left and right sides of (3.38), which fixes the constant of proportionality and moreover determines the intersection form $\eta^{(2)}$ to be the Cartan matrix of SU(3)

$$\eta^{(2)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\,. \tag{3.39}$$

Relations (3.6) and (3.38) represent the analogue of rigid special geometry for 4-folds with several moduli, in the same sense that special geometry for 3-folds manifests itself in the relations $F = Z^i F_i(Z)/2$ for the prepotential F, where Z^i and F_i are periods of the holomorphic 3-form w.r.t. a symplectic basis of $H_3(X_3,\mathbb{Z})$ and $C_{ijk} = \partial_i\partial_j\partial_k F$ for the three-point Yukawa couplings, where the derivatives are w.r.t. the special projective coordinates Z^i. They are the rigid limit of the structure found in [11] for the non-rigid case.

4. Conclusion

We have investigated the field theory limit of type IIA string compactification on a Calabi-Yau 4-fold whose relevant part for the purpose of extracting field theoretic properties consists of two intersecting $\mathbb{P}^1$'s fibered over a common base $\mathbb{P}^2$. The rigid limit of the local mirror is a complex surface that generalizes the Seiberg-Witten curve and on which there exist two holomorphic 2-forms that stem from the same meromorphic 2-form as derivatives w.r.t. the two moduli.

[3]These computations are straightforward but somehow unhandy. Therefore we refer to the appendix for details and present the verification of (3.38) only for one particular four-point coupling.

The effective field theory that is the appropriate description in the infrared is an $N = (2,2)$ supersymmetric gauge theory in two dimensions with abelian gauge group $\mathrm{U}(1)^2$. Its twisted chiral potential is of the form $\widetilde{W} = \nu \cdot \eta^{(2)} \cdot \sigma_{D1}$, where ν is a vector of 4-fluxes, $\eta^{(2)}$ an intersection form and σ_{D1} a period vector. By explicit computation of the period integrals as solutions of the Picard-Fuchs equations and of the four-point Yukawa couplings we were able to exhibit the generalization of rigid special geometry to 4-folds in a non-trivial example with two moduli. This structure manifests itself in the relation $C_{(4)} = (\partial^2 \sigma_{D1}) \cdot \eta^{(2)} \cdot (\partial^2 \sigma_{D1})$ between these four-point functions and derivatives of the middle periods w.r.t. flat coordinates.

We briefly mention a number of conclusions about two-dimensional gauge theories as derived from type IIA compactifications on 4-folds, which have already been discussed in [2] but apply to our example as well. The major novelty of 4-folds is the rôle played by the primitive subspace of $H^{2,2}_{\bar\partial}(X)$ and its dual 4-cycles, respectively. They lead to new discrete moduli of the gauge theories in two dimensions, the 4-fluxes. If they all vanish, the theory just exhibits a non-trivial Kähler potential. But once the 4-fluxes are switched on, the structure of the theory becomes richer, as a twisted chiral potential, FI couplings and a scalar potential are generated. Generically the last seems to break supersymmetry, as was the case in the one-modulus example of [2]. The coefficient of the logarithmic term in the FI couplings can furthermore be interpreted as indicating the presence of massive chiral matter we had not accounted for in the geometrical set-up. Finally, the choice of base is not unambiguous, in contrast to the case of 3-folds, but the instanton series depends on this choice. We do not know how to interpret or resolve this ambiguity.

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A. The four-point Yukawa couplings

In this appendix we discuss some details of the steps occuring in our computation of the rigid limit of the four-point Yukawa couplings, the general idea of which has already been explained above (3.37). Subsequently we present the verification of (3.38) for one particular coupling.

First we compute[4] the four-point Yukawa couplings of the B model on X^*, which is the mirror to $X_{36}(18,12,3,1,1,1)$. We write them in the rescaled algebraic variables (x, y, z) and in order to shorten the exposition we introduce the following

[4]These computations were performed with the help of the program Lop4f.m, written by Albrecht Klemm.

abbreviations for the components of the discriminant of (3.35)

$$\Delta = (1+y)((-1+z)^3 + yz^3)((-(-1+x)^2 + x^2 z)^3 + x^6 y z^3) \equiv \Delta_1 \Delta_2 \Delta_3 \,.$$

Furthermore $C^{(4,0,0)}$ stands for C_{xxxx}, $C^{(2,1,1)}$ for C_{xxyz} and so forth.

$$\begin{aligned}
C^{(4,0,0)} &= \frac{(-1+x)^2}{x^4\Delta_3}\,, \qquad C^{(3,0,1)} = -\frac{(-1+x)^3}{2x^3 z\Delta_3}\,,\\
C^{(2,0,2)} &= \frac{(-1+x)^4}{4x^2 z^2\Delta_3}\,, \qquad C^{(1,0,3)} = -\frac{(-1+x)^5}{8xz^3\Delta_3}\,,\\
C^{(0,0,4)} &= -\frac{(-1+2x)}{16z^3\Delta_2\Delta_3}\Big[3-3z+(1+y)z^2-6x^3\Big(2-3z+(1+y)z^2\Big)-\\
&\qquad -4x\Big(3-3z+(1+y)z^2\Big)+3x^4\Big(1-2z+(1+y)z^2\Big)+\\
&\qquad +x^2\Big(18-21z+7(1+y)z^2\Big)\Big]\,,\\
C^{(3,1,0)} &= \frac{(-1+x)\Big(-1+2x+x^2(-1+z)\Big)}{6x^3y\Delta_3}\,,\\
C^{(2,1,1)} &= -\frac{(-1+x)^2\Big(-1+2x+x^2(-1+z)\Big)}{12x^2yz\Delta_3}\,,\\
C^{(1,1,2)} &= \frac{(-1+x)^3\Big(-1+2x+x^2(-1+z)\Big)}{24xyz^2\Delta_3}\,,\\
C^{(0,1,3)} &= \frac{(-1+2x)}{48yz^2\Delta_2\Delta_3}\Big[-2+3z-(1+y)z^2+4x\Big(2-3z+(1+y)z^2\Big)-\\
&\qquad -2x^3\Big(-4+9z-3(2+y)z^2+(1+y)z^3\Big)+\\
&\qquad +x^2\Big(-12+21z-(10+7y)z^2+(1+y)z^3\Big)+\\
&\qquad +x^4\Big(-2+6z-3(2+y)z^2+2(1+y)z^3\Big)\Big]\,,\\
C^{(2,2,0)} &= \frac{\Big(-1+2x+x^2(-1+z)\Big)^2}{36x^2y^2\Delta_3}\,,\\
C^{(1,2,1)} &= -\frac{(-1+x)\Big(-1+2x+x^2(-1+z)\Big)^2}{72xy^2z\Delta_3}\,,\\
C^{(0,2,2)} &= -\frac{(-1+2x)}{144y^2z\Delta_2\Delta_3}\Big[1-2z+(1+y)z^2-4x\Big(1-2z+(1+y)z^2\Big)+\\
&\qquad +x^2\Big(6-14z+(10+7y)z^2-2(1+y)z^3\Big)+\\
&\qquad +x^4(-1+z)\Big(-1+3z-3(1+y)z^2+(1+y)z^3\Big)+\\
&\qquad +2x^3\Big(-2+6z-3(2+y)z^2+2(1+y)z^3\Big)\Big]\,,\\
C^{(1,3,0)} &= -\frac{(-1+x)\Big(1-4x-3x^2(-2+z)+x^3(-4+6z)+x^4(1-3z+3z^2)\Big)}{216xy^2\Delta_1\Delta_3}\,,
\end{aligned}$$

$$
\begin{aligned}
C^{(0,3,1)} &= -\frac{(-1+2x)\Big(1-4x+x^2(7-3z)+6x^3(-1+z)+3x^4(-1+z)^2\Big)z^2}{432y^2\Delta_2\Delta_3}\,, \\
C^{(0,4,0)} &= -\frac{(-1+2x)z}{1296y^3\Delta_1\Delta_2\Delta_3}\Big[-1+3z+(-2+y)z^2- \\
&\qquad -4x\Big(-1+3z+(-2+y)z^2\Big)+ \\
&\qquad +x^2\Big(-6+21z+(-20+7y)z^2+(5-4y)z^3\Big)+ \\
&\qquad +2x^3\Big(2-9z-3(-4+y)z^2+(-5+4y)z^3\Big)+ \\
&\qquad +x^4\Big(-1+6z+3(-4+y)z^2-2(-5+4y)z^3+ \\
&\qquad\qquad +(-3+6y)z^4\Big)\Big]\,.
\end{aligned}
$$

Using the transformation property

$$
C_{w_iw_jw_kw_l} = \sum_{m,n,p,q}\frac{\partial z_m}{\partial w_i}\frac{\partial z_n}{\partial w_j}\frac{\partial z_p}{\partial w_k}\frac{\partial z_q}{\partial w_l}C_{z_mz_nz_pz_q}
$$

of the four-point Yukawa couplings and (3.36) we transform the above couplings to the field theory variables (u,v). We make sure that we use corresponding gauges for the holomorphic 4-forms in the local and global construction by rescaling the global periods by a holomorphic function $1/f(u,v)$, such that the Picard-Fuchs system (3.35), when transformed to this gauge, reduces in the field theory limit to the rigid system (3.18). This implies that we have to rescale the Yukawa couplings by $1/f^2(u,v)$ after which we can take the rigid limit.

Next we have to change from variables (u,v) to (σ_1,σ_2). Since we are going to compare leading terms in series expansions, it is easiest to work with $\tilde\sigma$ and $\tilde\sigma_{D1}$, which are given in (3.25) and (3.26) and are related to the true σ and σ_{D1} by the transformations (3.28), and to express everything in the variables (z_2,z_3) as given in (3.16). For (3.38) this implies that we replace (σ,σ_{D1}) by $(\tilde\sigma,\tilde\sigma_{D1})$ whereby $\eta^{(2)}$ gets transformed to $\tilde\eta^{(2)}=c_1^2M^t\eta^{(2)}M$ with c_1 and M as in (3.29).

Let us verify (3.38) for $C_{\tilde\sigma_1\tilde\sigma_1\tilde\sigma_1\tilde\sigma_1}$. Performing the above mentioned steps we find

$$
\begin{aligned}
(-1)^{7/6}\frac{8}{3}C_{\tilde\sigma_1\tilde\sigma_1\tilde\sigma_1\tilde\sigma_1} &= z_2^{1/3}z_3^{2/9}\Bigg(-162+1620z_2-\frac{4184}{9}z_3-18306z_2^2+\frac{242000}{9}z_2z_3- \\
&\qquad -\frac{4584286}{6561}z_3^2+212652z_2^3-\frac{7880872}{9}z_2^2z_3+ \\
&\qquad +\frac{670041316}{6561}z_2z_3^2-\frac{117589832812}{129140163}z_3^3+\cdots\Bigg)\,.
\end{aligned}
$$

From (3.25) and (3.26) we have

$$
\begin{aligned}
\partial^2_{\tilde\sigma_1}\tilde\sigma_{D1,1} &= z_3^{1/9}\Bigg(-9+36z_2-\frac{274}{81}z_3-315z_2^2+\frac{25252}{81}z_2z_3-\frac{152633}{59049}z_3^2+ \\
&\qquad +3120z_2^3-\frac{890726}{81}z_2^2z_3+\frac{42870113}{59049}z_2z_3^2-\frac{2555818340}{1162261467}z_3^3+\cdots\Bigg)
\end{aligned}
$$

and

$$\partial^2_{\tilde\sigma_1}\tilde\sigma_{D1,2} = z_2^{1/3} z_3^{1/9}\biggl(9 - 54z_2 - \frac{202}{9}z_3 + 486z_2^2 - 1072z_2z_3 + \frac{547126}{19683}z_3^2 - 4860z_2^3 + \\ + 30524z_2^2z_3 - \frac{23885459}{6561}z_2z_3^2 + \frac{1355705192}{43046721}z_3^3 + \cdots\biggr).$$

Thus (3.38) holds for $C_{\tilde\sigma_1\tilde\sigma_1\tilde\sigma_1\tilde\sigma_1}$ if

$$\tilde\eta^{(2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This implies the validity of (3.38) for $C_{\sigma_1\sigma_1\sigma_1\sigma_1}$ with $\eta^{(2)}$ the Cartan matrix of SU(3) as stated in (3.39). Using the same method one easily checks that the same constant of proportionality and the same intersection form $\eta^{(2)}$ work for all four-point Yukawa couplings, thus verifying (3.38).

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