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Non-perturbative Temperature Instabilities in $N = 4$ Strings*

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Abstract

We derive a universal thermal effective potential, which describes all possible high-temperature instabilities of the known $N = 4$ superstrings, using the properties of gauged $N = 4$ supergravity. These instabilities are due to three non-perturbative thermal dyonic modes, which become tachyonic in a region of the thermal moduli space. The latter is described by three moduli, s, t, u, which are common to all non-perturbative dual-equivalent strings with $N = 4$ supersymmetry in five dimensions: the heterotic on $T^4 \times S^1$, the type IIA on $K_3 \times S^1$, the type IIB on $K_3 \times S^1$ and the type I on $T^4 \times S^1$. The non-perturbative instabilities are analysed. These strings undergo a high-temperature transition to a new phase in which five-branes condense. This phase is described in detail, using both the effective supergravity and non-critical string theory in six dimensions. In the new phase, supersymmetry is perturbatively restored but broken at the non-perturbative level. In the infinite-temperature limit the theory is topological with an $N = 2$ supersymmetry based on a topologically non-trivial hyper-Kähler manifold.

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1 Introduction

In a physical system where the density of states grows exponentially with the energy,

$$\rho(E) \sim E^{-k} e^{bE} , \tag{1.1}$$

there is a critical temperature, $\beta^{-1} \equiv T = T_H = b^{-1}$, at which various thermodynamical quantities diverge [1]. In particular, the partition function Z and the mean energy U develop power pole singularities:

$$\begin{aligned} Z(\beta) &= \int dE\, \rho(E) e^{-\beta E} \sim \frac{1}{(\beta - b)^{(k-1)}} , \\ U(\beta) &= -\frac{\partial}{\partial\beta} \ln Z \sim (k-1)\frac{1}{\beta - b} + \text{regular} . \end{aligned} \tag{1.2}$$

An alternative interpretation of the mean energy pole singularity follows from the identification of the temperature with the inverse radius of a compactified Euclidean time on S^1. In this representation, the partition function is given by the (super)trace over the thermal spectrum of the theory in one dimension less:

$$\ln Z = \text{Str} \ln \mathcal{M}(\beta) . \tag{1.3}$$

The pole singularity is then a manifestation of a thermal state that becomes massless at the critical temperature. Thus, the knowledge of the thermal spectrum of the theory $\mathcal{M}(\beta)$, as a function of the S^1 radius $R = \beta/2\pi$, determines the critical temperature [2]–[5].

Perturbative string theory provides an example of an exponentially growing density of states, with k in Eq.(1.1) equal to the dimension of space-time, and the exponent b^{-1}, and thus the Hagedorn temperature, given as a theory-dependent constant in terms of the string scale $(\alpha')^{-1/2}$ [6, 7, 8, 4, 5]. In the picture where temperature is regarded as a compactification on a circle of radius R, one can exactly construct the partition function $Z(R)$ and identify the state that becomes tachyonic at the critical temperature [3, 5]. This state has necessarily a non-zero winding number n, as perturbative quantum field theory is not able to generate a similar critical behaviour. A detailed discussion of this phenomenon in perturbative string theory will be the subject of the next section.

It is interesting that one can go a long way into the discussion of thermal instabilities due to non-perturbative string states, and then also, of non-perturbative field theory states. The first observation is that, in $N_4 = 4$ supersymmetric strings[1] (or $N_6 = 2$), the *perturbative* string states becoming tachyonic above the Hagedorn temperature are (thermally-shifted) BPS states that preserve half of the supersymmetries ($N_4 = 2$ or $N_6 = 1$). In these theories, the masses of *non-perturbative* BPS states are also known from $N_4 = 4$ supersymmetry [9, 10, 11] and one can identify among them the states that are able to induce a thermal instability and the critical temperature at which they become tachyonic. We will develop this argument in Section 3, using heterotic–type II duality [12]–[15].

Notice that considering only those $N_4 = 4$ BPS states preserving $N_4 = 2$ supersymmetries (1/2-BPS) is certainly sufficient to study thermal instabilities in, for instance, non-perturbative heterotic strings in space-time dimensions $D \geq 6$. BPS states preserving less supersymmetries only arise in lower dimensions (1/4-BPS, with $N_4 = 1$, for $D \leq 5$ or 1/8-BPS, with $N_2 = 1$, for $D \leq 3$). In dimension six or higher, it is expected that thermal instabilities due to $N_4 = 2$ BPS states are similar to those of perturbative winding states, with an entropy growing linearly with the mass. This statement, as we will see in the following sections, can be checked in six dimensions, since heterotic–type II duality allows a non-perturbative behaviour on one side to be turned to a perturbative one on the dual side. In dimensions lower than six, the 1/4- or 1/8-BPS states singularities have to correspond to an entropy growing with the mass faster than linearly. This statement is supported by the behaviour of black hole entropy in four and five dimensions, which follows the area law [16]. In this case, the temperature is fixed and the canonical ensemble does not exist [8]. However, we stress that the analysis given in this paper is fully general in dimensions greater than (or equal to) six, while it only applies to 1/2-BPS states in lower dimensions.

With this limitation in mind, the main result of this paper is a computation of the exact effective potential for all the potentially tachyonic states, as a function of the (universal[2]) temperature. It reproduces all known Hagedorn temperatures for heterotic and type II strings in the appropriate limits. The exact potential has a global minimum in a domain of the six-dimensional string coupling that includes the perturbative heterotic regime. In this phase, supersymmetry is perturbatively restored, the temperature is fixed, $T^{-1} = \pi\sqrt{2\alpha'_H}$ and space-time geometry is that of the heterotic

[1] N_D is the number of D-dimensional supersymmetries.

[2] Invariant under string dualities.

or type IIA five-brane. More precisely, the system loses four units of central charge. It describes a non-critical string in six dimensions with massless thermal excitations extending the concept of particles with infinite correlation length in finite-temperature field theory with a second-order phase transition. On the type II side, this phase is characterized by a condensation of five-branes.

The present paper is organized as follows. In Section 2, we recall the aspects of perturbative strings at finite temperature, which will be used in the non-perturbative discussion. In Section 3, we discuss the temperature modification to the perturbative and non-perturbative BPS spectra in $D-1=5$ and $D-1=4$ dimensions[3]. Section 4 presents the derivation of the effective Lagrangian for the potentially tachyonic states, as a four-dimensional supergravity theory, and the discussion of the minima of the scalar potential. Section 5 provides a detailed discussion of the high-temperature phase found in perturbative heterotic strings, using the effective supergravity theory. We show that a linear dilaton is a background of the effective supergravity, we study the structure of the mass spectrum, the fate of supersymmetry, and consider various limits in this background. This phase is further discussed in Section 6, in the framework of non-critical strings with $\mathcal{N}_{sc}=2$ or $\mathcal{N}_{sc}=4$ superconformal symmetry. We demonstrate the existence of massless excitations in twenty-eight $N_4=2$ hypermultiplets. We conclude in Section 7.

2 Perturbative analysis

To construct the thermal partition function of a system of fields, spin-statistics requires the boundary condition around the S^1 circle to be modified according to

$$\Psi(t+2L\pi R)=(-1)^{La}\Psi(t)\,.$$

Under a 2π rotation ($L=1$) of Euclidean time, bosons ($a=0$) are periodic while fermions ($a=1$) are antiperiodic. The generalization to (perturbative) string theory is dictated by modular invariance. It replaces the above sign with [3]–[5]

$$(-1)^{La+nb+\delta Ln}$$

for a state with winding numbers L and n along the two non-contractible loops on the world-sheet torus. Here, a and b denote the fermionic spin structures along these

[3]As already mentioned, a D-dimensional theory at finite temperature can as well be studied as a $(D-1)$-dimensional theory, hence this notation.

two cycles. Modular invariance indicates that the parameter δ is equal to one for the heterotic string and zero for the type IIA and IIB strings[4]. It can be seen that the consequence of this phase is to shift the lattice momenta of the S^1 string coordinate according to the rule[5] [4, 5]

$$P_{L,R} = \frac{1}{R}\left[m + \frac{a}{2} - \frac{n\delta}{2} \pm \frac{nR^2}{\alpha'} \right], \tag{2.1}$$

and to reverse the GSO projection in the odd winding number sector.

It turns out that string theories with D-dimensional space-time supersymmetry look at finite temperature as if supersymmetry were spontaneously broken in $D-1$ dimensions. Indeed, with a redefinition of m, a can be identified with the (D dimensional) helicity operator: $Q = \text{integer} + a/2$. Then, the states of the thermal theory, viewed as $(D-1)$-dimensional, are mapped to those of a supersymmetric theory compactified on S^1, without the temperature spin-statistics factor $(-1)^{LQ+nb+\delta Ln}$ that induces helicity shifts in the momenta (2.1). Explicitly, a state of the latter with momentum, winding and helicity charges $(m,\ n,\ Q)$ is mapped in the thermal case to

$$n \to n' = n, \qquad m \to m' = m + \vec{e}\cdot\vec{Q} - \vec{e}\cdot\vec{e}\,\frac{n}{2}, \qquad \vec{Q} \to \vec{Q}' = \vec{Q} - \vec{e}\,n\,, \tag{2.2}$$

where the helicity vector $\vec{Q}$ is constructed in terms of the left- and right-moving string helicities $\vec{Q} = (Q_L,\ Q_R)$. The vector $\vec{e} = (1,0)$ in the heterotic string and $\vec{e} = (1,1)$ in type II theories, and the inner product is Lorentzian: $\vec{A}\cdot\vec{B} = A_L B_L - A_R B_R$. Note that $Q' \equiv Q'_L + Q'_R$ is the helicity operator in $D-1$ dimensions. The above shift of charges follows from a Lorentzian boost, which keeps invariant the combination $\frac{\alpha'}{2}\,\vec{P}\cdot\vec{P} + \vec{Q}\cdot\vec{Q}$ and thus preserves modular invariance.

The perturbative superstring mass formula can be read from the left-movers, which carry world-sheet supersymmetry:

$$\frac{1}{2}\alpha'\mathcal{M}^2 = \sum_{i=1}^{4} Q_i^2 - 1 + \frac{1}{2}\alpha' P_L^2 + \frac{1}{2}\alpha'\mathcal{M}^2_{others}, \tag{2.3}$$

where we have dropped the subscript L for notational simplicity, $\mathcal{M}^2_{others}$ denotes the contributions from oscillator modes as well as from the momenta of the remaining part of the lattice, and P_L is as in Eq.(2.1). The four (left-) charges Q_i are the eigenvalues

[4]The thermal singularities of Type I strings are the same as for Type IIB, so that we will not refer to open strings in the sequel.

[5]Whenever α' is not explicitly mentioned in a formula, our convention is $\alpha' = 2$.

under the four $U(1)$ helicities acting on the world-sheet fermions. One of them can be identified with the contribution of left-movers to the (four-dimensional) space-time helicity, Q_L, introduced in the discussion following Eq.(2.2). These charges Q_i are integers for space-time bosons (NS states), and half-integers for space-time fermions (R states). The supersymmetric GSO projection implies that $\sum_{i=1}^4 Q_i$ is an odd integer for NS states, while it is an even or odd integer for R states, depending on a free choice of chirality. But in any case, $\sum_{i=1}^4 Q_i^2$ is an odd integer. The lowest BPS states of the supersymmetric theory have $\mathcal{M}_{others} = 0$ and $\sum_{i=1}^4 Q_i^2 = 1$.

At finite temperature, the GSO projection is modified as Q_i gets shifted according to Eq.(2.2), which also affects the momentum P_L. Notice that for an even winding number n, the thermal modification of P_L defined in Eq.(2.2) can be regarded as a shift of m and Q compatible with the (supersymmetric) GSO projection. As a consequence, the spectrum in even n sectors is not different in the thermal and supersymmetric cases, the mass formula for the (lightest) BPS fermions, gauge bosons and scalars with even windings n remains $\mathcal{M}^2 = P_L^2$, with m modified as in Eq.(2.2), and tachyonic states are not present. The situation is not the same for states with odd winding number n. In this case the BPS mass formula becomes $\frac{1}{2}\alpha'\mathcal{M}^2 = \frac{1}{2}\alpha' P_L^2 + n(n-2Q_L)$. From the GSO condition $\sum_{i=1}^4 Q_i^2 = 1$, it follows that the only states that can become tachyonic are those with $n = \pm 1$ and $Q_L = \pm 1 (= -Q_R$ for type II) [5]. They correspond to $(D-1)$-dimensional scalars coming from the longitudinal components of the D-dimensional metric.

The Hagedorn temperature is identified with the critical value of the radius at which the first tachyonic state appears, as $2\pi R = T^{-1}$ decreases. From the above mass formula, its charges are:

$$\begin{array}{rrcl} \text{heterotic :} & (m,n,Q) & = & \pm(-1,1,1), \\ \text{type II :} & (m,n,Q_L,Q_R) & = & \pm(0,1,1,-1). \end{array} \tag{2.4}$$

The Hagedorn temperatures are $T_H = \frac{1}{\sqrt{2\alpha'}\pi}(\sqrt{2}-1)$ for the heterotic string[6] and $T_H = \frac{1}{2\sqrt{2\alpha'}\pi}$ for type II theories.

The appearance of tachyons cannot take place in a perturbative supersymmetric field theory, which behaves like the zero-winding sector of strings; all masses (squared) are increased by finite temperature corrections, $\mathcal{M}^2 = P^2$, and a thermal instability is

[6] There is a second (higher) critical temperature due to temperature duality, $R \to \alpha'/R$, $T \to (4\pi^2\alpha' T)^{-1}$.

never generated by a state becoming tachyonic at high temperature. However, as we will see below, in non-perturbative supersymmetric field theories such an instability can arise from thermal dyonic modes, which behave as the odd winding string states. Indeed, in theories with $N_4 = 4$ supersymmetries the BPS mass formula is determined by the central extension of the corresponding superalgebra [9]–[11] and dyonic field theory states are mapped to string winding modes [14, 11]. Using heterotic–type II duality, one can argue that the thermal shift of the BPS masses modifies only the perturbative momentum charge m. In both heterotic and type II perturbative strings, the thermal winding number n is not affected by the temperature shifts [see Eq.(2.2)]. Since, in dimensions lower than six, heterotic–type II duality exchanges the winding numbers n of the two theories, and since the winding number of the one theory is the magnetic charge of the other, it is inferred that field theory magnetic numbers are not shifted at finite temperature. This in turn indicates how to modify the BPS mass formula at finite temperature. In Section 4, we will give an independent argument based only on spontaneously broken $N_4 = 4$ supersymmetry and the nature of BPS states.

3 String duality and BPS spectrum

Our goal is to study six- and five-dimensional string theories with $N_4 = 4$ supersymmetry, at finite temperature. The heterotic string is then compactified on T^4 and type II theories on K_3. In six dimensions, there is an S-duality that relates heterotic and type IIA strings. Upon compactification to five dimensions on a circle, type IIA and IIB theories are related by a T-duality.

In view of the observation that the thermal spectrum is obtained by the modification (2.2) applied to the spectrum of the supersymmetric theory with the temperature replaced by an ordinary circle, we start by describing the supersymmetric BPS spectrum in five and four dimensions. The states that can induce a thermal instability are charged under the Kaluza-Klein $U(1)$. Their mass depends on the temperature radius $R = (2\pi T)^{-1}$. The mass formula from the heterotic point of view and in α'_H units is

$$\mathcal{M}^2 = \left(\frac{m}{R} + \frac{nR}{\alpha'_H} + \frac{\ell R}{\lambda_H^2 \alpha'_H}\right)^2 , \tag{3.1}$$

where m and n are the circle momentum and winding numbers, ℓ is the non-perturbative wrapping number for the heterotic five-brane around $T^4 \times S^1$, with tension λ_H^{-2} in α'_H units, and λ_H is the string coupling in six dimensions[7]. The combination

$$g_5^2 = \frac{\alpha'_H}{R}\lambda_H^2 \tag{3.2}$$

is the five-dimensional string coupling.

Performing an S-duality in Eq.(3.1),

$$\lambda_H = \frac{1}{\lambda_{IIA}}, \qquad \lambda_H^2 \alpha'_H = \alpha'_{II}, \tag{3.3}$$

we find the mass formula for type IIA strings:

$$\mathcal{M}^2 = \left(\frac{m}{R} + \frac{nR}{\alpha'_{II}\lambda_{IIA}^2} + \frac{\ell R}{\alpha'_{II}} \right)^2 . \tag{3.4}$$

The momentum and winding numbers are now m and ℓ, while n is the wrapping number for the Neveu-Schwarz type IIA five-brane around $K_3 \times S^1$.

From the six-dimensional viewpoint, the first term, m/R, is the Kaluza–Klein momentum, while the last two terms correspond to BPS (dyonic) strings with tension

$$T_{p,q} = \frac{p}{\alpha'_H} + \frac{q}{\alpha'_{II}} , \tag{3.5}$$

where p, q are relatively primes, so that $(n, \ell) = k\,(p, q)$. The common divisor k defines the wrapping of the $T_{p,q}$ string around S^1. On the type IIA side, q is the charge of the fundamental string and p the magnetic charge of the solitonic string obtained by wrapping the NS five-brane around K_3. These $T_{p,q}$ strings cannot become tensionless since they are never associated to vanishing cycles of the internal manifold. Consequently, their tension is always positive and p, q must be non-negative integers. On the other hand, for $\ell = 0$, the heterotic GSO projection implies $mn \geq 0$, while for $n = 0$ the type IIA projection implies $ml \geq 0$. More generally, the $T_{p,q}$ string implies that $mk \geq 0$ [10].

Following the procedure described above, the temperature deformation transforms Eq.(3.1) into:

$$\mathcal{M}_T^2 = \left(\frac{m + Q' + \frac{kp}{2}}{R} + k\; T_{p,q}\; R \right)^2 - 2\; T_{p,q}\; \delta_{k,\pm 1}\; \delta_{Q',0}\, , \tag{3.6}$$

[7] The six-dimensional gravitational action in the string frame is $-\frac{1}{2}\,(\alpha'_H)^{-2} \int d^6x\, \lambda_H^{-2} eR$, so that λ_H is dimensionless. Traditionally, λ_H is related to the dilaton by $\lambda_H^2 = e^{2\phi}$.

where Q' is the helicity operator in $D-1=5$ dimensions [see Eq.(2.2)]. In fact, this formula reproduces the perturbative result for both heterotic and type IIA theories, as specified by Eq.(2.2). In the heterotic perturbative limit $\lambda_H \to 0$, only the $\ell = 0 = q$ states survive, while in the type IIA perturbative limit $\lambda_{II} \to 0$, only the $n = 0 = p$ states survive. Note that in the general case of a $T_{p,q}$ string with the temperature deformation, the condition $mk \geq 0$ becomes $mk \geq -1$ because of the inversion of the GSO projection.

It follows from Eq.(3.6) that if the heterotic coupling λ_H is smaller than the critical value

$$\lambda_H^c = \frac{\sqrt{2}+1}{2} \tag{3.7}$$

the first tachyon has $(m, n, \ell, Q') = \pm(-1, 1, 0, 0)$ and it appears at $R = \sqrt{\alpha'_H/2}(\sqrt{2}+1)$, which corresponds to the heterotic Hagedorn temperature. On the other hand, if the heterotic theory is strongly-coupled, $\lambda_H > \lambda_H^c$, the first tachyon has $(m, n, \ell, Q') = \pm(0, 0, 1, 0)$, and the critical radius is $R = \sqrt{2\alpha'_H}\,\lambda_H = 2\sqrt{\alpha'_{II}/2}$, which corresponds to the type IIA Hagedorn temperature. Besides the above two would-be tachyons, mass formula (3.6) leads in general to two series of potentially tachyonic states with $m = -1$:

$$\begin{array}{llll} p = 1, & \forall q\ : & R & = \left(\dfrac{\sqrt{2}\pm 1}{\sqrt{2}}\right) \dfrac{1}{\sqrt{T_{1,q}}}, \\ p = 2, & \forall q \text{ odd}\ : & R & = \sqrt{\dfrac{2}{T_{2,q}}} \end{array} \tag{3.8}$$

(which includes the first heterotic tachyon with $p = 1, q = 0$). The critical temperature $(2\pi R)^{-1}$ for each of the states in both series is always higher than the lowest Hagedorn heterotic temperature while, as discussed above, the type IIA Hagedorn temperature first appears when the heterotic coupling exceeds the critical value (3.7).

In order to include type IIB strings, we need to discuss five-dimensional theories at finite temperature, taking into account the compactification radius R_6 from six to five dimensions. Type IIA and IIB strings are then related by the inversion of R_6. The extension to four dimensions of the mass formula (3.6) is straightforward. It depends on three parameters, the string coupling g_H, the temperature radii R and R_6. It is convenient to introduce the three combinations

$$t = \frac{RR_6}{\alpha'_H}, \qquad u = \frac{R}{R_6}, \qquad s = g_H^{-2} = \frac{t}{\lambda_H^2}, \tag{3.9}$$

in terms of which the BPS mass formula in the $N_4 = 4$ supersymmetric case reads [11]:

$$\begin{aligned}\mathcal{M}^2 &= \frac{|m + ntu + i(m'u + n't) + is\,[\tilde{m} + \tilde{n}tu - i(\tilde{m}'u + \tilde{n}'t)]|^2}{\alpha'_H tu} \\ &= \left[\frac{m}{R} + \frac{nR}{\alpha'_H} + g_H^{-2}\left(\frac{\tilde{m}'}{R_6} + \frac{\tilde{n}'R_6}{\alpha'_H}\right)\right]^2 + \left[\frac{m'}{R_6} + \frac{n'R_6}{\alpha'_H} + g_H^{-2}\left(\frac{\tilde{m}}{R} + \frac{\tilde{n}R}{\alpha'_H}\right)\right]^2 .\end{aligned} \tag{3.10}$$

In this expression, the integers m, n, m', n' are the four electric momentum and winding numbers, corresponding to the four $U(1)$ charges from T^2 compactification. The numbers $\tilde{m}, \tilde{n}, \tilde{m}', \tilde{n}'$ are their magnetic non-perturbative partners, from the heterotic point of view.

The mass formula (3.10) has been defined for heterotic variables. To exhibit the relation with the type IIB theory, we rewrite the above mass formula in terms of type IIA variables (3.3) and perform a T-duality

$$R_6 = \frac{\alpha'_{II}}{R_6^B}, \qquad \lambda_{IIA} = \lambda_{IIB}\frac{\sqrt{\alpha'_{II}}}{R_6^B}. \tag{3.11}$$

However, since[8]

$$R^2 = \alpha'_H tu = 2\kappa^2 stu$$

and R is by construction identical in all three string theories, the mass formula (3.10) is invariant under the exchanges $s \leftrightarrow t$, $s \leftrightarrow u$ and $t \leftrightarrow u$. These operations correspond respectively to heterotic–IIA, IIA–IIB and heterotic–IIB dualities. The mass formula will then apply to all three theories, provided s, t and u are defined as in Eqs.(3.9), but in terms of the appropriate variables α', R_6 and λ in each theory.

The five-dimensional IIA formula (3.1) is reobtained by choosing first $m' = n' = 0$, which removes the second torus, and then taking the limit $R_6 \to \infty$ with λ_H kept fixed. This limit implies $\tilde{m} = \tilde{n} = \tilde{n}' = 0$, and $\tilde{m}'$ is identified with ℓ in Eq.(3.1).

The five-dimensional IIB theory is obtained by taking $\tilde{m} = \tilde{n} = 0$ and the limit $R_6^B \to \infty$, which implies $n = m' = n' = 0$. We reobtain Eq.(3.4) with IIB variables, $\ell = \tilde{n}'$, $n = \tilde{m}'$ and m unchanged. Similarly, the finite temperature mass formula is identical to Eq.(3.6) with the same identification.

Finally the four-dimensional thermal mass formula is obtained from Eq.(3.10) by

[8] κ is the *four-dimensional* gravitational coupling, $\kappa = \sqrt{8\pi}M_P^{-1} = (2.4 \times 10^{18}\ \text{GeV})^{-1}$.

replacing m by $m + Q' + n/2$:

$$\mathcal{M}_T^2 = \left(\frac{m + Q' + \frac{kp}{2}}{R} + k\ T_{p,q,r}\ R \right)^2 - 2\ T_{p,q,r}\ \delta_{|k|,1}\ \delta_{Q',0}\ , \tag{3.12}$$

where we have set $m' = n' = \tilde{m} = \tilde{n} = 0$ corresponding to the lightest states, and we defined k as before, as the common divisor of $(n, \tilde{m}', \tilde{n}') \equiv k(p, q, r)$. Then $T_{p,q,r}$ is an effective string tension

$$T_{p,q,r} = \frac{p}{\alpha'_H} + \frac{q}{\lambda_H^2 \alpha'_H} + \frac{r R_6^2}{\lambda_H^2 (\alpha'_H)^2}\ .$$

Note that $\tilde{m}' = kq$ corresponds to the wrapping number of the heterotic five-brane around $T^4 \times S^1_R$ as in five dimensions, while $\tilde{n}' = kr$ corresponds to the same wrapping number after performing a T-duality along the $S^1_{R_6}$ direction, which is orthogonal to the five-brane. As we discussed in the previous section, all winding numbers $n, \tilde{m}', \tilde{n}'$ correspond to magnetic charges from the field theory point of view. Their masses are proportional to the temperature radius R and are not thermally shifted.

A nicer expression of the effective string tension $T_{p,q,r}$ is:

$$T_{p,q,r} = \frac{p}{\alpha'_H} + \frac{q}{\alpha'_{IIA}} + \frac{r}{\alpha'_{IIB}}\ , \tag{3.13}$$

where the various α' are

$$\alpha'_H = 2\kappa^2 s, \qquad \alpha'_{IIA} = 2\kappa^2 t, \qquad \alpha'_{IIB} = 2\kappa^2 u, \tag{3.14}$$

when expressed in Planck units. Note that α'_{IIB} defines a new type II theory obtained by heterotic T-duality with respect to R_6, in contrast to the type IIA–IIB T-duality (3.11). In the following we will refer to this new theory as perturbative IIB.

We stress here that p, q, r are all non-negative relatively prime integers. This follows from the constraints $n\tilde{m}' \geq 0$, $n\tilde{n}' \geq 0$ and $\tilde{m}'\tilde{n}' \geq 0$, which are a consequence of the BPS conditions and the $s \leftrightarrow t \leftrightarrow u$ duality symmetry in the undeformed supersymmetric theory. Futhermore, $mk \geq -1$ because of the inversion of the GSO projection in the temperature-deformed theory. Using these constraints, it is straightforward to show that in general there are two potential tachyonic series with $m = -1$ and $p = 1, 2$, generalizing the five-dimensional result (3.8):

$$\begin{aligned} &p = 1, \qquad \forall (q,r) \text{ relat. primes :} \qquad && R = \left(\frac{\sqrt{2} \pm 1}{\sqrt{2}} \right) \frac{1}{\sqrt{T_{1,q,r}}}, \\ &p = 2, \qquad \forall (p,q,r) \text{ relat. primes :} \qquad && R = \sqrt{\frac{2}{T_{2,q,r}}} \end{aligned} \tag{3.15}$$

One of the perturbative heterotic type IIA, or type IIB potential tachyons correspond to a critical temperature that is always lower than the above two series. The perturbative Hagedorn temperatures are:

$$\begin{array}{lll} \text{heterotic tachyon :} & (m,n,Q') = \pm(-1,1,0), & 2\pi T = \left(\sqrt{2}-1\right)\sqrt{\frac{2}{\alpha'_H}}; \\ \text{type IIA tachyon :} & (m,\tilde{m}',Q') = \pm(0,1,0), & 2\pi T = \frac{1}{\sqrt{2\alpha'_{IIA}}}; \\ \text{type IIB tachyon :} & (m,\tilde{n}',Q') = \pm(0,1,0), & 2\pi T = \frac{1}{\sqrt{2\alpha'_{IIB}}}. \end{array}$$

This discussion shows that the temperature modification of the mass formula inferred from perturbative strings and applied to the non-perturbative BPS mass formula produces the appropriate instabilities in terms of Hagedorn temperature. We will now proceed to show that it is possible to go beyond the simple enumeration of Hagedorn temperatures. We will construct an effective supergravity Lagrangian that allows a study of the nature of the non-perturbative instabilities and the dynamics of the various thermal phases.

4 Four-dimensional effective supergravity

In the previous section, we have studied, at the level of the mass formula for $N_4 = 4$ BPS states, the appearance of tachyonic states generating thermal instabilities. To obtain information on dynamical aspects of these instabilities, we now construct the full temperature-dependent effective potential associated with the would-be tachyonic states.

Our procedure to construct the effective theory is as follows. We consider five-dimensional $N_4 = 4$ theories at finite temperature. They can then effectively be described by four-dimensional theories, in which supersymmetry is spontaneously broken by thermal effects. Since we want to limit ourselves to the description of instabilities, it is sufficient to only retain, in the full $N_4 = 4$ spectrum, the potentially massless and tachyonic states. This restriction will lead us to consider only spin 0 and 1/2 states, the graviton and the gravitino[9]. This sub-spectrum is described by an $N_4 = 1$ supergravity with chiral multiplets[10].

[9] The four gravitinos remain degenerate at finite temperature; it is then sufficient to retain only one of them.

[10] When considering six-dimensional theories at finite temperature, one is similarly led to consider an $N_4 = 2$ theory with vector and hypermultiplets. We will briefly return to this point later in Section 4.3.

The scalar manifold of a generic, *unbroken*, $N_4 = 4$ theory is [17]–[20]

$$\left(\frac{Sl(2,R)}{U(1)}\right)_S \times G/H, \qquad G/H = \left(\frac{SO(6,r+n)}{SO(6)\times SO(r+n)}\right)_{T_I,\phi_A}. \tag{4.1}$$

The manifold G/H of the $N_4 = 4$ vector multiplets naturally splits into a part that includes the $6r$ moduli T_I, and a second part which includes the infinite number $n \to \infty$ of BPS states ϕ_A.

In the manifold G/H, we are only interested in keeping the six BPS states $Z_A^{\pm}$, $A = 1, 2, 3$, which, according to our discussion in the previous section, generate thermal instabilities in heterotic, IIA and IIB strings. For consistency, these states must be supplemented by two moduli T and U among the T_I's. We consider heterotic and type II strings respectively on $T^4 \times S^1_6 \times S^1_5$ and $K_3 \times S^1_6 \times S^1_5$, where S^1_6 is a trivial circle and S^1_5 is the temperature circle. The moduli T and U describe the $T^2 \equiv S^1_5 \times S^2_6$ torus. Thus, $r + n = 8$ in the $N_4 = 4$ manifold (4.1). To construct the appropriate truncation of the scalar manifold G/H, which only retains the desired states of $N_4 = 1$ chiral multiplets, we use a $Z_2 \times Z_2$ subgroup contained in the $SO(6)$ R-symmetry of the coset G/H. This symmetry can be used as the point group of an $N_4 = 1$ orbifold compactification, but we will only use it for projecting out non-invariant states of the $N_4 = 4$ theory[11] with $r + n = 8$.

A single Z_2 would split $H = SO(6)\times SO(8)$ in $[SO(2)\times SO(2)]\times[SO(4)\times SO(6)]$, and the scalar manifold would become

$$\begin{aligned}&\left(\frac{Sl(2,R)}{U(1)}\right)_S \times \left(\frac{SO(2,2)}{SO(2)\times SO(2)}\right)_{TU} \times \left(\frac{SO(4,6)}{SO(4)\times SO(6)}\right)_{\phi_A} \\ =& \left(\frac{Sl(2,R)}{U(1)}\right)_S \times \left(\frac{Sl(2,R)}{U(1)}\right)_T \times \left(\frac{Sl(2,R)}{U(1)}\right)_U \times \left(\frac{SO(4,6)}{SO(4)\times SO(6)}\right)_{\phi_A},\end{aligned} \tag{4.2}$$

At this stage, the theory would have $N_4 = 2$ supersymmetry and the first three factors in the scalar manifold are vector multiplet couplings with prepotential $\mathcal{F} = iSTU/X_0$. The last one is a quaternionic coupling of hypermultiplets. The second Z_2 projection acts on this factor and reduces it to

$$\left(\frac{SO(2,3)}{SO(2)\times SO(3)}\right)_{Z_A^+} \times \left(\frac{SO(2,3)}{SO(2)\times SO(3)}\right)_{Z_A^-}, \qquad A = 1, 2, 3. \tag{4.3}$$

[11] Only untwisted states would contribute to thermal instabilities.

This is a Kähler manifold for chiral multiplets coupled to $N_4 = 1$ supergravity [21]. The second Z_2 projection also truncates $N_4 = 2$ vector multiplets into $N_4 = 1$ chiral multiplets.

The structure of the truncated scalar manifold indicates that the Kähler potential can be written as

$$\begin{aligned} K &= -\log(S+S^*) - \log(T+T^*) - \log(U+U^*) \\ &\quad - \log Y(Z_A^+, Z_A^{+*}) - \log Y(Z_A^-, Z_A^{-*}), \end{aligned} \tag{4.4}$$

with

$$Y(Z_A^\pm, Z_A^{\pm *}) = 1 - 2Z_A^\pm Z_A^{\pm *} + (Z_A^\pm Z_A^\pm)(Z_B^{\pm *} Z_B^{\pm *}). \tag{4.5}$$

This choice is a solution to the $N_4 = 4$ constraints. For the S-manifold $SU(1,1)/U(1)$ $\sim Sl(2,R)/U(1)$, the constraint is

$$|\varphi_0| - |\varphi_1|^2 = 1/2. \tag{4.6}$$

The solution we use reads

$$\varphi_0 - \varphi_1 = \frac{1}{(S+\overline{S})^{1/2}}, \qquad \varphi_0 + \varphi_1 = \frac{S}{(S+\overline{S})^{1/2}}. \tag{4.7}$$

For an $SO(2,n_I)/SO(2)\times SO(n_I)$ manifold, the constraints are

$$\begin{aligned} |\sigma_I^1|^2 + |\sigma_I^2|^2 - |\vec{\phi}_I|^2 &= 1/2, \\ (\sigma_I^1)^2 + (\sigma_I^2)^2 - (\vec{\phi}_I)^2 &= 0, \end{aligned} \tag{4.8}$$

where $\vec{\phi}_I$ has n_I components and we introduced the index I, with values $0, +, -$, because we have three such manifolds, $I = 0, n_I = 2$ for the moduli T and U, $I = \pm, n_I = 3$ for the winding states Z_A^+ and Z_A^- $(A = 1,2,3 = n_\pm)$. The standard parametrization for the TU manifold, analogous to the choice of S, corresponds to the solution

$$\sigma_0^1 = \frac{1+TU}{2Y_{TU}^{1/2}}, \qquad \sigma_0^2 = i\frac{T+U}{2Y_{TU}^{1/2}}, \qquad \phi_0^1 = \frac{1-TU}{2Y_{TU}^{1/2}}, \qquad \phi_0^2 = i\frac{T-U}{2Y_{TU}^{1/2}}, \tag{4.9}$$

with

$$Y_{TU} = (T+\overline{T})(U+\overline{U}).$$

For the $I = \pm$ manifolds $SO(2,3)/SO(2)\times SO(3)$, a convenient parametrization is

$$\sigma_\pm^1 = \frac{1+(Z_A^\pm)^2}{2Y_\pm^{1/2}}, \qquad \sigma_\pm^2 = i\frac{1-(Z_A^\pm)^2}{2Y_\pm^{1/2}}, \qquad \phi_\pm^A = \frac{Z_A^\pm}{Y_\pm}, \qquad A = 1,2,3, \tag{4.10}$$

where $Y_\pm$ has been defined in Eq.(4.5). The Kähler function, which defines the $N_4 = 1$ supergravity theory, can be determined by directly comparing the gravitino mass terms in the $N_4 = 1$ Lagrangian with the similar term obtained after $Z_2 \times Z_2$ truncation of the $N_4 = 4$ theory:

$$e^{K/2} W = (\varphi_0 - \varphi_1) f_{ijk} \Phi_0^i \Phi_+^j \Phi_-^k + (\varphi_0 + \varphi_1) \tilde{f}_{ijk} \Phi_0^i \Phi_+^j \Phi_-^k, \tag{4.11}$$

where

$$\Phi_{0,\pm}^i = (\, \sigma_{0,\pm}^1 \,,\, \sigma_{0,\pm}^2 \,,\, \vec{\phi}_{0,\pm}\,).$$

The structure constants f_{ijk} and $\tilde{f}^{ijk}$ characterize the self-couplings of the $N_4 = 4$ vector multiplets. In this sense, they define the *gauging* of the $N_4 = 4$ theory [19, 22, 23, 24]. They induce a scalar potential that can, when appropriately chosen, spontaneously break supersymmetry [5].

The solutions (4.7), (4.9) and (4.10) to the $N_4 = 4$ constraints indicate that the non-analytic contribution to the gravitino mass (4.11) is $[(S+\overline{S}) Y_{TU} Y_+ Y_-]^{-1/2}$. This is identified with $e^{K/2}$ and leads to the expression (4.4) of the $N_4 = 1$ Kähler potential. The analytic superpotential then is

$$W = [(S+\overline{S}) Y_{TU} Y_+ Y_-]^{1/2} \left[(\varphi_0 - \varphi_1) f_{ijk} \Phi_0^i \Phi_+^j \Phi_-^k + (\varphi_0 + \varphi_1) \tilde{f}_{ijk} \Phi_0^i \Phi_+^j \Phi_-^k \right],$$

using the solutions to the $N_4 = 4$ constraints, once the gauging has been specified.

The superpotential for generic $N_4 = 4$ strings, after the $Z_2 \times Z_2$ truncation to $N_4 = 1$, is:

$$\begin{aligned} W_{susy} \;=\;& [m_1(\sigma_0^1 + \phi_0^1) + n_1(\sigma_0^1 - \phi_0^1)]\,(\varphi_0 - \varphi_1)\phi_+^{(m_1,n_1)}\phi_-^{(m_1,n_1)} \\ &+ [m_2(\sigma_0^2 + \phi_0^2) + n_2(\sigma_0^2 - \phi_0^2)]\,(\varphi_0 - \varphi_1)\phi_+^{(m_2,n_2)}\phi_-^{(m_2,n_2)} \\ &+ [\tilde{m}_1(\sigma_0^1 + \phi_0^1) + \tilde{n}_1(\sigma_0^1 - \phi_0^1)]\,(\varphi_0 + \varphi_1)\tilde{\phi}_+^{(\tilde{m}_1,\tilde{n}_1)}\tilde{\phi}_-^{(\tilde{m}_1,\tilde{n}_1)} \\ &+ [\tilde{m}_2(\sigma_0^2 + \phi_0^2) + \tilde{n}_2(\sigma_0^1 - \phi_0^1)]\,(\varphi_0 + \varphi_1)\tilde{\phi}_+^{(\tilde{m}_2,\tilde{n}_2)}\tilde{\phi}_-^{(\tilde{m}_2,\tilde{n}_2)}. \end{aligned} \tag{4.12}$$

The contributions proportional to $\varphi_0 - \varphi_1 = (S+\overline{S})^{-1/2}$ give rise to the perturbative T^2 heterotic string spectrum, provided the numerical coefficients m_1, m_2, n_1, n_2 are equal to the momentum and winding charges [22, 5]. These contributions define the structure constants f_{ijk} in Eq.(4.11). The contributions proportional to $\varphi_0 + \varphi_1 = S(S+\overline{S})^{-1/2}$ provide the mass spectrum of the non-perturbative magnetic T^2 torus, and correspond to the structure constants $\tilde{f}_{ijk}$. This superpotential summarizes the complete BPS mass spectrum valid for all (truncated) $N_4 = 4$ strings (heterotic, type IIA and type IIB). It is worth recalling that the expression of the superpotential, together with the Kähler potential K, defines not only mass terms, but the full scalar sector and its coupling to $N_4 = 1$ supergravity. This allows to examine in principle the vacuum structure far from large (or small) values of $S + \overline{S}$.

The superpotential (4.12) does not, however, break supersymmetry and is not appropriate to a finite-temperature theory. In general, breaking $N_4 = 4$ supergravity requires a gauging with non-zero structure constants cubic in the compensating fields $\sigma^{1,2}_{0+-}$. The appropriate finite-temperature gauging can be found either in field theory from the $N_4 = 4$ thermal spectrum or by examining the heterotic string spectrum at finite temperature, which corresponds to the Scherk–Schwarz gauging [23, 24, 5]. The result is to add

$$\delta W = e(\varphi_0 - \varphi_1)(\sigma^1_0 + \phi^1_0)\sigma^1_+\sigma^1_- \tag{4.13}$$

to W_{susy}. The numerical coefficient e is fixed by the thermal mass of the gravitino. In addition, the coefficient m_1 is shifted at finite temperature, according to the rule (2.2) discussed in Section 2. Inserting the representation of the scalar field, and truncating the spectrum to retain only the odd winding states, we find the superpotential

$$\begin{aligned} W &= 2\sqrt{2}\Big[\tfrac{1}{2}(1 - Z^+_A Z^+_A)(1 - Z^-_B Z^-_B) \\ &\quad + (TU-1)Z^+_1 Z^-_1 + SU Z^+_2 Z^-_2 + ST Z^+_3 Z^-_3\Big]. \end{aligned} \tag{4.14}$$

As a check, the same result can be derived at the $N_4 = 2$ level, considering a single Z_2 truncation of the $N_4 = 4$ theory. The vector multiplets ate S, T and U with manifold $[Sl(2,R)/U(1)]^3$ with prepotential [25, 26]

$$\mathcal{F}(S,T,U) = i\frac{STU}{X_0}, \tag{4.15}$$

where X_0 is the compensating scalar in the (superconformal) vector multiplet describing the $N_4 = 2$ graviphoton. The superpotential in $N_4 = 1$ language has the general form [27]

$$W = \gamma(m_I X^I - n^I \mathcal{F}_I)\Phi^i_+\Phi^i_-, \qquad \mathcal{F}_I = \frac{\partial}{\partial X^I}\mathcal{F},$$

with a numerical constant γ. The index I runs over X^0, S, T and U. After performing the algebra, we take the Poincaré gauge $X^0 = 1$. The part of the superpotential that leaves supersymmetry unbroken corresponds to the ϕ^A_{+-} terms in $\Phi^i_+\Phi^i_-$. They provide the entire BPS mass terms, electric and magnetic. The compensator contributions $\sigma_+^i\sigma_-^i$ ($i=1,2$) provide the desired breaking terms and correspond to δW, Eq.(4.13). The $N_4=2$ formulation can be useful to examine the finite temperature non-perturbative behaviour of theories based upon more general prepotentials than (4.15), such as $K3$ compactifications of heterotic strings or Calabi–Yau threefolds of type II strings.

4.1 The scalar potential

We now analyse the thermal effective potential and its instabilities. It follows from the general expression of $N_4=1$ supergravity coupled to chiral multiplets, for a given Kähler potential K and superpotential W.

Positivity of the kinetic energies and the form of the Kähler potential (4.4) impose that $s,t,u>0$, as well as non-trivial conditions on $Z_A^\pm$. In particular,

$$\sum_A (\mathrm{Re}\ Z_A^\pm)^2 < 1.$$

The scalar potential[12]

$$V = \kappa^{-4} e^K \left[(K^{-1})^i_j (W_i + W K_i)(\overline{W}^j + \overline{W} K^j) - 3|W|^2 \right],$$

is of course complicated. It can, however, be written in a closed form, which is given in the Appendix. From the analysis of the mass matrices around the vacuum $Z_A^\pm=0$, it is apparent that the discussion of thermal instabilities and of possible phase transitions only relies upon the scalar field directions

$$s = \mathrm{Re}\, S, \qquad t = \mathrm{Re}\, T, \qquad u = \mathrm{Re}\, U, \tag{4.16}$$

and, in the winding modes sector,

$$\mathrm{Re}\, Z_A^+ = \mathrm{Re}\, Z_A^- \equiv z_A.$$

Important simplifications in the potential occur then. For instance, the winding mode Kähler metric becomes diagonal:

$$(K^\pm)^A_B = \frac{\partial^2 K}{\partial Z_A^\pm \partial \overline{Z}_B^\pm} = \frac{2}{(1-x^2)^2}\,\delta^B_A, \qquad\qquad x^2 = \sum_A z_A^2,$$

[12] Using the standard notation $K_i = \frac{\partial K}{\partial z^i}$, ... Scalar fields are dimensionless.

and the kinetic terms of the scalars z_A are

$$\frac{4}{(1-x^2)^2}(\partial_\mu z_A)(\partial^\mu z_A). \tag{4.17}$$

It is interesting to observe that the resulting scalar potential is a simple fourth-order polynomial when expressed in terms of new field variables H_A, taking values on the entire real axis,

$$H_A = \frac{z_A}{1-x^2}, \qquad\qquad A = 1,2,3. \tag{4.18}$$

Defining also

$$\xi_1 = tu, \qquad \xi_2 = su, \qquad \xi_3 = st \tag{4.19}$$

($\xi_i > 0$), the potential can be nicely rewritten as

$$\begin{aligned} V &= V_1 + V_2 + V_3, \\ \kappa^4 V_1 &= \frac{4}{s}\left[(\xi_1 + \xi_1^{-1})H_1^4 + \frac{1}{4}(\xi_1 - 6 + \xi_1^{-1})H_1^2\right], \\ \kappa^4 V_2 &= \frac{4}{t}\left[\xi_2 H_2^4 + \frac{1}{4}(\xi_2 - 4)H_2^2\right], \\ \kappa^4 V_3 &= \frac{4}{u}\left[\xi_3 H_3^4 + \frac{1}{4}(\xi_3 - 4)H_3^2\right]. \end{aligned} \tag{4.20}$$

This expression displays the duality properties

$\xi_1 \;\rightarrow\; \xi_1^{-1}$: heterotic temperature duality;

$t \;\leftrightarrow\; u$, $H_2 \;\leftrightarrow\; H_3$: IIA–IIB duality.

Since at $H_i = 0$, the Kähler metric is $4\delta^A_B$, the scalar potential is normalized according to $V = 4\kappa^2 \sum_A m_A^2 H_A^2 + \ldots$ The masses m_A^2 correspond to the mass formula for the heterotic, IIA and IIB tachyons respectively:

$$\begin{aligned} m_1^2 &= \frac{1}{4\kappa^2 s}\left[\xi_1^{-1} + \xi_1 - 6\right] = \frac{1}{2\alpha'_H}\left[\frac{\alpha'_H}{2R^2} + \frac{2R^2}{\alpha'_H} - 6\right] && \text{(heterotic)}, \\ m_2^2 &= \frac{1}{4\kappa^2 t}\left[\xi_2 - 4\right] = \frac{1}{2\alpha'_{IIA}}\left[\frac{2R^2}{\alpha'_{IIA}} - 4\right] && \text{(IIA)}, \\ m_3^2 &= \frac{1}{4\kappa^2 u}\left[\xi_3 - 4\right] = \frac{1}{2\alpha'_{IIB}}\left[\frac{2R^2}{\alpha'_{IIB}} - 4\right] && \text{(IIB)}. \end{aligned}$$

The three α' scales are clearly as in Eqs.(3.14). Notice that the variables s, t and u we are using in the finite-temperature case are, with respect to the case of unbroken supersymmetry, scaled by a factor $\sqrt{2}$ when expressed in terms of stringy quantities:

$$\xi_1 = \frac{2R^2}{\alpha'_H}, \qquad \xi_2 = \frac{2R^2}{\alpha'_{IIA}}, \qquad \xi_3 = \frac{2R^2}{\alpha'_{IIB}}. \tag{4.21}$$

4.2 Phase structure of the thermal effective theory

The scalar potential (4.20) derived from our effective supergravity possesses four different phases corresponding to specific regions of the s, t and u moduli space. Their boundaries are defined by critical values of the moduli s, t, and u (or of ξ_i, $i = 1, 2, 3$), or equivalently by critical values of the temperature, the (four-dimensional) string coupling and the compactification radius R_6. These four phases are:

1. The *low-temperature* phase:
   $T < (\sqrt{2}-1)^{1/2}/(4\pi\kappa)$;
2. The *high-temperature heterotic* phase:
   $T > (\sqrt{2}-1)^{1/2}/(4\pi\kappa)$ and $g_H^2 < (2+\sqrt{2})/4$;
3. The *high-temperature type IIA* phase:
   $T > (\sqrt{2}-1)^{1/2}/(4\pi\kappa)$, $g_H^2 > (2+\sqrt{2})/4$ and $R_6 > \sqrt{\alpha'_H}$;
4. The *high-temperature type IIB* phase:
   $T > (\sqrt{2}-1)^{1/2}/(4\pi\kappa)$, $g_H^2 > (2+\sqrt{2})/4$ and $R_6 < \sqrt{\alpha'_H}$.

The distinction between phases 3 and 4 is, however, somewhat academic, since there is no phase boundary at $R_6 = \sqrt{\alpha'_H}$.

4.2.1 Low-temperature phase

This phase, which is *common* to all three strings, is characterized by

$$H_1 = H_2 = H_3 = 0, \qquad V_1 = V_2 = V_3 = 0. \tag{4.22}$$

The potential vanishes for all values of the moduli s, t and u, which are then restricted only by the stability of the phase, namely the absence of tachyons in the mass spectrum of the scalars H_i. This mass spectrum is analysed in detail in the appendix. This leads to:

$$\xi_1 > \xi_H = (\sqrt{2}+1)^2, \qquad \xi_2 > \xi_A = 4, \qquad \xi_3 > \xi_B = 4. \tag{4.23}$$

From the above condition, it follows in particular that the temperature must verify

$$T = \frac{1}{2\pi\kappa}\left(\frac{1}{\xi_1\xi_2\xi_3}\right)^{1/4} < \frac{(\sqrt{2}-1)^{1/2}}{4\pi\kappa}. \tag{4.24}$$

Since the (four-dimensional) string couplings are

$$s = \sqrt{2} g_H^{-2}, \qquad t = \sqrt{2} g_A^{-2}, \qquad u = \sqrt{2} g_B^{-2},$$

this phase exists in the perturbative regime of all three strings. The relevant light thermal states are just the massless modes of the five-dimensional $N_4 = 4$ supergravity, with thermal mass scaling like $1/R \sim T$.

4.2.2 High-temperature heterotic phase

This phase is defined by

$$\xi_H > \xi_1 > \frac{1}{\xi_H}, \qquad \xi_2 > 4, \qquad \xi_3 > 4, \tag{4.25}$$

with $\xi_H = (\sqrt{2}+1)^2$, as in Eq.(4.23). The inequalities on ξ_2 and ξ_3 eliminates type II instabilities. In this region of the moduli, and after minimization with respect to H_1, H_2 and H_3, the potential becomes

$$\kappa^4 V = -\frac{1}{s} \frac{(\xi_1 + \xi_1^{-1} - 6)^2}{16(\xi_1 + \xi_1^{-1})}.$$

It has a stable minimum for fixed s (for fixed α'_H) at the minimum of the self-dual[13] quantity $\xi_1 + \xi_1^{-1}$:

$$\xi_1 = 1, \qquad H_1 = \frac{1}{2}, \qquad H_2 = H_3 = 0, \qquad \kappa^4 V = -\frac{1}{2s}. \tag{4.26}$$

The transition from the low-temperature vacuum is due to a condensation of the heterotic thermal winding mode H_1, or equivalently by a condensation of type IIA NS five-brane in the type IIA picture.

At the level of the potential only, this phase exhibits a runaway behaviour in s. We will show in the next section that a stable solution to the effective action exists with non-trivial metric and/or dilaton.

In heterotic language, s, t and u are particular combinations of the four-dimensional gauge coupling g_H, the temperature $T = (2\pi R)^{-1}$ and the compactification radius from six to five dimensions R_6. The relations are

$$\begin{aligned} s &= \sqrt{2} g_H^{-2}, \qquad t = \sqrt{2} \frac{R R_6}{\alpha'_H}, \qquad u = \sqrt{2} \frac{R}{R_6}, \\ \xi_1 &= tu = \frac{2R^2}{\alpha'_H}, \qquad \xi_2 = \frac{2R}{g_H^2 R_6}, \qquad \xi_3 = \frac{2 R R_6}{\alpha'_H g_H^2}. \end{aligned} \tag{4.27}$$

[13]With respect to heterotic temperature duality.

As expected, ξ_2 and ξ_3 are related by radius inversion, $R_6 \rightarrow \alpha'_H R_6^{-1}$. Then, in Planck units,

$$R = \frac{1}{2\pi T} = \kappa\sqrt{stu} = \kappa[\xi_1\xi_2\xi_3]^{1/4}, \qquad R_6 = \kappa\left(\frac{2st}{u}\right)^{1/2} = \frac{\sqrt{2}\kappa\xi_3}{[\xi_1\xi_2\xi_3]^{1/4}}. \tag{4.28}$$

The first equation indicates that the temperature, when expressed in units of the *four-dimensional* gravitational coupling constant κ is invariant under string–string dualities.

In terms of heterotic variables, the critical temperatures (4.25) separating the heterotic phases are

$$\begin{aligned} \xi_1 &= \xi_H : \qquad & 2\pi T_H^< &= \frac{g_H}{2^{1/4}\kappa}(\sqrt{2}-1), \\ \xi_1 &= \tfrac{1}{\xi_H} : \qquad & 2\pi T_H^> &= \frac{g_H}{2^{1/4}\kappa}(\sqrt{2}+1). \end{aligned} \tag{4.29}$$

In addition, heterotic phases are separated from type II instabilities by the following critical temperatures:

$$\begin{aligned} \text{IIA} : \qquad \xi_2 &= 4, \qquad & 2\pi T_A &= \frac{R_6}{4\sqrt{2}\kappa^2}, \\ \text{IIB} : \qquad \xi_3 &= 4, \qquad & 2\pi T_B &= \frac{1}{2g_H^2 R_6}. \end{aligned} \tag{4.30}$$

Then the domain of the moduli space that avoids type II instabilities is defined by the inequalities $\xi_{2,3} > 4$. In heterotic variables,

$$2\pi T < \frac{1}{2\alpha'_H g_H^2}\min\left(R_6\ ;\ \alpha'_H/R_6\right) = \frac{1}{4\sqrt{2}\kappa^2}\min\left(R_6\ ;\ \alpha'_H/R_6\right). \tag{4.31}$$

Type II instabilities are unavoidable when $T > T_{\text{self-dual}}$, with

$$2\pi T_{\text{self-dual}} = \frac{1}{2g_H^2\sqrt{\alpha'_H}} = \frac{2^{1/4}}{4\kappa g_H}.$$

The high-temperature heterotic phase cannot be reached[14] for any value of the radius R_6 if

$$T_H^< > T_{\text{self-dual}},$$

or

$$g_H^2 > \frac{\sqrt{2}+1}{2\sqrt{2}} \sim 0.8536. \tag{4.32}$$

[14]From low heterotic temperature.

In this case, $T_H^<$ always exceeds T_A and T_B. Only type II thermal instabilities exist in this strong-coupling regime and the value of $R_6/\sqrt{\alpha'_H}$ decides whether the type IIA or IIB instability will have the lowest critical temperature, following Eq.(4.30).

If on the other hand the heterotic string is weakly coupled,

$$g_H^2 < \frac{\sqrt{2}+1}{2\sqrt{2}}, \tag{4.33}$$

the high-temperature heterotic phase is reached for values of the radius R_6 verifying $T_H^< < T_A$ and $T_H^< < T_B$, or

$$2\sqrt{2}g_H^2(\sqrt{2}-1) < \frac{R_6}{\sqrt{\alpha'_H}} < \frac{1}{2\sqrt{2}g_H^2(\sqrt{2}-1)}. \tag{4.34}$$

The large and small R_6 limits, with fixed coupling g_H, again lead to either type IIA or type IIB instability.

4.2.3 High-temperature type IIA and IIB phases

These phases are defined by inequalities:

$$\xi_2 < 4 \qquad \text{and/or} \qquad \xi_3 < 4. \tag{4.35}$$

In this region of the parameter space, either H_2 or H_3 become tachyonic and acquire a vacuum value:

$$H_2^2 = \frac{4-\xi_2}{8\xi_2}, \qquad \kappa^4 V_2 = -\frac{1}{t}\frac{(4-\xi_2)^2}{16\xi_2}, \tag{4.36}$$

and/or

$$H_3^2 = \frac{4-\xi_3}{8\xi_3}, \qquad \kappa^4 V_3 = -\frac{1}{u}\frac{(4-\xi_3)^2}{16\xi_3}. \tag{4.37}$$

In contrast with the high-temperature heterotic phase, the potential does not possess stationary values of ξ_2 and/or ξ_3, besides the critical $\xi_{2,3} = 4$.

Suppose for instance that $\xi_2 < 4$ and $\xi_3 > 4$. The resulting potential is then V_2 only and ξ_2 slides to zero. In this limit,

$$V = -\frac{1}{stu\kappa^4},$$

and the dynamics of $\phi \equiv -\log(stu)$ is described by the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = -\frac{e}{2\kappa^2}\left[R + \frac{1}{6}(\partial_\mu\phi)^2 - \frac{2}{\kappa^2}e^{\phi}\right].$$

Other scalar components $\log(t/u)$ and $\log(s/u)$ have only derivative couplings, since the potential only depends on ϕ. They can be taken to be constant and arbitrary. The dynamics only restricts the temperature radius $\kappa^{-2}R^2 = e^{-\phi}$, R_6 and the string coupling are not constrained, besides inequalities (4.35).

In conformally flat gravity background, the equation of motion of the scalar ϕ is

$$\Box\phi = -\frac{6}{\kappa^2}e^{\phi}.$$

The solution of the above and the Einstein equations defines a non-trivial gravitational ϕ-background. This solution will correspond to the high-temperature type II vacuum. We will not study this solution further here. Instead, we will examine in detail in Sections 5 and 6 the high-temperature heterotic phase.

4.3 Five- and six-dimensional limits

Since we have constructed the effective theory of five-dimensional strings at finite temperature, an appropriate large radius limit should lead to a six-dimensional theory at finite temperature. There should also be a small radius limit leading to a six-dimensional theory at finite temperature, since torus compactification implies a radius inversion symmetry. These decompactification limits should, however, be distinguished from those on R_6 which are taken with fixed four-dimensional coupling g_H.

The large radius R_6, type IIA limit keeps the temperature radius R and the five-dimensional coupling

$$g_5^2 = R_6 g_H^2,$$

fixed. It corresponds to

$$t \to \infty, \qquad tu \quad \text{and} \quad t/s \quad \text{fixed.} \tag{4.38}$$

On the other hand, the type IIB, small R_6 limit keeps R and the coupling

$$g_5^2 = \frac{\alpha'_H}{R_6} g_H^2$$

fixed. It corresponds to

$$u \to \infty, \qquad tu \quad \text{and} \quad u/s \quad \text{fixed.} \tag{4.39}$$

In both limits, the inequality that separates heterotic and type II instabilities is

$$g_5^2 < \sqrt{\frac{\alpha'_H}{2}}\frac{\sqrt{2}+1}{2}. \tag{4.40}$$

This relation is similar to Eq.(4.33) and follows directly from inequalities (4.34). The analysis of the five-dimensional finite-temperature mass formula has been done in Section 3, in terms of the six-dimensional string coupling λ_H. Inequality (4.40) is indeed equivalent to the bound (3.7), by simply defining the dimensionless λ_H, as in Eq.(3.2):

$$\lambda_H^2 = g_5^2 \frac{R}{\alpha'_H} < \frac{\sqrt{2}+1}{2} \frac{R}{\sqrt{2\alpha'_H}} < \left(\frac{\sqrt{2}+1}{2} \right)^2 , \tag{4.41}$$

in both type IIA and IIB theories.

As mentioned in Section 3, the above type IIB theory is defined by T-duality from the heterotic side. It differs from the type IIB theory obtained by a T-duality from type IIA [see Eq.(3.11)]. The five-dimensional limit of the latter corresponds to a limit where the heterotic string coupling λ_H goes to infinity. This follows from the duality relations (3.3) and (3.11) in the type IIB decompactification limit

$$R_6^B \to \infty \qquad \text{with} \qquad \lambda_{IIB} \qquad \text{and} \qquad \alpha'_{II} \qquad \text{fixed.}$$

This takes us outside the bound (4.41), where the non-trivial high-temperature heterotic phase is defined. A separate analysis is then needed, which is beyond the scope of this work.

An alternative type IIB theory can, however, be defined directly in five dimensions by a T-duality from type IIA, reversing the temperature radius:

$$R \to \frac{\alpha'_{II}}{R}, \qquad \lambda_{II} \to \lambda_{II} \frac{\sqrt{\alpha'_{II}}}{R} .$$

The high-temperature heterotic phase fixes $R = \sqrt{\alpha'_H}$ or $R_B = \sqrt{\alpha'_{II}} \lambda_{IIA}$ in type IIA units. In the above type IIB units, this corresponds to $\lambda_B = 1$, while the temperature radius remains undetermined. However, in order to remain in the high temperature heterotic phase, the bound (4.41) implies

$$R_B < \frac{\sqrt{2}+1}{2} \sqrt{\alpha'_{II}} . \tag{4.42}$$

In the high-temperature heterotic phase, valid in the region (4.41), the temperature is fixed in heterotic units:

$$R = \sqrt{\alpha'_H} = \sqrt{\alpha'_{II}} \lambda_{IIA} = \frac{\alpha'_{II}}{R_B}, \qquad \lambda_H = \frac{1}{\lambda_{IIA}} = \frac{\sqrt{\alpha'_{II}}}{R} = \frac{R_B}{\sqrt{\alpha'_{II}}}, \qquad \lambda_B = 1 . \tag{4.43}$$

Thus, in this phase, the *only way* to change the temperature is by varying the heterotic string tension. In particular, the infinite temperature limit $R \to 0$ is defined by $\alpha'_H \to 0$, which corresponds to the zero slope field-theory limit of the corresponding string vacuum. As we will see in the next sections, the latter is described by a non-critical superstring in six dimensions, whose zero-slope limit contains a finite number of $N_4 = 2$ massless hypermultiplets. This result supports the conjecture that the high-temperature phase is described by a topological theory [3]. From the type IIA side, one may in principle take the infinite temperature limit by keeping the string tension fixed and sending its coupling to zero. However, this correspond to $\lambda_H \to \infty$, which lies outside the domain of validity of the new phase.

Another interesting limit is the infinite type IIB temperature $R_B \to 0$, with its string tension fixed. From Eq.(4.43), this corresponds to a zero temperature heterotic theory ($R \to \infty$) with vanishing tension and zero coupling but keeping the product $\lambda_H^2 \alpha'_H$ fixed. Notice the similarity of this limit to the large-N limit in Yang–Mills theory, with the Regge slope playing the role of the effective number of degrees of freedom. This is a non-trivial limit since all genera in principle contribute. We will return to the above limits in Section 6.

5 Analysis of the high-temperature heterotic phase

The thermal phase relevant to weakly coupled, high-temperature heterotic strings at intermediate values of the radius R_6 [see inequalities (4.33) and (4.34)] has an interesting interpretation; we study this here, using the information contained in its effective theory, which is characterized by Eqs.(4.26):

$$tu = 1, \qquad H_1 = \frac{1}{2}, \qquad H_2 = H_3 = 0. \tag{5.1}$$

These values solve the equations of motion of all scalar fields with the exception of $s = \mathrm{Re}\, S$. The resulting bosonic effective Lagrangian describing the dynamics of s and $g_{\mu\nu}$ is

$$\mathcal{L}_{\rm bos} = -\frac{1}{2\kappa^2} eR - \frac{e}{4\kappa^2}(\partial_\mu \ln s)^2 + \frac{e}{2\kappa^4 s}. \tag{5.2}$$

For all (fixed) values of s, the cosmological constant is negative since $V = -(2\kappa^4 s)^{-1}$ and the apparent geometry is anti-de Sitter. But the effective theory (5.1) does not stabilize s.

To study the bosonic Lagrangian, we first rewrite it in the string frame. Defining the dilaton as

$$e^{-2\phi} = s, \tag{5.3}$$

and rescaling the metric according to

$$g_{\mu\nu} \quad \longrightarrow \quad \frac{2\kappa^2}{\alpha'_H} e^{-2\phi} g_{\mu\nu}, \tag{5.4}$$

one obtains[15]

$$\mathcal{L}_{\text{string frame}} = \frac{e^{-2\phi}}{\alpha'_H}\left[-eR + 4e(\partial_\mu\phi)(\partial^\mu\phi) + \frac{2e}{\alpha'_H}\right]. \tag{5.5}$$

The equation of motion for the dilaton then is

$$R + 4(\partial_\mu\phi)(\partial^\mu\phi) - 4\Box\phi = \frac{2}{\alpha'_H}. \tag{5.6}$$

Comparing with the two-dimensional sigma-model dilaton β-function [28] with central charge deficit $\delta c = D - 26$, which leads to

$$R + 4(\partial_\mu\phi)(\partial^\mu\phi) - 4\Box\phi = -\frac{\delta c}{3\alpha'_H}, \tag{5.7}$$

we find a central charge deficit $\delta c = -6$, or, for a superstring[16],

$$\delta\hat{c} = \frac{2}{3}\delta c = -4. \tag{5.8}$$

In the string frame, a background for theory (5.5) has flat (sigma-model) metric[17] $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ and linear dilaton dependence [29] on a spatial coordinate, say x^1:

$$\tilde{\phi} = Qx^1, \qquad\qquad Q^2 = \frac{\delta\hat{c}}{8\alpha'_H} = \frac{1}{2\alpha'_H}. \tag{5.9}$$

In the flat background, the Lagrangian density for the dilaton expanded up to quadratic order in $\varphi = \phi - Qx^1$ is

$$\mathcal{L}_{dil.} = \frac{16Q}{\alpha'_H} e^{-2Qx^1}\varphi(\partial_1\varphi) + \frac{4}{\alpha'_H} e^{-2Qx^1}(\partial_\mu\varphi)(\partial^\mu\varphi) - \frac{8}{\alpha'^2_H} e^{-2Qx^1}\varphi^2,$$

omitting a φ–independent contribution. Defining then the rescaled field

$$\hat{\varphi} = \varphi\, e^{-Qx^1}\ ,$$

[15] Since the rescaling $g_{\mu\nu} \to e^{-2\sigma} g_{\mu\nu}$ leads to $e[R + 6(\partial_\mu\sigma)^2] \to e^{-2\sigma} eR$.
[16] The same analysis in Ref.[5] is in error by a factor 2.
[17] The notation $\tilde{}$ is used for a background field.

one obtains the equivalent Lagrangian

$$\mathcal{L}_{\rm dil} = \frac{4}{\alpha'_H}\left[(\partial_\mu\hat\varphi)(\partial^\mu\hat\varphi) + \frac{1}{2\alpha'_H}\hat\varphi^2\right], \tag{5.10}$$

which indicates that a scalar field with mass

$$m^2_{\rm dil} = \frac{1}{2\alpha'_H} = Q^2 \tag{5.11}$$

propagates in the background.

A similar analysis can be applied to the axionic partner $\mathrm{Im}\, S = a$ of the supergravity dilaton $s = \mathrm{Re}\, S$. Its (bosonic) Lagrangian is simply

$$\mathcal{L}_a = -\frac{e}{4s^2\kappa^2}(\partial^\mu a)(\partial_\mu a),$$

in the Einstein frame and

$$\mathcal{L}_{a,\rm string} = -\frac{e}{2\alpha'_H}e^{-2\phi}(\partial^\mu a)(\partial_\mu a)$$

in the string frame, according to rescaling (5.4). In the linear dilaton background $\phi = \tilde\phi = Qx^1$, the rescaled axion $\hat a = e^{-Qx^1}a$ has quadratic Lagrangian

$$-\frac{1}{2\alpha'_H}\left[(\partial^\mu\hat a)(\partial_\mu\hat a) + Q^2\hat a^2\right],$$

and its mass squared is again Q^2. The same mass shift by the quantity $Q^2 = m^2_{3/2}$ will appear in all scalar masses computed in the linear dilaton background [29].

Before turning to the complete analysis of the mass spectrum in the high-temperature heterotic phase, we now establish the residual supersymmetries expected in the background chracterized by the linear dilaton dependence on x^1.

5.1 Broken supersymmetry

The linear dilaton background breaks both four-dimensional Lorentz symmetry and four-dimensional Poincaré supersymmetry. Since supersymmetry breaks spontaneously, one expects to find goldstino states in the fermionic mass spectrum and massive spin 3/2 states. And, because of the non-trivial background, the theory in the high-temperature heterotic phase is effectively a three-dimensional supergravity.

Local supersymmetry in three-dimensional space-time is notoriously difficult to establish in the presence of massive states. This is because [30, 31] masses induce asymptotically a conical geometry [32], making Noether supercharges hard to define. Supersymmetry of the vacuum does not necessarily imply supersymmetry of the massive spectrum. This phenomenon exists in (locally) flat background and the presence of the linear dilaton does not simplify the matter. In the following paragraphs, we will first consider the existence of goldstino fermions in the high-temperature vacuum, then compute the mass spectrum, which will turn out to be supersymmetric for moduli T and U (which are not massless) and perturbative heterotic windings $Z_1^\pm$. This supersymmetry in the spectrum will however be broken in the non-perturbative sector $Z_2^\pm$ and $Z_3^\pm$. Finally, by taking the five-dimensional limit discussed in the previous section, we will observe that this non-perturbative breaking of supersymmetry persists, indicating clearly, in five dimensions, broken supersymmetry.

To discuss the pattern of goldstino states, observe first that the supergravity extension of the bosonic Lagrangian (5.2) includes a non-zero gravitino mass term for all values of s since

$$m_{3/2}^2 = \kappa^{-2}\, e^{\mathcal{G}} = \frac{1}{4\kappa^2 s} = \frac{1}{2\alpha'_H} = Q^2. \tag{5.12}$$

Notice also for future use that the potential at the vacuum verifies

$$V = -\frac{2}{\kappa^4} e^{\mathcal{G}} = -\frac{1}{2\kappa^4 s} = -\frac{2}{\kappa^2}\, m_{3/2}^2. \tag{5.13}$$

Then, consider the transformation of fermions in the chiral multiplet (z^i, χ^i) [18]:

$$\delta\chi_{Li} = \frac{1}{2}\kappa(\partial\!\!\!/ z_i)\epsilon_R - \frac{1}{2}e^{\mathcal{G}/2}\,(\mathcal{G}^{-1})_i^j \mathcal{G}_j\, \epsilon_L + \ldots, \tag{5.14}$$

omitting fermion contributions. In the high-temperature heterotic phase,

$$\mathcal{G}_S = \frac{\partial}{\partial S}\mathcal{G} = -\frac{1}{2s}, \qquad\qquad \mathcal{G}_a = \frac{\partial}{\partial z^a}\mathcal{G} = 0, \tag{5.15}$$

and the Kähler metric is diagonal with $\mathcal{G}_S^S = (2s)^{-2}$. Since also

$$\partial\!\!\!/ s = -2Qs\gamma^1, \qquad e^{\mathcal{G}/2} = \kappa Q,$$

only the fermionic partner χ_s of the dilaton s participates in supersymmetry breaking, with the transformation

$$\delta\chi_s = \frac{\sqrt{s}}{2}(1-\gamma^1)\epsilon. \tag{5.16}$$

[18] The notation is as in Ref.[21], with sign-reversed $\mathcal{G}$ and $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. Indices $i, j, \ldots$, enumerate all chiral multiplets (z^i, χ^i).

Supersymmetries generated by $(1-\gamma^1)\epsilon$ are then broken in the linear dilaton background in the x_1 direction while those with parameters $(1+\gamma^1)\epsilon$ remain unbroken. Starting then from sixteen supercharges ($N_4 = 4$ supersymmetry) at zero temperature, the high-temperature heterotic vacuum has eight unbroken supercharges. Since the effective space-time symmetry is three-dimensional, the high-temperature phase has $N_3 = 4$ supersymmetry: the linear dilaton background acts identically with respect to the $N_4 = 4$ spinorial charges. It simply breaks one half of the charges in each spinor. Thus, the high-temperature phase is expected to be stable because of supersymmetry of its effective field theory and because of its superconformal content[19].

The pattern of supersymmetry breaking can be confirmed by computing the gravitino–χ_s quadratic couplings, which generate the super-Higgs phenomenon. In the linear dilaton background, the non-kinetic quadratic fermionic terms are:

$$e^{-1}\mathcal{L}_{3/2-1/2} = -Q\,\overline{\psi}_\mu\sigma^{\mu\nu}\gamma_5\psi_\nu - \frac{Q}{2s}\,\overline{\psi}_\mu(1+\gamma^1)\gamma^\mu\gamma_5\chi_s - \frac{3}{2}\frac{Q}{(2s)^2}\,\overline{\chi}_s\gamma^1\gamma_5\chi_s. \qquad (5.17)$$

We then separate the three-dimensional gravitino ψ_m, $m = 0, 2, 3$ from the spinor ψ_1, with the redefinition

$$\psi_m + \frac{1}{2}\gamma_m\gamma^1\psi_1 \quad\longrightarrow\quad \psi_m,$$

and the gravitino contributions become

$$-Q\,\overline{\psi}_m\sigma^{mn}\gamma_5\psi_n - \frac{Q}{2s}\,\overline{\psi}_m\gamma^m(1-\gamma^1)\gamma_5\chi_s.$$

These terms identify the goldstino fermion as

$$\psi_G \sim \frac{1}{2s}(1-\gamma^1)\gamma_5\chi_s, \qquad (5.18)$$

in agreement with the result (5.16), which indicates that supersymmetries with parameter $(1-\gamma^1)\epsilon$ are broken.

At the level of the background solution, one would conclude that one half of the supersymmetries remain unbroken ($N_3 = 4$). We now confront this statement with the mass spectrum in the scalar and spin 1/2 sectors of the effective supergravity.

5.2 Mass spectrum

We now analyse the complete mass spectrum of the effective supergravity theory in the linear dilaton background relevant to the high-temperature heterotic phase. This

[19] See next section.

spectrum naturally splits in two sectors. First, as already discussed in the previous paragraph, the heterotic dilaton multiplet, with scalar S and spin 1/2 partner χ_s, is actively involved in the fate of supersymmetry in the high-temperature phase and in the background. Secondly, all other chiral multiplets play a passive role in these respects. To simplify the notation, we will in this paragraph collectively denote the scalar fields T, U, Z_A^+ and Z_A^- by y_a, and their spin 1/2 partners by χ_a.

This splitting of the chiral multiplets arises because in the 'vacuum' defined by Eqs.(5.1), the Kähler function $\mathcal{G}$ does not induce any mixing of (S, χ_s) with (y_a, χ_a) [see Eqs.(5.15)] and supersymmetry breaking is entirely decided in the S sector. The Kähler metric is diagonal, $\mathcal{G}_a^S = \frac{\partial^2 \mathcal{G}}{\partial S^* \partial z^a} = 0$, for all values of the fields. In addition,

$$\mathcal{G}_{Sa} = \frac{\partial^2}{\partial S \partial y^a} \mathcal{G} = 0.$$

Notice, however, that $\mathcal{G}_{Sab} = \frac{\partial^3 \mathcal{G}}{\partial S \partial y^a \partial y^b}$ does not vanish: it will generate a contribution to the mass spectrum in the y^a sector.

The splitting of S and y^a does not exist in the low-temperature phase $H_1 = H_2 = H_3 = 0$, in which

$$\mathcal{G}_S = -(2s)^{-1}, \qquad \mathcal{G}_T = -(2t)^{-1}, \qquad \mathcal{G}_U = -(2u)^{-1}, \tag{5.19}$$

with

$$\psi_G = \frac{1}{2s}\chi_s + \frac{1}{2t}\chi_t + \frac{1}{2u}\chi_u$$

as goldstino state[20]. The low-temperature phase is symmetric in the moduli s, t and u: it is common to the three dual strings, in their perturbative and non-perturbative domains. In contrast, the high-temperature heterotic phase only exists in the perturbative domain of the heterotic string, where s is the dilaton, and, by duality, in non-perturbative type II regimes.

We have already considered the gravitino states and fields of the chiral multiplet $(S = s + ia, \chi_s)$. We now turn to the fermions χ_a and the scalars y_a. It is useful to recall that the Kähler metric in the high-temperature phase is diagonal and particularly simple:

$$\begin{aligned} &\mathcal{G}_S^S = (2s)^{-2}, \qquad \mathcal{G}_T^T = (2t)^{-2}, \qquad \mathcal{G}_U^U = (2u)^{-2}, \qquad (tu = 1), \\ &\frac{\partial^2 \mathcal{G}}{\partial Z_A^\pm \partial Z_B^{\pm *}} = \frac{2}{(1 - z_1^2)^2}\delta_{AB} = \frac{1}{2z_1^2}\delta_{AB}, \qquad (A, B = 1, 2, 3). \end{aligned} \tag{5.20}$$

[20]Expressed using non-normalized fermions. Canonical normalization of the spinors would lead to $\psi_G = \chi_s + \chi_t + \chi_u$.

The last equality follows from $H_1 = z_1/(1-z_1^2) = 1/2$. These results will be used to canonically normalize massive fields.

- *Fermions χ_a:*

The mass matrix of the spin 1/2 partners of T, U, Z_A^+ and Z_A^-, with inverse Kähler metric factors to canonically normalize fields, is simply

$$(\mathcal{M}_{1/2})_{ab} = \kappa^{-1}\, e^{\mathcal{G}/2}\, (\mathcal{G}^{-1/2})_a^c \left(\mathcal{G}_{cd} + \frac{1}{3}\mathcal{G}_c\mathcal{G}_d - \mathcal{G}_e {\mathcal{G}^{-1}}^e_f \mathcal{G}^f_{cd}\right) (\mathcal{G}^{-1/2})^d_b.$$

Since the Kähler metric is diagonal, $\mathcal{G}_b = 0$ and $\mathcal{G}^S_{cb} = 0$, the mass matrix simplifies to

$$(\mathcal{M}_{1/2})_{ab} = \kappa^{-1}\, e^{\mathcal{G}/2}\, (\mathcal{G}^{-1/2})_a^c \mathcal{G}_{cd} (\mathcal{G}^{-1/2})^d_b = m_{3/2}\, (\mathcal{G}^{-1/2})_a^c \mathcal{G}_{cd} (\mathcal{G}^{-1/2})^d_b. \tag{5.21}$$

Mixings can only arise from non-zero values of $\mathcal{G}_{ab}$ due to superpotential contributions. Since W includes a term proportional to $TUZ_1^+Z_1^-$, these four fields, which are non-zero at the vacuum, will get mixed. Masses will be completely determined (in $m_{3/2}$ units) since all parameters are fixed in this sector. On the other hand, the two fermion masses in the $Z_2^\pm$ sector are $m_{3/2}[su \pm 1]$ and $m_{3/2}[st \pm 1]$ in the $Z_3^\pm$ sector.

- *Scalars y_a:*

The high-temperature 'vacuum' is a minimum of the potential for the scalars y^a, $V_a = \frac{\partial V}{\partial y^a} = 0$, and also $\mathcal{G}_a = 0$, according to Eq.(5.15). Again, since

$$\frac{\partial^2 V}{\partial S \partial y^a} = \frac{\partial^2 V}{\partial S \partial y^*_a} = 0,$$

at the minimum, the scalar mass matrix splits into mass terms for S and a mass matrix for the scalars y^a, which is given by

$$\mathcal{M}_0^2 = \kappa^{-2} \left([\mathcal{G}^{-1/2}]^a_e\ [\mathcal{G}^{-1/2}]^f_c\right) \begin{pmatrix} V^e_g & V^{eh} \\ V_{fg} & V^h_f \end{pmatrix} \begin{pmatrix} [\mathcal{G}^{-1/2}]^g_b \\ [\mathcal{G}^{-1/2}]^d_h \end{pmatrix}.$$

The metric factors $[\mathcal{G}^{-1/2}]$ normalize the fields canonically. Each term can be computed at the high-temperature vacuum and one obtains

$$\begin{aligned}\mathcal{M}_0^2 &= m_{3/2}^2 \left([\mathcal{G}^{-1/2}]^a_e\ [\mathcal{G}^{-1/2}]^f_c\right) \begin{pmatrix} \mathcal{G}^{en} {\mathcal{G}^{-1}}^r_n \mathcal{G}_{rg} & -2sW^{-1}W^{ehS} \\ -2sW^{-1}W_{fgS} & \mathcal{G}_{fm} {\mathcal{G}^{-1}}^m_p \mathcal{G}^{ph} \end{pmatrix} \begin{pmatrix} [\mathcal{G}^{-1/2}]^g_b \\ [\mathcal{G}^{-1/2}]^d_h \end{pmatrix} \\ & \quad - m_{3/2}^2 \begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^d_c \end{pmatrix}.\end{aligned}$$

As already observed, the linear dilaton background further shifts this mass matrix by a quantity that precisely cancels the last contribution. The physical scalar mass matrix then becomes

$$\mathcal{M}_0^2 \;=\; m_{3/2}^2 \left([\mathcal{G}^{-1/2}]^a_e\ [\mathcal{G}^{-1/2}]^f_c\right) \begin{pmatrix} \mathcal{G}^{en}\mathcal{G}^{-1}{}^r_n\mathcal{G}_{rg} & -2sW^{-1}W^{ehS} \\ -2sW^{-1}W_{fgS} & \mathcal{G}_{fm}\mathcal{G}^{-1}{}^m_p\mathcal{G}^{ph} \end{pmatrix} \begin{pmatrix} [\mathcal{G}^{-1/2}]^g_b \\ [\mathcal{G}^{-1/2}]^d_h \end{pmatrix}. \tag{5.22}$$

Comparing with the fermion mass matrix (5.21), one observes that the spectrum would be supersymmetric[21] without the off-diagonal term proportional to $2sW^{-1}W^{Sij}$. Since supersymmetry breaks in the S direction, these off-diagonal contributions generate O'Raifeartaigh-type bosonic mass shifts for states that couple in the superpotential to S: these are the heterotic dyonic states $Z_2^\pm$ and $Z_3^\pm$, which generate type II thermal instabilities. As observed in Ref.[5], heterotic perturbative states have a supersymmmetric spectrum.

These supersymmetry-breaking contributions to the scalar mass spectrum imply the existence of non-perturbative modes lighter than their fermionic partners. Explicitly, since

$$-2sW^{-1}\frac{\partial^3 W}{\partial S\partial Z_2^+\partial Z_2^-} = -\frac{su}{z_1^2}, \qquad -2sW^{-1}\frac{\partial^3 W}{\partial S\partial Z_3^+\partial Z_3^-} = -\frac{st}{z_1^2}, \qquad (t=1/u),$$

the scalar mass matrix in the Z_2 sector reads

$$\begin{array}{l} Z_2^+: \\ Z_2^-: \\ Z_2^{+*}: \\ Z_2^{-*}: \end{array} \qquad m_{3/2}^2 \begin{pmatrix} (su)^2+1 & -2su & 0 & -2su \\ -2su & (su)^2+1 & -2su & 0 \\ 0 & -2su & (su)^2+1 & -2su \\ -2su & 0 & -2su & (su)^2+1 \end{pmatrix}.$$

The eigenvalues are

$$m_{3/2}^2[(su-1)^2 \pm 2su], \qquad\qquad m_{3/2}^2[(su+1)^2 \pm 2su],$$

to be compared with the fermion masses $|su-1|m_{3/2}$ and $|su+1|m_{3/2}$. The mass pattern in the $Z_3^\pm$ sector is obtained by substituting u for t in the $Z_2^\pm$ sector.

To summarize, the spectrum is supersymmetric in the perturbative heterotic and moduli sector $(T, U, Z_1^\pm)$, and with O'Raifeartaigh pattern in the non-perturbative

[21] In the sense of equal boson and fermion masses.

sectors:

$$Z_2^{\pm}: \quad m^2_{bosons} \;=\; m^2_{fermions} \pm 2su\, m^2_{3/2},$$
$$Z_3^{\pm}: \quad m^2_{bosons} \;=\; m^2_{fermions} \pm 2st\, m^2_{3/2}.$$

This phenomenon persists in the five-dimensional type IIA [and type IIB] limit (4.38) [and (4.39)], in which the $Z_3^{\pm}$ [$Z_2^{\pm}$] states become superheavy and decouple while the $Z_2^{\pm}$ [$Z_3^{\pm}$] scalar masses are shifted by a non-perturbative amount, since in this limit $su = 1/\lambda_H^2$ [$st = 1/\lambda_H^2$]. Thus, supersymmetry appears to be broken by non-perturbative effects. Note again, however, that this statement may not hold in the case of the four-dimensional background solution with a dilaton motion in one direction. In this case, there is only an effective three-dimensional Poincaré invariance, which *does not* imply in general mass degeneracy within a massive multiplet, even if local supersymmetry is unbroken [30, 31].

In the special infinite heterotic temperature limit discussed at the end of Section 4, where $\alpha'_H \to 0$, all massive states decouple and consequently one recovers $N_4 = 2$ unbroken (rigid) supersymmetry in the effective (topological) field theory of the remaining massless hypermultiplets.

6 The high-T Heterotic Phase Transition

As we discussed in Sections 4 and 5, the high-temperature phase of $N_4 = 4$ strings is described by a non-critical string with central charge deficit $\delta\hat{c} = -4$, provided the (six-dimensional) heterotic string is in the weakly coupled regime with $\lambda_H \leq (\sqrt{2}+1)/2$. One possible description is in terms of the (5+1) super-Liouville theory compactified (at least) on the temperature circle with radius fixed at the fermionic point $R = \sqrt{\alpha'_H}$. The perturbative stability of this ground state is guaranteed when there is at least $\mathcal{N}_{\rm sc} = 2$ superconformal symmetry on the world-sheet, implying at least $N_4 = 1$ supersymmetry in space-time. However, our analysis of the previous section shows that the boson–fermion degeneracy is lost at the non-perturbative level, although the ground state remains supersymmetric.

An explicit example with $\mathcal{N}_{\rm sc} = 4$ superconformal was given in Ref.[33, 34]. It is obtained when together with the temperature circle there is an additional compactified coordinate on S^1 with radius $R_6 = \sqrt{\alpha'_H}$. These two circles are equivalent to a compactification on $[SU(2) \times SU(2)]_k$ at the limiting value of level $k = 0$. Indeed, at

$k = 0$, only the 6 world-sheet fermionic $SU(2) \times SU(2)$ coordinates survive, describing a $\hat{c} = 2$ system instead of $\hat{c} = 6$ of $k \to \infty$, consistently with the decoupling of four supercoordinates, $\delta\hat{c} = -4$. The central charge deficit is compensated by the linear motion of the dilaton associated to the Liouville field, $\phi = Q^\mu x_\mu$ with $Q^2 = 1/(2\alpha'_H)$ so that $\delta\hat{c}_L = 8\alpha'_H\, Q^2 = 4$. Using the techniques developed in Refs.[35, 34], one can derive the one-loop (perturbative) partition function in terms of the left- and right-moving degrees of freedom on the world-sheet. Namely:

1. The four left- and right-moving non-compact coordinates x_μ (which include the Liouville coordinate) together with the reparametrization ghosts $b, c; \bar{b}, \bar{c}$. Their contribution to the partition function is:

$$Z\{x_\mu, b, c;\, \bar{b}, \bar{c}\} \;=\; \frac{\mathrm{Im}\tau^{-1}}{\eta^2\, \bar{\eta}^2}\,; \tag{6.1}$$

2. The two left- and right-moving coordinates $\phi_1,\ \phi_2$ compactified on $S^1_R \times S^1_{R_6}$ at the fermionic point, $R = R_6 = \sqrt{\alpha'_H}$. By fermionization the currents $\partial\phi_1$, $\partial\phi_2$ and $\bar{\partial}\phi_1$, $\bar{\partial}\phi_2$ are equivalent to four left- and four right-moving world-sheet fermions χ_I, and $\bar{\chi}_I$, $(I = 1, 2, 3, 4)$ giving rise to an $SO(4)_{\mathrm{left}} \times SO(4)_{\mathrm{right}}$ current algebra. Their contribution to the partition function is given in terms of the $SO(4)_{\mathrm{left}} \times SO(4)_{\mathrm{right}}$ characters:

$$Z\{\chi_I,\ \bar{\chi}_I\} = \frac{1}{\eta^2\, \bar{\eta}^2} \theta^2 \left[{}^{\alpha+h}_{\beta+g} \right]\ \bar{\theta}^2 \left[{}^{\bar{\alpha}+h}_{\bar{\beta}+g} \right]\,, \tag{6.2}$$

where the spin structures α, β, $\bar{\alpha}$, $\bar{\beta}$, h and g take values 0 or 1.

3. The remaining left-moving degrees of freedom Ψ_μ, $\mu = 1, \ldots, 6$, are the super-partners of the coordinates x_μ, ϕ_1, ϕ_2, and the super-reparametrization ghosts $\beta,\ \gamma$. Their partition function reads:

$$Z\{\Psi_\mu, \beta, \gamma\} \;=\; \frac{1}{\eta^4}\, \theta^2 \left[{}^{\alpha}_{\beta} \right]. \tag{6.3}$$

4. The right-moving degrees of freedom also include a conformal system, which can be described by 28 fermions $\bar{\Psi}_A$, with central charge:

$$c_R[\bar{\Psi}_A] \;=\; 26 - \;4 \text{ (from } \;x_\mu) - \;2 \text{ (from } \;\phi_1, \phi_2) - \;6 \;\left(\text{from } \;\frac{3}{2}\delta\hat{c}\right) \;=\; 14\,.$$

Their contribution to the partition function in terms of $SO(28)$ characters is:

$$Z\{\bar{\Psi}_A\} \;=\; \frac{1}{\bar{\eta}^{14}}\, \bar{\theta}^{14} \left[{}^{\bar{\alpha}}_{\bar{\beta}} \right]. \tag{6.4}$$

Assembling the above conformal blocks, one obtains the partition function of the (5+1)-dimensional Liouville model, with the desired $\mathcal{N}_{sc} = 4$ superconformal symmetry:

$$Z^{\rm Liouv}[SU(2)\times SU(2)]_{k=0} \;=\; \frac{{\rm Im}\tau^{-1}}{\eta^6\,\overline{\eta}^{18}}\,\frac{1}{8}\sum_{\alpha,\beta,\overline{\alpha},\overline{\beta},h,g}(-)^{\alpha+\beta+\alpha\beta}$$
$$\times\,\theta^{\,2}\left[{}^{\alpha}_{\beta}\right]\;\theta\left[{}^{\alpha+h}_{\beta+g}\right]\;\theta\left[{}^{\alpha-h}_{\beta-g}\right]\;\;\overline{\theta}\left[{}^{\overline{\alpha}+h}_{\overline{\beta}+g}\right]\;\overline{\theta}\left[{}^{\overline{\alpha}-h}_{\overline{\beta}-g}\right]\;\overline{\theta}^{\,14}\left[{}^{\overline{\alpha}}_{\overline{\beta}}\right]. \tag{6.5}$$

This partition function encodes a number of properties, which deserve some comments:

- The initial $N_4 = 4$ supersymmetry is reduced to $N_4 = 2$ (or $N_3 = 4$) because of the Z_2 projection generated by (h, g). This agrees with our effective field theory analysis of the high-temperature phase given in Section 5. The (perturbative) bosonic and fermionic mass fluctuations are degenerate due to the remaining $N_4 = 2$ supersymmetry.

- The $h = 0$ sector does not have any massless fluctuations due to the linear dilaton background or to the temperature coupling. The linear dilaton background shift the bosonic masses (squared) by $m^2_{3/2}$, so that all bosons in this sector have masses larger than or equal to $m_{3/2}$. This is again in agreement with our effective theory analysis. Similarly, fermion masses are shifted by the same amount because of the S^1_R temperature modification.

- In the $h = 1$ "twisted", sector there are massless excitations as expected from the (5+1) super-Liouville theory [36, 37, 34].

- The 5+1 Liouville background can be regarded as a Euclidean five-brane solution wraped on $S^1 \times S^1$ preserving one-half of the space-time supersymmetries ($N_4 = 2$).

- The massless space-time fermions coming from the $h = 1$ sector are six-dimensional spinors constructed with the Ψ_μ and β, γ. They are also spinors under the $SO(4)_{\rm right}$ constructed using $\overline{\chi}_I$, and vectors under $SO(28)$ constructed with $\overline{\Psi}_A$.

- The massless space-time bosons are in the same right-moving representation *e.g.* $SO(4)_{\rm right}$ spinors and $SO(28)$ vectors. In addition, they are spinors under $SO(4)_{\rm left}$ constructed with χ_I. Together with the massless fermions, they form 28 $N_4 = 2$ hypermultiplets.

These 28 massless hypermultiplets are the only states that survive in the zero-slope limit and their effective field theory is described by a $N_4 = 2$ sigma-model on a hyper-Kähler manifold. This topological theory corresponds to the infinite temperature limit of the $N_4 = 4$ strings after the heterotic Haggedorn phase transition.

Although the $5+1$ Liouville background is perturbatively stable due to the $\mathcal{N}_{sc} = 4$ superconformal symmetry, its stability is not ensured at the non-perturbative level when the heterotic coupling is large:

$$g_H^2(x_\mu) = e^{2(\phi_0 - Q^\mu x_\mu)} > \frac{\sqrt{2}+1}{2\sqrt{2}} \sim 0.8536. \tag{6.6}$$

Indeed, as we explained in Paragraph 4.2, the high-temperature heterotic phase only exists if $g_H^2(x_\mu)$ is lower than a critical value separating the heterotic and Type II high-temperature phases. Thus one expects a domain wall in space-time, at $x_\mu^0 = 0$, separating these two phases: $g_H^2(Q^\mu x_\mu^0) \sim 0.8536$. This domain wall problem can be avoided by replacing the $(5+1)$ super-Liouville background with a more appropriate one with the same superconformal properties, $\mathcal{N}_{\rm sc} = 4$, obeying however the additional perturbative constraint $g_H^2(x_\mu) << 1$ in the entire space-time.

Exact superstring solutions based on gauged WZW two-dimensional models with $\mathcal{N}_{\rm sc} = 4$ superconformal symmetries have been studied in the literature [38, 39, 33, 34, 40]. We now consider the relevant candidates with $\delta\hat{c} = -4$.

The first one is the $5+1$ super-Liouville with $\delta\hat{c} = 4$, already examined above. It is based on the $2d$-current algebra:

$$\begin{aligned} & U(1)_{\delta\hat{c}=4} \times U(1)^3 \times U(1)_{R^2=\alpha'_H} \times U(1)_{R_6^2=\alpha'_H} \\ \equiv\ & U(1)_{\delta\hat{c}=4} \times U(1)^3 \times SO(4)_{k=1}. \end{aligned} \tag{6.7}$$

Another class of candidate background is made up of the non-compact parafermionic spaces described by gauged WZW models:

$$\begin{aligned} & \left[\tfrac{SL(2,R)}{U(1)_{V,A}}\right]_{k=4} \times \left[\tfrac{SL(2,R)}{U(1)_{V,A}}\right]_{k=4} \times U(1)_{R^2=\alpha'_H} \times U(1)_{R_6^2=\alpha'_H} \\ \equiv\ & \left[\tfrac{SL(2,R)}{U(1)_{V,A}}\right]_{k=4} \times \left[\tfrac{SL(2,R)}{U(1)_{V,A}}\right]_{k=4} \times SO(4)_{k=1}\,, \end{aligned} \tag{6.8}$$

where indices A and B stand for the "axial" and "vector" WZW $U(1)$ gaugings.

Then, many backgrounds can be obtained by marginal deformations of the above, preserving at least $\mathcal{N}_{\rm sc} = 2$, or also by acting with S- or T-dualities on them.

As already explained, the appropriate background must verify the weak-coupling constraint:

$$g_H^2(x_\mu) = e^{2\phi} <<\sim\ 0.8536\,, \tag{6.9}$$

in order to avoid the domain-wall problem, and in order to trust the perturbative validity of the heterotic string background. This weak-coupling limitation is realized in the "axial" parafermionic space. In this background, $g_H^2(x_\mu)$ is bounded in the entire non-compact four-dimensional space, with coordinates $x_\mu = \{z, z^*, w, w^*\}$, provided the initial value of $g_0^2 = g_H^2(x_\mu = 0)$ is small.

$$\frac{1}{g_H^2(x_\mu)} = e^{-2\phi} = \frac{1}{g_0^2}\ (1 + zz*)\ (1 + ww*)\ \geq\ \frac{1}{g_0^2}. \tag{6.10}$$

The metric of this background is everywhere regular:

$$ds^2 = \frac{4dzdz^*}{1+zz^*} + \frac{4dwdw^*}{1+ww^*}\ . \tag{6.11}$$

The Ricci tensor

$$R_{z\,z^*} = \frac{1}{(1+zz^*)^2}\ , \qquad R_{w\,w^*} = \frac{1}{(1+ww^*)^2}\ . \tag{6.12}$$

The scalar curvature

$$R = \frac{1}{4(1+zz^*)} + \frac{1}{4(1+ww^*)}$$

vanishes for asymptotically large values of $|z|$ and $|w|$ (asymptotically flat space). This space has maximal curvature when $|z| = |w| = 0$. This solution has a behaviour similar to that of the Liouville solution in the asymptotic regime $|z|,\ |w|\ \to\ \infty$. In this limit, the dilaton ϕ becomes linear when expressed in terms of the flat coordinates x_i:

$$\phi = -\,\mathrm{Re}[\log z] - \mathrm{Re}[\log w] = -Q^1|x_1| - Q^2|x_2|, \tag{6.13}$$

where

$$x_1 = -\mathrm{Re}[\log z], \quad x_2 = -\mathrm{Re}[\log w], \quad x_3 = \mathrm{Im}[\log z], \quad x_4 = \mathrm{Im}[\log w],$$

and the line element is $ds^2 = 4(dx_i)^2$. The important point here is that, for large values of $|x_1|$ and $|x_2|$, $\phi \ll 0$, in contrast to the Liouville background in which $\phi = Q^1 x_1 + Q^2 x_2$, the dilaton becomes positive and arbitrarily large in one half of the space, violating the weak-coupling constraint (6.9).

We then conclude that the high-temperature phase is described by the above parafermionic space, which is stable because of $N_4 = 2$ supersymmetry. Since it is perturbative everywhere, the perturbative massive bosonic and fermionic fluctuations are always degenerate, while the non-perturbative ones are superheavy and decouple in the limit of vanishing coupling.

7 Conclusions

In this work we studied string theory at finite temperature T and the issue of Hagedorn transition, using the recent non-perturbative understanding of the theory based on string dualities. For simplicity, we restricted ourselves to the simplest case of $N_4 = 4$ compactifications in $D = 6$ dimensions, obtained by compactifying the heterotic string on T^4 or the type II string on K_3. As usual, the thermodynamics can be described by introducing an additional compactification of the (Euclidean) time on a circle of radius $R = 1/(2\pi T)$. In this context, finite temperature boundary conditions correspond to a particular gauging of the $N_4 = 4$ supersymmetry, while Hagedorn instabilities of different perturbative string descriptions appear as thermal dyonic 1/2-BPS modes that become massless (and then tachyonic) at (above) the corresponding Hagedorn temperature.

Going to four dimensions and using techniques of $N_4 = 4$ supergravity, we were able to compute the exact effective potential of all potential tachyonic modes, describing all three perturbative instabilities of $N_4 = 4$ strings (heterotic, type IIA and type IIB) simultaneously. We find that this potential has a global stable minimum in a region where the heterotic string is weakly coupled, so that the 6D string coupling $\lambda_H < (\sqrt{2} + 1)/2$. At the minimum, the temperature is fixed in terms of the heterotic string tension (or in terms of the string coupling in type IIA units), the four internal supercoordinates decouple, and the system is described by a non-critical superstring in six dimensions. Supersymmetry, although restored in perturbation theory, appears to be broken at the non-perturbative level.

On the heterotic or type IIA side, the high-temperature limit corresponds to a topological theory described by an $N_4 = 2$ supersymmetric sigma-model on a non-trivial hyper-Kähler manifold. On the type IIB side, on the other hand, the high-temperature phase corresponds to a tensionless string defined by a limit that generalizes the large-

N limit of Yang–Mills theory. It is very interesting to study in detail the properties of these theories describing the high-temperature phase of string theory, to generalize these results to other compactifications with a lower number of supersymmetries, and to study possible physical implications, e.g. in cosmology as well as in the case of TeV strings.

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Appendix

The $N = 1$ supergravity scalar potential generated by the Kähler function (4.4) and the superpotential (4.14) can be written

$$V = \kappa^{-4} e^K \left(\Delta_S + \Delta_T + \Delta_U + \Delta_+ + \Delta_-\right), \tag{A.1}$$

with

$$\begin{aligned}
\Delta_S &= |W - 2sW_S|^2 - |W|^2, \\
\Delta_T &= |W - 2tW_T|^2 - |W|^2, \\
\Delta_U &= |W - 2uW_U|^2 - |W|^2, \\
\Delta_+ &= |W|^2 + \tfrac{1}{2} Y^+ W_{A^+} (W_{A^+})^* + 2|Z^+|^2 |W - W_{A^+} Z_A^+|^2 \\
&\quad + [1 + Z^{+2} (Z^{+2})^*] |W_{A^+} Z_A^{+*}|^2 - |W - W_{A^+} Z_A^+ - Z^{+2} W_{A^+} Z_A^{+*}|^2, \\
\Delta_- &= |W|^2 + \tfrac{1}{2} Y^- W_{A^-} (W_{A^-})^* + 2|Z^-|^2 |W - W_{A^-} Z_A^-|^2 \\
&\quad + [1 + Z^{-2} (Z^{-2})^*] |W_{A^-} Z_A^{-*}|^2 - |W - W_{A^-} Z_A^- - Z^{-2} W_{A^-} Z_A^{-*}|^2.
\end{aligned} \tag{A.2}$$

Each of the above quantities is a polynomial in the fields S, T, U, $Z_A^\pm$. The notation is

$$\begin{aligned}
W_S &= \tfrac{\partial W}{\partial S}, & W_T &= \tfrac{\partial W}{\partial T}, \\
W_U &= \tfrac{\partial W}{\partial U}, & W_{A^\pm} &= \tfrac{\partial W}{\partial Z_A^\pm},
\end{aligned} \tag{A.3}$$

repeated indices are summed over $A, B = 1,2,3$, $Y^{\pm}$ has been defined in Eq.(4.5),

$$|Z^{\pm}|^2 = Z_A^{\pm} Z_A^{\pm *}, \qquad\qquad Z^{\pm 2} = Z_A^{\pm} Z_A^{\pm}$$

and $e^K = (8stuY^+Y^-)^{-1}$. Expressions (A.1 and (A.2) follow from the Kähler function K only; the structure of the superpotential has not been used. Notice also that V depends on *quadratic* combinations of the fields $Z_A^{\pm}$ and their conjugates. It is then invariant under $Z_A^{\pm} \to -Z_A^{\pm}$ and stationary at $Z_A^{\pm} = 0$ with respect to these fields. Since $V(S,T,U,Z_A^{\pm} = 0) \equiv 0$, $Z_A^{\pm} = 0$ is a stable extremum for all values of S, T and U. We will in this appendix analyse this vacuum, which corresponds to the low-temperature phase common to the three strings, compare the scalar spectrum with masses of perturbative string states, and identify the truncation relevant to the study of thermal instabilities that is used in the body of the paper.

The calculation of the scalar mass matrices is a straightforward exercise, using

$$\mathcal{G}_{A^{\pm}} = \frac{\partial \mathcal{G}}{\partial Z_A^{\pm}} = 0, \qquad \mathcal{G}_S = -\frac{1}{2s}, \qquad \mathcal{G}_T = -\frac{1}{2t}, \qquad \mathcal{G}_U = -\frac{1}{2u},$$

with $\mathcal{G} = K + \log|W|^2$. The mass matrix splits into four sectors: $Z_1^{\pm}$ (heterotic winding modes), $Z_2^{\pm}$ (IIA windings), $Z_3^{\pm}$ (IIB windings), S, T, U (moduli). As already mentioned, the moduli sector is trivially massless since the potential at $Z_A^{\pm} = 0$ is flat.

1) $Z_1^{\pm}$:

In terms of the gravitino mass

$$m_{3/2}^2 = \kappa^{-2} e^{\mathcal{G}} = \frac{1}{4\kappa^2 stu} = \frac{1}{2\alpha'_H tu} = \frac{1}{2\alpha'_{IIA} su} = \frac{1}{2\alpha'_{IIB} st}, \tag{A.4}$$

the mass matrix in the $Z_1^{\pm}$ sector is:

$$\begin{array}{c} Z_1^+ : \\ Z_1^- : \\ Z_1^{+*} : \\ Z_1^{-*} : \end{array} \qquad m_{3/2}^2 \begin{pmatrix} (tu-1)^2+2 & -2(tu-1) & -2 & -2(tu+1) \\ -2(tu-1) & (tu-1)^2+2 & -2(tu+1) & -2 \\ -2 & -2(tu+1) & (tu-1)^2+2 & -2(tu-1) \\ -2(tu+1) & -2 & -2(tu-1) & (tu-1)^2+2 \end{pmatrix}. \tag{A.5}$$

Using mass formula (3.12), the eigenstates and their masses can be identified with (perturbative) heterotic states with momentum and winding numbers m and n:

$$\begin{array}{llll}
\frac{1}{2}\,\mathrm{Re}(Z_1^+ + Z_1^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_H}\left[tu + (tu)^{-1} - 6\right], & m = \pm 1, & n = \mp 1; \\
\frac{1}{2}\,\mathrm{Im}(Z_1^+ - Z_1^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_H}\left[tu + (tu)^{-1} - 2\right], & m = 0, & n = \pm 1; \\
\frac{1}{2}\,\mathrm{Re}(Z_1^+ - Z_1^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_H}\left[tu + (tu)^{-1} + 2\right], & m = 0, & n = \pm 1; \\
\frac{1}{2}\,\mathrm{Im}(Z_1^+ + Z_1^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_H}\left[tu + 9(tu)^{-1} - 2\right], & m = \pm 1, & n = \pm 1.
\end{array}$$

The first state is the lowest would-be tachyon at the origin of the heterotic Hagedorn temperatures. The other three states cannot become tachyonic and it is then sufficient to truncate the spectrum to $\frac{1}{2}\,\mathrm{Re}(Z_1^+ + Z_1^-)$ to study the thermal instabilities induced by perturbative heterotic states.

2) $Z_2^\pm$:

The mass matrix is:

$$\begin{array}{l} Z_2^+: \\ Z_2^-: \\ Z_2^{+*}: \\ Z_2^{-*}: \end{array} \qquad m_{3/2}^2 \begin{pmatrix} (su)^2+2 & -2su & -2 & -2su \\ -2su & (su)^2+2 & -2su & -2 \\ -2 & -2su & (su)^2+2 & -2su \\ -2su & -2 & -2su & (su)^2+2 \end{pmatrix}.$$

Again, using mass formula (3.12), the eigenstates and their masses can be identified with (perturbative) type IIA states with momentum and winding numbers m and $\tilde{m}'$:

$$\begin{array}{llll}
\frac{1}{2}\,\mathrm{Re}(Z_2^+ + Z_2^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_{IIA}}\left[su - 4\right], & m = 0, & \tilde{m}' = \pm 1; \\
\frac{1}{2}\,\mathrm{Im}(Z_2^+ + Z_2^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_{IIA}}\left[su + 4(su)^{-1}\right], & m = \pm 1, & \tilde{m}' = \pm 1; \\
\frac{1}{2}\,\mathrm{Im}(Z_2^+ - Z_2^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_{IIA}}\left[su + 4(su)^{-1}\right], & m = \pm 1, & \tilde{m}' = \pm 1; \\
\frac{1}{2}\,\mathrm{Re}(Z_2^+ - Z_2^-): & \mathrm{mass}^2 = \frac{1}{2\alpha'_{IIA}}\left[su + 4\right], & m = 0, & \tilde{m}' = \pm 1.
\end{array}$$

The first state is the lowest potential tachyon and one can truncate the theory to this direction only for the study of thermal instabilities induced by type IIA perturbative states.

3) $Z_3^\pm$:

The scalar mass matrix is obtained by replacing u by t in the $Z_2^\pm$ mass matrix, as a result of IIA–IIB duality. The discussion of the mass spectrum is then similar for string states with momentum number m and IIB winding $\tilde{n}'$. Again, thermal instabilities are generated in the field direction $\frac{1}{2}\,\mathrm{Re}(Z_3^+ + Z_3^-)$ only.

In the body of the paper, we have used, in general, the scalar potential truncated to directions $Z_A^+ = Z_A^- = (Z_A^+)^* = (Z_A^-)^*$ only to enumerate the thermal phases of the theory. But we have also checked by computing the complete mass matrices that this phase structure is not modified by tachyons arising in other directions in the scalar field space.

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