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# On the Thermodynamics of a Gas of AdS Black Holes and the Quark-Hadron Phase Transition

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## Abstract

We discuss the thermodynamics of a gas of black holes in five-dimensional anti-de-Sitter (AdS) space, showing that they are described by a van der Waals equation of state. Motivated by the Maldacena conjecture, we relate the energy density and pressure of this non-ideal AdS black-hole gas to those of four-dimensional gauge theory in the unconfined phase. We find that the energy density rises rapidly above the deconfinement transition temperature, whilst the pressure rises more slowly towards its asymptotic high-temperature value, in qualitative agreement with lattice simulations.

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# 1 Introduction

The striking conjecture of Maldacena [1] on the equivalence of large- $N_c$  superconformal quantum gauge theories on  $d$ -dimensional Minkowski space  $M_d$  - considered as the boundary of  $(d + 1)$ -dimensional anti-de Sitter space  $\text{AdS}_{d+1}$  - to classical supergravity in the AdS bulk, has opened a new dialogue between students of non-perturbative gauge theories and string theorists. Quantities in the strong-coupling limit of gauge theory may be calculable using classical correlators in AdS (super)gravity, and/or non-perturbative aspects of string theory may be related to correlators in gauge theories [2, 3], in a holographic spirit [4]. In particular, the AdS approach was used in [5] to relax the assumption of four-dimensional supersymmetry by starting from a supersymmetric theory in six dimensions, which was one of the cases for which the conjecture was thought to be valid, and compactifying appropriately two of the dimensions. The resulting compactification led to a high-temperature regime for the four-dimensional boundary theory, which had broken supersymmetry. In this way, confinement at low temperatures and deconfinement at high temperatures could be demonstrated. However, the gauge theory was still conformal, and asymptotic freedom was therefore not present. Nevertheless, this approach has motivated intriguing estimates of glueball masses [6], the quark-antiquark potential [7] and QCD vacuum condensates [8] that agree surprisingly well with lattice and other phenomenological estimates.

Two of us (J.E. and N.E.M.) have proposed [9] a generalization of this holographic approach to the  $\text{AdS}_{d+1}/M_d$  correspondence which is based on Liouville string theory [10, 11], in which conformal symmetry and supersymmetry need not be assumed, provided world-sheet defects [12] are taken into account properly in the Liouville-dressed theory. The Liouville field itself provides an extra bulk dimension, and the AdS structure is induced by the recoil of the world-sheet defect, when considered in interaction with a closed-string loop [13]. Within this approach, it was possible to demonstrate the formation of a condensate of world-sheet defects at low temperatures, which was related to the condensation of magnetic monopoles in target space and induced confinement [9]. It was also possible to demonstrate the logarithmic running of the gauge coupling strength. Although this Liouville approach is somewhat heuristic, it opens up a new way to discuss non-supersymmetric QCD in the strong-coupling regime and at finite temperature. In particular, the target-space quark-hadron deconfinement-confinement transition may be viewed as a Berezinskii-Kosterlitz-Thouless phase transition of world-sheet vortices [14, 12], which can also be related to the phase transition of black holes in AdS [15].

In this paper we embark on a heuristic attempt to use this approach to model aspects of quark-hadron phase transition. Lattice analyses [16] indicate that the free energy of pure QCD rises relatively rapidly above the critical temperature to approach the asymptotic ideal-gas value as predicted in perturbative QCD. On the other hand, the pressure is calculated [16] to rise much more slowly towards its asymptotic ideal-gas value, and one possible interpretation is that massive effective degrees of freedom are important close to the transition, causing a larger departure

from the ideal-gas picture for the pressure than is the case for the free energy. Calculations close to the phase transition necessarily require non-perturbative techniques, such as the lattice, and our hope is that the  $M_4/\text{AdS}_5$  correspondence may also prove useful in this region.

Specifically, we model aspects of the gluon plasma using a non-ideal gas of black holes in  $\text{AdS}_5$ , interacting via forces of van der Waals type, and described by an effective van der Waals equation of state that we derive in this paper.

In order to set this approach up in the most reliable way, and to relate it most closely to previous work [5], we first consider the high-temperature limit. Here the black holes in  $\text{AdS}_5$  are stable as well as massive, and hence suitable for interpretation as ‘molecules’ of a non-relativistic gas with small velocities  $|u_i| \ll 1$ . We demonstrate in this limit how the AdS structure of the ambient bulk space-time itself may be translated into non-ideal-gas interactions between the massive black-hole ‘gas molecules’. The question then arises how to extend this description down to lower temperatures.

According to the world-sheet point of view, target-space black holes may be viewed as ‘spike’ defects on the world sheet [12], which are dual to ‘vortex’ defects, that in turn correspond to  $D$  particles in space-time [17]. In the above-mentioned high-temperature limit, these  $D$  particles are light, and difficult to treat using the Liouville approach. However, the Liouville approach is well-adapted for a discussion of a dual limit, in which the  $D$  particles become very heavy [9].

It should be emphasized, though, that both of these limits correspond to temperatures that are high compared to those in the confining phase. In both QCD [18] and AdS gravity [15], three distinct transition temperatures have been identified:  $T_0$ , below which only the confined phase of the gauge theory exists and black holes condense leaving a residual gas of radiation;  $T_1$ , at which the free energies of the confined and deconfined phases (or black-hole and radiation phases) are equal; and  $T_2$ , beyond which only the deconfined and stable-black-hole phases exist. In the world-sheet picture, these correspond to the temperatures of Berezinskii-Kosterlitz-Thouless transitions for vortex and spike condensation [9]. The high-temperature limit [5] we take corresponds to  $T \sim T_2$  and above, and our lower-temperature limit corresponds to  $T \gtrsim T_0$ . We aim to establish a judicious interpolation between these two limits that describes qualitatively correctly the intermediate region  $T \sim T_1$ .

The layout of this paper is as follows. In section 2 we discuss in more detail general features of our approach to the thermodynamics of AdS black holes. Then, in section 3 we discuss the high-energy limit in which the  $D$  particles are light [5], and section 4 contains an application of Liouville string theory [9] to the lower-temperature limit. Finally, in sections 5 and 6 we pull together a general picture of the quark-hadron phase transition using the information we have obtained from our studies of the thermodynamics of AdS black holes, and relate it to lattice results [16].

## 2 Thermodynamics of a Gas of AdS Black Holes

We consider a homogeneous gas of AdS black holes, each of mass  $M$ , and restrict ourselves to the case where the characteristic velocity of a generic black hole in the ensemble is either zero or very small:  $|u_i| \ll 1$ , corresponding to the case of very massive black holes. We consider first the static case:  $u_i = 0$ . This will help in gaining insight into the  $u_i \neq 0$  case, which we treat later as a perturbation of the static case.

We consider an ensemble of  $N$  indistinguishable Schwarzschild black holes in a five-dimensional AdS space-time with radius  $b$ , which is related to the critical temperature  $T_1$  above which massive black holes are stable:  $T_1 = 1/(\pi b)$ . The invariant line element is taken to be (in Minkowskian signature) [15, 5]:

$$ds^2 = - \left( 1 + \frac{r^2}{b^2} - \frac{\omega_4 M}{r^2} \right) dt^2 + \left( 1 + \frac{r^2}{b^2} - \frac{\omega_4 M}{r^2} \right)^{-1} dr^2 + r^2 d\Omega_3 \quad (1)$$

where the AdS radius  $b$  is related to the negative cosmological constant  $\Lambda$  by  $b = \sqrt{-3/\Lambda}$ , and  $\omega_4 \equiv 8G_N/3\pi$ , where  $G_N$  is the five-dimensional Newton constant that is related to the Planck length  $\ell_P$  via  $G_N = \ell_P^3$ , and  $M$  is the ADM mass of the black hole. The outer horizon of the black hole is defined to be the larger positive root  $r_+$  of the equation

$$1 + \frac{r_+^2}{b^2} - \frac{\omega_4 M}{r_+^2} = 0, \quad (2)$$

namely

$$r_+ = b \left( -\frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\omega_4 M/b^2} \right)^{1/2}. \quad (3)$$

For the purposes of calculating the partition function and other thermodynamic quantities of the ensemble, a Wick rotation to a Euclidean AdS geometry:  $t \rightarrow it$ , will always be understood. The Euclideanized AdS-Schwarzschild space-time has been found to be smooth [15], provided the Euclidean time direction is periodic at a particular radius  $\beta_0$ :

$$\beta_0 \equiv T_H^{-1} = \frac{4\pi b^2 r_+}{4r_+^2 + 2b^2}. \quad (4)$$

where  $T_H$  is the Hawking temperature of the black hole.

In subsequent sections of this paper, we consider *two* limits of (4) that correspond to high Hawking temperatures  $T_H \gtrsim T_0$ , namely (i)  $b^2 \ll r_+^2$  and (ii)  $b^2 \gg r_+^2$ , where we assume in each case that  $\ell_P^2 \ll r_+^2, b^2$ . It is easy to check using (3) that in the limit (i) we also have  $\omega_4 M/r_+^2 \sim r_+^2/b^2$ , whereas in the limit (ii) we also have  $r_+^2 \simeq \omega_4 M \gg \ell_P^2$ , so that we are consistent with the static limit in both cases.

According to the analysis of [15], the thermodynamic ensemble of black holes is stable in the limit (i):  $\ell_P^2 \ll b^2 \ll r_+^2 \ll \omega_4 M$ , which was considered in [5]. This corresponds to the region  $T \sim T_2$ . The limit (ii):  $\ell_P^2 \ll r_+^2 \sim \omega_4 M \ll b^2$ , where the radius of the AdS space-time is large compared to the outer horizon, was studied perturbatively in [9], using the Liouville approach where  $b \sim \delta^{-2} \rightarrow \infty$ , with  $\delta \rightarrow 0^+$

a parameter appropriately defined to regulate the recoil operators. According to [15], this limit corresponds to a temperature  $T \gtrsim T_0$ . However, the intermediate regime of the phase transition where  $T \sim T_1$ , lying between the regions (i) and (ii), cannot be studied reliably by analytic methods, and we resort later to continuity arguments in order to describe the energy and pressure curves in this region.

### 3 Is there a Phase Transition?

To answer this question and to investigate its order, one should examine the equation of state for an ensemble of AdS black holes in appropriate regimes of the parameters. We consider first the limit (i) above.

Using standard General Relativity, the effective static potential  $U(r)$  between two black holes in the ensemble with a radial separation  $r$  is given by the temporal metric component  $g_{00}$ , that can be obtained from (1):

$$g_{00} = -1 - 2U(r), \quad U(r) = \frac{r^2}{2b^2} - \frac{\omega_4 M}{2r^2} \quad (5)$$

provided the potential is weak. We note that the potential (5) vanishes at a radius  $r_0$ :

$$U(r_0) = 0 \rightarrow r_0 = (\omega_4 M b^2)^{1/4}. \quad (6)$$

Using (3), we see that  $r_0 \simeq r_+ + \frac{b^2}{4r_+}$ . Therefore, if we restrict ourselves to a thin shell outside the horizon:  $r_+ \leq r \leq r_+ + \varepsilon$ , the potential is indeed small, and one can take  $\varepsilon = (r_0 - r_+) \simeq b^2/4r_+$  to a good approximation. Since the potential varies very little over the range  $r_+ \leq r \leq r_+ + \varepsilon$ , we make the second approximation  $U(r) \simeq \bar{U}$ , where  $\bar{U}$  is a constant, justified because  $\delta U \sim \varepsilon$ . The fact that the absolute value of the constant  $\bar{U}$  is a small number also justifies our weak-field approximation. These statements can be checked more precisely using the following formulae:

$$U(r) \simeq \bar{U} = \text{const} \equiv \kappa N / \Delta\Omega$$

$$\kappa = (1/N) \int_{r_+}^{r_+ + \varepsilon} d^4x U(r) \simeq \frac{\pi^2}{4N} \left[ \frac{\varepsilon r_+^5}{b^2} - \omega_4 M \varepsilon r_+ \right] \quad (7)$$

where the effective volume  $\Delta\Omega = \Omega_4 - \Omega_0$  is the size of the four-volume of the shell  $r_+ \leq r \leq r_+ + \varepsilon$  and  $N$  is the number of black holes in the ensemble. We denote by  $\Omega_0$  the four-volume inside the horizon. From the expression (7) for  $\kappa$ , one can easily check that  $\bar{U} < 1$ . We shall see that the above approximations lead to analytic expressions for thermodynamic quantities of our AdS black-hole system.

We first evaluate the classical partition function  $Z$  of the system, assuming that it is in thermal equilibrium at a Hawking temperature  $T \equiv \beta^{-1}$ :

$$Z = \frac{1}{N!} \left( \int \frac{d^4x d^4p}{(2\pi)^4} e^{-\beta[p^2/2M + MU(r)]} \right)^N$$

$$= \frac{1}{N!} \left( \frac{M}{2\pi\beta} \right)^{2N} \left( \int d^4x e^{-\beta U(r)M} \right)^N = \frac{1}{N!} \left( \frac{M}{2\pi\beta} \right)^{2N} \Delta\Omega^N e^{-\beta \bar{U} M N}. \quad (8)$$

Some remarks are in order at this point. First, in the static case which we are considering now, the kinetic term of the black holes has no physical meaning and serves only as a cut-off for the momentum integrals. Secondly, the spatial integral is understood to be taken over the prescribed shell  $r_+ \leq r \leq r_+ + \varepsilon$ , which give rise to the volume factor  $\Delta\Omega$ . Finally, we underline that we have made explicit use of the approximations mentioned above.

Before proceeding to compute other thermodynamic quantities, we also examine the low-velocity non-static case. As we shall see, the difference between the static and the non static cases can be described effectively by a renormalisation shift of the mass term  $M \rightarrow M + T$  in the partition function above (8). To prove this, we note that in the non-static case, where the heavy black holes move with a small velocity  $|u_i| \ll 1$  relative to each other, we may employ both the small-velocity and weak-potential approximations simultaneously. To obtain the velocity corrections to the potentials, we use the Minkowski-signature Lagrangian:

$$L = -M \frac{ds}{dt}, \quad ds^2 = -g_{AB} dx^A dx^B, \quad A, B = 0 \dots 4 \quad (9)$$

where  $ds^2$  is given by (1). We then write

$$\frac{ds}{dt} = \sqrt{-g_{00} - g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} = \sqrt{-g_{00} - g_{ij} u^i u^j}. \quad (10)$$

and set

$$g_{00} = -1 - 2U(r), \quad g_{ij} = \delta_{ij} + h_{ij}, \quad r^2 \equiv \sum_{i=1}^4 x_i^2. \quad (11)$$

from which we see that  $ds/dt = \sqrt{1 + 2U - |u_i|^2 - h_{ij} u^i u^j}$ . Expanding in powers of both  $U$  and  $u_i^2$  to leading non-trivial order, we find:

$$L \simeq -M \left( 1 + U - \frac{1}{2} |u_i|^2 - \frac{1}{2} u^i h_{ij} u^j + \frac{1}{2} U |u_i|^2 + \dots \right). \quad (12)$$

Parametrising with a generalized velocity-dependent potential  $\tilde{U}$  in the Lagrangian:  $L = -M + \frac{1}{2} M |u_i|^2 - M \tilde{U}(\bar{x}, u)$ , we get

$$\tilde{U}(\bar{x}, u) = U + \frac{1}{2} U |u_i|^2 - \frac{1}{2} u^i h_{ij} u^j \quad (13)$$

which clearly reduces to  $U$  in the static limit  $u_i \rightarrow 0$ .

The partition function for slowly-moving black holes can be computed in a straightforward manner. First, we note that the generalized momenta are given by

$$p_i = \frac{\partial L}{\partial u^i} = M \left( u_i - u_i U + h_{ij} u^j \right), \quad (14)$$

giving rise to the Hamiltonian:

$$H = p_i u^i - L = M(1 + U) + \frac{1}{2} p_i u^i. \quad (15)$$

Taking into account the facts that  $h_{ij} = 2Ux_ix_j/r^2$  and  $u_i \simeq \frac{p_i}{M}[\delta_{ij}(1+U) - h_{ij}]$ , we can re-express the Hamiltonian as:

$$H = \frac{1}{2}p_i\mathcal{K}_{ij}p_j + MU, \quad \mathcal{K}_{ij} = \frac{1}{M}[\delta_{ij}(1+U) - h_{ij}] \quad (16)$$

which resembles the Hamiltonian of a particle with momenta  $p_i$  in a ‘curved space’ whose ‘metric’  $\mathcal{K}_{ij}$  depends solely on the potential  $U$  and not on the velocity  $u_i$  (to this order). The resulting partition function is

$$\begin{aligned} Z &= \frac{1}{N!} \left( \int \frac{d^4x d^4p}{(2\pi)^4} e^{-\beta[\frac{1}{2}p_i\mathcal{K}_{ij}p_j + MU(r)]} \right)^N \\ &= \frac{1}{N!} \left( \frac{M}{2\pi\beta} \right)^{2N} \left( \int d^4x \frac{1}{\sqrt{\det(KM)}} e^{-\beta U(r)M} \right)^N \end{aligned} \quad (17)$$

where

$$\sqrt{\det(KM)} = [\text{Det}((1+U)\delta_{ij} - h_{ij})]^{1/2} \simeq \exp \left[ \frac{1}{2} \text{Tr}(U\delta_{ij} - h_{ij}) + \dots \right] \simeq e^U. \quad (18)$$

Thus, one has, upon approximating  $U \simeq \bar{U}$  (c.f. (7)):

$$Z \simeq \frac{1}{N!} \left( \frac{M}{2\pi\beta} \right)^{2N} \Delta\Omega^N e^{-N(\beta M + 1)\bar{U}} \quad (19)$$

which, when compared with (8), demonstrates the aforementioned renormalisation shift in the mass by  $T$ . In view of this simple change, from now on we shall deal with the general velocity-dependent case.

The energy of the system is defined as

$$\bar{E} = \frac{\partial}{\partial\beta} (-\text{Ln}Z + \beta\mu N) \quad (20)$$

where the ‘ground-state energy’ due to the chemical potential  $\mu = \partial\text{Ln}Z/\partial N$  has been appropriately added. The pressure of the system is defined as:

$$P \equiv \frac{1}{\beta} \frac{\partial\text{Ln}Z}{\partial\Delta\Omega}, \quad (21)$$

which upon using (8) yields the following equation of state:

$$NT = \left( P - \kappa\tilde{M} \frac{N^2}{(\Omega_4 - \Omega_0)^2} \right) (\Omega_4 - \Omega_0), \quad (22)$$

where the Boltzmann constant has been put to unity and  $\tilde{M}$  represents the shifted mass appearing in the partition function (19).

The relation (22) is nothing other than a *van der Waals equation of state*. In our view, this leads to the *prima facie* expectation that a first-order phase transition takes place in the bulk, though this remains to be verified.

For the non-static case (19), the quantity  $\bar{E}$  is

$$\begin{aligned} \bar{E} = & -N\text{Ln}N - N - 2N\frac{2\pi\beta}{M} \left( \frac{\partial M}{\partial\beta} \frac{1}{2\pi\beta} - \frac{M}{2\pi\beta^2} \right) (1 - \beta) + 2N\text{Ln}\left(\frac{M}{2\pi\beta}\right) + \\ & N\text{Ln}\Delta\Omega + \frac{N}{\Delta\Omega} \frac{\partial\Delta\Omega}{\partial\beta} + (1 - 2\beta)MN\bar{U} - N\bar{U} . \end{aligned} \quad (23)$$

In the limit under consideration, we may use (4) to relate the black-hole mass to the temperature:

$$M \sim \frac{\pi^4}{\omega_4} b^6 T^4 \quad (24)$$

for a five-dimensional AdS-Schwarzschild black hole.<sup>1</sup> We assume (see later) that the AdS radius  $b$  scales with the temperature as

$$b \sim c_0 T^{-1}, \quad (25)$$

where  $c_0$  is taken sufficiently large to ensure that  $\omega_4 M \gg b^2$ . Thus  $\bar{E}$  is easily evaluated:

$$\bar{E} = \text{const} + 2NT - 6N\text{Ln}T + c''\bar{U}NT^{-2} - 2c''N\bar{U}T^{-3} \quad (26)$$

where  $\bar{U} < 0$ , and is assumed constant, and we have used  $\text{Ln}N! \simeq N\text{Ln}N$  for large  $N$ . The constant in (26) can be set to zero by an appropriate normalization of the energy, since only energy differences matter.

The energy density  $\rho$  for the *four-dimensional* system on the boundary of AdS<sub>5</sub> is obtained by dividing  $\bar{E}$  by the *three-volume* which, in view of the above discussion, scales like  $T^{-3}$ . Thus

$$\rho/T^4 \propto 2N - 6NT^{-1}\text{Ln}T + c''\bar{U}NT^{-3} - 2c''N\bar{U}T^{-4} \quad (27)$$

As for the energy density, the pressure in *four dimensions* (three space dimensions, one periodic temperature dimension), denoted by  $p$ , is computed from the equation of state (22) after writing it in the form

$$p \equiv bP = \text{const} \times \frac{1}{\Delta\Omega_3} NT, \quad (28)$$

using (7) and assuming that the variations of the potential with the volume are small. The quantity  $bP$  simply represents the fact that the three-space pressure should be computed with respect to a three-volume shell,  $\Delta\Omega_3$ , and *not* a four-volume shell,  $\Delta\Omega_4$ . The former scales with one length dimension less compared to the latter, and

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<sup>1</sup>In the case of an  $(n + 1)$ -dimensional AdS-Schwarzschild black hole of large mass  $M$ , the temperature scales as  $T^n$ .



thus the quantity  $bp$  in (22) has the right scaling with  $T$ . With in this mind, we remark that  $\Delta\Omega_3$  scales like  $T^{-3}$  in the very-high-temperature regime, beyond the upper phase transition, and hence that

$$p/T^4 \sim \text{const} . \quad (29)$$

The constant term in (29) may be fixed by the fact that in the very-high-temperature regime the system is supposed to represent a gas of massless gluons, and hence, from the classical statistical mechanics of a ideal gas of massless Bose particles, the energy density is three times the pressure.

The energy density curve is plotted in Figure 1. We observe that the qualitative features of QCD are correctly captured by our classical statistical system of AdS black holes. The energy density drops sharply as we approach low temperatures, and it is tempting to identify this region might with the deconfined region of QCD, approached from the high-temperature unconfined phase. Our approximate calculation exhibits a bump in the energy-density curve before the ‘confined’ region is reached, due to the  $-T^{-1}\text{Ln}T$  term in (27). However, the limit (i) that we have used above is valid only for high temperatures, and should not be trusted quantitatively in this region. On the other hand, the appearance of this bump may indicate the existence of a thermodynamic instability, given that the ‘bump’ region is followed by a sharp drop in the energy density as the ‘confined’ region is approached.

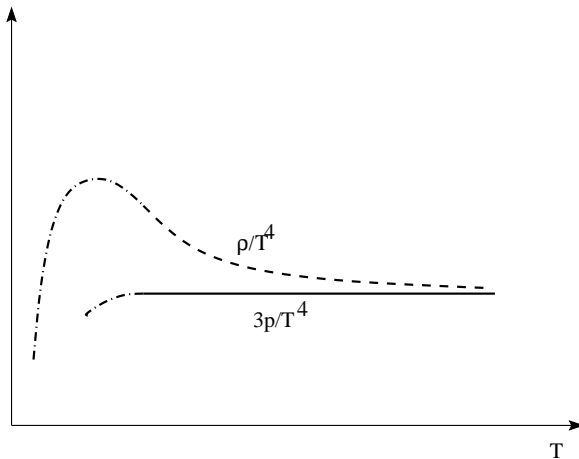


Figure 1: *The scaled energy density  $\rho/T^4$  (dashed line) and pressure  $3p/T^4$  (continuous line) in a gauge theory, plotted as functions of the temperature  $T$ , as calculated in the high-temperature limit (i)  $b \ll r_+$  [5] using a typical set of parameters for  $N$  indistinguishable AdS black holes. The bump in the energy density is reminiscent of the transition from a gas of pions to a deconfined quark-gluon plasma in the QCD case, but the approximations made in the limit (i) are not reliable in the regions where the lines are dotted.*

## 4 Another View of the Phase Transition

In this section we shall look at the phase transition from the opposite viewpoint, described by the limit (ii) defined above:  $\ell_P^2 \ll r_+^2 \simeq \omega_4 M \ll b^2$ . The approximations made in the analysis of the previous section are also valid in this parameter regime, though in a different way. When there is a large separation between any pair of black holes in the ensemble:  $r_+ \ll r \ll b$ , it is again a good approximation to take  $U(r) \simeq \bar{U}$ , where  $\bar{U}$  is a positive constant, because the potential varies very little over the range  $r_+ \ll r \ll b$ . Not only that, but the constant  $\bar{U}$  is also a small number and hence one can again make a weak-field approximation. The analogues of the formulae (7) are in this case:

$$U(r) \simeq \bar{U} = \text{const} \equiv \kappa N / \Omega$$

$$\kappa = (1/N) \int_{r_+}^b d^4x U(r) = \frac{\pi^2}{24N} \left[ b^4 - \frac{r_+^6}{b^2} - 3\omega_4 M (b^2 - r_+^2) \right] \quad (30)$$

where, as before, the effective volume  $\Omega = \Omega_4 - \Omega_0$  is the size of the four-volume of the shell  $r_+ \ll r \ll b$ , and  $N$  is the number of black holes in the ensemble. Given that  $\Omega \sim b^4$ , and using the above formula (30) for  $\kappa$ , one can easily check that  $\bar{U} < 1$ . Notice that the potential is *attractive* in the region  $r_+ \ll r < r_0$  and *repulsive* in the region  $r_0 < r \ll b$ . In this region, (4) tells us that the black-hole mass is related to the temperature by:

$$M \sim \frac{\beta^2}{4\pi\omega_4}. \quad (31)$$

One can perform calculations for the partition function, energy and pressure of the ensemble that are similar to those described in the previous section, which we do not reproduce in detail here.

As in the limit (i), we find again in the limit (ii) a *van der Waals equation of state*, as in (22).

To understand qualitatively the physics in the lower end of the transition region  $T \sim T_1$ , we recall that, as we approach the transition region from above, we enter a regime where the Liouville theory takes over. In this theory the radius  $b$  may be assumed to be independent of temperature [9], and large. In this limit of large  $b$  and  $T$ -independence, a different approximation is needed to capture correctly the features of the transition region, since the classical description of a gas of stable black-hole particles breaks down in this case. However, we can still obtain qualitative ideas of the dynamics by applying the above statistical-mechanical approach to this case. Notice that the smallness of  $\omega_4 M$  compared to  $b$  implies that the AdS space is regular for large  $r$ . This is the regime discussed in the Liouville approach of [9].

We have calculated the energy density  $\rho$  and the pressure  $3p$ , with the results shown in Figure 2. We observe that the pressure is almost constant near the transition region, whilst the energy increases and exhibits a bump. As compared to our results in limit (i), this bump is rather smoother. The constant value of the pressure is again fixed by the fact that at low temperatures the system again enters an

ideal-gas regime, where in this case the physical degrees of freedom are the bound states, i.e., massless pions in the case of QCD, so that the relation  $\rho = 3p$  should again be valid.

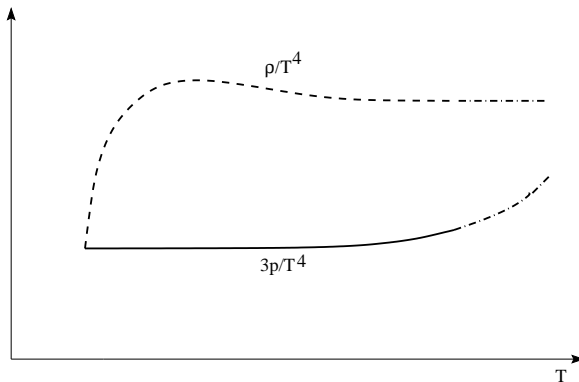


Figure 2: As in Fig. 1, but in the limit (ii):  $r^+ \ll b$  [9], conjectured to represent the start of the phase transition regime. The pressure curve lies below the energy density curve and is almost constant: the bump in the energy density is less marked than indicated in the limit (i).

## 5 Relating the Two Descriptions

The regime (i) which describes the high-temperature tail of the phase transition must connect smoothly with the regime (ii) as the scales cross:  $r_+ \leftrightarrow b$ . Since one expects that the temperature should rise after the phase transition, we assume that  $T_{(ii)} \ll T_{(i)}$ . At the boundary of the two regions, one has  $r_+^B \sim \sqrt{\omega_4 M_B} \sim b_B$ , and the temperature  $T_B \sim 1/b_B$  should therefore lie in the range  $T_{(ii)} \ll T_B \ll T_{(i)}$ . A natural way to arrange this crossover is to keep  $r_+$  fixed at the boundary value and study the variations of the other parameters as we go from one regime to the other. Clearly this puts the following bounds on the other parameters:

$$b_{(i)} \ll b_B \ll b_{(ii)}, \quad M_{(ii)} \sim M_B \ll M_{(i)}. \quad (32)$$

These bounds seem natural and consistent with the definitions of the limits (i) and (ii). In particular, one outcome of this crossover, namely that the black-hole degrees of freedom are more massive in region (i) and hence decouple from infrared physics, seems consistent for describing the region just above the phase transition, which, according to (32), is associated with lower-mass black holes:  $M_{(ii)} \sim M_B \ll M_{(i)}$  in region (ii).

We would also like to comment on the behaviour of the AdS radius  $b$  in the transition region. The scaling (25) is justified in the Liouville approach of [9], in which the recoil-induced AdS radius  $b$  is proportional to a homotopic ‘time’ variable. In the analysis of [9], this homotopic time was identified with the target time  $X^0$

in a *real-time* formulation of Liouville QCD. In this real-time formalism, the time  $X^0$  should not be confused with the temperature. However, from the equivalence of the real-time and Matsubara formalisms, where one identifies the temperature with the inverse radius of a compactified Euclidean time, it is natural to assume that, at least in the high-temperature regime where one assumes thermal equilibrium with a heat bath of temperature  $T$ , the scaling (25) should be valid.

An alternative way to justify the scaling (25) is to notice that, in order to arrive at the regime where the analysis of [9] is valid, one needs to go to very low temperatures, where  $b$  is huge. This result is not in contradiction with our above procedure of identifying  $b$  with  $1/T$ . However, in the low-temperature regime  $b$  is almost constant [9], and *not* scaling with temperature. We conjecture that there are in general competing contributions to  $b$ , so that

$$b \sim \delta^{-2} + \mathcal{O}(1/T), \quad (33)$$

and that the  $\delta^{-2}$  term dominates in the low-temperature regime, whilst  $\delta$  is comparatively large in the high-temperature regime, and the  $1/T$  term dominates. In [9],  $\delta$  was identified with the area of the Wilson loop that generated the world sheet of the string. This is consistent with the above picture: for low temperatures in the confining regime, the dominant degrees of freedom are related to large world-sheet areas, in the sense that the (temporal Polyakov or spatial Wilson) loops that define the order parameters relevant for confinement are large. It is these quark-antiquark loops that can be described by weakly-coupled string theory, for which the analysis of [9] is valid. At high temperatures, on the other hand, the areas defined by the dominant order parameters (Wilson and/or Polyakov loops) are relatively small or microscopic, as remarked in [9]. This corresponds to the pure stringy limit  $\delta^{-2} \rightarrow 0$ . In that limit the perturbative string theory approach of [9] is invalid, and should be replaced by the above semi-classical picture [5] of a gas of very massive black holes. We now remark that, in our approximate treatment of near-horizon distances  $r$ , where the potential is *weak*, one obtains the typical order of magnitude estimate

$$r^4 \simeq \omega_4 M b^2 + \mathcal{O}(b^4 \sqrt{\omega_4 M b^{-2}}) \quad (34)$$

Combining (3),(24) and (25), we find that the volume  $\Omega_0 \propto r_+^4$  varies with  $T$  as  $T^{-4}$ , and hence

$$\Delta\Omega \equiv \Omega_4 - \Omega_0 \propto r^4 - r_+^4 \simeq c' T^{-4} \quad (35)$$

where  $c'$  is a constant.

As commented above and in [15], the phase transition of the five-dimensional black-hole system is expected to be of first order. Moreover, it is here identified with a deconfining transition in gauge theory. However, at present our analysis cannot determine the precise order of these associated transitions, and this remains an open issue. A related issue is whether holography [4] survives the first-order phase transitions associated with the boundary and bulk dynamics. Based on the Liouville renormalization argument given above, we would expect so, but this issue is also open.

## 6 Comparison with QCD

Here we comment on the temperature-dependence of the pressure, and relate it to what is known for QCD from lattice simulations. We recall from the discussion of section 4 that in low-temperature limit (i) of the phase transition region (see Fig. 3), where  $b$  is roughly  $T$ -independent and the mass of the black hole  $M \sim T^4$ , there is no difference in scaling between the four- and five-dimensional pressures, and hence  $3p/T^4$  in (28) is initially approximately constant and then increases slightly (due to the smallness of  $\delta$ ) as the temperature increases. Thus the pressure curve does not increase as abruptly as the energy density, and always lies beneath it as long as it can be calculated reliably.

As the temperature increases towards the upper end of the phase transition, the increase in the pressure may be obtained from terms that have been ignored so far in deriving (22). These include terms that express fluctuations of the potential  $\bar{U}$  with the volume  $\Omega_4 - \Omega_0$ . These are required by continuity between the two asymptotic regimes for the pressure computed above. The generic (approximate) form of such terms may be found by representing the potential fluctuations as

$$U \simeq \bar{U} + \epsilon' \frac{N}{\Omega_4 - \Omega_0} \quad (36)$$

where  $\epsilon'$  is small and positive. Such a dependence of the potential on  $\Omega_4$  results in extra terms on the right-hand side of the equation of state. Thus, for example, in the high-temperature phase we expect the the boundary pressure to have the form:

$$p/T^4 \sim \text{const} \frac{1}{\Delta\Omega_3 T^4} \left[ NT + \epsilon' N^2 (M + T) / (\Omega_4 - \Omega_0) \right] . \quad (37)$$

We now recall that, on the high-temperature side of the phase transition,  $\Delta\Omega_3$  scales like  $T^{-3}$ ,  $(\Omega_4 - \Omega_0)$  scales like  $T^{-4}$ , and the mass of the black hole scales like  $T^{-2}$ . The mass is sufficiently large that the  $M$ -dependent term is still dominant. Hence, from (36) one obtains a linear increase for  $3p/T^4$ :

$$3p/T^4 \sim \text{const} + \mathcal{O}(\text{const}' \times \epsilon' N^2 T) . \quad (38)$$

As the temperature increases, the  $\epsilon'$  term becomes smaller and smaller, and one recovers a constant temperature at the end of the of the transition region. The proportionality constants are again fixed by requiring that this scaling should describe at high temperature an almost ideal gas of massless particles, in which case we have the relation  $\rho = 3p$  in three space dimensions.

We display in Fig. 3 heuristic interpolations of  $\rho$  and  $3p$  between the high- and low-temperature limits (i) and (ii). These curves can be compared with those calculated for QCD on the lattice [16]. In both cases, there appears to be a sharp jump of the energy density at the onset of the deconfining phase transition, from the value where the system is equivalent to a gas of pions, towards that where the system is described by an almost ideal gas of quarks and gluons. On the other hand, the

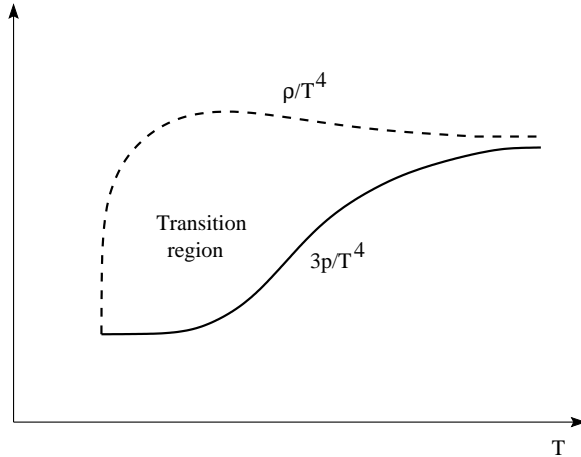


Figure 3: *Interpolations of the scaled energy density  $\rho/T^4$  (dashed line) and pressure  $3p/T^4$  (continuous line), including the transition region between the limits (i) and (ii) shown in Figs. 1 and 2, respectively. The behaviours of the energy density and pressure in the intermediate-temperature region are reminiscent of lattice calculations [16]: in particular, the pressure curve rises more slowly than that of the energy density.*

increase in the pressure is much smoother in both the lattice and our AdS calculations. This is related in our approach to the weak-field assumption for the potential:  $|U| \ll 1$ , which is valid for near-horizon AdS geometries in the high-temperature phase.

We should repeat that our analysis in the limit (ii) is not yet quantitative at low temperatures. However, the Liouville world-sheet approach of [9], which describes the dynamics of world-sheet defects via  $D$  particles, describes this regime qualitatively correctly, leading in particular to confinement as a low-temperature property. In this case, the space-time obtained from  $D$ -particle recoil is indeed of AdS type with  $M \rightarrow 0$  in (1) [13, 9]. We have shown in this paper that this approach has, moreover, a plausible regular continuation to the high-temperature limit (i) explored in [5]. This gives us further reason to hope that the Liouville string approach may be suitable for development into a reliable tool for describing non-perturbative gauge dynamics, and therefore may contribute to the new avenue for non-perturbative gauge-theory calculations opened up in [1, 5, 6, 7, 8].

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