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On Central Charges and Hamiltonians for 0-brane dynamics*

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Abstract

We consider general properties of central charges of zero-branes and associated duality invariants, in view of their double role, on the bulk and on the world volume (quantum mechanical) theory.

A detailed study of the BPS condition for the mass spectrum arising from toroidal compactifications is given for 1/2, 1/4 and 1/8 BPS states in any dimension. As a byproduct, we retrieve the U-duality invariant conditions on the charge (zero mode) spectrum and the orbit classification of BPS states preserving different fractions of supersymmetry.

The BPS condition for 0-branes in theories with 16 supersymmetries in any dimension is also discussed.

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1 Introduction.

In recent time, the role of duality symmetries of a dynamical theory encompassing quantum gravity has received increasing attention in several contexts.

Particular examples where the duality takes an important role, especially in connection with non perturbative properties, is the AdS/CFT correspondence [1], related to the horizon geometry of p-branes and their world-volume conformal field theory description.

Another example is the connection between M-theory compactified on a torus T^d [2, 5, 6, 3, 4] and $(d+1)$ Yang-Mills theory compactified on the dual torus $\tilde{\mathrm{T}}^d$.

More closely related to the latter is the recent investigation of D-brane Born-Infeld actions and the role played by duality in explaining several properties of their Hamiltonian formulation and the corresponding energy spectrum of BPS states [7]. In this framework it is believed that Born-Infeld non abelian gauge theories with non trivial R-R backgrounds, are naturally described by some generalization of gauge theories on non commutative tori [8].

The framework of non commutative geometry offers for instance, a new interpretation of the T-duality group $O(d,d;\mathbf{Z})$ of quantum mechanical systems obtained by compactifying the Born-Infeld action of D-branes on T^d. The latter occurs in type II string theory compactified on T^d [9, 10, 11, 12].

These quantum mechanical systems have also been shown [13], at least for $d \leq 4$, to exhibit the full extended U-duality symmetry[1] $\mathrm{E}_{d+1(d+1)}$, rather than the smaller symmetry $\mathrm{E}_{d(d)}$ present in matrix gauge theory on T^d, where it appears as an extension of the geometrical symmetry $\mathrm{SL}(d)$ [4, 6].

In previous investigations, the central charge matrix Z for 0-branes played a role, not only as central extension of supersymmetry algebra in theories with non trivial 0-brane background metric, but also as effective potential of the geodesic action of a one-dimensional Lagrangian system derived from the bulk Einstein-Maxwell Lagrangian, in presence of moduli fields $\{\phi\}$ and quantized charges q_A of zero-branes [14, 15].

The critical points of this potential were seen to determine the Bekenstein-Hawking entropy formula as the extremization of the Weinhold potential [15].

$$W = \frac{1}{2}\mathrm{Tr}(ZZ^\dagger) \tag{1.1}$$

or equivalently of the BPS mass $m_{BPS} = |Z_h|$ where $|Z_h|$ is the highest eigenvalue of $\sqrt{ZZ^\dagger}$ [17].

In the world-volume description of 0-branes, the very same function W appears as Hamiltonian of the 0-brane quantum mechanics [7, 13, 16]

$$H_\phi(\hat{q}) = \sqrt{\frac{1}{N}\mathrm{Tr}(ZZ^\dagger(\phi,\hat{q}))}, \tag{1.2}$$

where the quantized charges are replaced by a set of Hamiltonian variables $\hat{q}$, which belong to the same duality multiplet as the quantized charges of the bulk supergravity theory in presence of zero-brane sources.

[1] In this paper U-duality will mean both the classical and quantum U-duality

The appearence of the central charge in the 0-brane action in arbitrary $D=4$ supergravity backgrounds has recently been shown to occur as a consequence of κ-supersymmetry [18].

In this framework the energy spectrum of the Hamiltonian (1.2) is given by the BPS mass formula of the effective supergravity theory [7]

$$m_{BPS} = |Z_h(\phi, q_0)|, \tag{1.3}$$

where the hamiltonian variables $\hat{q}$ are replaced by their zero mode part q_0 which eventually coincide with the same duality multiplet of the quantized charges of the bulk theory, but now with the interpretation of "fluxes" and "momenta" of the world-volume hamiltonian description [4, 6].

These zero modes fill representations of the U-duality group $E_{d+1(d+1)}(\mathbf{Z})$ for systems with maximal supersymmetry and the BPS spectrum preserves some fraction of supersymmetry depending on the particular orbit of the charge vector state [19, 20, 21].

Note that the BPS energy $|Z_h(\phi, q_0)|$ is not the same as replacing in $\sqrt{ZZ^\dagger(\phi, \hat{q})}$ the zero mode q_0 of q, unless the states are 1/2 BPS [7], which, as we will see, can only occur if the charge duality multiplet satisfies some duality invariant conditions. At the classical level, where the charges are continuous , this is equivalent to say that the zero-mode part belongs to a particular orbit of the charge vector representation of the duality group G.

It is the aim of the present investigation to derive general formulas of the energy spectrum for any torus $\mathrm{T}^d,\ d = 1, \ldots 6$ and provide a new derivation of the different BPS conditions in terms of the U-duality invariant constraints, retrieving then the analysis of Maldacena and one of the authors [19] as well as the classification of Gunaydin and one of the authors [20] and Lu, Pope and Stelle [21]. We deal also with the case of 0-branes in theories in any dimension with 16 supersymmetries. This is interesting because it is related to heterotic strings compactified on T^d or Type II theories compactified on more general manifolds (such as K_3).

The paper is organized as follows:

In Section 3 we consider systems compactified on $\mathrm{T}^d,\ d = 1, \ldots 4$ where only 1/2 and 1/4 BPS states occur.

In Section 4 we consider the richer structure occurring for d=5,6 where a complete understanding of the world volume theory is still missing.

In Section 5 the BPS conditions are derived for the case of theories with sixteen supersymmetries in any dimension.

2 Central charges and geometrical tools of coset spaces.

In the present section we review the central charges for 0-branes in theories with maximal supersymmetry and the BPS conditions on the U-multiplet of quantized charges which entail different orbits of the duality group which preserve different fractions of supersymmetry. The analysis for theories with 16 supersymmetries will be considered in the last section.

The supergravity theories describing these systems can be obtained in three different ways, by compactifying M theory on T^{d+1} ($(d+1)$-dimensional torus) or type IIA and type IIB string theories on T^d. We will consider here supergravity theories compactified down up to $D = 4$ space-time dimensions ($d = 1, \ldots 6$).

Some of the results presented here overlap with previous analysis for $d = 1, \ldots 4$, when only 1/2 or 1/4 BPS states are present [5, 7, 16]. The analysis of $d = 5, 6$ is essentially novel although the BPS conditions for 1/2, 1/4 and 1/8 BPS states were previously discussed in the literature and the orbit classifications derived [19, 20, 21].

2.1 R-Symmetry and U-duality.

The supersymmetry algebra of type II string theory compactified on T^d down to $10 - d$ dimensions has an R-symmetry group and a continuous duality group which depends on d. The R-symmetry is given below [22]:

R-symmetry group H

$$
\begin{array}{ll}
d=1 & \mathrm{U}(1) \\
d=2 & \mathrm{SU}(2)\times\mathrm{U}(1) \\
d=3 & \mathrm{USp}(4)\approx\mathrm{O}(5) \\
d=4 & \mathrm{USp}(4)\times\mathrm{USp}(4)\approx\mathrm{O}(5)\times\mathrm{O}(5) \\
d=5 & \mathrm{USp}(8) \\
d=6 & \mathrm{SU}(8)
\end{array}
\tag{2.4}
$$

The U-duality groups G are $\mathrm{E}_{d+1(d+1)}$ [22], and the R-symmetry groups are their maximal compact subgroups. The quantum U-duality groups are $\mathrm{E}_{d+1(d+1)}(\mathbf{Z})$ [23]. Because of the connection between M-theory and string theory, the groups $\mathrm{E}_{d+1(d+1)}$ contain, both

$$\mathrm{Gl}(d+1) \subset \mathrm{E}_{d+1(d+1)} \tag{2.5}$$

which is the classical isometry group of the moduli space of a T^{d+1} torus in M-theory and

$$\mathrm{O}(1,1)\times\mathrm{O}(d,d) \subset \mathrm{E}_{d+1(d+1)} \quad (d\neq 6), \quad \mathrm{Sl}(2)\times\mathrm{O}(6,6)\subset\mathrm{E}_{7(7)} \quad \text{for} \quad d=6, \tag{2.6}$$

which is the S-T duality group of string theory [24, 25, 23].

In string theory the $\mathrm{O}(d,d)$ group combines the geometric isometry of the T^d torus $\mathrm{GL}(d)$ with the shift of the antisymmetric tensor $B_{ij} \mapsto B_{ij} + N_{ij}$ while the O(1,1) factor corresponds to the dilaton shift. The group $\mathrm{E}_{d+1(d+1)}$ emerges from the combination of $\mathrm{Sl}(d+1)$ with $\mathrm{O}(d,d)$ and this operation gives rise to non perturbative symmetries which combine the N-S-NS and R-R fields in supermultiplets.

The spinorial charges of the supersymmetry algebra transform in representations of the R-symmetry group and this implies that the central charges of interest to us have certain symmetry and reality properties.

In the case of Lorentz scalar central charges, appropriate to 0-branes, the classification goes as follows: the central charge matrix $Z(\phi, q)$ is in the same representation of the R-symmetry as the vector fields A_μ of the corresponding theory. This gives the following result,

Central charge representation of the R-symmetry

$d=1$	**3** of O(2), (real symmetric tensor).
$d=2$	**3(+)** of SU(2)×U(1) , (complex triplet).
$d=3$	**10** of USp(4), (real antisymmetric tensor).
$d=4$	**16** of USp(4)×USp(4), (bispinor (4,4) of O(5)×O(5)).
$d=5$	**27** of USp(8), (Ω-traceless symplectic antisymmetric tensor).
$d=6$	**28** of SU(8), (complex antisymmetric tensor). (2.7)

The previous results follow both, from a dynamical reduction of the 11 or 10 dimensional supergravities with 32 supercharges or by an analysis of extended superalgebras in the appropriate dimensions [26, 27].

Since in the original IIA theory there is only one D 0-brane (one scalar central charge) [26], all the charges in lower dimensions come by wrapping branes on the torus cycles, other than momenta and string windings. In the type IIB on T^d, 0-branes emerge as momenta, string windings, and D-branes wrapped on the torus cycles.

If we want to discuss quantum mechanical systems emerging from $d+1$ Born-Infeld Lagrangians compactified on T^d, we must consider IIA D-branes compactified on even dimensional tori and IIB D-branes compactified on odd dimensional tori.

The world volume description of the central charges Z and quantized charges q is fairly well understood for the case of T^d with $d=1,\ldots 4$. The U-duality multiplets of the quantized charges q correspond to fluxes, momenta, instanton number and rank of the gauge groups in the world volume Yang-Mills theory. The moduli dependent central charge determines the hamiltonian of the quantum mechanical system as well as the energy spectrum of the BPS states [7, 6].

The main role played by duality is that the central charge vector extends the representation of the R-symmetry group to a representation of the full duality group $\mathrm{E}_{d+1(d+1)}$ acting on the vector field strength. The relevant extensions are as follows [22]

0-brane representation of U-duality group

$d=1$	**2+1** of E_2=SL(2)× O(1,1)
$d=2$	**(3,2)** of E_3=Sl(3)×Sl(2)
$d=3$	**10** of E_4=Sl(5)
$d=4$	**16** of E_5=O(5,5)
$d=5$	**27** of $\mathrm{E}_{6(6)}$
$d=6$	**56** of $\mathrm{E}_{7(7)}$ (2.8)

The moduli space of these theories is G/H. At the string level this space can be modded out further by $\mathrm{E}_{d+1}(\mathbf{Z})$ [23, 24] something similar to the fundamental domain versus the half plane for the prototype $\mathrm{Sl}(2,R)/\mathrm{O}(2)$.

In Table(2.1) we present the U-duality multiplets for 0-branes in the bulk and world volume description [7].

The gauge fields of type II supergravity theory, as well as the Yang-Mills world-volume fluxes complete U-duality multiplets of $\mathrm{E}_{d+1(d+1)}$ for $d=1,\ldots 4$. These multiplets are obtained by the $E_{d(d)}$ flux and momenta multiplets of matrix gauge theory on T^d and by adding an $E_{d(d)}$ singlet, the rank of the gauge group. For $d = 5, 6$, the Yang-Mills theory side misses some states corresponding to a NS five brane and K-K monopoles [4, 6].

d	**Supergravity Vector Fields**	**Born-Infeld Y-M fluxes**
	IIA	
2	$Z_\mu, g_{\mu i}, b_{\mu i}, A_{\mu ij}$	$\int \mathrm{Tr} F_{ij}$, $\int$ TrP_i $\int \mathrm{Tr} E_i$, rank
4	$Z_\mu, g_{\mu i}, b_{\mu i}, A_{\mu ij}$ $A^D_{\mu ijkl}$	$\int \mathrm{Tr} F_{ij}F_{kl}$, $\int \mathrm{Tr} P_i$, $\int \mathrm{Tr} E_i$,$\int \mathrm{Tr} F_{ij}$ rank
6	$Z_\mu, g_{\mu i}, b_{\mu i}, A_{\mu ij}$ $A^D_{\mu ijkl}, Z^D_{\mu ijklpq}$ $b^{NS}_{\mu ijklp}, g^D_{\mu i}$	$\int \mathrm{Tr} F_{ij}F_{kl}F_{pq}$, $\int \mathrm{Tr} P_i$, $\int \mathrm{Tr} E_i$, $\int \mathrm{Tr} F_{ij}F_{kl}$ $\int \mathrm{Tr} F_{ij}$, rank
	IIB	
1	$g_{\mu 1}, b_{\mu 1}, b^C_{\mu 1}$	$\int \mathrm{Tr} P$, $\int \mathrm{Tr} E$, rank
3	$g_{\mu i}, b_{\mu i}, b^C_{\mu i}, A_{\mu ijk}$	$\int \mathrm{Tr} P_i$, $\int \mathrm{Tr} E_i$, $\int \mathrm{Tr} F_{ij}$, rank
5	$g_{\mu i}, b_{\mu i}, b^C_{\mu i}, A_{\mu ijk}$ $b^D_{\mu ijklp}$ $b^{NS}_{\mu ijklp}$	$\int \mathrm{Tr} P_i$,$\int \mathrm{Tr} E_i$, $\int \mathrm{Tr} F_{ij}F_{kl}$, $\int \mathrm{Tr} F_{ij}$ rank

Table 2.1

The coset spaces G/H ($G \equiv U, H \equiv R$, in our case) can be described by choosing a representative in each equivalence class. If ϕ denotes the coordinates of a point in G/H, then the coset representative will be given by an element $L(\phi) \in G$, in the equivalence class correspondig to ϕ, that is, $L(\phi)$ is a local section in the principal bundle G over G/H with structure group H . Under the action of $g \in G$ on G/H we have $\phi \mapsto \phi_g$ and the coset representative $L(\phi)$ will be mapped to $gL(\phi)$ which is on the fiber over ϕ_g, so necessarily $L(\phi_g) = gL(\phi)h(\phi)$. Taking a representation of the group G, (which is also a representation of H) we obtain $L(\phi)$ as a matrix $L^\Lambda_a(\phi)$ where the indexes a and Λ run in principle over the same representation space, the different names being used to remind the covariant properties of L. If the representation of G is reducible under H, one can project down a the subspace where the representation of H is irreducible, so the index a will be understood as running on that subspace. We will use the representations appropriated to 0-brane multiplets.

The central charge is given by

$$Z_a(\phi, q) = (q^T L)_a = (q^T)_\Lambda L^\Lambda_a(\phi) \tag{2.9}$$

q is a vector transforming under the contravariant representation of G,

$$q^g = (g^{-1})^T q \tag{2.10}$$

so the central charge is U-duality covariant in the sense that under a transformation

$$\phi \mapsto \phi_g, \quad (q^{gT})_\Lambda = (q^T)_\Sigma (g^{-1})^\Sigma_\Lambda, \tag{2.11}$$

then $Z \mapsto Zh$. It then follows that any H-invariant function $I(Z)$ is also U-duality invariant in the sense that

$$I(\phi_g, q^g) = I(\phi, q), \quad \text{or} \quad I(Zh) = I(Z) \tag{2.12}$$

Among the duality invariant combinations there are some which are "topological",i. e., they do not depend on the moduli [28, 30]. This happens when the H-invariant is also G-invariant with respect to the right action of G. In fact, since $Z = q^T L$, with $L \in G$, it is obvious that if $I(Z)$ is G-invariant for $Z \mapsto Zg$, then

$$I(Zg) = I(Z) = I(q) \tag{2.13}$$

Such objects exist for $d = 5, 6$ [17, 31], but not for $d = 1, \ldots 4$, with the implication that a Bekenstein-Hawking entropy formula for 0-branes exist only in 4 and 5 dimensions [32].

We can make a generalization of this analysis to obtain other moduli invariant conditions. Let us consider now a covariant expression,

$$E_\alpha(Z) = E_\alpha(\phi, q), \tag{2.14}$$

where the index α runs over the space of some representation T of G (and H). The covariance property means that under a left G transformation

$$E_\alpha(Zg) = E_\beta(Z) T(g)^\beta_\alpha \tag{2.15}$$

It follows that an equation of the form $E_\alpha(Z) = E_\alpha(\phi, q) = 0$ is moduli independent, so $E_\alpha(q) = 0$.

Now, assume that the representation T admits an H-invariant norm (which is positive since H is compact).

$$\|E_\alpha(Z)\|^2 = g^{\alpha\beta} E_\alpha E_\beta. \tag{2.16}$$

An equation like

$$\|E_\alpha(Z)\|^2 = 0 \tag{2.17}$$

is in principle moduli dependent since this expression is not G-invariant. But the constraint $\|E_\alpha(Z)\| = 0$ implies that $E_\alpha(Z) = 0$, which is moduli independent so $E_\alpha(q) = 0$.

The central charges satisfy some differential identities that are inherited from the coset representatives. To see this, let us consider the algebra valued Maurer-Cartan form in G, usually expressed for a matrix group as

$$\alpha_{MC} = g(x)^{-1} dg(x) \tag{2.18}$$

where x denotes coordinates on the group G. The components of the Maurer-Cartan form are left invariant forms, and one can take the pullback to G/H by the local section $L(\phi)$, giving a local, algebra valued left invariant form on G/H

$$\Omega(\phi) = L^{-1}(\phi) dL(\phi). \tag{2.19}$$

Consider now the Cartan decomposition of the Lie algebra of G as $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$, where $\mathbf{h}$ is the Lie algebra of H, the maximal compact subgroup of G, and $\mathbf{k}$ can be identified with the tangent space at the identity coset. Since our coset spaces are symmetric spaces, the following properties are satisfied,

$$[\mathbf{h}, \mathbf{h}] \subset \mathbf{h}, \quad [\mathbf{h}, \mathbf{k}] \subset \mathbf{k}, \quad [\mathbf{k}, \mathbf{k}] \subset \mathbf{h}. \tag{2.20}$$

The second equation in (2.20) means that by the adjoint, $\mathbf{h}$ acts on $\mathbf{k}$ as a representation R of dimension equal to $\dim(G/H)$. We can write Ω according to this decomposition of the algebra,

$$\Omega = \omega^i T_i + P^\alpha T_\alpha \tag{2.21}$$

where $\{T_i\}$ form a basis of $\mathbf{h}$ and $\{T_\alpha\}$ form a basis of $\mathbf{k}$. The projection of Ω on $\mathbf{h}$, $\omega^i T_i$ is a G invariant connection on the bundle with fiber $\mathbf{k}$ and basis G/H, associated to the principal bundle $G(G/H)$ by the representation R of H. We call this bundle $E(G/H)$. The connection is expressed in an open set as

$$(\omega)^\alpha_\beta = \omega^i C^\alpha_{i\beta} \tag{2.22}$$

The other components of Ω, P^α, provide us with the local expresion of a homomorphism between $E(G/H)$ and the tangent bundle $T(G/H)$. By means of this homomorphism we can pull back the invariant Riemannian connection on G/H which coincides with ω. Finally, the invariant metric on G/H, induced by the Cartan-Killing metric on G can be locally represented as

$$g_{\mu\nu} = \mathrm{Tr}(T_\alpha T_\beta) P^\alpha_\mu P^\beta_\nu. \tag{2.23}$$

(The indices (μ, ν) are 1-form indices on G/H).

We want now to write (2.19) using a representation of G (and H) labeled as we explained above indistinctely by indices of the type Λ or a, then we have

$$dL^\Lambda_a = L^\Lambda_b \omega^b_a + L^\Lambda_b P^b_a. \tag{2.24}$$

By defining as usual the covariant derivative with respect to H

$$\nabla_H L^\Lambda_a = dL^\Lambda_a - L^\Lambda_b \omega^b_a, \tag{2.25}$$

we obtain

$$\nabla_H L^\Lambda_a = L^\Lambda_b P^b_a. \tag{2.26}$$

Suppose now that the representation of H is reducible and we want to project onto an irreducible factor. Since ω^b_a is block diagonal, the indices of type a in $\nabla_H L^\Lambda_a$ can be understood as running on the smaller representation, while P^b_a will have in general off diagonal components, so we could denote it by $P^{b'}_a$, b' running on the large representation space, but still specifying the covariant properties under H. This happens when matter fields are present. In that case the H group is a direct product $H_R \times H_M$. H_R is the R-symmetry group and H_M is some matter flavour symmetry. We assume now (as it will happen in all our examples) that the representation of G decomposes under H as $(\mathbf{1}, \mathbf{T}_M)$ $+(\mathbf{T}_R, \mathbf{1})$, where $\mathbf{T}_M$ is a representation of H_M and $\mathbf{T}_R$ is a representation of H_R. Then, there is a basis where the generic index Λ splits into (a, I), where a runs over the vector space representation of $\mathbf{T}_R$ and I runs over the representation space of $\mathbf{T}_M$. Then (2.26) decomposes as

$$\begin{aligned} \nabla_{H_R} L^\Lambda_a &= L^\Lambda_b P^b_a + L^\Lambda_I P^I_a \\ \nabla_{H_M} L^\Lambda_I &= L^\Lambda_a P^a_I + L^\Lambda_J P^I_J, \end{aligned} \tag{2.27}$$

where

$$\begin{aligned}\nabla_{H_R} L_a^\Lambda &= dL_a^\Lambda - L_b^\Lambda \omega_a^b \\ \nabla_{H_M} L_I^\Lambda &= dL_I^\Lambda - L_J^\Lambda \omega_I^J .\end{aligned} \tag{2.28}$$

The central charges $Z_a = (q^T L)_a$ and matter charges $Z_I = (q^T L)_I$ satisfy consequently the identities

$$\begin{aligned}\nabla_{H_R} Z_a &= Z_b P_a^b + Z_I P_a^I \\ \nabla_{H_M} Z_I &= Z_a P_I^a + Z_J P_I^J .\end{aligned} \tag{2.29}$$

In the forthcoming sections we will see that these properties enter in the discussion of the BPS conditions and their duality invariant character.

2.2 Orbit classification of BPS states.

In order to further study properties of central charges which will be useful in the following section, we would like to remind the orbit classification of 0-brane BPS configurations [20, 21]. To do so we will state some results of matrix algebra that will be useful for our analysis. We consider matrices over the real, complex and quaternion fields, M^r, M^c, M^q. For each of these matrices, the following polar decomposition holds

$$\begin{aligned}M^r &= \sqrt{M^r M^{rT}} O \\ M^c &= \sqrt{M^c M^{c\dagger}} U \\ M^q &= \sqrt{M^q M^{q\dagger}} U_q\end{aligned} \tag{2.30}$$

where the matrices $\sqrt{MM^\dagger}$ are hermitian and O, U and U_q are orthogonal, unitary and quaternionic unitary (unitary symplectic), respectively.

From this decomposition it then follows that if M^r is symmetric, M^c hermitian and M^q symplectic hermitian, they can be diagonalized by an appropriate transformation which is respectively orthogonal, unitary and unitary symplectic. Instead, for general matrices we can bring them to a diagonal form in the following way,

$$M_D = U_1 M U_2^\dagger \tag{2.31}$$

where U_1 and U_2 belong to O(n), U(n) or USp(n) in each case.

It can also be shown that any antisymmetric matrix can be brought to a skew-diagonal form (normal form) by a transformation [33]

$$M_{SD} = U M U^T \tag{2.32}$$

where as before, U belongs to the appropriate group. .

Since the central charge vector is a 2-tensor representation of H, we can always apply one of the above results.

From the structure of the R-symmetry group listed above and the representation properties from (2.7) we will see that it follows that with an R-rotation we can always diagonalize (or skew-diagonalize) the matrix Z. For $d = 1, \dots 4$ there will be only two eigenvalues, three eigenvalues for for $d = 5$ and four eigenvalues for $d = 6$.

The richer structure occurring for $d = 5, 6$ is the why also 1/8 BPS states occur, instead of the two possibilities of $d = 1, \dots 4$.

In the following table we list the orbits of the representations in Eq.(2.1) corresponding to the 0-brane BPS configurations [20, 21].

Orbits	**1/2 BPS**	**1/4 BPS**	**1/8 BPS**
$d = 1$	Sl(2) or $\mathbf{R}$	Sl(2)$\times\mathbf{R}$	
$d = 2$	Sl(3) $\times$ Sl(2)/Gl(2) $\propto \mathbf{R}^3$	Sl(3) $\times$ Sl(2)/Sl(2) $\propto \mathbf{R}^2$	
$d = 3$	Sl(5)/(Sl(3) $\times$ Sl(2)) $\propto \mathbf{R}^6$	Sl(5)/O(2,3) $\propto \mathbf{R}^4$	
$d = 4$	O(5,5)/Sl(5) $\propto \mathbf{R}^{10}$	O(5,5)/O(3,4) $\propto \mathbf{R}^8$	
$d = 5$	$\mathrm{E}_{6(6)}$/O(5,5) $\propto \mathbf{R}^{16}$	$\mathrm{E}_{6(6)}$/O(4,5) $\propto \mathbf{R}^{16}$	$\mathrm{E}_{6(6)}/\mathrm{F}_{4(4)}$
$d = 6$	$\mathrm{E}_{7(7)}/\mathrm{E}_{6(6)} \propto \mathbf{R}^{27}$	$\mathrm{E}_{7(7)}$/(O(5,6) $\propto \mathbf{R}^{32}$) $\times$ $\mathbf{R}$	$\mathrm{E}_{7(7)}/\mathrm{F}_{4(4)} \propto \mathbf{R}^{26}$, $\mathrm{E}_{7(7)}/\mathrm{E}_{6(2)}$

Table 2.2

These orbits correspond, for $d = 1, \dots 4$ to the possibility of having two coinciding eigenvalues (1/2 BPS) or not (1/4 BPS) for the central charge matrix. For $d = 5, 6$, 1/2 BPS correspond to 3 and 4 coinciding eigenvalues respectively, 1/4 BPS orbits correspond to 2 equal eigenvalues and 2 pairs of equal eigenvalues respectively and 1/8 BPS orbits correspond to all different eigenvalues. In $d = 6$ there are there two kinds of 1/8 BPS orbits depending whether the quartic invariant vanishes or not (light-like or time-like orbit)[19, 20].

In the following section we will see that, in spite of the fact that such statements look moduli dependent, they are actually moduli independent and therefore U-duality invariant, as expected from physical considerations.

3 BPS spectrum for 0-branes in type II string theory compactified on $\mathbf{T}^d$, $d = 1, \dots 4$

In the present section we consider the BPS spectrum and the central charge matrix for 0-branes in the cases when only 1/2 and 1/4 BPS states exist. These are the cases where the central charge has only two eigenvalues, as it happens for $d = 1, \dots 4$.

Let us consider the relevant anticommutators, containing the scalar central charge,

$$\begin{aligned}
&d = 1 \qquad \{Q_i, Q_j\} &&= Z_{ij} \quad (i, j = 1, 2);\ Z_{ij} \text{ real symmetric} \\
&d = 2 \qquad \{Q_A, Q_B\} &&= Z_I \sigma^I_{AB} = Z_{AB} \quad (A, B = 1, 2); \\
& && \quad Z^I \text{ complex}, \quad Z_{AB} \text{ symmetric}, \\
&d = 3 \qquad \{Q_a, Q_b\} &&= Z_{IJ} (\gamma^{IJ})_{ab} = Z_{ab} \quad (a, b = 1, \dots 4); \\
& && \quad Z_{ab} \text{ symmetric and symplectic} \\
&d = 4 \qquad \{Q_a, Q'_b\} &&= Z_{ab'} \quad (a, b' = 1, \dots 4);\ Z_{ab'} \text{ symplectic}
\end{aligned} \tag{3.33}$$

where σ^I are the Pauli matrices and $\gamma^{IJ} = 1/2[\gamma^I, \gamma^J]$ (γ^I are the O(5) gamma matrices). We say that Z_{ab} is a symplectic (or quaternionic) matrix if:

$$\bar{Z} = -\Omega Z \Omega \tag{3.34}$$

where Ω is the bilinear form invariant under USp(4) which satisfies

$$\Omega = \bar{\Omega} = -\Omega^T = -\Omega^{-1} \tag{3.35}$$

The indices a, b in the gamma matrices are raised and lowered with Ω.

Cases $d = 1, 2$**.** In these cases the chare matrix Z is 2×2 and has two independent eigenvalues. The 1/2 BPS conditions correspond to these eigenvalues equal in magnitude. We consider separately both cases.

• For $d = 1$, Z is real and symmetric. We can decompose it as

$$Z_{ij} = Y\delta_{ij} + Z^\alpha (T_\alpha)_{ij} \tag{3.36}$$

where $\alpha = 1, 2$, $T_1 = \sigma_1$ and $T_2 = \sigma_3$ (the Pauli matrices). The characteristic equation for Z is

$$\lambda^2 - \mathrm{Tr}Z\,\lambda + \det Z = 0 \tag{3.37}$$

where

$$\mathrm{Tr}Z = 2Y \tag{3.38}$$

$$\det Z = Y^2 - Z^\alpha Z_\alpha \tag{3.39}$$

It then follows that the two solutions of (3.37) satisfy $|\lambda_1| = |\lambda_2|$ if either

$$\mathrm{Tr}Z = 0 \tag{3.40}$$

or

$$\det Z = 1/4(\mathrm{Tr}Z)^2 \tag{3.41}$$

then implying

$$YZ^\alpha Z_\alpha = 0. \tag{3.42}$$

It is obvious that condition (3.42) is only O(2) or R-invariant, but the unique solution $YZ_\alpha = 0$, is an SL(2)×O(1,1) or U-invariant. This is the first example of a condition of the type (2.17). So we retrieve the result of Ref.[19] for $d = 1$, 1/2 BPS states

$$q = 0 \quad \text{or} \quad q_\alpha = 0 \tag{3.43}$$

• In the $d = 2$ case the hermiticity condition is lacking, but still the matrix $Z^I\sigma_I$ can be diagonalized with real eigenvalues using (2.31), where the difference between U_1 and U_2 is simply a phase. We just use a transformation of the R-symmetry group U(2), with SU(2) acting on the σ-matrices and U(1) acting as a phase on Z^I.

In fact, note that

$$Z_I Z^I = (A_I + iB_I)(A^I + iB^I) = A_I A^I - B_I B^I + 2iA_I B^I \tag{3.44}$$

Therefore, with a U(1) transformation we bring $A_I B^I$ to zero, which means that $\vec{A}$ and $\vec{B}$ are orthogonal vectors, so by an orthogonal transformation we can bring them to coincide with the axes, and only two real numbers (related to the two eigenvalues) are left.

We proceed by diagonalizing the hermitian matrix $ZZ^\dagger$. The square root of the eigenvalues will be the eigenvalues of Z, that we denote by λ_1, λ_2. (In this way we include both cases, when the eigenvalues are equal and when they are opposite in sign).We have :

$$\mathrm{Tr} ZZ^\dagger = \lambda_1^2 + \lambda_2^2, \quad \mathrm{Tr}(ZZ^\dagger)^2 = \lambda_1^4 + \lambda_2^4, \quad \det ZZ^\dagger = \lambda_1^2 \lambda_2^2 \tag{3.45}$$

where λ_i are the eigenvalues obtained as in (2.31). From the characteristic equation for $ZZ^\dagger$ we have

$$\lambda_{1,2}^2 = \frac{1}{2}[\mathrm{Tr} ZZ^\dagger \pm \sqrt{2\mathrm{Tr}(ZZ^\dagger)^2 - (\mathrm{Tr} ZZ^\dagger)^2}] \tag{3.46}$$

Using now the properties of the σ matrices,

$$ZZ^\dagger = Z^I \sigma_I \bar{Z}^J \sigma_J = Z^I \bar{Z}_I \mathbf{I} + i\epsilon_{IJK} Z^I \bar{Z}^J \sigma^K \tag{3.47}$$

We denote $\hat{Z}_K = i\epsilon_{IJK} Z^I \bar{Z}^J$. Then we have

$$\mathrm{Tr} ZZ^\dagger = 2Z^I \bar{Z}_I, \quad \mathrm{Tr}(ZZ^\dagger)^2 = 2[(Z^I \bar{Z}_I)^2 + \hat{Z}_I \hat{Z}_I], \tag{3.48}$$

Hence, the discriminant in (3.46) is given by

$$\mathrm{Tr}(ZZ^\dagger)^2 - \frac{1}{2}(\mathrm{Tr} ZZ^\dagger)^2 = 2\hat{Z}^I \hat{Z}_I. \tag{3.49}$$

We set $Z_1^I = A^I$, $Z_2^I = B^I$. The 1/2 BPS condition $\hat{Z}_I \hat{Z}^I = 0$, can be written as

$$\|\epsilon^{\alpha\beta} Z_\alpha^I Z_\beta^J \epsilon_{KIJ}\| = 0 \Rightarrow \epsilon^{\alpha\beta} Z_\alpha^I Z_\beta^J \epsilon_{KIJ} = 0. \tag{3.50}$$

where $\| \ \|$ is the O(3)$\times$O(2) invariant norm. As before, the condition obtained is actually invariant under SL(3)$\times$SL(2), so we obtain the moduli independent condition of Ref.[19]:

$$\epsilon^{\alpha\beta} q_\alpha^I q_\beta^J \epsilon_{KIJ} = 0. \tag{3.51}$$

Cases $d = 3, 4$. In these cases the matrix Z is 4-dimensional, but because of its symplectic property (3.34) there are only two independent eigenvalues (two pairs of equal eigenvalues), $\lambda_{1,2}$. Indeed, we find

$$\begin{aligned} \mathrm{Tr} ZZ^\dagger &= 2(\lambda_1^2 + \lambda_2^2) \\ \mathrm{Tr}(ZZ^\dagger)^2 &= 2(\lambda_1^4 + \lambda_2^4). \end{aligned} \tag{3.52}$$

The characteristic equation (or better, its square root) is

$$\lambda^2 - \frac{1}{2}\mathrm{Tr} ZZ^\dagger \lambda + (\det ZZ^\dagger)^{1/2} = 0, \tag{3.53}$$

with

$$(\det ZZ^\dagger)^{1/2} = \frac{1}{8}(\mathrm{Tr} ZZ^\dagger)^2 - \frac{1}{4}\mathrm{Tr}(ZZ^\dagger)^2. \tag{3.54}$$

The roots are

$$\lambda_{1,2}^2 = \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}ZZ^\dagger \pm \sqrt{\mathrm{Tr}(ZZ^\dagger)^2 - \frac{1}{4}(\mathrm{Tr}ZZ^\dagger)^2}\right) \tag{3.55}$$

We consider now the two cases separately,

• In the $d = 3$ case we can switch from Sp(4) to O(5) indices by setting:

$$Z_{ab} = Z_{IJ}(\gamma^{IJ})_{ab}, \quad (I, J = 1, \ldots 5) \tag{3.56}$$

where Z_{IJ} is real and antisymmetric and

$$\gamma^{IJ} = \frac{1}{2}[\gamma^I, \gamma^J]. \tag{3.57}$$

It is clear that Z_{IJ} can be skew-diagonalized with an O(5) transformation, so Z_{ab} can be diagonalized with a USp(4) transformation. From the relation

$$\frac{1}{2}\{\gamma^{IJ}, \gamma^{KL}\} = (g^{IJ}g^{KL} - g^{JK}g^{IL} + \epsilon^{IJKLP}\gamma_P) \tag{3.58}$$

it follows that

$$ZZ^\dagger = Z^2\mathbf{I} + Z^P\gamma_P, \tag{3.59}$$

where

$$Z^2 = 2Z^{PQ}Z_{PQ}, \quad Z^P = \epsilon^{PIJKL}Z_{IJ}Z_{KL}. \tag{3.60}$$

From this, we have

$$\begin{aligned} \mathrm{Tr}ZZ^\dagger &= 4Z^2 \\ \mathrm{Tr}(ZZ^\dagger)^2 &= 4Z^4 + 4Z^PZ_P. \end{aligned} \tag{3.61}$$

The 1/2 BPS condition becomes:

$$\mathrm{Tr}(ZZ^\dagger)^2 - \frac{1}{4}(\mathrm{Tr}ZZ^\dagger)^2 = 4Z^PZ_P = 0 \tag{3.62}$$

which is o(5) invariant and implies

$$Z^P = \epsilon^{PIJKL}Z_{IJ}Z_{KL} = 0 \tag{3.63}$$

Equation(3.63) is SL(5) invariant when Z^{IJ} is in the 10-dimensional representation of SL(5), and therefore it is moduli independent, giving the result of Ref. [19],

$$\epsilon^{PIJKL}q_{IJ}q_{KL} = 0 \tag{3.64}$$

• The $d = 4$ case was already discussed in Ref.[5, 7], but we outline it here for completeness. In this case, the matrix $Z_{ab'}$ is a general O(5) bispinor. However, since its square is hermitian it decomposes as

$$\begin{aligned} ZZ^\dagger &= Z^2\mathbf{I} + Z_{(l)}{}^P\gamma_P, \quad p = 1, \ldots 5 \\ Z^\dagger Z &= Z^2\mathbf{I} + Z_{(r)}{}^P\gamma_P \quad \text{with} \quad Z_{(l)}{}^PZ_{(l)P} = Z_{(r)}{}^PZ_{(r)P}. \end{aligned} \tag{3.65}$$

where the subindices l, r refer to the two $O(5)$ factors of the $R-$symmetry group. It follows that

$$\mathrm{Tr} Z Z^\dagger = 4Z^2, \quad \mathrm{Tr}(ZZ^\dagger)^2 = 4Z^4 + 4{Z_{(l)}}^P Z_{(l)P}. \tag{3.66}$$

The 1/2 BPS condition is then :

$$\mathrm{Tr}(ZZ^\dagger)^2 - 1/4(\mathrm{Tr}ZZ^\dagger)^2 = 4{Z_{(l)}}^P Z_{(l)P} = 0. \tag{3.67}$$

The equations ${Z_{(l)}}^P Z_{(l)P} = {Z_{(r)}}^P Z_{(r)P} = 0$ imply that the O(5) vectors $Z^P_{(r)}$, $Z^P_{(l)}$ vanish. This is an O(5,5) invariant statement. $(Z, Z^\dagger)$ form the 16 dimensional (chiral spinor) representation of O(5,5) and the O(5,5) 10 dimensional (light-like vector) $(\mathrm{Tr}\gamma Z Z^\dagger, \mathrm{Tr}\gamma Z^\dagger Z)$ then vanishes when $|Z_{(l)}| = |Z_{(r)}| = 0$. We then retrieve the condition of Ref.[19] on the quantized charges in the spinor representation of O(5,5).

4 BPS spectrum for the $d{=}5, 6$ dimensional cases.

In this section we will examine the more interesting cases of $d = 5, 6$, corresponding to supergravity compactified down to $D = 4, 5$ dimensions respectively.

The different BPS states, preserving some fraction of supersymmetry, are classified by the orbits of $E_{6(6)}$ and $E_{7(7)}$ respectively as given in Table 2.2

To put this analogy in perspective it is useful to parametrize the set of BPS charges allowed by the duality constraints by their eigenvalues and some angular variables (which can be removed by an R-symmetry transformation [20]). The duality constraints which follow from the BPS conditions are precisely those constraints which do not depend on these extra angular variables and which can be removed by an H transformation in G. These constraints will give different orbits corresponding to different BPS conditions on the 0-brane charges.

Orbits of the BPS energy levels for $d = 5, 6$

	Orbit	dim.	eigenv.	angles
d=5				
1/2 BPS	$E_{6(6)}/O(5,5) \propto \mathbf{R}^{16}$	17	1	16=dim(USp(8)/O(5)×O(5))
1/4 BPS	$E_{6(6)}/O(4,5) \propto \mathbf{R}^{16}$	26	2	24=dim(USp(8)/O(4)×O(4))
1/8 BPS	$E_{6(6)}/F_{4(4)}$	26+1	3	24=dim(USp(8)/USp(2)^4)
d=6				
1/2 BPS	$E_{7(7)}/E_{6(6)} \propto \mathbf{R}^{27}$	28	1	27=dim(SU(8)/USp(8))
1/4 BPS	$E_{7(7)}/(O(5,6) \propto \mathbf{R}^{32}) \times \mathbf{R}$	45	2	43=dim(SU(8)/USp(4)^2)
1/8 BPS	$E_{7(7)}/F_{4(4)} \propto \mathbf{R}^{26}$	55	4	51=dim(SU(8)/USp(2)^4)
	$E_{7(7)}/E_{6(2)}$	55+1	5	51=dim(SU(8)/USp(2)^4)

Table 4.1

The different orbits of different BPS levels will correspond to different solutions of the characteristic equation of the central charge matrix (or its square). These different solutions will be characterized by invariant constraints which are moduli independent in spite of the fact that the eigenvalues of the matrix are moduli dependent. Becuse of this, the orbits are simply given by invariant constraints on the "quantized" charges, as found in Ref.[19].

Case $d = 6$. We consider the E_7 quartic invariant [22, 30, 31]

$$I = 4\text{Tr}(Z\bar{Z})^2 - (\text{Tr}Z\bar{Z})^2 + 2^4(PfZ + Pf\bar{Z}) \tag{4.68}$$

where

$$PfZ = \frac{1}{2^4 4!}\epsilon^{ABCDRPGH} Z_{AB}Z_{CD}Z_{RP}Z_{GH}. \tag{4.69}$$

We want to consider second derivatives of the above quartic invariant that could give us covariant equations. The antisymmetric matrix Z_{AB} is in the 28-dimensional representation of SU(8), while we can express symbolically $Z_{\mathbf{56}} = (Z_{AB}, \bar{Z}^{AB})$, in the 56-dimensional representation of E_7 ($\mathbf{56} = \mathbf{28} + \bar{\mathbf{28}}$). Taking the the second derivative

$$\frac{\partial^2 I}{\partial Z_{\mathbf{56}}\partial Z_{\mathbf{56}}}\Big|_{Adj_{E_7}}, \tag{4.70}$$

will give us a quadratic polynomial which is a symmetric tensor, in the $(\mathbf{56}\times\mathbf{56})_S = \mathbf{1596}$ representation of E_7 which is not irreducible and decomposes as $\mathbf{1463} + \mathbf{133}$. $\mathbf{133}$ is the Adj_{E_7}, so we can project on that space as indicated above (4.70). Since $\mathbf{133}$ decomposes as $\mathbf{63}+\mathbf{70}$ under SU(8), the expression (4.70) splits into the two following SU(8) covariant polynomials

$$\frac{\partial^2 I}{\partial Z_{AB}\partial\bar{Z}^{CB}}\Big|_{Adj_{SU(8)}} \approx (Z_{AB}\bar{Z}^{CB} - \frac{1}{8}\delta_A^C Z_{PQ}\bar{Z}^{PQ}) = V_A^C. \tag{4.71}$$

$$\frac{\partial^2 I}{\partial Z_{[AB}\partial Z_{CD]}} - \frac{1}{4!}\epsilon^{ABCDPQRS}\frac{\partial^2 I}{\partial\bar{Z}^{[AB}\partial\bar{Z}^{CD]}} = V^+_{[ABCD]}. \tag{4.72}$$

The 1/2 BPS condition is the E_7 invariant statement $V_A^C = 0$ and $V^+_{[ABCD]} = 0$. This is the constraint imposed in Ref.[19] on the quantized 56 electric and magnetic charges defining a 1/2 BPS configuration. The equation $V_A^C = 0$ implies that the matrix $ZZ^\dagger$ has four coinciding eigenvalues (that is, it is a multiple of the identity), while the equation $V^+_{[ABCD]} = 0$ implies that the eigenvalues of Z are real.

The vanishing of (4.72) follows from the vanishing of (4.71) and the differential relations (2.29) satisfied by Z_{AB} [28], which in this case take the form

$$\nabla_{SU(8)}Z_{AB} = \frac{1}{2}\epsilon_{ABCD}\bar{Z}^{CD}. \tag{4.73}$$

We now want to consider more general cases. The characteristic equation (or better, its square root) is given by

$$\sqrt{\det(ZZ^\dagger - \lambda\mathbf{I})} = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \tag{4.74}$$

where

$$\begin{aligned}
a &= -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \\
&= -\frac{1}{2}\text{Tr}ZZ^\dagger \\
b &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}[\frac{1}{2}(\mathrm{Tr}ZZ^\dagger)^2 - \mathrm{Tr}(ZZ^\dagger)^2] \\
c &= -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4) \\
&= -\frac{1}{6}\left(\frac{1}{8}(\mathrm{Tr}ZZ^\dagger)^3 + \mathrm{Tr}(ZZ^\dagger)^3 - \frac{3}{4}\mathrm{Tr}ZZ^\dagger\mathrm{Tr}(ZZ^\dagger)^2\right) \\
d &= \lambda_1\lambda_2\lambda_3\lambda_4 \\
&= \frac{1}{4}\left(\frac{1}{96}(\mathrm{Tr}ZZ^\dagger)^4 + \frac{1}{8}(\mathrm{Tr}(ZZ^\dagger)^2)^2 + \frac{1}{3}\mathrm{Tr}(ZZ^\dagger)^3\mathrm{Tr}ZZ^\dagger \right. \\
&\left. -\frac{1}{2}\mathrm{Tr}(ZZ^\dagger)^4 - \frac{1}{8}(\mathrm{Tr}ZZ^\dagger)^2\mathrm{Tr}(ZZ^\dagger)^2\right) \qquad (4.75)
\end{aligned}
$$

In the case of two pairs of equal roots we have

$$\prod_{i=1}^{4}(\lambda - \lambda_i) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2. \qquad (4.76)$$

This implies the following relations among the coefficients

$$
\begin{aligned}
c &= \frac{1}{2}a(b - \frac{1}{4}a^2) \\
d &= \frac{1}{4}(b - \frac{1}{4}a^2)^2 \qquad (4.77)
\end{aligned}
$$

which imply the following relations among the invariants,

$$
\begin{aligned}
&\frac{32}{3}\mathrm{Tr}(ZZ^\dagger)^3 = 4\mathrm{Tr}ZZ^\dagger\mathrm{Tr}(ZZ^\dagger)^2 - \frac{1}{3}(\mathrm{Tr}ZZ^\dagger)^3 \\
&(\det ZZ^\dagger)^{1/2} = \frac{1}{64}[\mathrm{Tr}(ZZ^\dagger)^2 - \frac{1}{4}(\mathrm{Tr}ZZ^\dagger)^2]^2. \qquad (4.78)
\end{aligned}
$$

The eigenvalues are given by the expression

$$\lambda_{1,2} = \frac{1}{8}\mathrm{Tr}ZZ^\dagger \pm \frac{1}{2}\sqrt{\frac{1}{2}\mathrm{Tr}(ZZ^\dagger)^2 - \frac{1}{16}(\mathrm{Tr}ZZ^\dagger)^2} \qquad (4.79)$$

being the BPS mass, m^2_{BPS} the highest eigenvalue (+ sign).

We want now to show how the 1/4 BPS condition follows from the E_7 invariance. Let us consider the E_7 covariant constraint

$$\frac{\partial I}{\partial Z_{AB}} = 0 \quad (\Rightarrow \frac{\partial I}{\partial \bar{Z}^{AB}} = 0). \qquad (4.80)$$

where I is the invariant from (4.68). From this, the following quartic SU(8) invariant equations follow,

$$
\begin{aligned}
\frac{\partial I}{\partial Z_{AB}}Z_{AB} + \frac{\partial I}{\partial \bar{Z}^{AB}}\bar{Z}^{AB} &= 4I = 0 \qquad (4.81) \\
\frac{\partial I}{\partial Z_{AB}}Z_{AB} - \frac{\partial I}{\partial \bar{Z}^{AB}}\bar{Z}^{AB} &= 0. \qquad (4.82)
\end{aligned}
$$

The second equation implies that the Pfaffian of Z is real, so

$$Pf Z = Pf Z^\dagger \qquad (4.83)$$

and therefore

$$(PfZ)^2 = (\mathrm{det}ZZ^\dagger)^{1/2}. \tag{4.84}$$

Plugging (4.84) into (4.81) and squaring, it gives $(\mathrm{det}ZZ^\dagger)^{1/2}$ as in (4.78).

In the same way one can show that the equation giving $\mathrm{Tr}(ZZ^\dagger)^3$ as in (4.78) is the SU(8) invariant equation

$$\frac{\partial I}{\partial Z_{AB}}\frac{\partial I}{\partial \bar{Z}^{AB}} = 0. \tag{4.85}$$

In the generic case the 1/8 BPS states will correspond to 4 different eigenvalues. They are explicitely given as follows. Define the quantities

$$\begin{aligned}
u &= b^2 + 12d - 3ca \\
v &= 2b^3 + 27c^2 - 72bd - 9abc + 27da^2 \\
w &= \left(\frac{v + \sqrt{v^2 - 4u^3}}{2}\right)^{1/3} \\
s &= \sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{u}{3w} + \frac{w}{3}}
\end{aligned} \tag{4.86}$$

Then,

$$\begin{aligned}
\lambda_{1,2} &= -\frac{a}{4} + \frac{s}{2} \pm \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{a^3 - 4ab + 8c}{4s} - \frac{u}{3w} - \frac{w}{3}} \\
\lambda_{3,4} &= -\frac{a}{4} - \frac{s}{2} \pm \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{a^3 - 4ab + 8c}{4s} - \frac{u}{3w} - \frac{w}{3}}.
\end{aligned} \tag{4.87}$$

The BPS mass, m^2_{BPS} is λ_1. This is the actual detemination of the energy spectrum for 1/8 BPS states in terms of the duality invariant quantities (4.75).

It is amusing that analytic expressions for the roots of a polynomial exist only up to quartic equations, as found by Galois [29], and this is precisely what is required by maximal supersymmetry ($N = 8$ at $D = 5, 6$).

Case $d = 5$. The central charge $\hat{Z}_{AB}$ is a symplectic, Ω-traceless antisymmetric matrix;

$$\bar{\hat{Z}} = -\Omega\hat{Z}\Omega, \quad \hat{Z}^T = -\hat{Z}, \quad \mathrm{Tr}\hat{Z}\Omega = 0. \tag{4.88}$$

This implies that the matrix

$$Z = \hat{Z}\Omega \tag{4.89}$$

is hermitian traceless. The characteristic equation for Z becomes

$$\sqrt{\mathrm{det}Z - \lambda\mathbf{I}} = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 + b\lambda^2 + c\lambda + d = 0 \tag{4.90}$$

where

$$\begin{aligned}
b &= -\frac{1}{4}\mathrm{Tr}Z^2 \\
c &= -\frac{1}{6}\mathrm{Tr}Z^3 \\
d &= \frac{1}{8}\left(\frac{1}{4}(\mathrm{Tr}Z^2)^2 - \mathrm{Tr}Z^4\right)
\end{aligned} \tag{4.91}$$

A 1/4 BPS state is a state for which $c = 0$. This is an E_6 invariant statement since $c = I_3$ is the E_6 cubic invariant. In this case we get

$$2\lambda_{1,2}^2 = \frac{1}{4}\mathrm{Tr}Z^2 \pm \sqrt{\frac{1}{2}\mathrm{Tr}Z^4 - \frac{1}{16}(\mathrm{Tr}Z^2)^2} \tag{4.92}$$

The discriminant is related to the modulus of the USp(8) (and $E_{6(6)}$) vector

$$V_B^A = \frac{\partial I}{\partial Z_A^B} \approx Z_A^C Z_C^B - \frac{1}{8} Z_D^C Z_C^D \delta_A^B \tag{4.93}$$

Indeed,

$$\mathrm{Tr}V^2 = \mathrm{Tr}Z^4 - \frac{1}{8}(\mathrm{Tr}Z^2)^2. \tag{4.94}$$

The condition for 1/2 BPS is that the discriminant vanishes. Therefore, this implies, by positivity, $V = 0$, which is an E_6 invariant statement

$$\frac{\partial I}{\partial Z_A^B} = 0 \tag{4.95}$$

We therefore have retrieved the results of Maldacena and one of the authors [19].

For the 1/8 BPS state the 4 roots are given by

$$\begin{aligned} \lambda_{1,2} &= \frac{s}{2} \pm \frac{1}{2}\sqrt{\frac{-4b}{3} - \frac{2c}{s} - \frac{u}{3w} - \frac{w}{3}} \\ \lambda_{3,4} &= -\frac{s}{2} \pm \frac{1}{2}\sqrt{\frac{-4b}{3} + \frac{2c}{s} - \frac{u}{3w} - \frac{w}{3}} \end{aligned} \tag{4.96}$$

where

$$\begin{aligned} u &= b^2 + 12d, \quad z = 2b^3 + 27c^2 - 72bd \\ w &= \left(\frac{z + \sqrt{z^2 - 4u^3}}{2}\right)^{1/3}, \quad s = \sqrt{\frac{w}{3} + \frac{u}{3w} - \frac{2b}{3}} \end{aligned} \tag{4.97}$$

The BPS mass is therefore given by the highest root, λ_1.

5 BPS conditions for theories with 16 supersymmetries

In this last section we will extend our analysis to theories with 16 supersymmetries. These theories are obtained in three different ways: by compactifying Heterotic string theory on T^d ($1 \leq d \leq 6$), from M theory compactified on K_3 ($D = 7$) and from Type IIA theory compactified on K_3 ($D = 6$).

In the theories where matter vector fields exist, the duality group G depends on the matter content and on the space-time dimension D. Its maximal compact subgroup is $H_R \times H_M$ where H_R is the R-symmetry and H_M is the group acting on the matter multiplets. In our case, H_M=O(n), where n is the number of matter multiplets. G is of the form O(10-D,n)×O(1,1) for $5 \leq D \leq 9$ while for $D = 4$ it is SL(2)×O(6,n). The R-symmetry groups are O(10-D) for $5 \leq D \leq 9$ and O(6)×O(2)≈SU(4)×U(1) for D=4. The last result can easily been understood from the geometric symmetry of Heterotic string on T^6, where G is enlarged by the electric-magnetic duality for 0-branes.

The G and H_R representations of the 0-branes are given in the following tables.

Central charge representation of H_R.

$$\begin{array}{lcl} d=1 & \mathbf{1} & \mathrm{O}(1)=\mathbf{I} \\ d=2 & \mathbf{1^c}\ \text{complex} & \mathrm{U}(1)\approx \mathrm{O}(2) \\ d=3 & \mathbf{3}\ \text{real} & \mathrm{SU}(2)\approx \mathrm{USp}(2) \\ d=4 & \mathbf{4}\ \text{real} & \mathrm{O}(4)\approx \mathrm{USp}(2)\times \mathrm{USp}(2) \\ d=5 & \mathbf{1}+\mathbf{5}\ \text{real} & \mathrm{O}(5)\approx \mathrm{USp}(4) \\ d=6 & \mathbf{6^c}\ \text{complex} & \mathrm{O}(6)\times \mathrm{O}(2)\approx \mathrm{SU}(4)\times \mathrm{U}(1) \end{array} \tag{5.98}$$

From the above table, and according our previous analysis, it follows that the central charge matrix Z_a has only one independent eigenvalue for $d = 1, \ldots 4$ and two independent eigenvalues for $d = 5, 6$. Therefore, for $d = 1, \ldots 4$ only 1/2 BPS states can occur while for $d = 5, 6$ both, 1/2 and 1/4 BPS states can occur.

0-brane representation of G

$$\begin{array}{rcl} d=1,\ldots 4 & \mathbf{d}+\mathbf{n}\ \text{real vector} & \mathrm{O}(d,n)\times \mathrm{O}(1,1) \\ d=5 & (\mathbf{1},\mathbf{2})+(\mathbf{5}{+}\mathbf{n},-\mathbf{1})\ \text{(singlet+vector)} & \mathrm{O}(5,n)\times \mathrm{O}(1,1) \\ d=6 & (\mathbf{2},\mathbf{6}{+}\mathbf{n}) & \mathrm{Sl}(2)\times \mathrm{SO}(6,n) \end{array} \tag{5.99}$$

We consider now separately the two cases $d = 5, 6$.

Case $d = 5$. This is the case which corresponds to heterotic string on T^5 or M theory (Type IIA, Type IIB) on K$_3$×T^2 (K$_3$×S^1). In such compactifications, $n = 21$, so G=O(5,21)× O(1,1) but our analysis is independent of this specific number n.

The central charge $\hat{Z}$ is an antisymmetric symplectic matrix. The hermitian matrix, $Z = \hat{Z}\Omega$ decomposes as

$$Z = Z^a \gamma_a + Z^0 \mathbf{I} \tag{5.100}$$

where γ_a are the O(5) γ-matrices and Z^a, Z^0 are real. It follows that

$$\begin{aligned} &\mathrm{Tr} Z = 4Z^0 \\ &(\mathrm{detZ})^{1/2} = Z^{0^2} - \vec{Z}^2 = \frac{1}{8}(\mathrm{Tr} Z)^2 - \frac{1}{4}\mathrm{Tr} Z^2 \end{aligned} \tag{5.101}$$

The characteristic equation (or better, its square root) is

$$\lambda^2 - \frac{1}{2}\mathrm{Tr} Z\,\lambda + (\det Z)^{1/2} = 0 \tag{5.102}$$

implying that Z has two coinciding eigenvalues (in absolute value) either if

$$\mathrm{Tr}Z = 0 \quad \text{or} \quad \frac{1}{4}(\mathrm{Tr}Z)^2 = 4(\det Z)^{1/2} \tag{5.103}$$

Using (5.101), the above equation directly implies

$$Z_0 Z_a = 0. \tag{5.104}$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}Z \pm \sqrt{\mathrm{Tr}Z^2 - \frac{1}{4}(\mathrm{Tr}Z)^2}\right), \tag{5.105}$$

being the plus sign the mass squared of the BPS state.

We discuss now the covariance of (5.104). Since $Z_0 = e^{2\sigma}m$ where $e^{2\sigma}$ parametrizes O(1,1) and m is the charge associated to Z_0, $Z_0 = 0$ implies $m = 0$ which is an O(5,n) singlet, so it is G-invariant.

According to table (5) we write the projection of the coset representative over the **(5+n, -1)** representation as $e^{-\sigma}L_a^\Lambda$ where σ parametrizes O(1,1) and L_a^Λ is the coset representative of O(5+n)/O(5)×O(n). If Z_I, $I = 1, \dots n$ are the matter charges associated to the n matter multiplets, we have that, because of (2.29)

$$\nabla_{O(5)} Z_a = \frac{1}{4}\mathrm{Tr}(\gamma_a P_I)Z^I - Z_a d\sigma \tag{5.106}$$

therefore $Z_a = 0$ implies $Z_I = 0$. This is also an O(5,n) invariant statement since, it comes by differentiating the quadratic invariant polynomial

$$I = \sum_{a=1}^{5} Z_a Z^a - \sum_{I=1}^{M} Z_I Z^I. \tag{5.107}$$

Therefore, $Z_a = Z_I = 0$ implies $q^\Lambda = 0$ where q^Λ, $\Lambda = 1, \dots 5+n$, is a fixed charge vector of O(5,M), as found in [19].

Case $d = 6$, ($D = 4$). We now consider theories with 16 supersymmetries in $D = 4$, as heterotic string compactified on T^6, TypeII on $K_3 \times T^2$ or M theory on $K_3 \times T^3$. The new phenomenon which occurs here is the electric-magnetic duality of 0-branes which are assigned to the $(2, 6+n)$ representation of SU(1,1)×O(6,n).

The central charge is a 4 dimensional complex matrix Z_{AB}, antisymmetric in the SU(4)≈O(6) indices. Therefore, the matrix $ZZ^\dagger$ has two independent eigenvalues, given by the characteristic equation

$$\begin{aligned} &\left(\det(ZZ^\dagger - \lambda\mathbf{I})\right)^{1/2} = 0 \\ &\lambda^2 - \frac{1}{2}\mathrm{Tr}ZZ^\dagger\,\lambda + (\det ZZ^\dagger)^{1/2} = 0 \end{aligned} \tag{5.108}$$

with solution

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{1}{2}\mathrm{Tr}ZZ^\dagger \pm \sqrt{\mathrm{Tr}(ZZ^\dagger))^2 - \frac{1}{4}(\mathrm{Tr}ZZ^\dagger))^2}\right). \tag{5.109}$$

A generic 1/4 BPS state has m^2_{BPS} equal to the eigenvalue with + sign above. The 1/2 BPS configuration corresponds to a vanishing discriminant, i.e. $\lambda_1 = \lambda_2$. We would like to show how this condition is SU(1,1)×O(6,n) invariant in the sense that it is moduli independent in spite of the fact that the discriminant is moduli dependent.

For this purpose we proceed like for the maximally supersymmetric case in $D = 4$. If the two eigenvalues of $ZZ^\dagger$ coincide, then the hermitian traceless matrix

$$V_A^C = Z_{AC}\bar{Z}^{BC} - \frac{1}{4}\delta_A^B Z_{PQ}\bar{Z}^{PQ} \tag{5.110}$$

vanishes. (The discriminant is just TrV^2, the invariant norm of the SU(4) vector V_A^C).

Consider now the SU(1,1)×O(6,n) quartic invariant ,

$$I = I_1^2 - I_2\bar{I}_2 \tag{5.111}$$

where

$$\begin{aligned} I_1 &= Z_{AB}\bar{Z}^{AB} - Z_I\bar{Z}^I \\ I_2 &= \frac{1}{4}\epsilon^{ABCD}Z_{AB}Z_{CD} - \bar{Z}_I\bar{Z}^I. \end{aligned} \tag{5.112}$$

The fact that I is an invariant was derived in Ref.[30] and can be easily understood from the fact that $(I_1, I_2, \bar{I}_2)$ is a triplet of SU(1,1)≈O(1,2), each of the entries being O(6,n) invariant.

The equation (5.110) can be seen as the second derivative of I projected onto the adjoint representation of SU(4).

$$V_A^C \approx \frac{\partial^2 I}{\partial Z_{AB}\partial\bar{Z}^{CD}}\Big|_{Adj_{SU(4)}} \tag{5.113}$$

Indeed, let us call U the $(\mathbf{2}, \mathbf{6}+\mathbf{n})$ of Sl(2) vector constructed with (Z_{AB}, Z_I) and its complex conjugate. The quantity

$$\frac{\partial^2 I}{\partial U\partial U} \tag{5.114}$$

is in the symmetric product $((\mathbf{2}, \mathbf{6}+\mathbf{n}) \times (\mathbf{2}, \mathbf{6}+\mathbf{n}))\,|_S$, which decomposes under O(6,M) as **(3, Sym)+(1,Adj**$_{O(6,n)}$**)= (3,1)+ (3,TrSym)+(1,Adj**$_{O(6,n)}$**)**, where **Sym** is the two fold symmetric representation, **TrSym** is the traceless symmetric representation. To show that the $V_A^C = 0$ is a G-invariant statement we use the fact that **Adj**$_{O(6,n)}$ decomposes under O(6)×O(n) as **Adj**$_{O(6,n)}\mapsto$ **(Adj**$_{O(6)}$**,1)+(1, Adj**$_{O(n)}$**) +(6,n)**. We will show that the vanishing of the projection onto **Adj**$_{O(6)}$ ≈**Adj**$_{SU(4)}$ of (5.114) implies the vanishing of the projection onto **(1, Adj**$_{O(n)}$**)** and **(6,n)**. In fact, differentiating $V_A^C = 0$ and using the differential identities (2.29) one also finds

$$Z_I\bar{Z}_J - \bar{Z}_I Z_J = 0, \tag{5.115}$$

$$Z_{AB}Z_J - \frac{1}{4}\epsilon_{ABCD}\bar{Z}^{CD}\bar{Z}_J = 0. \tag{5.116}$$

The vanishing of the three equations $V_A^C = 0$, (5.115) and (5.116) implies that the projection of (5.114) on **Ad**$_{O(6,n)}$ vanishes. This is a SU(1,1)×SO(6,n) invariant and therefore the moduli dependence drops out. These three equations can be rewritten in terms of the fixed charges (q_Λ, p_Λ), in the **(6+n)** of O(6,n) times the fundamental representation of Sl(2)≈SU(1,1)) as

$$T^{(A)}_{\Lambda\Sigma} = q_\Lambda p_\Sigma - p_\Lambda q_\Sigma = 0. \tag{5.117}$$

Note that in this basis the projection of (5.114) onto the representation **(3,Sym)** is

$$T^{(S)}_{\Lambda\Sigma} = (q_\Lambda q_\Sigma,\ p_\Lambda p_\Sigma,\ \frac{1}{2}(q_\Lambda p_\Sigma + p_\Lambda q_\Sigma)) \tag{5.118}$$

whose trace part is the Sl(2) triplet $(q^2.p^2, q \cdot p)$. It can be written as a matrix

$$T^{(0)} = \begin{pmatrix} q^2 & q \cdot p \\ q \cdot p & p^2 \end{pmatrix}. \tag{5.119}$$

The invariant I can be written either as $T^{(A)}_{\Lambda\Sigma} T^{(A)\Lambda\Sigma}$ or as $\det T^0$, and its square root is the entropy formula for 1/4 BPS 0-branes in theories with sixteen supersymmetries [17, 32].

As a final remark, let us comment on the orbits of the O(5,n) and O(6,n) vectors for BPS configurations discussed above.

For 1/2 BPS states at $d = 5$ we have $mq_\Lambda = 0$, so either m or q_Λ vanish. In the former case, the BPS condition requires q_Λ to be time-like or light-like ($q_\Lambda q^\Lambda \geq 0$) [19] so the orbit is either O(5,n)/O(4)×O(n) or O(5,n)/IO(4,$n-1$). If $m \neq 0$ then $q_\Lambda = 0$, so the orbit is a point since the little group is O(5,n) itself.

Let us consider now the $d = 6$ case. The BPS condition corresponds to the statement that the matrix $T^{(0)}$ is positive semidefinite. This implies

$$\det T^{(0)} = q^2 p^2 - (q \cdot p)^2 \geq 0, \quad \mathrm{Tr} T^{(0)} = q^2 + p^2 \geq 0. \tag{5.120}$$

From this it follows that $q^2 \geq 0$ and $p^2 \geq 0$.

$\det T^{(0)} = 0$ corresponds to 1/2 BPS states; this happens when $q = \lambda p$, $(\lambda \geq 0)$.

For $\det T^{(0)} > 0$, $q^2 > 0$ and $p^2 > 0$ and the generic 1/4 BPS configuration will depend on five parameters, since p, q, by an O(2) transformation in SL(2) can be made orthogonal ($q_\Lambda P^\Lambda = 0$). Indeed, the first vector can be put in the form $(p_1, 0, \cdots, 0, p_{n+1}, 0, \cdots, 0)$ and the second in the form $(q_1, q_2, 0, \cdots, 0, q_{n+1}, q_{n+2}, 0, \cdots, 0)$ by an O(6)×O(n) transformation. The orthogonality condition is used to eliminate one of the six parameters. The remaining $7 + 2n$ parameters are the "angles" in O(2)×O(6)×O(n)/O(4)×O(n-2). The little group in G of the two time-like vectors is O(4)×O(n).

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References

[1] J. Maldacena, *Adv. Theor. Math. Phys.* **2** (1997) 231.

[2] T. Banks, W. Fischler, S. Shenker and L. Susskind, *Phys. Rev.* **D55** (1997) 5112.

[3] D. Bigatti and L. Susskind, hep-th/9712072 (and references therein).

[4] N. A. Obers and B. Pioline hep-th/9809039 (and references therein).

[5] R. Dijkgraaf, E. Verlinde and H. Verlinde *Nucl. Phys.* **B 486** (1997) 89.

[6] S Elitzur, A. Giveon, D. Kutasov and E. Rabinovici *Nucl. Phys.* **B 509** (1998) 122, hep-th/9707217.

[7] C. Hofman, E. Verlinde and G. Zwart, hep-th/9808128.

[8] A. Connes, M. Douglas and A. Schwarz, *JHEP* 9802 (1998) 003..

[9] B. Morariu and B. Zumino, Proc. R. Arnowitt Fest: A symposium on Supersymmetry and Gravitation, College Station, 1998 (to be published by World Scientific).

[10] D. Brace and B. Morariu, hep-th/9810185.

[11] A. Konechny and A. Schwarz, hep-th/9811159 hep-th/9901077.

[12] M. Douglas, hep-th/9901146.

[13] C. Hofman and E. Verlinde, *JEHP* 9812 (1994) 010.

[14] G.W. Gibbons, R. Kallosh and B. Kol, *Phys. Rev. Lett.* **77** (1996) 4992.

[15] S. Ferrara, G. W. Gibbons and R. Kallosh, *Nucl. Phys.* **B 500** (1997) 75.

[16] C. Hofman and E. Verlinde, *JHEP* 9812 (1999) 010, and hep-th/9810219.

[17] S. Ferrara, R. Kallosh and A. Strominger, *Phys. Rev.* **D52** (1995) 5412;
A. Strominger. *Phys. Lett. B* **383** (1996) 39;
S. Ferrara and R. Kallosh, *Phys. Rev.* **D54** (1996) 1514;
S. Ferrara and R. Kallosh, *Phys. Rev.* **D54** (1996) 1525.

[18] M. Billó, S. Cacciatori, F. Denef, P. Fré, A. Van Proeyen and D. Zanon, hep-th/9902100.

[19] S. Ferrara and J. Maldacena, (19) *Class. Quant. Grav.* **15** (1998) 749.

[20] S. Ferrara and M. Gunaydin *Int. Jour. Mod. Phys.* **A13** (1998) 2075.

[21] H. Lu, K. S. Stelle and C. N. Pope, *Class. Quant. Grav.* **15** 537.

[22] E. Cremmer, in *Supergravity '81*, eds. S. Ferrara and J. G. Taylor, p. 313;
B. Julia in *Superspace and Supergravity*, eds. S. W. Hawking and M. Rocek (Cambridge, 1981), p. 331.

[23] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B451** (1995) 525.

[24] E. Witten, *Nucl. Phys.* **B443** (1995) 85.

[25] L. Andrianopoli, R. D'Auria and S Ferrara, P. Fré, R. Minasian, M. Trigiante, *Nucl. Phys.* **B493** (1997) 249.

[26] P.K. Townsend, PKT Proc. 19th Johns Hopkins Workshop on Current Problems in Particle Thoery and 5th PASCOS Interdisciplinary Symposium, Baltimore, 1995, ed. J. Bagger (World Scientific, Singapore, 1996).

[27] L. Andrianopoli, R. D'Auria and S. Ferrara, *Int. Mod. Phys.* **A13** (1998) 431.

[28] L. Andrianopoli, R. D'Auria and S. Ferrara, *Int. Mod. Phys.* **A12** (1997) 3759.

[29] Harold M. Edwards, *Galois Theory.* Springer Verlag (1984).

[30] L. Andrianopoli, R. D'Auria and S. Ferrara, *Phys. Lett.* **B403** (1997) 12.

[31] R. Kallosh and B. Kol, *Phys. Rev.* **D53** (1996) 5344.

[32] M. Cvetic and C. M. Hull, *Nucl. Phys.* **B480** (1996) 296;
M.J. Duff, R.R. Khuri and J.X. Lu, *Phys. Rep.* **259** (1995) 213;
M. Cvetic and D. Youm, *Phys. Rev.* **D53** (1996) 584;
M. Cvetic and A. A. Tseytlin, *Phys. Rev.* **D53** (1996) 5619.

[33] Hua Loo Keng *Am. J. Math.* **66** (1944) 470;
B. Zumino *Math. Phys.* **3** (1962) 1056.