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NON-PERTURBATIVE ASPECTS OF HOT QCD

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Abstract

I discuss some non-perturbative aspects of hot gauge theories as related to the unscreened static magnetic interactions. I first review some of the infrared divergences which cause the breakdown of the perturbation theory. Then I show that kinetic theory, as derived from quantum field theory, is a powerful tool to construct effective theories for the soft modes, which then can be treated non-perturbatively. The effective theory at the scale gT follows from a collisionless kinetic equation, of the Vlasov type. The effective theory at the scale g^2T is generated by a Boltzmann equation which includes the collision term for colour relaxation.

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I discuss some non-perturbative aspects of hot gauge theories as related to the unscreened static magnetic interactions. I first review some of the infrared divergences which cause the breakdown of the perturbation theory. Then I show that kinetic theory, as derived from quantum field theory, is a powerful tool to construct effective theories for the soft modes, which then can be treated non-perturbatively. The effective theory at the scale gT follows from a collisionless kinetic equation, of the Vlasov type. The effective theory at the scale g^2T is generated by a Boltzmann equation which includes the collision term for colour relaxation.

1 Introduction

At high temperature, the non-Abelian gauge theories describe weakly coupled plasmas, which, in a first approximation, are very much alike the ordinary, electromagnetic, plasmas^{1,2}. The plasma constituents, e.g., quarks and gluons for hot QCD, have typical momenta $k \sim T$, and take part in collective excitations which typically develop on a space-time scale $\lambda \sim 1/gT$. (T is the temperature, and g is the gauge coupling, assumed to be small.) Such excitations are similar to the familiar charge oscillations of the electromagnetic plasmas³, and can indeed be described by simple kinetic equations of the Vlasov type^{4,1}.

But this simple analogy breaks down at the softer scale g^2T . There, the non-Abelian plasmas enter a new non-perturbative regime where the coupling constant is small but the field strengths are large, so that perturbation theory breaks down because of large non-linear effects. Indeed, gluons obey Bose-Einstein statistics, so the population of the soft ($k \ll T$) gluon modes is strongly enhanced in thermal equilibrium:

$$N_0(k) \equiv \frac{1}{e^{\beta k} - 1} \simeq \frac{T}{k} \gg 1, \quad \text{for } k \ll T. \quad (1)$$

Thus, the long wavelength thermal fluctuations with $\lambda \sim 1/g^2T$ involve many quanta $N_0 \sim 1/g^2$ and behave in many respects as classical colour fields A_a^μ with large amplitudes. We shall verify later that, typically, $|A| \equiv \sqrt{\langle A^2 \rangle} \sim gT$. This is a large fluctuation in the sense that the two terms in the soft covariant derivative $D_x \equiv \partial_x + igA$ are of the same order in g , $\partial_x \sim gA \sim g^2T$, so that the non-linear effects are indeed non-perturbative. In perturbation theory, such

effects show up as infrared divergences in relation with the mutual interactions of the soft ($k \sim g^2 T$) magnetic gluons (cf. Sec. 2.a below).

Moreover, infrared divergences are also associated with interactions among the hard ($k \sim T$) particles, as mediated by the exchange of soft magnetic gluons (or photons, in QED). This occurs, for instance, in the calculation of the quasiparticles damping rates in both QCD and QED (cf. Sec. 2.b).

In order to deal with such problems, one has to go beyond ordinary perturbation theory. For an Abelian plasma, it is possible to eliminate the infrared problem of the damping rate by a specific resummation of the perturbation theory, based on the Bloch-Nordsieck (or eikonal) approximation⁵. For non-Abelian plasmas, the non-linear effects in the soft magnetic sector must be treated exactly, which requires numerical methods. An useful strategy in this sense — which is especially well suited for real-time calculations — is to first construct an *effective theory* for the soft fields, by integrating out the hard fields in perturbation theory, and then study the effective theory non-perturbatively, e.g., as a classical theory on a lattice^{6,7,8,9,10,11,12,13,14}. (The classical approximation will be discussed in Sec. 3 below.) This strategy can be seen as an extension to real time of the “dimensional reduction” generally performed in static calculations¹⁵.

What I would like to show you here, is that kinetic theory is a powerful tool for constructing such effective theories. This has been first demonstrated for the collective dynamics at the scale gT , where we have shown⁴ that simple, collisionless, kinetic equations resum an infinite number of one-loop diagrams with soft external lines and hard loop momenta, the so-called “hard thermal loops”^{16,17}. These equations will be reviewed below, in Sec. 4, together with the effective classical theory they generate¹⁰, in Sec. 5. Then, in Sec. 6, I shall extend this approach to describe collective colour excitations at the softer scale $g^2 T$. This involves a Boltzmann equation^{12,18} which generates Bödeker’s effective theory¹¹ (see also Refs.^{19,14}).

2 Infrared problems in perturbation theory

(a) Higher-order corrections to the free energy

The infrared (IR) complications are most easily seen in the calculation of static quantities, like the pressure, as performed in the imaginary time formalism^b. There, the energies of the gluon modes are purely imaginary and

^bI consider a purely Yang-Mills plasma; indeed, quarks are not important for the infrared physics to be discussed here.

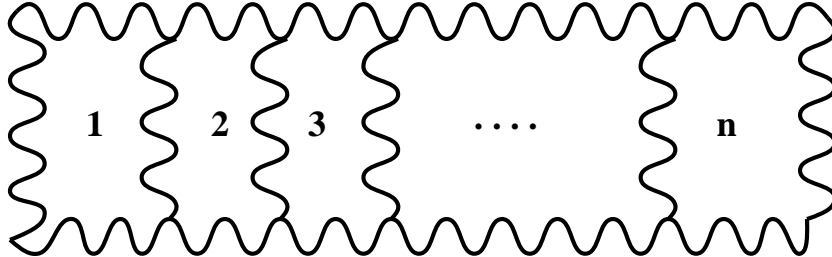


Figure 1: An n -loop ladder diagram contributing to the free energy in hot QCD.

discrete, $k_0 = i\omega_n \equiv i2\pi nT$, with integer n (“Matsubara frequencies”), which provides a large “screening mass” $|\omega_n| \sim T$ for all the non-static ($n \neq 0$) modes. Thus, the IR problems can be associated only with the static ($\omega_n = 0$) Matsubara modes, and the most severe IR divergences are expected to come from diagrams involving the static modes alone.

For instance, when computing higher order corrections to the free energy in hot QCD, one finds strong IR divergences from the ladder diagrams depicted in Fig. 1. In this diagram, all the propagators are static, and the loop integrations are three-dimensional. By power counting, this can be estimated as $F^{(n)} \sim g^6 T^4 (g^2 T/\mu)^{n-4}$ (for $n \geq 4$ loops), where μ is an ad-hoc IR cutoff.

For the *electric* gluons, an IR cutoff $\sim gT$ is indeed generated dynamically, via Debye screening. In the *magnetic* sector, however, screening can only occur non-perturbatively, at the scale $g^2 T$. With $\mu \sim g^2 T$, all the ladder diagrams with four or more loops will contribute to the *same* order in g , namely, to order g^6 . Thus, perturbation theory breaks down, in the sense that we lose the usual connection between powers of the coupling constant and the number of loops.

As mentioned in the Introduction, this breakdown is associated with long wavelength ($\lambda \sim 1/g^2 T$) thermal fluctuations with large amplitudes, $A \sim gT$. To see this, consider the free propagator of the magnetic gluon, in imaginary time:

$$\langle A(\tau, \mathbf{x}) A(0) \rangle = T \sum_n \int \frac{d^3 k}{(2\pi)^3} e^{-i\omega_n \tau + i\mathbf{k} \cdot \mathbf{x}} \frac{1}{k^2 + \omega_n^2}. \quad (2)$$

By letting $\tau \rightarrow 0$, $x \rightarrow 0$, and keeping only the contribution of the static modes ($\omega_n = 0$) with soft momenta $k \sim g^2 T$, one obtains:

$$\langle A^2 \rangle \simeq T \int \frac{d^3 k}{k^2} \sim g^2 T^2, \quad (3)$$

so that $|A| \equiv \sqrt{\langle A^2 \rangle} \sim gT$, as anticipated.

A different perspective on these IR problems follows by noticing that the diagram in Fig. 1 is actually a graph of three-dimensional QCD with coupling constant $g_3 = g\sqrt{T}$. That is, the leading IR behaviour of hot QCD can be studied with the replacement (with $A_a^i(\mathbf{x}) \equiv T \int_0^\beta d\tau A_a^i(\tau, \mathbf{x})$) :

$$\begin{aligned} Z_4 &\equiv \int \mathcal{D}A_a^\mu(\tau, \mathbf{x}) \exp \left\{ - \int_0^\beta d\tau \int d^3x \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \right\} \longrightarrow \\ &\longrightarrow Z_3 \equiv \int \mathcal{D}A_a^i(\mathbf{x}) \exp \left\{ -\beta \int d^3x \frac{1}{4} F_a^{ij} F_{ij}^a \right\} \end{aligned} \quad (4)$$

which reduces the initial finite-temperature problem in four dimensions to an effective zero-temperature problem in three Euclidean dimensions. This is the crudest example of “dimensional reduction”, a strategy which consists in integrating out the non-static modes in perturbation theory in order to obtain an effective three-dimensional theory for the static modes¹⁵. This is convenient since three-dimensional lattice simulations are much easier to perform than in four dimensions. For instance, the magnetic mass, and the non-perturbative correction, of $O(g^6)$, to the pressure, have been computed in this way²⁰. Moreover, the reduction to three dimensions has also permitted for some analytic non-perturbative studies of the magnetic screening²¹.

(b) *Quasiparticle damping rates*

Non-perturbative aspects, as related to the unscreened magnetic gluons, appear also in the computation of some *real-time* correlation functions. The most celebrated example in that sense is the rate for anomalous baryon number violation at high temperature^{7,8,11,13}, or “hot sphaleron rate”, a quantity which has been extensively discussed at this conference (see, e.g., the contributions by P. Arnold, D. Bödeker, A. Krasnitz, G. Moore and L. Yaffe).

Here, however, I would like to discuss an example which is conceptually simpler (since it involves only the computation of a 2-point function), namely the calculation of the lifetime of the quasiparticles. A single particle excitation is created by adding a particle with momentum \mathbf{p} (e.g., a transverse gluon) to the plasma initially in equilibrium. The added particle will then scatter off the other particles in the thermal bath (see Fig. 2), thus changing its momentum and colour. That is, the initial excitation will decay, and this can be measured from the corresponding retarded propagator $D_R(t, \mathbf{p}) \equiv i\theta(t)\langle [A(t, \mathbf{p}), A(0, -\mathbf{p})] \rangle$. A usual expectation is that $D_R(t, \mathbf{p})$ decays exponentially in time, $|D_R(t, \mathbf{p})|^2 \sim e^{-2\gamma t}$, which identifies the lifetime

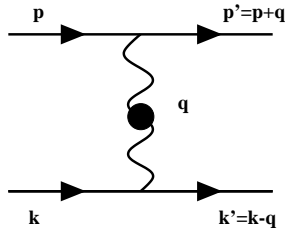


Figure 2: Elastic scattering in the (resummed) Born approximation. The continuous lines refer to hard gluons, while the wavy line is the soft gluon exchanged in the collision. The blob indicates the resummation of the screening effects.

of the single particle excitation as $\tau = 1/2\gamma$. It turns out, however, that the calculation of γ is afflicted with IR problems. Specifically^{16,5} :

$$\gamma \simeq \frac{g^4 N_c^2 T^3}{12} \int dq \int_{-q}^q \frac{dq_0}{2\pi} \left\{ |{}^* \mathcal{D}_l(q_0, q)|^2 + \frac{1}{2} \left(1 - \frac{q_0^2}{q^2}\right)^2 |{}^* \mathcal{D}_t(q_0, q)|^2 \right\}, \quad (5)$$

where ${}^* \mathcal{D}_l(q)$ and ${}^* \mathcal{D}_t(q)$ denote the propagators of the exchanged gluon in the electric and the magnetic channels, respectively. These are dressed so as to include the screening effects at the scale gT ; indeed, with a bare gluon propagator, the integral in eq. (5) would be quadratically IR divergent, showing that the damping rate is dominated by soft momentum transfers, $q \lesssim gT$. The dressed propagators have the following IR behaviour^{1,2} (below, m_D is the Debye mass, $m_D^2 = g^2 N_c T^2/3$):

$${}^* \mathcal{D}_l(q_0 \rightarrow 0, q) \simeq \frac{-1}{q^2 + m_D^2}, \quad {}^* \mathcal{D}_t(q_0 \ll q) \simeq \frac{1}{q^2 - i(\pi q_0/4q) m_D^2}, \quad (6)$$

which exhibits Debye screening in the electric sector and dynamical screening in the magnetic sector. The latter is due to Landau damping³, i.e., the thermal absorption of the space-like gluons, which gives an imaginary part to the self-energy of the magnetic gluon: $\text{Im} \Pi_t(q_0 \ll q) \simeq -(\pi q_0/4q) m_D^2$. This is proportional to the frequency q_0 since only the time-dependent ($q_0 \neq 0$) magnetic fields can transfer energy to the plasma constituents (as mechanical work), and thus get damped.

Because of Debye screening, the electric contribution to the damping rate γ_l is finite and of order $g^4 T^3/m_D^2 = O(g^2 T)$. However, even after including the dynamical screening, the magnetic contribution γ_t remains logarithmically

divergent (below, μ is an ad-hoc IR cutoff) :

$$\gamma_t \simeq \frac{g^2 N_c T}{4\pi} \int_{\mu}^{m_D} \frac{dq}{q} = \frac{g^2 N_c T}{4\pi} \ln \frac{m_D}{\mu}. \quad (7)$$

The remaining divergence is associated to the static magnetic interactions, which are not screened at the scale gT . Assuming magnetic screening at the scale $g^2 T$, it follows that $\gamma \simeq \alpha N_c T \ln(1/g)$ (with $\alpha \equiv g^2/4\pi$), where, however, the constant term under the logarithm cannot be determined (since sensitive to the non-perturbative screening mechanism). Thus, in QCD, the lifetime of the quasiparticles cannot be computed in perturbation theory beyond logarithmic accuracy.

The same IR problem occurs also in QED, in the calculation of the lifetime of the charged particles. There, this is even more intriguing, since there is no magnetic screening in the Abelian theories²³. However, as shown in Ref.⁵, the Abelian problem can be solved by a further resummation of the perturbation theory, with the peculiar result that the electron propagator has an anomalous, non-exponential, decay law: $G_R(t, \mathbf{p}) \sim \exp\{-\alpha T t \ln(m_D t)\}$ (see also Ref.²⁴).

3 The classical approximation

I come now to the main question to be addressed in this talk, which is, how to compute non-perturbative *real-time* correlations in hot gauge theories. Clearly, the standard lattice calculations, as formulated in imaginary time, are not appropriate for this problem. Fortunately, a fully quantum calculation is actually not needed: because of the Bose enhancement, the non-perturbative modes with $k \sim g^2 T$ have large thermal occupation numbers (cf. eq. (1)), which, by the correspondence principle, is a classical limit. For instance, the average energy per soft mode in thermal equilibrium, namely,

$$\varepsilon(k) = \frac{k}{e^{\beta k} - 1} \simeq T \quad \text{for } k \ll T. \quad (8)$$

is the same as expected from the classical equipartition theorem. Based on this observation, it has been suggested to compute the hot baryon number violation rate through lattice simulations of a *classical* thermal field theory^{6,7}. The only question is, what is the correct classical theory ?

It is well known that the classical approximation becomes meaningless at high momenta $k \gtrsim T$, where eq. (1) is not correct anylonger. At a first sight, one could expect this to be irrelevant for the problem at hand. Indeed, we are interested here in non-perturbative correlations, as determined by the dynamics at the scale $g^2 T$. Thus, one may expect such correlations not to be

sensitive to the hard plasma modes⁷. If this was the case, such quantities could be simply computed from the classical Yang-Mills theory at finite temperature, without worrying too much about its bad ultraviolet behaviour.

But the previous examples in Sec. 2 show that this argument is too naïve⁸: in the plasma, the soft and hard modes are coupled by the interactions, which results in screening effects which considerably modify the dynamics of the soft modes. For instance, eq. (6) for ${}^*D_t(q)$ shows that, for large enough frequencies q_0 , the soft magnetic fields are efficiently screened by Landau damping, and therefore decouple from the non-perturbative IR physics. For $q \sim g^2T$, only the modes with very low frequencies $q_0 \lesssim q^3/m_D^2 \sim g^4T$ can take part in non-perturbative phenomena. According to Arnold, Son and Yaffe⁸, this sets the time scale for non-perturbative phenomena to be $1/g^4T$ (see also Refs. ^{5,11,12}).

The classical Yang-Mills theory does not describe correctly the screening effects due to the hard particles. For instance, it yields a linearly divergent Debye mass^{6,8,10} $m_{cl}^2 \sim g^2T\Lambda$, rather than the correct, quantum, result $m_D^2 \sim g^2T^2$. A possible solution to this problem is to treat hard and soft modes on a different footing: first, the hard modes are integrated out in perturbation theory, which properly generates the screening corrections; then, the resulting effective theory for the soft modes is treated as a classical field theory, via non-perturbative methods (e.g., via lattice simulations). This strategy requires an unambiguous separation between hard and soft degrees of freedom, e.g., an intermediate cutoff μ , which moreover must be consistent with gauge symmetry and with the lattice implementation. If μ is chosen such as $gT \ll \mu \ll T$, then one obtains the “hard thermal loop” (HTL) effective theory^{16,17,4,6,10}, to be presented in the next two sections. If, on the other hand, $g^2T \ll \mu \ll gT$, then one obtains Bödeker’s effective theory^{11,12,14,18}, to be discussed in Sec. 6 below.

4 Effective theories from kinetic equations

To construct an effective theory for the soft modes, one needs to study the dynamics of the hard plasma constituents — here, transverse gluons with momenta $k \sim T$ — in the presence of soft ($q \lesssim gT$) background fields A_a^μ . This is an off-equilibrium situation: the plasma is perturbed away from equilibrium by the background fields which induce long wavelength ($\lambda \gtrsim 1/gT$) fluctuations in the colour density of the hard particles. Since $\lambda \gg \bar{r}$ (where $\bar{r} \sim 1/T$ is the mean interparticle distance), these are *collective* colour excitations. Since, furthermore, $\lambda \gg \lambda_T$ (where $\lambda_T \equiv 1/k \sim 1/T$ is the thermal wavelength of the hard particles), we expect such excitations to be described by kinetic theory^{1,3}. And, indeed, the equations to be presented below⁴ can be viewed as a

generalization of the Vlasov equation for ordinary plasmas³.

Specifically, the longwavelength colour excitations of the hard transverse gluons are described by a colour density matrix $N_{ab}(\mathbf{k}, x)$ which, to the order of interest, can be written in the form:

$$N_{ab}(\mathbf{k}, x) = N_0(\varepsilon_k)\delta_{ab} - gW_{ab}(x, \mathbf{v}) (dN_0/d\varepsilon_k), \quad (9)$$

where $N_0(\varepsilon_k) \equiv 1/(e^{\beta\varepsilon_k} - 1)$ is the equilibrium distribution (with $\varepsilon_k = |\mathbf{k}|$), and the function $W(x, \mathbf{v})$, which parametrizes the off-equilibrium deviation, is a colour matrix in the adjoint representation, $W(x, \mathbf{v}) \equiv W_a(x, \mathbf{v})T^a$, which depends upon the velocity $\mathbf{v} = \mathbf{k}/\varepsilon_k$ (a unit vector), but not upon the magnitude $k \equiv |\mathbf{k}|$ of the momentum. It satisfies the following simple equation:

$$(v \cdot D_x)_{ab}W^b(x, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}_a(x), \quad (10)$$

where $v^\mu \equiv (1, \mathbf{v})$, $D^\mu = \partial^\mu + igA_a^\mu T_a$ is the covariant derivative defined by the background field, and E_a^i is the chromoelectric mean field. The system is closed by the Yang-Mills equations for the soft fields A_a^μ , namely:

$$(D_\nu F^{\nu\mu})_a(x) = j_a^\mu(x), \quad (11)$$

with the induced current:

$$j_a^\mu(x) = m_D^2 \int \frac{d\Omega}{4\pi} v^\mu W_a(x, \mathbf{v}), \quad (12)$$

where the angular integral $\int d\Omega$ runs over the orientations of \mathbf{v} .

Eq. (10) is a collisionless kinetic equation. It has been obtained⁴ from the general Dyson-Schwinger equations for the off-equilibrium plasma, by neglecting the collisions among the hard particles and by performing a gauge-covariant gradient expansion which takes profit of the assumed separation of scales. Note that all these approximations are controlled by the same small parameter, the coupling strength g , so that eq. (10) is actually correct to leading order in g .

By formally solving eq. (10), we can express the current in terms of the gauge fields A_a^μ , and thus obtain an *effective* Yang-Mills equation which involves the soft fields alone:

$$D_\nu F^{\nu\mu} = m_D^2 \int \frac{d\Omega}{4\pi} \frac{v^\mu v^i}{v \cdot D} E^i. \quad (13)$$

Eq. (13) describes the propagation of soft colour fields in the high- T plasma. The hard particles are not explicit anymore, since they have been integrated to yield the induced current in the r.h.s. By expanding this current in powers of

the gauge fields one generates⁴ all the HTL's of Braaten and Pisarski¹⁶, which encompass the screening phenomena at the scale gT (cf. Sec. 2). However, because of the non-local structure of the current (note the covariant derivative in the denominator), eq. (13) is not very convenient for the construction of the classical thermal theory, to which I turn now.

5 The classical effective theory

I shall now use eqs. (10)–(12) to define a classical field theory at finite temperature¹⁰. As usual with gauge theories, this is most easily done in the temporal gauge $A_0^a = 0$, where the equations read:

$$\begin{aligned} E_i^a &= -\partial_0 A_i^a, \\ -\partial_0 E_i^a + \epsilon_{ijk}(D_j B_k)^a &= m_D^2 \int \frac{d\Omega}{4\pi} v_i W^a(x, \mathbf{v}), \\ (\partial_0 + \mathbf{v} \cdot \mathbf{D})^{ab} W_b &= \mathbf{v} \cdot \mathbf{E}^a, \end{aligned} \quad (14)$$

together with the constraint expressing Gauss' law (i.e., the $\mu = 0$ component of eq. (11)) :

$$G^a(\mathbf{x}) \equiv (\mathbf{D} \cdot \mathbf{E})^a + m_D^2 \int \frac{d\Omega}{4\pi} W^a(x, \mathbf{v}) = 0. \quad (15)$$

Note that eqs. (14) are not in canonical form: this is already obvious from the fact that we have an odd number of equations. Still, it can be verified that these equations are conservative; the corresponding, conserved energy functional has the gauge-invariant expression^{4,1} :

$$H = \frac{1}{2} \int d^3x \left\{ \mathbf{E}_a \cdot \mathbf{E}_a + \mathbf{B}_a \cdot \mathbf{B}_a + m_D^2 \int \frac{d\Omega}{4\pi} W_a(x, \mathbf{v}) W_a(x, \mathbf{v}) \right\}. \quad (16)$$

Eqs. (14)–(16) define an effective theory for the soft degrees of freedom. Besides the soft colour fields $A_a^i(x)$ and $E_a^i(x)$, these equations also involve the auxiliary fields $W_a(x, \mathbf{v})$ which simulate the hard thermal gluons (or, more precisely, their long wavelength colour fluctuations). We are interested in computing the (non-perturbative) real-time correlations of the fields A_a^i . To this aim, we need to construct the thermal partition function for this classical field theory.

Recall first how this is done in some generic theory: the thermal expectation values are obtained by first solving the classical equations of motion for given initial conditions, and then averaging over the initial conditions (i.e., over the classical “phase-space”) with the Boltzmann weight $\exp(-\beta H)$. Since the initial conditions (say $\phi(\mathbf{x})$ and $\dot{\phi}(\mathbf{x})$ for a scalar theory) depend only on the

spatial coordinate \mathbf{x} , the phase space integration is actually a *three-dimensional* functional integral, which can be implemented on a lattice in the standard way.

For the problem at hand, the classical phase-space is determined by the initial conditions to eqs. (14), that is,

$$A_a^i(0, \mathbf{x}) = \mathcal{A}_a^i(\mathbf{x}), \quad E_a^i(0, \mathbf{x}) = \mathcal{E}_a^i(\mathbf{x}), \quad W_a(0, \mathbf{x}, \mathbf{v}) = \mathcal{W}_a(\mathbf{x}, \mathbf{v}), \quad (17)$$

and the thermal weight is provided by the Hamiltonian in eq. (16). Then, the (real-time) thermal correlation functions of the fields A_a^i can be obtained from the following generating functional:

$$Z_{cl}[J_i^a] = \int \mathcal{D}\mathcal{E}_i^a \mathcal{D}\mathcal{A}_i^a \mathcal{D}\mathcal{W}^a \delta(\mathcal{G}^a) \exp \left\{ -\beta \mathcal{H} + \int d^4x J_i^a(x) A_i^a(x) \right\}, \quad (18)$$

where $A_a^i(t, \mathbf{x})$ is the solution to eqs. (14) with the initial conditions (17), and \mathcal{G}^a and \mathcal{H} are expressed in terms of the initial fields, cf. eqs. (15) and (16). Physically, the fluctuations in the initial conditions $\mathcal{W}(\mathbf{x}, \mathbf{v})$ for the auxiliary fields can be interpreted as a thermal noise due to the hard particles and which drives the soft fields toward thermal equilibrium. It is this noise which generates the Landau damping of the soft correlation functions¹⁰.

In particular, for $J = 0$, eq. (18) yields the following expression for the free energy of the classical thermal radiation:

$$Z_{cl} = \int \mathcal{D}\mathcal{A}_0^a \mathcal{D}\mathcal{A}_i^a \exp \left\{ -\beta \int d^3x \left(\frac{1}{4} (\mathcal{F}_{ij}^a)^2 + \frac{1}{2} (\mathcal{D}_i \mathcal{A}_0)^2 + \frac{m_D^2}{2} (\mathcal{A}_0^a)^2 \right) \right\} \quad (19)$$

which is also the result expected from dimensional reduction (see eq. (4)).

To complete the construction of the effective theory, eq. (18) must be supplemented with an UV cutoff μ , which is the scale separating hard from soft degrees of freedom: $gT \ll \mu \ll T$. Correspondingly, the Hamiltonian must be extended to include μ -dependent counterterms, chosen so as to cancel the cutoff dependence of the classical theory in any complete calculation. In practice, however, this “matching” turns out to be difficult to achieve^{10,8}, mainly because of the constraints of the lattice implementation. It is therefore rewarding that the results of the first lattice simulations²⁵ of eqs. (14)–(18) appear to be quite robust and insensitive to lattice artifacts. In particular, the hot sphaleron rate obtained in this way is consistent with the previous calculations in Ref.⁹.

6 A Boltzmann equation for colour

Since the “semi-hard” modes with $q \sim gT$ are also perturbative, it is possible to integrate them out as well, and thus get an effective theory involving only

the “ultrasoft” modes with $q \sim g^2 T$ ¹¹. Then, the collisions among the hard particles, as mediated by the semi-hard fields, must be included explicitly in the kinetic equation. That is, the previous equation (10) must be generalized so as to include the collisions terms. This is also necessary for consistency: for colour fluctuations at the scale $g^2 T$, the effects of the collisions among the plasma particles become as important as those of the mean fields¹⁸.

The resulting kinetic equation is a Boltzmann equation describing the propagation and relaxation of longwavelength ($\lambda \sim 1/g^2 T$) colour excitations. It allows, in particular, to compute the colour conductivity^{11,12,18}, thus clarifying some previous work on this subject¹⁹. To leading logarithmic accuracy (see below), this equation has been first obtained by Bodeker¹¹. Then, Arnold, Son and Yaffe have shown¹² that Bodeker’s theory can be generated by a rather simple Boltzmann equation, which has been physically motivated, but not rigorously proven, in Refs.¹². Recently, we have given a derivation of this equation¹⁸, starting from the quantum field equations. This has clarified the nature of the approximations involved, thus fixing its range of applicability.

Remarkably, even after the inclusion of the collision term, the density matrix $N_{ab}(\mathbf{k}, x)$ preserves the same structure as in the mean field approximation (cf. eq. (9)), but the functions $W_a(x, \mathbf{v})$ satisfy a more complicated equation (compare to eq. (10)) :

$$(v \cdot D_x)^{ab} W_b(x, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}^a(x) - \gamma \left\{ W^a(x, \mathbf{v}) - \frac{\langle \Phi(\mathbf{v} \cdot \mathbf{v}') W^a(x, \mathbf{v}') \rangle}{\langle \Phi(\mathbf{v} \cdot \mathbf{v}') \rangle} \right\} \quad (20)$$

The new feature here is the collision term in the r.h.s. This is proportional to the quasiparticle damping rate γ (cf. Sec. 2.b), which thus appears to set the scale for the colour relaxation time: $\tau_{col} \sim 1/\gamma \sim 1/(g^2 T \ln(1/g))$. The other notations above are as follows: the angular brackets in the collision term denote angular average over the directions of the unit vector \mathbf{v}' (as in eq. (22) below), and the quantity $\Phi(\mathbf{v} \cdot \mathbf{v}')$ is given by:

$$\Phi(\mathbf{v} \cdot \mathbf{v}') \equiv \int \frac{d^4 q}{(2\pi)^2} \delta(q_0 - \mathbf{q} \cdot \mathbf{v}) \delta(q_0 - \mathbf{q} \cdot \mathbf{v}') \left| {}^* \mathcal{D}_l(q) + (\mathbf{v}_t \cdot \mathbf{v}'_t) {}^* \mathcal{D}_t(q) \right|^2 \quad (21)$$

with ${}^* \mathcal{D}_l(q)$ and ${}^* \mathcal{D}_t(q)$ defined after eq. (5). Up to a normalization, $\Phi(\mathbf{v} \cdot \mathbf{v}')$ is the total interaction rate for two hard particles with momenta \mathbf{k} and \mathbf{p} (and velocities $\mathbf{v} \equiv \hat{\mathbf{k}}$ and $\mathbf{v}' \equiv \hat{\mathbf{p}}$) in the (resummed) Born approximation, as illustrated in Fig. 2 (\mathbf{v}_t and \mathbf{v}'_t are the transverse projections of the velocities with respect to the momentum \mathbf{q} of the exchanged gluon). The damping rate γ is obtained from $\Phi(\mathbf{v} \cdot \mathbf{v}')$ as follows (cf. eq. (7)) :

$$\gamma = \frac{g^4 N_c^2 T^3}{6} \int \frac{d\Omega'}{4\pi} \Phi(\mathbf{v} \cdot \mathbf{v}') \simeq \frac{g^2 N_c T}{4\pi} \ln(1/g). \quad (22)$$

There are two important remarks about the previous equations:

First, whereas most transport phenomena are dominated by large momentum transfers $gT \lesssim q \lesssim T$, so that the typical relaxation times are²² $\tau_{tr} \sim 1/(g^4 T \ln(1/g))$, the relaxation of colour excitations turns out to be dominated by soft gluon exchanges, $g^2 T \lesssim q \lesssim gT$, as the quasiparticle damping rate. There is a simple physical reason for that^{19,11,12}: unlike momentum fluctuations, which require a large angle scattering to relax, colour can be efficiently exchanged in any scattering, even a small angle one. This yields a colour conductivity^{19,11,12} $\sigma_c \sim T/\ln(1/g)$, to be contrasted with the usual, electric, conductivity²²: $\sigma_{el} \sim T/(e^2 \ln(1/e))$. Note also that it is the same physical process — namely, the scattering via one-gluon exchange in Fig. 2 — which provides relaxation for both single-particle and collective (momentum or colour) excitations. If, nevertheless, the relevant time scales turn out not to be the same (namely $\tau \sim \tau_{col} \gg \tau_{tr}$), it is because of specific cancellations among various collision terms¹⁸, which occur in the calculation of most transport coefficients, but not in that of the quasiparticle lifetime τ , or in that of the relaxation time of colour excitations τ_{col} .

Second, strictly speaking, the functional form of the collision term in eq. (20) is only valid to leading logarithmic accuracy, because of the approximations performed in its derivation. This limitation comes from the poor convergence of the gradient expansion when the range of the interactions becomes comparable to the scale of the system inhomogeneities¹⁸. Within this logarithmic accuracy, eq. (20) can be shown^{12,18} to reduce to Bödeker's equation¹¹.

Note finally that, even though conceptually interesting, the leading logarithmic approximation appears to be of little use for the calculation of the hot baryon number violation rate. Indeed, the numerical calculations^{9,13,25} show that, for realistic values of g , the constant term under the logarithm is sensibly larger than $\ln(1/g)$.

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