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Prepotentials from Symmetric Products

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Abstract

We investigate the prepotential that describes certain F^4 couplings in eight dimensional string compactifications, and show how they can be computed from the solutions of inhomogenous differential equations. These appear to have the form of the Picard-Fuchs equations of a fibration of $\mathrm{Sym}^2(K3)$ over $\mathbb{P}^1$. Our findings give support to the conjecture that the relevant geometry which underlies these couplings is given by a five-fold.

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1. Introduction

In string theories with extended supersymmetry, BPS-saturated amplitudes [1–3] play an important rôle for non-trivial tests of various kinds of dualities. They tend to be characterized by holomorphic quantities (*e.g.* prepotentials), and this is why one often can use geometrical methods to compute them exactly. Typically, the counting of BPS states that contribute to a given amplitude can be mapped to the computation of the Euler characteristic of a space of geometric moduli. For prepotentials this generically reduces to the counting of curves in some complex manifold X, and this manifold may, or may not have a concrete physical meaning in some appropriate dual formulation of the theory. In practice, this counting is often done via mirror symmetry [4], which boils down to computing the

Some of the most canonical BPS-saturated amplitudes involve an even number, n, of external gauge bosons in theories with $4n$ supercharges in $2n$ dimensions. These amplitudes arise in heterotic string compactifications on $Y \times T^2$, where Y is some $(4 - n)$-fold. In the following, we will focus only on the subsector of the theory that depends on the familiar torus moduli T and U (neglecting any Wilson lines), and consider couplings of the form $\Delta_{F_T{}^{n-m}F_U{}^m}(T,U)F_T \wedge ...F_T \wedge F_U \wedge ...F_U$, which are saturated by 1/2-BPS states. In the heterotic string formulation, the perturbative piece is given by a one-loop amplitude that involves [5] the heterotic elliptic genus [6] $\mathcal{A}_{-n}$ in $2n + 2$ dimensions, e.g.,

$$\Delta_{F_T{}^n} = \int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \overline{p}_R^n \, q^{\frac{1}{2}|p_L|^2}\overline{q}^{\frac{1}{2}|p_R|^2} \, \mathcal{A}_{-n}(\overline{q}) \ . \tag{1.1}$$

Here, $p_L = \frac{1}{\sqrt{2T_2U_2}}(m_1 + m_2\overline{U} + n_1\overline{T} + n_2\overline{TU})$ and $p_R = \frac{1}{\sqrt{2T_2U_2}}(m_1 + m_2\overline{U} + n_1T + n_2T\overline{U})$ are the usual Narain momenta of the compactification torus T^2.

By explicitly performing the modular integral in (1.1) for general n, we find (by extensive calculations generalizing methods developed in [7,1,8–10]) that these couplings satisfy non-trivial integrability conditions.[†] These imply that the couplings

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[†] An explicit demonstration of this for $n = 6$ is given in Appendix A.

$\Delta_{F_T{}^{n-m}F_U{}^m}(T,U)$ can be written as n-fold (covariant) derivatives with respect to T, U of the following holomorphic prepotentials:

$$
\begin{aligned}
f^{\{n\}}(T,U) \;=\; & -(-1)^{n/2}\frac{ic^{\{n\}}(0)\zeta(n+1)}{2^{n+2}\pi^{n+1}} - \frac{U^{n+1}}{(n+1)!} + \mathcal{Q}(T,U) \\
& -(-1)^{n/2}\frac{i}{(2\pi)^{n+1}}\sum_{(k,l)>0}c^{\{n\}}(k\,l)\mathcal{L}i_{n+1}\Big[q_T{}^k q_U{}^l\Big].
\end{aligned}
\tag{1.2}
$$

Here, $\mathcal{Q}(T,U)$ is some undetermined n-th order polynomial in T, U and TU (with real coefficients), $q_T \equiv e^{2\pi i T}$, $q_U \equiv e^{2\pi i U}$, and $\mathcal{L}i_a(z) = \sum_{p>0}\frac{z^p}{p^a}$ is the a-th polylogarithm. The sum runs over the positive roots $k>0$, $l\in\mathbb{Z}\;\wedge\;k=0,\,l>0$, and the coefficients, $c^{\{n\}}$, are simply the expansion coefficients of the corresponding elliptic genus, $\mathcal{A}_{-n}(q) =: \sum_{k\geq-1}c^{\{n\}}(k)q^k$, which is a modular form of weight $-n$.

Of course, for $n=2$ (*i.e.*, $N=2$ supersymmetry in four dimensions) the situation is well understood; the prepotential is nothing other than the effective lagrangian of special geometry [11]. A dual formulation is given by Type II A/B strings compactified on the familiar $K3$-fibered Calabi-Yau threefold $X_{24}(1,1,2,8,12)_3^{-480}$, and its mirror. The mirror symmetry allows to exactly compute the full non-perturbative prepotential $\mathcal{F}^{\{2\}}(S,T,U)$, which also involves the dilaton modulus, S. The one-loop prepotential $f^{\{2\}}(T,U)$ in (1.2), with

$$
\mathcal{A}_{-2}(q) \;\equiv\; \frac{E_4 E_6}{\eta^{24}}(q)\;,
\tag{1.3}
$$

is then reproduced [12] in the weak coupling limit, $S\to\infty$, where the non-perturbative corrections disappear.

On the other hand, the situation is much less well understood$^{\diamond}$ for $n=4$, which corresponds to $N=1$ supersymmetry (16 supercharges) in eight dimensions, and where

$$
\mathcal{A}_{-4}(q) \;\equiv\; \frac{E_4{}^2}{\eta^{24}}(q)\;.
\tag{1.4}
$$

An interesting issue is to find a geometrical computation that would lead to the prepotential $\mathcal{F}^{(4)}(T,U)$, in an analogous manner to the more familiar computation that leads to $\mathcal{F}^{(2)}(T,U)$.

$\diamond$ Not the least because an appropriate generalization of special geometry, in which $\mathcal{F}^{(4)}(T,U)$ would figure as a superspace lagrangian, is not known. However, see [13] for some recent progress in eight dimensional lagrangians.

Since the dual formulation of the eight-dimensional heterotic compactification on T^2 is given by F-theory [14] compactified on $K3$, one would expect that $\mathcal{F}^{(4)}(T,U)$ should be computable in terms of the geometrical data of $K3$. The main puzzle is that the prepotential $\mathcal{F}^{(4)}(T,U)$ does not seem to be in any obvious way related to $K3$, but rather looks like a prepotential that would canonically come from a five-fold. This is essentially because its fifth derivatives have exactly the structure as "world-sheet instanton corrected Yukawa couplings", *i.e.* $\partial_T{}^m \partial_U{}^{5-m} \mathcal{F}^{(4)}(T,U) = \text{const} + \sum_{k,l} c^{\{4\}}(k\,l) k^m l^{5-m} \frac{q_T{}^k q_U{}^l}{1-q_T{}^k q_U{}^l}$.

Some preliminary investigations in this direction have been presented in [9,15], and in particular in [15] evidence was found that the relevant five-fold should be given by the symmetric square, $\text{Sym}^2(K3)$, fibered over $\mathbb{P}^1$ (where the size of $\mathbb{P}^1$ is eventually taken to be infinite). This structure was uncovered by investigating certain other couplings (involving four external non-abelian gauge fields), for which no prepotential exists. It is the purpose of the present paper to extend this analysis to the couplings $\Delta_{F_T{}^m F_U{}^{4-m}}$ and their prepotential $\mathcal{F}^{(4)}(T,U)$, and gather further evidence that the relevant underlying quantum geometry is given by such a five-fold.

Here we will not, however, try to answer the question as to what the physical interpretation of this five-fold might be, if there is any at all. The situation is, in this respect, somewhat similar to $N = 2$ SYM theory in four dimensions, where the Riemann surfaces underlying the effective lagrangian were found in [16], and at the time the geometry appeared to be merely a convenient mathematical tool for encoding appropriate data. It was only later that the geometry was given a much deeper physical interpretation.[‡] In the same spirit, one may speculate that the five-folds that seem to emerge here may ultimately have an interpretation in terms of a yet unknown dual formulation of the theory, or, perhaps more likely, in terms of sigma-models describing the relevant 7-brane interactions [9] that lead to the requisite F^4 terms in the effective action. Indeed, sigma models on symmetric products of $K3$ do naturally appear in D-brane physics [19], so that there is hope that we may eventually learn something substantially new about how to do exact non-perturbative computations.

In the next section, we will review how the perturbative prepotential $\mathcal{F}^{(2)}(T,U)$ arises geometrically; in particular, we will derive the inhomogenous Picard-Fuchs equations that capture the relevant information of the $K3$ fibration in the large base

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[‡] For example, as part of world-volumina of type IIA [17] or M-theory [18] fivebranes.

space limit. The motivation is, of course, to subsequently apply the *reverse* of this procedure to the eight-dimensional situation, where we want to start from the known perturbative prepotential $\mathcal{F}^{(4)}(T, U)$, to arrive at a large base space limit of some fibration. This will be done in section 3, where we will find that the periods of the fiber are given by the squares of the ordinary $K3$ periods, *i.e.* by $(1, T, U, TU, T^2, U^2, T^2U^2)$. These are precisely the periods of the hyperkähler symmetric square of $K3$, which we denote by $\mathrm{Sym}^2(K3)$. In the appendix, we formally extend this reasoning to $n = 6$ external gauge bosons, and relate $\mathcal{F}^{(6)}(T, U)$ to cubic powers of the $K3$ periods. More generally, we conclude that the prepotentials $\mathcal{F}^{(n)}(T, U)$ can be formally related to $(n+1)$-folds, given by $\mathbb{P}^1$ fibrations of symmetric products, $\mathrm{Sym}^{n/2}(K3)$. Finally, we will present some comments on curve counting in $K3$.

2. The Prepotential $\mathcal{F}^{(2)}$ in the Large Base-Space Limit

The defining polynomial of the Calabi-Yau manifold $X_{24}(1, 1, 2, 8, 12)_3^{-480}$ is given by

$$p = x_1^2 + x_2^3 + x_3^{12} + x_4^{24} + x_5^{24} - 12\psi_0 x_1 x_2 x_3 x_4 x_5 - 2\psi_1 (x_3 x_4 x_5)^6 - \psi_2 (x_4 x_5)^{12} . \quad (2.1)$$

As described in [12], this Calabi-Yau manifold may be thought of as a fibration of a $K3$ family of type $X_{12}(1, 1, 4, 6)$ over the $\mathbb{P}^1$ base defined by the coordinates x_1, x_2. Moreover this $K3$ is itself an elliptic fibration over $\mathbb{P}^1$ with generic fiber $X_6(1, 2, 3)$.

The variables that are appropriate for describing the complex structure near the point of maximal unipotent monodromy in the large complex structure limit are: $x = -\frac{2\psi_1}{1728^2 \psi_0^6}$, $y = \frac{1}{\psi_2^2}$, $z = -\frac{\psi_2}{4\psi_1^2}$. In these variables the Picard-Fuchs (PF) system, which determines the three-fold periods, becomes [20]

$$\begin{aligned}
\mathcal{D}_1^{CY} &= \theta_x (\theta_x - 2\theta_z) - 12x(6\theta_x + 5)(6\theta_x + 1) , \\
\mathcal{D}_2^{CY} &= \theta_z (\theta_z - 2\theta_y) - z(2\theta_z - \theta_x + 1)(2\theta_z - \theta_x) , \\
\mathcal{D}_3^{CY} &= \theta_y^2 - y(2\theta_y - \theta_z + 1)(2\theta_y - \theta_z) ,
\end{aligned} \quad (2.2)$$

where $\theta_x \equiv x\frac{d}{dx}$ etc. For $y \to 0$ this system degenerates to the two moduli system of the $K3$ fiber:

$$\begin{aligned}
\mathcal{D}_1^{K3} &= \theta_x^2 - 12x(6\theta_x + 5)(6\theta_x + 1) , \\
\mathcal{D}_2^{K3} &= \theta_z^2 - z(2\theta_z - \theta_x + 1)(2\theta_z - \theta_x) .
\end{aligned} \quad (2.3)$$

Denoting the flat coordinates by S, T and U, in the usual manner, the prepotential of this Calabi-Yau manifold can be written in the form

$$\mathcal{F}^{\{2\}}(S,T,U) \; = \; STU \; + \; f^{\{2\}}(T,U) \; + \; \sum_{n=1}^{\infty} g_n(T,U)\, q_S{}^n \; , \qquad (2.4)$$

where $q_S = e^{-4\pi S}$, and $y \sim q_S$ as $S \to \infty$. In this expression, the first term is the classical part, and the second term, $f^{\{2\}}(T,U)$, may be thought of as the perturbative one-loop part of the prepotential that comes from the $K3$ fiber. The last sum is over world-sheet instantons that wrap the base, which gives the non-perturbative corrections from the heterotic string point of view. Our aim is to extract the function $f^{\{2\}}(T,U)$, and compare[♮] it with the heterotic one-loop prepotential given in (1.2). To do this we must carefully take the limit $S \to \infty$ in the PF system, keeping track all the divergent and finite parts.

Let π_0 and ϖ_0 be the fundamental periods of the Calabi-Yau and the $K3$, respectively. They are the unique solutions of (2.2) and (2.3) with finite limits at $x = 0$ and $z = 0$. Then the following represents the asymptotics (as $S \to \infty$) of the Calabi-Yau three-fold periods:

$$
\begin{aligned}
\pi_0 \; &\sim \; \varpi_0 \\
T\pi_0 \; &\sim \; T\varpi_0 \\
U\pi_0 \; &\sim \; U\varpi_0 \\
\mathcal{F}_S^{\{2\}}\pi_0 \; &\sim \; TU\varpi_0 \; , \\
S\pi_0 \; &\sim \; (\log(y) \; + \; \mu_0(T,U))\varpi_0 \\
\mathcal{F}_T^{\{2\}}\pi_0 \; &\sim \; (U(\log(y) \; + \; \mu_0(T,U)) \; + \; f_T^{\{2\}}(T,U))\varpi_0 \\
\mathcal{F}_U^{\{2\}}\pi_0 \; &\sim \; (T(\log(y) \; + \; \mu_0(T,U)) \; + \; f_U^{\{2\}}(T,U))\varpi_0 \\
\mathcal{F}_0^{\{2\}}\pi_0 \; &\sim \; (TU(\log(y) \; + \; \mu_0(T,U)) \; + \; f_0^{\{2\}}(T,U))\varpi_0 \; .
\end{aligned}
\qquad (2.5)
$$

We see that in this limit, the first four CY periods turn directly into the periods of the $K3$ fiber, which are the solutions of (2.3). On the other hand, the non-trivial function that we seek, $f^{\{2\}}(T,U)$, is encoded in the remaining half of the periods. These are governed by an *inhomogenous* Picard-Fuchs system [15], whose homogenous part is exactly the system (2.3) of the $K3$ fiber, and whose source part stems from θ_y in $\mathcal{D}_2^{CY}$

[♮] Of course, this has been already done before in [12]; our purpose here is to formulate the problem in a way that allows an easy generalization to eight dimensions.

hitting $\log(y)$ (which survives the $y \to 0$ limit). More precisely, it follows from (2.2) and (2.5) that if μ_{jk} are the solutions to

$$\mathcal{D}_1^{K3}(\mu_{jk}\varpi_0) \; = \; 0 \; , \qquad \mathcal{D}_2^{K3}(\mu_{jk}\varpi_0) \; = \; T^j \, U^k \, (\theta_z \, \varpi_0) \; , \tag{2.6}$$

then we have

$$\begin{aligned}
\mu_0 &= \mu_{00} \; , & \mu_{01} &= f_T^{\{2\}} + U\mu_{00} \; , \\
\mu_{10} &= f_U^{\{2\}} + T\mu_{00} \; , & \mu_{11} &= f_0^{\{2\}} + TU\mu_{00} \; ,
\end{aligned} \tag{2.7}$$

and in particular, from homogeneity:

$$f^{\{2\}}(T,U) \; = \; \mu_{11} - T\mu_{01} - U\mu_{10} + TU\mu_{00} \; , \tag{2.8}$$

which reflects the familiar relation $\mathcal{F} = \frac{1}{2}X^A\mathcal{F}_A$ of special geometry.

To explicitly see that (2.8) indeed coincides with the heterotic one-loop expression (1.2), we first need to simplify the PF system (2.2). To accomplish this, we make a change of variables to w_1, w_2, where:

$$\begin{aligned}
x &= \frac{1}{864} \left[1 \; - \; \sqrt{(1-w_1)\,(1-w_2)} \right] \; , \\
z &= \frac{w_1 w_2}{4} \, (w_1 + w_2 - w_1 w_2)^{-2} \left[1 \; + \; \sqrt{(1-w_1)\,(1-w_2)} \right]^2 \; .
\end{aligned} \tag{2.9}$$

From the explicit expressions given in [12] it follows that simply

$$w_1 = \frac{1728}{j(T)} \; , \qquad w_2 = \frac{1728}{j(U)} \; . \tag{2.10}$$

This effectively separates variables in the PF equations, and one finds

$$\begin{aligned}
\mathcal{D}_1^{K3} &= \frac{1728\,x}{w_1 - w_2} \left[w_1 \, \mathcal{L}_{w_1} \; - \; w_2 \, \mathcal{L}_{w_1} \right] \\
\mathcal{D}_2^{K3} &= -\frac{w_1 w_2}{w_1 - w_2} \left[\mathcal{L}_{w_1} \; - \; \mathcal{L}_{w_2} \right] \\
\theta_z &= -\frac{w_1 w_2}{w_1 - w_2} \left[(1-w_1)\frac{d}{dw_1} \; - \; (1-w_2)\frac{d}{dw_2} \right] \; ,
\end{aligned} \tag{2.11}$$

where $\mathcal{L}_w$ is the second order hypergeometric operator

$$\mathcal{L}_w \; \equiv \; \frac{1}{w} \left[\theta_w^2 \; - \; w \, (\theta_w + \frac{5}{12})(\theta_w + \frac{1}{12}) \right] \; . \tag{2.12}$$

The fundamental period ϖ_0 of the $K3$ must therefore satisfy $\mathcal{L}_{w_1}\varpi_0 = \mathcal{L}_{w_2}\varpi_0 = 0$, and hence it must have the form $\varpi_0 = \omega_0\widetilde{\omega}_0$, where ω_0 is given by the fundamental series solution of (2.12):

$$\omega_0(w) = \,_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1, w\right) = (E_4)^{1/4} . \tag{2.13}$$

with $w = w_1$, and $\widetilde{\omega}_0$ is the same function but with $w = w_2$. Using (2.11) the equations (2.6) can be rewritten as

$$\mathcal{L}_{w_a}(\mu_{jk}\,\varpi_0) = -\frac{1}{w_a}\frac{2w_1w_2}{w_1 - w_2}\left[(1 - w_1)\frac{d}{dw_1} - (1 - w_2)\frac{d}{dw_2}\right](T^jU^k\varpi_0) . \tag{2.14}$$

From this and (2.7) it follows, for example, that:

$$w_1\mathcal{L}_{w_1}(f_T^{\{2\}}\,\varpi_0) = \frac{w_1w_2}{w_1 - w_2}(1 - w_2)\frac{dU}{dw_2}\,\varpi_0 .$$

Using (2.10) and the identity[†] [15]

$$w\,\mathcal{L}_w\,(f(w)\omega_0(w)) = \frac{1}{E_4(T)}\,(\theta_{q_T}^2\,f(w(T)))\,\omega_0 \tag{2.15}$$

(for any function $f(w)$), we finally see that:

$$(\theta_{q_T}^2\,f_T^{\{2\}}) = -E_4(T)\frac{w_1w_2}{w_1 - w_2}(1 - w_2)\frac{dU}{dw_2} = \frac{E_4(T)\,E_4(U)E_6(U)}{[j(T) - j(U)]\eta^{24}(U)} . \tag{2.16}$$

This coincides exactly with the known [21,12] expression for $f_{TTT}^{\{2\}}(T,U)$.

Summarizing, we have shown how the perturbative component of the quantum prepotential can be obtained directly from the $K3$ Picard-Fuchs equations with properly chosen sources, and these sources are simply derivatives of the $K3$ periods.

We now briefly indicate how to reverse this process, and in the next section we will use this method to construct differential equations whose solutions lead to the other $f^{\{n\}}(T,U)$.

It turns out that the strongest single constraint on the form of the differential operators comes from the explicit form of the dilaton, which is essentially the difference

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† This identity is straightforward and is simply the result of a the change of variables (2.10).

(at large S) between the solution μ_{00} and the manifestly modular invariant quantity $\log(y)$. Since the dilaton is non-singular at $T = U$, this solution must have the form

$$\mu_{00} = 2\pi i \, f_{TU}^{\{2\}} - \log(j(T) - j(U)) \,. \qquad (2.17)$$

The general idea is to first obtain a differential equation for $f_{TU}^{\{2\}}$, by inserting it into the identity (2.15) with $w = w_1(T)$. The right-hand side of this equation, which represents the source part, is then given by $(1/E_4(T))(\partial_U f_{TTT}^{\{2\}})\omega_0$, which can be evaluated by using the known expression (2.16) for $f_{TTT}^{\{2\}}(T, U)$. After subtracting the logarithmic singularity, this leads precisely to the source term on the RHS of the Picard-Fuchs system (2.14).

3. Generalizations

Assuming that the Picard-Fuchs equations we seek for $n = 4$ generalize the structure we found above, we will try to construct differential equations for $f_{T^{n/2}U^{n/2}}^{\{n\}}(T, U)$ by applying the simple procedure outlined earlier. However, before doing that, we will first discuss some general features of the homogenous PF equations for arbitrary n.

3.1. Symmetric Powers of Picard-Fuchs Operators

Crucial to our arguments will be the following sequence of differential operators:

$$
\begin{aligned}
\mathcal{L}_w^{\otimes 1} &\equiv \frac{1}{w} \left[\theta_w^2 - w \, (\theta_w + \frac{5}{12})(\theta_w + \frac{1}{12}) \right] \equiv \mathcal{L}_w \,, \\
\mathcal{L}_w^{\otimes 2} &\equiv \frac{1}{w} \left[\theta_w^3 - w \, (\theta_w + \frac{5}{6})(\theta_w + \frac{1}{2})(\theta_w + \frac{1}{6}) \right] \,, \\
\mathcal{L}_w^{\otimes 3} &\equiv \frac{1}{w} \left[\theta_w^4 - 2w \, (\theta_w + \frac{1}{4})(\theta_w^3 + \frac{5}{4}\theta_w^2 + \frac{31}{36}\theta_w + \frac{5}{24}) \right. \\
&\qquad \left. + \; w^2 \, (\theta_w + \frac{5}{4})(\theta_w + \frac{11}{12})(\theta_w + \frac{7}{12})(\theta_w + \frac{1}{4}) \right] \,, \quad \text{etc.,}
\end{aligned}
\qquad (3.1)
$$

where $\mathcal{L}_w^{\otimes 1} \equiv \mathcal{L}_w$ is identical to the hypergeometric operator in (2.12). The $(m+1)^{\text{th}}$ order operator $\mathcal{L}_w^{\otimes m}$ is what is called the "m^{th} symmetric power" of the basic operator $\mathcal{L}_w$, the reason being that its solution space is the mth symmetric product of the solution space of $\mathcal{L}_w$. The notion of symmetric powers of differential operators has been discussed in the mathematical literature, *e.g.* in [22] and in [23,24], where also a systematic procedure for computing them has been described.

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More explicitly, while the fundamental solutions to $\mathcal{L}_w \omega_i(w) = 0$ are given by the periods

$$\omega_0(w) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1, w\right) = (E_4)^{1/4}, \quad \omega_1(w) = T\,\omega_0 = T\,(E_4)^{1/4}, \quad (3.2)$$

the solutions of $\mathcal{L}_w^{\otimes m}$ are given by

$$\begin{aligned}
\omega_j^{\otimes 2}(w) &= \omega_{j-i}\,\omega_i = T^j\,(E_4)^{1/2}, & j &= 0, 1, 2 \\
\omega_k^{\otimes 3}(w) &= \omega_{k-j}\,\omega_{j-i}\,\omega_i = T^k\,(E_4)^{3/4}, & i, j, k &= 0, 1, 2, 3,
\end{aligned} \quad (3.3)$$

and so on. Moreover, we find that these operators satisfy certain identities when filtered through the mirror map, $w = 1728/j(T)$: for any function $f(z)$ one has

$$\begin{aligned}
w\,\mathcal{L}^{\otimes 1}\,(f(w)\omega_0(w)) &= \frac{1}{E_4(T)}\,\left(\theta_{q_T}{}^2\,f(w(T))\right)\omega_0, \\
w\,\mathcal{L}^{\otimes 2}\,(f(w)\omega_0(w)^2) &= \frac{1}{E_6(T)}\,\left(\theta_{q_T}{}^3\,f(w(T))\right)\omega_0{}^2, \\
w\,\mathcal{L}^{\otimes 3}\,(f(w)\omega_0(w)^3) &= \frac{1}{E_4^2(T)}\,\left(\theta_{q_T}{}^4\,f(w(T))\right)\omega_0{}^3, \quad etc.
\end{aligned} \quad (3.4)$$

These identities will prove important momentarily.

3.2. Determination of the source terms

Note that the prepotentials (1.2) have the property that $\partial_T^{n+1} f^{\{n\}}(T, U)$ is a good modular function of weights $(n+2, -n)$ in (T, U), and must have a simple pole at $T = U$ (which reflects gauge symmetry enhancement to $SU(2)$). From this one can deduce the functional form. For example, one has:

$$\begin{aligned}
\partial_T^3\,f^{\{2\}}(T, U) &= \frac{E_4(T)\,E_4(U)E_6(U)}{[J(T) - J(U)]\,\eta^{24}(U)}, \\
\partial_T^5\,f^{\{4\}}(T, U) &= \frac{E_6(T)\,E_4^2(U)}{[J(T) - J(U)]\,\eta^{24}(U)}, \\
\partial_T^7\,f^{\{6\}}(T, U) &= \frac{E_4^2(T)\,E_6(U)}{[J(T) - J(U)]\,\eta^{24}(U)}, \quad etc.
\end{aligned} \quad (3.5)$$

Suppose we set $w = w_1$ in (3.4), and take the f to be $\partial_T^m \partial_U^m f^{\{2m\}}(T, U)$. The right-hand side of the mth equation in (3.4) can then be rewritten using m U-derivatives of the mth identity in (3.5). The resulting right-hand side is completely modular of T,

and almost modular in U. Indeed, the right-hand side is m^{th} order in $E_2(U)$. These factors of E_2 may be traded for derivatives of the fundamental periods as follows: One first notes that the fundamental periods of the various PF systems can be written as $\varpi_0^m \equiv \omega_0^m \widetilde{\omega}_0^m$, where $\omega_0^m \equiv E_4(T)^{m/4}$ and $\widetilde{\omega}_0^m \equiv E_4(U)^{m/4}$. Therefore, one can express the w_2-derivatives of the periods in terms of U derivatives to obtain:

$$\theta_{w_2}\widetilde{\omega}_0^m = \frac{m\widetilde{\omega}_0^m}{4E_4}\,\theta_{w_2}(E_4) = \frac{m\widetilde{\omega}_0^m}{12E_4}\,w_2\left(\frac{dw_2}{dU}\right)^{-1}(E_2E_4 - E_6) = \frac{m\widetilde{\omega}_0^m}{12}\left(\frac{E_2E_4}{E_6} - 1\right).$$

More generally, $(\theta_{w_2})^p\,\widetilde{\omega}_0^m$ may be written in terms of a polynomial of degree p in $E_2(U)$. Conversely, a polynomial of degree p in $E_2(U)$ may be expressed as a linear differential operator of order p in w_2, acting on ϖ_0^m. In this way, one can use (3.5) and (3.4) to determine the right-hand sides of $\mathcal{L}_{w_a}^{\otimes(n/2)}\big[\partial_T^{\,n/2}\partial_U^{\,n/2}f^{\{n\}}\,\varpi_0^{n/2}\big]$. The resulting expressions have poles in $(w_1 - w_2)$ of orders up to $(n/2+1)$. To arrive at a PF system similar to (2.11) one can tolerate at most single poles in the source terms (as in (2.14); this ensures that the "dilaton" period will be non-singular at $T = U$, c.f. eq. (2.17)). The leading pole can be cancelled by the addition of a suitable multiple of $\log(w_1 - w_2)$. The subleading poles can then be cancelled by the addition of multiples of $\partial_T^{\,k/2}\partial_U^{\,k/2}f^{\{k\}}$ for $k < n$.

At the end of this iterative procedure, one arrives at a pair of inhomogenous Picard-Fuchs equations of the general form,

$$\mathcal{L}_{w_a}^{\otimes(n/2)} \cdot \mu_{00}^{\{n\}}\varpi_0^{n/2} = \mathcal{M}_a^{\{n/2\}} \cdot \varpi_0^{n/2}\,, \qquad a = 1, 2, \tag{3.6}$$

which generalizes (2.14) and whose source part involves some $(n/2)^{\text{th}}$-order operators $\mathcal{M}_a^{\{n/2\}}$. The homogenous, "fiber" part consists of two copies of the symmetric product of $\mathcal{L}_w$, whose solutions look, after dividing out the fundamental period $\varpi_0^{n/2}$, like

$$(1, T, U, TU, T^2, U^2, ..., (TU)^{n/2})\,. \tag{3.7}$$

These are the periods of the $n/2$-fold symmetric product, $\text{Sym}^{n/2}(K3)$.

3.3. Explicit Results for $n = 4$

By following the steps described above, we find for $n = 4$ (which corresponds to the eight-dimensional compactification) that

$$\mu_{00}^{\{4\}} = 2\pi i (f_{TTUU}^{\{4\}} + 3f_{TU}^{\{2\}}) - 2\log(w_1 - w_2) \,,$$

satisfies the following inhomogenous PF equation:

$$\mathcal{L}_{w_1}^{\otimes 2} \cdot \mu_{00}^{\{4\}} \varpi_0{}^2 = \frac{6w_2}{(w_1 - w_2)} \left[\mathcal{L}_{w_1} + \mathcal{L}_{w_2} + w_1(1 - w_2)\frac{d^2}{dw_1 dw_2} - \tfrac{5}{72} \right] \cdot \varpi_0{}^2 \,, \quad (3.8)$$

along with the corresponding equation for $\mathcal{L}_{w_2}^{\otimes 2}(\mu_{00}^{\{4\}} \varpi_0{}^2)$ obtained by interchanging w_1 and w_2. Since (3.8) only involves a simple pole in $w_1 - w_2$, one can take sums or differences of the equations for $\mathcal{L}_{w_1}^{\otimes 2}$ and $\mathcal{L}_{w_2}^{\otimes 2}$ so as to cancel the pole, and obtain a form that more closely resembles the PF system of a manifold. In particular, one can write:

$$\left(w_1 \mathcal{L}_{w_1}^{\otimes 2} + w_2 \mathcal{L}_{w_2}^{\otimes 2}\right) \cdot \mu_{00}^{\{4\}} \varpi_0{}^2 = 6\frac{d^2}{dw_1 dw_2}\varpi_0{}^2 \,,$$

$$\left((1 - w_1)\mathcal{L}_{w_1}^{\otimes 2} + (1 - w_2)\mathcal{L}_{w_2}^{\otimes 2}\right) \cdot \mu_{00}^{\{4\}} \varpi_0{}^2 = 12 \left[\mathcal{L}_{w_1} + \mathcal{L}_{w_2} - \tfrac{5}{72} \right] \cdot \varpi_0{}^2 \,. \quad (3.9)$$

Having now obtained equations for the "fundamental" inhomogenous solution $\mu_{00}^{\{4\}}$, we can now investigate the full set of solutions $\mu_{jk}^{\{4\}}$, for which $\varpi_0{}^2$ on the right-hand side of (3.8) or (3.9) is replaced by $T^j U^k \varpi_0{}^2$. One can then verify that the partial derivatives of $f^{\{4\}}$ are related to $\mu_{jk}^{\{4\}}$ in a manner completely analogous to (2.7). Explicitly, abbreviating $\mu \equiv \mu^{\{4\}}$, one has:

$$
\begin{aligned}
\mu_{01} - U\mu_{00} &= -6\pi i \ (f_T^{\{2\}} + f_{TTU}^{\{4\}}) \\
\mu_{10} - T\mu_{00} &= -6\pi i \ (f_U^{\{2\}} + f_{TUU}^{\{4\}}) \\
\mu_{02} - 2U\mu_{01} + U^2\mu_{00} &= -24\pi i \ f_{TT}^{\{4\}} \\
\mu_{20} - 2T\mu_{10} + T^2\mu_{00} &= -24\pi i \ f_{UU}^{\{4\}} \\
\mu_{11} - U\mu_{10} - T\mu_{01} + TU\mu_{00} &= -6\pi i \ (f^{\{2\}} + 3f_{TU}^{\{4\}}) \\
(\mu_{12} - 2U\mu_{11} + U^2\mu_{10}) - T(\mu_{02} - 2U\mu_{01} + U^2\mu_{00}) &= -72\pi i \ f_T^{\{4\}} \\
(\mu_{21} - 2T\mu_{11} + T^2\mu_{01}) - U(\mu_{20} - 2T\mu_{10} + T^2\mu_{00}) &= -72\pi i \ f_U^{\{4\}} \,,
\end{aligned}
\quad (3.10)
$$

and in particular:

$$216\pi i \; f^{\{4\}}(T,U) \;\; = \;\; (\mu_{22} - 2U\mu_{21} + U^2\mu_{20}) - 2T(\mu_{12} - 2U\mu_{11} + U^2\mu_{10})+$$
$$T^2(\mu_{02} - 2U\mu_{01} + U^2\mu_{00}) \; . \tag{3.11}$$

One can prove these relations by first differentiating both sides sufficiently often with respect to T until the left-hand side can be simplified using (3.4) combined with the differential equations satisfied by the $\mu_{jk}^{\{4\}}$, while the right-hand side is simplified using (3.5). This process is then repeated for the U-derivatives of the (3.10). The success of this procedure critically depends on the proper form of the (3.8) and provides a significant number of non-trivial tests upon the form of (3.8).

3.4. Periods of a Five-Fold?

Eq. (3.11) is a direct analog of the classic special geometry relation (2.8), and reflects how the periods of the suspected five-fold would assemble into the prepotential. It thus appears as a good starting point for unraveling the analog of special geometry in eight dimensions.

In this context, it is instructive to go one step further and try to infer how (3.9) and the prepotential $f^{\{4\}}(T,U)$ could arise from a PF system of a 5-fold and a corresponding prepotential $\mathcal{F}(S,T,U)$, respectively. Recall that for $f^{\{2\}}(T,U)$ and the 3-fold the periods are $\pi_0, S\pi_0, T\pi_0, U\pi_0$ and $\mathcal{F}_S\pi_0, \mathcal{F}_T\pi_0, \mathcal{F}_U\pi_0, \mathcal{F}_0\pi_0$, and in the $S \to \infty$ limit $\pi_0, T\pi_0, U\pi_0$ and $\mathcal{F}_S\pi_0$ become the periods of the $K3$ fiber, while the finite parts of $\mathcal{F}_T\pi_0, \mathcal{F}_U\pi_0$ and $\mathcal{F}_0\pi_0$ satisfy the $K3$ PF system with sources, and give rise to $f^{\{2\}}(T,U)$ and its first derivatives.

Based upon this, and remembering the structure (3.7) of the homogenous solutions, we conjecture (in line with the findings of [15]) that the 5-fold is the hyper-Kähler 4-fold $\mathrm{Sym}^2(K3)$ fibered over a $\mathbb{P}^1$ base. As mentioned above, the periods of the fiber are $T^j U^k \varpi_0^2$, $j,k = 0,1,2$, and these arise in the 5-fold as the $S \to \infty$ limit of $\pi_0, S\pi_0, T\pi_0, U\pi_0$ and $\mathcal{F}_S^{\{4\}}\pi_0, \mathcal{F}_{ST}^{\{4\}}\pi_0, \mathcal{F}_{SU}^{\{4\}}\pi_0, \; \mathcal{F}_{STT}^{\{4\}}\pi_0, \mathcal{F}_{STU}^{\{4\}}\pi_0, \mathcal{F}_{SUU}^{\{4\}}\pi_0$. Thus only the fiber periods that are linear in T and U are realized directly. From (3.10) it appears that only the derivatives $\mathcal{F}_{TTU}^{\{4\}}, \mathcal{F}_{TUU}^{\{4\}}$, and by extension $\mathcal{F}_{STT}^{\{4\}}\pi_0, \mathcal{F}_{STU}^{\{4\}}\pi_0$ and $\mathcal{F}_{SUU}^{\{4\}}\pi_0$ will actually appear directly as 5-fold periods. Moreover, as with $\mathcal{F}_0^{\{2\}}$, lower order derivatives of $\mathcal{F}^{\{4\}}$ will appear in the periods as combinations like $\mathcal{F}_{TT}^{\{4\}} + \frac{1}{2}U(\mathcal{F}_T^{\{2\}} + \mathcal{F}_{TTU}^{\{4\}})$. The proper combinations are inferred from how the source equations arise in the $S \to \infty$ limit of the 3-fold, and based upon this we

expect that the combinations that would arise from a 5-fold will be those of the form $\mu_{jk}^{\{4\}} - T^j U^k \mu_{00}^{\{4\}}$.

We could not explicitly verify this conjecture, simply because there is no known algebraic representation of $\mathrm{Sym}^2(K3)$, and even less, of the relevant $\mathbb{P}^1$ fibration of it. An algebraic or toric representation would however be necessary for obtaining the Picard-Fuchs system. The closest one seems to be able to get at, is the beautiful construction of Beauville and Donagi [25], which leads to the periods and Picard-Fuchs equations of the holomorphic $(2,0)$-form of $\mathrm{Sym}^2(K3)$. Unfortunately, there does not seem to be any simple way to obtain from this the periods of the $(5,0)$-form of the $\mathbb{P}^1$ fibration.

In the absence of such explicit algebraic representations, we can thus far only conclude that our results provide further evidence for the conjectured five-fold, augmenting the findings of ref. [15]. Summarizing, our main results supporting this structure are: a) the form (3.7) of the homogenous solutions, which corresponds to a fibration of $\mathrm{Sym}^2(K3)$, and b) the writing (3.11) of the prepotential $f^{\{4\}}(T, U)$ in terms of the inhomogenous solutions of the PF equations.

4. Some remarks on curve counting

In the compactification to four dimensions, sending $S \to \infty$ corresponds to the large base space limit of the $K3$ fibration. Therefore, the coefficients $c^{\{2\}}$ of $\mathcal{A}_{-2}(q)$ in (1.3) must correspond to counting certain "rational curves" in the $K3$ fiber. However, it is known that other $K3$ fibrations lead to different counting functions, see, for example, [26]. Moreover, a generic $K3$ has no rational curves at all. Counting rational curves in $K3$ thus depends upon how one broadens the concept. By considering $\mathcal{A}_{-2}(q) = E_4 E_6 / \eta^{24}$ we count the 2-cycles in $K3$ that become rational curves in our particular choice of fibration over $\mathbb{P}^1$.

The most canonical way to count rational curves in $K3$ was presented in [27], where one counts certain singular curves that are holomorphic in a given, fixed complex structure; the relevant counting function in this instance is simply given by η^{-24}. As was shown in [28], this can be obtained by the trivial fibration $K3 \times \mathbb{P}^1$, where $\mathbb{P}^1$ corresponds to the twistor family of complex structures in the hyper-Kähler $K3$. This reasoning does not involve mirror symmetry, and indeed $K3 \otimes \mathbb{P}^1$ is not even a Calabi-Yau space.

Our point is that it is in eight dimensions where one can compute the counting function η^{-24} via mirror symmetry. More precisely, in our computation the counting function was $\mathcal{A}_{-4}(q) = E_4{}^2/\eta^{24}$ (1.4), and the difference as compared to four dimensions is that the E_4's can be removed by incorporating the $E_8 \times E_8$ Wilson lines $\vec{V}$ in the prepotential. That is, as mentioned in [9], extending the sum over the $E_8 \times E_8$ lattice one can write

$$f^{\{4\}}(T, U, \vec{V}) \ \sim \ \sum_{\substack{(k,l,\vec{r}) > 0 \\ \vec{r} \in \Lambda_{E_8 \times E_8}}} \widetilde{c}^{\{4\}}(kl - \vec{r}^{\,2}/2) \, \mathcal{L}i_5\big[e^{2\pi i(kT + lU + \vec{r}\cdot\vec{V})}\big] \ , \qquad (4.1)$$

where

$$\eta(q)^{-24} \ \equiv \ \frac{1}{q} \prod_{l \geq 1}(1 - q^l)^{-24} \ =: \ \sum_{n \geq -1} \widetilde{c}^{\{4\}}(n) q^n \qquad (4.2)$$

is exactly the counting function of [27,28]. This function is known to count $1/2$-BPS states in $K3$ compactifications of the IIA theory [19]. Here we find that it also counts $1/2$-BPS states in F-theory on $K3$, in line with the arguments in [2] for the heterotic string in eight dimensions.

Thus, what we have been arguing in this paper is, essentially, how to determine this counting function via the mirror map.[†] While on the one hand $K3 \times \mathbb{P}^1$ is not a Calabi-Yau space, and on the other, non-trivial $K3$ fibrations over $\mathbb{P}^1$ do not lead to η^{-24}, it appears that the appropriate geometry to obtain (4.2) from mirror symmetry is a fibration of $\mathrm{Sym}^2(K3)$ over $\mathbb{P}^1$.

Acknowledgements

[†] To do this completely would require incorporating the 16 Wilson line moduli in the differential equations.

Appendix A. Formal extension to $n = 6$

The mathematical structure of the prepotentials (1.2) can be considered for any value of n, and for any modular form $\mathcal{A}_{-n}(q)$ of weight $-n$. From the physical point of view the generalization appears to be purely formal. On the "heterotic side" we would be need to start in $2n+2$ dimensions, and consider a toroidal compactification to give an amplitude $(\mathrm{Tr}\,(F^n))$ in $2n$-dimensions. Of course, there are no such superstrings for $n > 4$. However, the situation is reminiscent of anomaly cancellation [29], and is indeed related to it: the mathematical mechanism is very general, being just based on modular properties of the elliptic genus, and works in "string theories" in any dimension, no matter how pathological their physical meaning.

We demonstrate here that the prepotential (1.2) makes formally sense for $n = 6$, even though there is no known consistent string theory whose amplitudes it would describe. Just from modular properties we must have that $\mathcal{A}_{-6}(q) = E_6/\eta^{24}$ and so the relevant "one-loop amplitudes" are of the form:

$$\Delta_{F_T^6} = \frac{(U - \overline{U})^3}{(T - \overline{T})^3} \int \frac{d^2\tau}{\tau_2} \sum_{(p_L, p_R)} \overline{p}_R^6 \; q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \; \frac{\overline{E}_6}{\overline{\eta}^{24}} \;,$$

$$\Delta_{F_T^3 F_U^3} = \int \frac{d^2\tau}{\tau_2} \sum_{(p_L, p_R)} \left[|p_R|^6 - \frac{9}{2\pi\tau_2}|p_R|^4 + \frac{9}{2\pi^2\tau_2^2}|p_R|^2 - \frac{3}{4\pi^3\tau_2^3} \right] q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \frac{\overline{E}_6}{\overline{\eta}^{24}}$$

$$\tag{A.1}$$

and similar expressions for $\Delta_{F_T^5 F_U}$, $\Delta_{F_T^4 F_U^2}$. These couplings integrate to one and the same holomorphic prepotential $f^{\{6\}}(T, U)$, given by (1.2) for $n = 6$. Explicitly:

$$\begin{aligned}
\Delta_{F_T^6} = & -32\pi i \left(\partial_T + \frac{4}{T - \overline{T}} \right) \left(\partial_T + \frac{2}{T - \overline{T}} \right) \partial_T \\
& \times \left(\partial_T - \frac{2}{T - \overline{T}} \right) \left(\partial_T - \frac{4}{T - \overline{T}} \right) \left(\partial_T - \frac{6}{T - \overline{T}} \right) f^{\{6\}}(T, U) \\
& + 32\pi i \frac{(U - \overline{U})^6}{(T - \overline{T})^6} \left(\partial_{\overline{U}} - \frac{4}{U - \overline{U}} \right) \left(\partial_{\overline{U}} - \frac{2}{U - \overline{U}} \right) \partial_{\overline{U}} \\
& \times \left(\partial_{\overline{U}} + \frac{2}{U - \overline{U}} \right) \left(\partial_{\overline{U}} + \frac{4}{U - \overline{U}} \right) \left(\partial_{\overline{U}} + \frac{6}{U - \overline{U}} \right) \overline{f}^{\{6\}}(T, U)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\Delta_{F_T^3 F_U^3} = & -32\pi i \left(\partial_U - \frac{2}{U - \overline{U}} \right) \left(\partial_U - \frac{4}{U - \overline{U}} \right) \left(\partial_U - \frac{6}{U - \overline{U}} \right) \\
& \times \left(\partial_T - \frac{2}{T - \overline{T}} \right) \left(\partial_T - \frac{4}{T - \overline{T}} \right) \left(\partial_T - \frac{6}{T - \overline{T}} \right) f^{\{6\}}(T, U) + h.c. \; .
\end{aligned}$$

The correction $\Delta_{F_T^6}$ represents a function of weights $(w_T, w_U) = (6, -6)$ and $(w_{\overline{T}}, w_{\overline{U}}) = (0, 0)$, respectively. While it is not fully harmonic, a holomorphic, covariant quantity may be obtained via an additional T–modulus insertion, by considering

$$f^{\{6\}}_{TTTTTTT} = \frac{i}{16} \frac{(U - \overline{U})^3}{(T - \overline{T})^4} \int d^2\tau \sum_{(p_L, p_R)} p_L \overline{p}_R^7 \, q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \frac{\overline{E}_6}{\overline{\eta}^{24}} \ .$$

It is a non-trivial feature that this integral indeed yields a holomorphic covariant quantity:

$$f^{\{6\}}_{TTTTTTT} = \prod_{k=-3}^{3} \left(\partial_T - \frac{2k}{T - \overline{T}}\right) f^{\{6\}} = \frac{E_4(T)^2 E_6(U)}{[J(T) - J(U)]\eta^{24}(U)} \ , \tag{A.3}$$

and similarly for the other couplings. For example,

$$f^{\{6\}}_{TTTTUUU} = \frac{1}{2\pi i}\partial_T \log\left[J(T) - J(U)\right] + \frac{1}{2\pi i}\partial_T \ln \Psi_0(T, U) \ , \tag{A.4}$$

where

$$\Psi_0(T, U) = q_T \prod_{(k,l)>0} \left(1 - q_T^{\ k} q_U^{\ l}\right)^{d(kl)} \ . \tag{A.5}$$

The cusp form Ψ_0 stays finite everywhere in the moduli space, i.e., $d(-1) = 0 = d(0)$, and the exponents are generated by $\sum_{n>0} d(n)q^n = \left(\frac{5}{72}E_2^3 E_6 + \frac{5}{24}E_2^2 E_4^2 + \frac{3}{8}E_2 E_4 E_6 - \frac{11}{36}E_6^2 - \frac{25}{72}E_4^3\right)/\eta^{24}$.

Moreover, as we have indicated above, the relationship between the functions $f^{\{n\}}(T, U)$ and PF systems with sources also appears to generalize in a natural manner. As discussed above, $\partial_T^7 \, f^{\{6\}}(T, U)$ is given by (3.5). Following the algorithm outlined above we find the function:

$$\mu_{00}^{\{6\}} = 2\pi i(f^{\{6\}}_{TTTUUU} + 5 f^{\{4\}}_{TTTUU} + 9 f^{\{2\}}_{TU}) - 5 log(w_1 - w_2) \ .$$

It satisfies (3.6) for $n = 6$ in which the homogenous part, $\mathcal{L}_{w_a}^{\otimes 3}$ is given by (3.1), and the source part by:

$$\begin{aligned}
\mathcal{M}_1^{\{6\}} := & -\frac{20w_2}{(w_1 - w_2)} \left[(1 - w_1)\left(\mathcal{L}_{w_1}^{\otimes 2} - \frac{5}{48}\theta_{w_1} - \frac{5}{144}\right) - \right. \\
& (1 - w_2)\left(\mathcal{L}_{w_2}^{\otimes 2} - \frac{5}{48}\theta_{w_2} - \frac{5}{144}\right) - \\
& w_1(1 - w_2)\frac{d}{dw_2}\left(\mathcal{L}_{w_1}^{\otimes 1} - \frac{5}{72}\right) + \\
& \left. w_1(1 - w_1)\frac{d}{dw_1}\left(\mathcal{L}_{w_2}^{\otimes 1} - \frac{5}{72}\right)\right] \\
& + 5 \left(w_2 \mathcal{L}_{w_2}^{\otimes 1} + \frac{1}{6}(1 - w_2)\theta_{w_2} - \frac{1}{12}w_2\right) \ .
\end{aligned} \tag{A.6}$$

The structure of the homogenous equations is indeed that of the PF equation of $\text{Sym}^3(K3)$.

References

[1] J. Harvey and G. Moore, *Algebras, BPS States, and Strings,* Nucl. Phys. **B463** (1996) 315-368, `hep-th/9510182`.

[2] C. Bachas and E. Kiritsis, F^4 *terms in N=4 string vacua,* Nucl. Phys. Proc. Suppl. **55B** (1997) 194, `hep-th/9611205`.

[3] For a comprehensive review, see: E. Kiritsis, *Introduction to nonperturbative string theory,* `hep-th/9708130`.

[4] See e.g., *Essays and mirror manifolds,* (S. Yau, ed.), International Press 1992; *Mirror symmetry II,* (B. Greene et al, eds.), International Press 1997.

[5] W. Lerche, *Elliptic index and superstring effective actions,* Nucl. Phys. **B308** (1988) 102.

[6] A. Schellekens and N. Warner, *Anomalies, characters and strings,* Nucl. Phys. **B287** (1987) 317;
E. Witten, *Elliptic genera and quantum field theory,* Commun. Math. Phys. **109** (1987) 525;
W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, Nucl. Phys. **B299** (1988) 91.

[7] L. Dixon, V. Kaplunovsky and J. Louis, *Moduli dependence of string loop corrections to gauge coupling constants,* Nucl. Phys. **B355** (1991) 649-688.

[8] K. Förger and S. Stieberger, *String amplitudes and $N = 2$, $d = 4$ prepotential in heterotic $K3 \times T^2$ compactifications,* Nucl. Phys. **B514** (1998) 135, `hep-th/9709004`.

[9] W. Lerche and S. Stieberger, *Prepotential, mirror map and F-theory on K3,* `hep-th/9804176`.

[10] K. Foerger, and S. Stieberger, *Higher derivative couplings and heterotic type I duality in eight-dimensions,* `hep-th/9901020`.

[11] A. Strominger, *Special geometry,* Commun. Math. Phys. **133** (1990) 163-180;
S. Ferrara and A. Strominger, *N=2 spacetime supersymmtry and Calabi-Yau moduli space,* in: Proceedings of College Station Workshop (1989) 245.

[12] A. Klemm, W. Lerche and P. Mayr, *K3 Fibrations and Heterotic-Type II String Duality,* Phys. Lett. **B357** (1995) 313-322, `hep-th/9506112`.

[13] L. Baulieu and S. Shatashvili, *Duality from topological symmetry,* `hep-th/9811198`.

[14] C. Vafa, *Evidence for F theory,* Nucl. Phys. **B469** (1996) 403-418, `hep-th/9602022`;
D. Morrison and C. Vafa, *Compactifications of F theory on Calabi-Yau three-folds I,* Nucl. Phys. **B473** (1996) 74-92, `hep-th/9602114`;
Compactifications of F theory on Calabi-Yau threefolds II, Nucl. Phys. **B476** (1996) 437-469, `hep-th/9603161`.

[15] W. Lerche, S. Stieberger and N. Warner, *Quartic gauge couplings from K3 geometry,* `hep-th/9811228`.

[16] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,* Nucl. Phys. $\underline{B426}$ (1994) 19, `hep-th/9407087`.

[17] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Self-dual strings and N=2 supersymmetric field theory,* Nucl. Phys. **B477** (1996) 746-766, `hep-th/9604034`.

[18] E. Witten, *Solutions of four-dimensional field theories via M-theory,* Nucl. Phys. **B500** (1997) 3-42, `hep-th/9703166`.

[19] M. Bershadsky, V. Sadov and C. Vafa, *D-Branes and Topological Field Theories,* Nucl. Phys. **B463** (1996) 420-434, `hep-th/9511222`.

[20] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces,* Commun. Math. Phys. **167** (1995) 301-350, `hep-th/9308122`.

[21] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Perturbative couplings of vector multiplets in $N = 2$ heterotic string vacua,* Nucl. Phys. **B451** (1995) 53-95, `hep-th/9504006`;
I. Antoniadis, S. Ferrara, E. Gava, K. Narain and T. Taylor, *Perturbative prepotential and monodromies in $N = 2$ heterotic superstring,* Nucl. Phys. **B447** (1995) 35-61, `hep-th/9504034`.

[22] B. Lian and S. Yau, *Arithmetic properties of mirror map and quantum coupling,* Commun. Math. Phys. **176** (1996) 163-192, `hep-th/9411234`;
Mirror maps, modular relations and hypergeometric series 1, `hep-th/9507151`;
Mirror maps, modular relations and hypergeometric series. 2, `hep-th/9507153`.

[23] M. Lee, *Picard-Fuchs equations for elliptic modular varieties,* Appl. Math. Letters 4, no.5 (1991) 91-95.

[24] C. Doran, *Picard-Fuchs Uniformization: Modularity of the Mirror Map and Mirror-Moonshine*, `math.AG/9812162`.

[25] A. Beauville and R. Donagi, *La variété des droits d'une hypersuface cubique de dimension 4*, C. R. Acad. Sci. Paris 301 (1985) 703-706.

[26] G. Moore, *String duality, automorphic forms, and generalized Kac–Moody algebras*, `hep-th/9710198`;
M. Marino and G. Moore, *Counting higher genus curves in a Calabi-Yau manifold*, `hep-th/9808131`.

[27] S. Yau and E. Zaslow, *BPS states, string duality, and nodal curves on K3*, Nucl. Phys. **B471** (1996) 503-512, `hep-th/9512121`.

[28] L. Bryan and N. Leung, *The enumerative geometry of K3 surfaces and modular forms*, `alg-geom/9711031`.

[29] A. N. Schellekens, N. P. Warner, *Anomalies and modular invariance in string theory*, Phys. Lett. **177B** (1986) 317;
A. N. Schellekens, N. P. Warner, *Anomalies, characters and strings*, Nucl. Phys. **B287** (1987) 317.