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Classification of the $N = 2$, $Z_2 \times Z_2$-symmetric type II orbifolds and their type II asymmetric duals

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Abstract

Using free world-sheet fermions, we construct and classify all the $N = 2$, $Z_2 \times Z_2$ four-dimensional orbifolds of the type IIA/B strings for which the orbifold projections act symmetrically on the left and right movers. We study the deformations of these models out of the fermionic point, deriving the partition functions at a generic point in the moduli of the internal torus $T^6 = T^2 \times T^2 \times T^2$. We investigate some of their perturbative and non-perturbative dualities and construct new dual pairs of type IIA/type II asymmetric orbifolds, which are related non-perturbatively and allow us to gain insight into some of the non-perturbative properties of the type IIA/B strings in four dimensions. In particular, we consider some of the (non-)perturbative gravitational corrections.

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1. Introduction

During the recent years, duality has played a fundamental role in the progress of string theory. However, despite the huge amount of work done in this field, in most of the cases duality has remained a conjecture, based more on field theory and supersymmetry/supergravity considerations than on tests made directly at the string level [1]. Actually, in order to perform string-loop computations, it is necessary to go to special points of the moduli space, in which it is possible to solve the two-dimensional conformal field theory. In this paper, we study a class of four-dimensional compactifications of type II strings, with two space-time supersymmetries, for which this is possible. Our main interest is in what we call $Z_2 \times Z_2$ symmetric orbifolds, namely orbifold constructions in which the $N = 8$ supersymmetry is reduced to $N = 2$ by two Z_2 projections that act symmetrically on the left and right movers. These orbifolds are of particular interest because they can be easily realized through a free fermion construction [2]–[5]. In this framework, the various constraints and requirements of a consistent string theory construction are collected in a set of rules, which can be easily handled. In particular, we show that it is possible to write a general formula for the GSO projections, which allows us to give a complete classification of such orbifolds.

All of these constructions can be seen as compactifications on singular limits of CY manifolds. For all the models, the scalar manifolds are coset spaces:

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2+N_V)}{SO(2)\times SO(2+N_V)} \quad \text{and} \quad \frac{SO(4,4+N_H)}{SO(4)\times SO(4+N_H)}, \tag{1.1}$$

describing respectively the space of the $N_V + 3$ moduli in the vector multiplets and that of the $N_H + 4$ in the hypermultiplets. For each pair (N_V, N_H) there always exists a construction for which N_V and N_H are exchanged, corresponding to a compactification on the mirror manifold. Some of them, namely the models with $N_V = N_H = 16, 8, 0$, correspond to compactifications on CY manifolds already investigated, although in slightly different contexts [6]–[9].

For each model, we write the (one-loop) partition function, which encodes all the information about its perturbative physics. We then establish the exact equivalence, for this class of orbifolds, of the fermionic construction and a geometric construction based on bosons compactified at special radii. In this way, we show that, once these orbifolds are constructed at the fermionic point, it is possible to switch on some moduli, namely the moduli T^i, U^i, $i = 1, 2, 3$, which on type IIA are associated respectively with the Kähler class moduli and the complex structure moduli of the three tori into which the compact space is factorized by the orbifold projections. The derivation of the partition functions, for any such construction, at a generic point in the space of these moduli, constitutes one of the main results of this paper. This allows us to investigate some deformations of the models. In particular, from the analysis of certain helicity supertraces [10]–[14], which distinguish between various BPS and non-BPS states, we read off the presence of perturbative Higgs and super-Higgs phenomena. The first account for the appearance of new massless states in particular corners of the moduli space, while the second, besides that, determine the restoration of a certain number of supersymmetries. Such properties play a key role in the search for dual constructions.

In particular, it is possible to recognize which models correspond to compactifications on orbifold limits of K3 fibrations [15, 16]. In these cases, the heterotic dual constructions can be easily identified [17]–[19]. In this paper, we focus our attention on the duality between type IIA/B and type II "asymmetric" constructions, in which all the supersymmetries come from the left movers only, the supersymmetries of the other chirality being projected out by $(-)^{\mathrm{F_R}}$, the right fermion number operator. In these constructions, as in the heterotic $N = 2$ compactifications, the dilaton–axion field belongs to a vector multiplet, and the type IIA/type II asymmetric dual pairs are related by a U-duality similar to the duality of the type IIA/heterotic strings (examples of such dual pairs were previously considered in [9, 19]). This implies that a perturbative computation performed on one side gives information on the non-perturbative physics of the dual. In this paper we present in detail the construction of such type II asymmetric duals. Then, as in [18, 19], we consider R^2 corrections, which serve both as a test of duality and as the actual computation of a quantity that is non-perturbative in the type II asymmetric duals. On the other hand, an investigation of the perturbative super-Higgs phenomena present in the type II asymmetric models tells us about the presence of an analogous phenomenon also in the type IIA/B duals, in which it is entirely non-perturbative and could not be seen from an analysis of the helicity supertraces.

The paper is organized as follows:

In Section 2 we present the fermionic construction of the $N = 2$, type IIA/B $Z_2 \times Z_2$ symmetric orbifolds and we discuss the analysis of the massless spectrum. The reader can find a short reminder of the rules of the fermionic construction in Appendix A, while more details on the massless spectrum are given in Appendix B. At the end of the section we explain our method of classification of such constructions, quoting in Appendix C the general formulae for the GSO projections.

In Section 3 we derive the partition functions of the various models. We establish the equivalence of world-sheet fermions and bosons, thereby deriving the partition functions at a generic point in the toroidal moduli T^i, U^i. The classification of the partition functions is given in Appendix D.

In Section 4 we compute the helicity supertraces and interpret the various orbifold operations in terms of stringy Higgs and super-Higgs phenomena.

In Section 5 we discuss the mirror symmetry, in the context of these symmetric orbifolds, and the non-perturbative dualities relating some of these models to heterotic duals and/or to type II asymmetric constructions. The type IIA/type II asymmetric dual pairs are discussed in detail in Sections 5.2 and 5.3, where we quote also the partition function for the type II asymmetric orbifolds. We discuss the corrections to the R^2 term. The dual pairs are then compared in Section 5.4, in which we discuss some of their non-perturbative aspects. A detailed discussion of shifted lattice sums and their integrals over the fundamental domain is in Appendix E, while in Appendix F we discuss the computation of helicity supertraces for the type II asymmetric orbifolds.

Our comments and conclusions are given in Section 6.

2. Type II $N=2$ symmetric orbifolds in the fermionic construction

The use of free world-sheet fermions turns out to be convenient for the analysis of the massless spectrum and the general classification of the $Z_2\times Z_2$ type II symmetric orbifolds. In order to construct them, we start from the $N=8$ type II string, which is described, in the light-cone gauge, by 8 world-sheet left/right moving bosonic and fermionic coordinates, $X_i^{L,R}$ and $\psi_i^{L,R}$ $(i=1,...,8)$. In our notation, the coordinates $\psi_\mu^{L,R}$ and $X_\mu^{L,R}$ $(\mu=1,2)$ represent the space-time transverse degrees of freedom, whereas the remaining ones correspond to the internal degrees of freedom. The $N=8$ string has therefore four space-time supersymmetries originating from the left-moving sector and four from the right-moving sector. We then introduce Z_2 projections, which act symmetrically on left/right moving coordinates, reducing the number of supersymmetries to $N=2$, one coming from the left and one from the right movers. In the fermionic construction [2]–[4], the $X_i^{L,R}$, $(i=3,...,8)$ are replaced by the pairs of Majorana–Weyl spinors $\omega_I^{L,R}$ and $y_I^{L,R}$, $(I=1,...,6)$. To follow the standard notation of the fermionic construction [5], we rename the internal components of the fields $\psi_i^{L,R}$ as $\chi_I^{L,R}$. The construction of string models then amounts to a choice of boundary conditions for the fermions, which satisfies local and global consistency requirements. A model is defined by a basis of sets α_i $(i=1,...,n)$ of fermions and by a modular-invariant choice of $n(n-1)/2+1$ phases (modular coefficients) $C_{(\alpha_i|\alpha_j)}$, which determine the GSO projections (we refer the reader to Appendix A for more details). In this language, the $N=8$ model is constructed by introducing three basis sets, namely F, which contains all the left- and right-moving fermions:

$$F=\left\{\begin{array}{c}\psi_\mu^L,\chi_I^L,y_I^L,\omega_I^L\\ \psi_\mu^R,\chi_I^R,y_I^R,\omega_I^R\end{array}\right\},\quad(\mu=1,2;\ I=1,...,6),\tag{2.1}$$

and the sets S and $\bar S$, which contain only eight left- or right-moving fermions, and distinguish the boundary conditions of the left- and right- moving world-sheet superpartners:

$$S=\left\{\psi_\mu^L,\chi_1^L,\dots,\chi_6^L\right\},\qquad \bar S=\left\{\psi_\mu^R,\chi_1^R,\dots,\chi_6^R\right\}.\tag{2.2}$$

In order to obtain a $Z_2\times Z_2$ symmetric orbifold, we add to the basis the two sets b_1 and b_2:

$$b_1\ =\ \left\{\begin{array}{c}\psi_\mu^L,\chi_{1,2}^L,y_{3,\dots,6}^L\\ \psi_\mu^R,\chi_{1,2}^R,y_{3,\dots,6}^R\end{array}\right\},\tag{2.3}$$

$$b_2\ =\ \left\{\begin{array}{c}\psi_\mu^L,\chi_{3,4}^L,y_{1,2}^L,y_{5,6}^L\\ \psi_\mu^R,\chi_{3,4}^R,y_{1,2}^R,y_{5,6}^R\end{array}\right\}.\tag{2.4}$$

These sets assign Z_2 boundary conditions, thereby introducing new projections, which break the $N=8$ supersymmetry.

The definition of the model is completed by the choice of the following modular coefficients, which fix the GSO projections and determine the chirality of the spinors:

	F	S	$\bar{S}$	b_1	b_2
F	1	−1	−1	1	1
S	−1	1	1	−1	−1
$\bar{S}$	−1	1	1	−1	−1
b_1	1	1	1	1	1
b_2	1	1	1	1	1

Table 2.1: The coefficient $C_{(\alpha_i|\alpha_j)}$ is given by the (i, j) entry of the matrix.

This choice corresponds to a type IIA compactification[1]. Six of the eight gravitinos are projected out and we are left with only two supersymmetries, whose generators can be read-off from the $-1/2$ picture vertex operator representation of the surviving gravitinos, given in (B.2).

It is easy to check that the massless spectrum fits into representations of the $N = 2$ supersymmetry. This is done by constructing the vertex operator representation of the states, which we quote in Appendix B. The $N = 2$ spectrum is in fact characterized by the $SU(2)$ symmetry under which the two supercharges form a doublet. In Appendix B we discuss in detail the construction of the generators of this $SU(2)$ symmetry. By looking at the $SU(2)$ charge of the scalars, we identify the ones belonging to the vector multiplets and the ones belonging to the hypermultiplets: the scalars of a hypermultiplet do transform under the $SU(2)$ symmetry of $N = 2$ [20]. In particular, it is easy to see that the pair dilaton–pseudoscalar is charged and therefore belongs to a hypermultiplet. Furthermore, it is also easy to see that all the scalars belonging to hypermultiplets are charged also under a second $SU(2)$. This allows us to conclude that the quaternionic manifold has an $SO(4)$ symmetry, and is given by the coset

$$\frac{SO(4, 4 + N_H)}{SO(4) \times SO(4 + N_H)} , \tag{2.5}$$

where N_H is the number of hypermultiplets that originate from the twisted sectors (in this case, these are the sectors b_1, b_2 and $FS\bar{S}b_1b_2$, which, for the choice of projections specified in Table 2.1, provide the scalars of $N_H = 12$ hypermultiplets[2]. A similar analysis, on the scalars uncharged under the $SU(2)$ of the $N = 2$ supersymmetry, allows us to conclude that the scalars belonging to vector multiplets span the coset:

$$\frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2 + N_V)}{SO(2) \times SO(2 + N_V)}, \tag{2.6}$$

[1]The type IIA↔B exchange is realized by changing the chirality of, say, the right-moving spinors. In Appendix B we explain how this is implemented in the fermionic construction.

[2]The scalars of the $S\bar{S}$ (Ramond–Ramond) sector are charged also under two other $SU(2)$'s. They therefore have an $SO(4) \times SO(4)$ symmetry. The four $SU(2)$'s are the remnant of the $SU(8)$ symmetry of the massless spectrum of the $N = 8$ theory, which is broken to $SU(2)^4$ by the orbifold projections.

with $N_V = 12$ in this particular case.

We can construct other $Z_2 \times Z_2$ symmetric orbifolds, with a higher or lower number of massless states originating from the twisted sectors, by varying the sets b_1 and b_2 and/or adding more sets to the basis. However, it is easy to see that, once required that the breaking of the $N = 8$ supersymmetry to $N = 2$ be symmetric in the left- and right- movers, the untwisted sector is automatically fixed to be the same as for the orbifold considered above. By constructing then, for this class of orbifolds, the vertex operator representation of the massless states of the twisted sectors, it is easy to check that there is always an $SO(4) = SU(2) \times SU(2)$ symmetry common to all the scalars of the hypermultiplets. On the other hand, the complex scalars of the vector multiplets possess the $SO(2) \approx U(1)$ symmetry of complex conjugation, as the scalars of (2.6). As a consequence, the scalar manifolds are uniquely specified by the numbers N_H and N_V of hyper and vector multiplets provided by the twisted sectors, and are always expressed by (2.5) and (2.6).

In order to give an exhaustive classification of such orbifolds, we notice that, instead of varying the sets b_1, b_2, we can equivalently keep them fixed and add to the fermion basis the sets e_i:

$$e_i = \left\{ y_i^L, \omega_i^L \mid y_i^R, \omega_i^R \right\} \qquad (i = 1, ..., 5), \tag{2.7}$$

which factorize the six circles of the compact space (e_6 is generated by the product $FS\bar{S}\ e_1\ e_2\ e_3\ e_4\ e_5$) by introducing independent Z_2 boundary conditions for all of them.

With such a basis, we can construct any $Z_2 \times Z_2$ orbifold, provided we properly choose the modular coefficients. In fact, with these fermion sets, we can construct 48 massless twisted sectors[3], that is as many twisted sectors as the maximal number of fixed points a $Z_2 \times Z_2$ symmetric orbifold can have. Each such fixed point gives rise either to a vector- or to a hypermultiplet. Any specific choice of the modular coefficients amounts to a choice of GSO projections, which act by excluding some sectors and determining whether the states of the remaining sectors fit into vector- or hypermultiplets.

We therefore proceed by expressing N_V and N_H, for each twisted sector, as functions of the modular coefficients (we quote the general formula of the GSO projections on the 48 twisted sectors in Appendix C). Then we fix the coefficients that determine the GSO projections onto the untwisted sector (RR sector included), because they amount to an arbitrary choice of the chirality of the spinors; we then vary all the other GSO projections, by allowing a change in the coefficients $C_{(b_1|e_1)}$, $C_{(b_1|e_2)}$, $C_{(b_2|e_3)}$, $C_{(b_2|e_4)}$, $C_{(e_1|e_2)}$, $C_{(e_1|e_3)}$, $C_{(e_1|e_4)}$, $C_{(e_1|e_5)}$, $C_{(e_2|e_3)}$, $C_{(e_2|e_4)}$, $C_{(e_2|e_5)}$, $C_{(e_3|e_4)}$, $C_{(e_3|e_5)}$, $C_{(e_4|e_5)}$, $C_{(b_1|Fe_3e_4)}$, $C_{(b_2|Fe_1e_2)}$ and $C_{(b_1b_2|e_5)}$. In this way we obtain all the possible (N_V, N_H) pairs. The coefficient $C_{(b_1|b_2)}$ determines, instead, the general projection onto the chirality of the bispinors of the twisted sectors. Under a change of sign of this coefficient, N_V and N_H get exchanged. As a consequence, each pair (N_V, N_H) appears accompanied by its mirror (N_H, N_V). We list the pairs (N_V, N_H) in Table D.1. Indeed, what we obtain is much more than a simple classification of the possible massless spectra: having performed such an analysis on the single twisted sectors, we actually obtain a complete classification of the possible orbifold projections, something that, as we will see in the following, allows us to reconstruct the one-loop partition function of each

[3]They are quoted in Appendix C.

orbifold, even away from the fermionic point.

3. The partition functions

In the type II $Z_2 \times Z_2$ symmetric constructions, the degrees of freedom of the compact space can be equivalently described by compactified bosons. The conditions of existence of the two world-sheet supercurrents (A.1) allow in fact different Z_2 boundary conditions to be assigned not to single fermions but only to sets of bilinears of fermions. The symmetry between left and right movers implies that such bilinears must always appear paired in such a way as to form compact bosons. Indeed, we want to show in the following that all the above models can be constructed as orbifolds by using the symmetries of the conformal theory of six bosons compactified on a torus T^6 at the point of moduli for which it is described by a product of circles $T^6 = S^1 \times \ldots \times S^1$ [4]. In this approach, the dependence on the geometrical moduli of T^6 is explicit[5]. In order to see what is the partition function at a generic point in $\mathcal{T}^6$, we use identities satisfied by the modular forms and recast the partition function of free fermions as a sum over lattice momenta and windings, as in the case of a single boson. By substituting generic values of moduli in the lattice sums, wethen get the partition function of the model at any value in the orbifold moduli space.

The partition function, at the fermionic point (see Appendix A), is given by the integral over the modular parameter τ, with modular-invariant measure $(\,\mathrm{Im}\ \tau)^{-2} d\tau d\bar{\tau}$, of:

$$Z^{\rm string} = \frac{1}{\mathrm{Im}\,\tau |\eta(\tau)|^4}\ \frac{1}{4} \sum_{(H_1,G_1,H_2,G_2)} \left(\frac{1}{2}\right)^6 \sum_{(\gamma,e_i,\delta,d_i)} C\begin{bmatrix}\gamma, e_i, H_j\\ \delta, d_i, G_j\end{bmatrix}\ Z_{\rm L}^F\ Z_{\rm R}^F\ Z_{6,6}\ , \tag{3.1}$$

where $Z^F_{L,R}$ contain the contribution of the world-sheet fields $\psi_\mu^{L,R}$, $\chi_a^{L,R}$ (the sets S and $\bar{S}$); $Z_{6,6}$ encodes the contribution of the $c = (6, 6)$ internal space, i.e. of the fields $\omega_I^{L,R}$, $y_I^{L,R}$ (the fields of the sets $\Gamma \equiv FS\bar{S}$, e_i, $i = 1, ..., 5$ and their products) and $C\left[{\gamma,e_i,h_j \atop \delta,d_i,g_j}\right]$ is a modular covariant phase (discrete torsion). We have:

$$Z_{\rm L}^F = \frac{1}{2}\sum_{(a,b)} \frac{{\rm e}^{i\pi\varphi_{\rm L}\left(a,b,\vec{H},\vec{G}\right)}}{\eta^4} \vartheta\begin{bmatrix}a\\ b\end{bmatrix} \vartheta\begin{bmatrix}a+H_1\\ b+G_1\end{bmatrix} \vartheta\begin{bmatrix}a+H_2\\ b+G_2\end{bmatrix} \vartheta\begin{bmatrix}a-H_1-H_2\\ b-G_1-G_2\end{bmatrix}\ , \tag{3.2}$$

$$Z_{\rm R}^F = \frac{1}{2}\sum_{(\bar{a},\bar{b})} \frac{{\rm e}^{i\pi\varphi_{\rm R}\left(\bar{a},\bar{b},\vec{H},\vec{G}\right)}}{\bar{\eta}^4} \vartheta\begin{bmatrix}\bar{a}\\ \bar{b}\end{bmatrix} \vartheta\begin{bmatrix}\bar{a}+H_1\\ \bar{b}+G_1\end{bmatrix} \vartheta\begin{bmatrix}\bar{a}+H_2\\ \bar{b}+G_2\end{bmatrix} \vartheta\begin{bmatrix}\bar{a}-H_1-H_2\\ \bar{b}-G_1-G_2\end{bmatrix}\ , \tag{3.3}$$

[4] In some cases, the factorization $T^6 = T^2 \times T^2 \times T^2$ is sufficient.

[5] In the fermionic construction the moduli dependence was not manifest, because compactified bosons can be fermionized only for some particular values of moduli. For instance, in the case of a single boson, the fermionic partition function corresponds to the bosonic one when the radius of compactification R is 1. The fermionic construction must, however, be considered as describing a model at a particular point in moduli space.

with

$$\varphi_{\rm L}\left(a,b,\vec{H},\vec{G}\right) = a+b+\frac{1}{2}\left(1-C_{(S|S)}\right)ab+\frac{1}{2}\left(1-C_{(S|S\bar{S}b_1)}\right)(aG_1+bH_1) + \frac{1}{2}\left(1-C_{(S|S\bar{S}b_2)}\right)(aG_2+bH_2) \quad (3.4)$$

and an analogous expression for $\varphi_{\rm R}\left(\bar{a},\bar{b},\vec{H},\vec{G}\right)$, obtained from $\varphi_{\rm L}$ through the substitutions $(a,b)\to(\bar{a},\bar{b})$ and $S\to\bar{S}$. The contribution of the compact bosons is:

$$\begin{aligned} Z_{6,6} &= \frac{1}{|\eta|^4}\left|\vartheta\begin{bmatrix}\gamma+e_1\\ \delta+d_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_1+H_2\\ \delta+d_1+G_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_2\\ \delta+d_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_2+H_2\\ \delta+d_2+G_2\end{bmatrix}\right| \\ &\times \frac{1}{|\eta|^4}\left|\vartheta\begin{bmatrix}\gamma+e_3\\ \delta+d_3\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_3+H_1\\ \delta+d_3+G_1\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_4\\ \delta+d_4\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_4+H_1\\ \delta+d_4+G_1\end{bmatrix}\right| \\ &\times \frac{1}{|\eta|^4}\left|\vartheta\begin{bmatrix}\gamma+e_5\\ \delta+d_5\end{bmatrix}\vartheta\begin{bmatrix}\gamma+e_5+H_1+H_2\\ \delta+d_5+G_1+G_2\end{bmatrix}\vartheta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\vartheta\begin{bmatrix}\gamma+H_1+H_2\\ \delta+G_1+G_2\end{bmatrix}\right| . \end{aligned} \quad (3.5)$$

In this notation, the pairs (a,b) and $(\bar{a},\bar{b})$ specify the boundary conditions, in the directions $\mathbf{1}$ and τ of the world-sheet torus, of the sets S and $\bar{S}$, while (γ,δ), (e_i,d_i) refer respectively to the sets Γ and e_i; (H_1,G_1) and (H_2,G_2) refer to the sets b_1, b_2. When a field belongs to the intersection of many sets, its boundary conditions are specified by the sum of the boundary conditions of the sets it belongs to. The modular coefficients appear in the phases $\varphi_{\rm L}$, $\varphi_{\rm R}$ in $Z^F_{L,R}$ and in

$$C\begin{bmatrix}\gamma,e_i,h_j\\ \delta,d_i,g_j\end{bmatrix} = \exp\ \frac{i\pi}{2}\sum_{k,\ell}\left(1-C_{(X_k|X_\ell)}\right)\alpha_k\beta_\ell\ , \quad (3.6)$$

where

$$X_k,\ X_\ell\ \in\ \{\Gamma,b_1,b_2,e_i\} \quad (3.7)$$

and (α_k,β_ℓ) indicate the corresponding boundary conditions in the two directions of the world-sheet torus. For the specific choice of Table 2.1 (type IIA), we have

$$\varphi_{\rm L} = a+b+ab\ , \quad (3.8)$$
$$\varphi_{\rm R} = \bar{a}+\bar{b}+\bar{a}\bar{b}\ . \quad (3.9)$$

For the type IIB choice specified in Appendix B.2, $\varphi_{\rm R}$ is, instead:

$$\varphi_{\rm R} = \bar{a}+\bar{b}\ . \quad (3.10)$$

The partition function is the sum of five terms:

1. the $N=8$ sector, specified by $(H_1,G_1)=(H_2,G_2)=(0,0)$;
2. the $N=4$ sector specified by $(H_1,G_1)\neq(0,0)$, $(H_2,G_2)=(0,0)$;
3. the $N=4$ sector with $(H_2,G_2)\neq(0,0)$, $(H_1,G_1)=(0,0)$;

4. the $N = 4$ sector with $(H_1, G_1) = (H_2, G_2) \neq (0, 0)$ and $(H_1 + H_2, G_1 + G_2) = (0, 0)$;

5. the $N = 2$ sector, which contains all the terms for which $(H_1, G_1) \neq (0, 0)$, $(H_2, G_2) \neq (0, 0)$, $(H_1 + H_2, G_1 + G_2) \neq (0, 0)$.

The $N = 8$ sector is universal: it is the same for any orbifold, since it is proportional to the unprojected partition function of the $N = 8$ string. In the $N = 2 = (1, 1)$ sector all the bosons of the compact space are twisted and/or projected: this implies that the part of the partition function that corresponds to this sector is the same at any point in the moduli space of the orbifold. The only non-trivial moduli dependence is contained in the $N = 4$ sectors: in the following, we will therefore concentrate on these.

In each $N = 4$ sector the moduli dependence is contained in the untwisted $c = (2, 2)$ conformal block. The latter corresponds to the complex planes (1,2) (for the first $N = 4$ sector), (3,4) in the second sector and (5,6) in the third sector. We want to rewrite such blocks in terms of sums over lattice windings and momenta. To this purpose, we make use of the identity:

$$\Gamma^w_{2,2}\begin{bmatrix} h_1, & h_2 \\ g_1, & g_2 \end{bmatrix}(T(w), U(w)) = \sum_{a_1,b_1,a_2,b_2} \mathrm{e}^{i\pi(a_1g_1+b_1h_1+h_1g_1)}\mathrm{e}^{i\pi(a_2g_2+b_2h_2+h_2g_2)} \left| \vartheta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \vartheta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \right|^2 , \tag{3.11}$$

which generalizes the equivalence of the partition functions of two Weyl–Majorana fermions and one boson at radius 1 to the case of two bosons toroidally compactified, with generic lattice shifts (h_1, h_2, g_1, g_2) in the momenta and windings in the two circles. Here $w \equiv (w_1, w_2)$ stays for a pair of lattice vectors w_1, w_2, which specify the directions of the shifts (see Appendix E). We do not need to specify the particular value, which depends on the shift vectors, of the toroidal Kähler and complex structure moduli T and U for which the equivalence (3.11) is valid. This, however, can be easily computed, and we refer to Appendix E for this detail. In our case, the pairs (a_1, b_1), (a_2, b_2) are substituted by $(\gamma + e_1, \delta + d_1)$, $(\gamma + e_2, \delta + d_2)$ in the first $N = 4$ sector, $(\gamma + e_3, \delta + d_3)$, $(\gamma + e_4, \delta + d_4)$ in the second, and $(\gamma + e_5, \delta + d_5)$, $(\gamma + e_6, \delta + d_6)$ in the third.

The shifts (h_1, g_1), (h_2, g_2), for any specific case, are read-off from (3.1) and (3.6). Owing to the fact that in a twisted/shifted lattice character $\Gamma\left[\begin{smallmatrix} h|h' \\ g|g' \end{smallmatrix}\right]$, when $(h, g) \neq (0, 0)$ the twist (h, g) imposes a constraint on the shift (h', g') ($(h', g') = (0, 0)$ or $(h', g') = (h, g)$), it turns out that in the $N = 4$ sectors all the shifts can be expressed in terms of the two Z_2 supersymmetry-breaking projections introduced by the sets b_1 and b_2 (these may or may not act freely, by translating some of the coordinates of the compact space), and in terms of the projections associated to the symmetries of each $c = (1, 1)$, S^1/Z_2 orbifold, generated by the two elements [21]:

$$D : \quad (\sigma_+, \sigma_-, V_{nm}) \;\to\; (\sigma_-, \sigma_+, (-)^m V_{nm}), \tag{3.12}$$

$$\tilde{D} : \quad (\sigma_+, \sigma_-, V_{nm}) \;\to\; (-\sigma_+, \sigma_-, (-)^n V_{nm}). \tag{3.13}$$

Here σ_+, σ_- are the two twist fields of S^1/Z_2, and V_{nm} are the untwisted vacua labelled by the momentum m and the winding number n. The orbifolds we are considering indeed possess such symmetries. In fact, the presence, in the fermionic basis, of the sets e_i, $i = 1, \ldots, 5$ corresponds to a choice of coordinates for which the compact space is described by a product of circles: $T^6 = S^1 \times \ldots \times S^1$. Any one of the $Z_2\,(b_i)$ projections ($i = 1, 2, 3 = 1 + 2$), then creates a $c = (4, 4)$ twisted block that corresponds to an orbifold $(S^1)^4/Z_2\,(b_i)$. (The construction with complex planes corresponds instead to the separation $[T^2 \times T^2]/Z_2\,(b_i)$. At this point in the moduli space, it is possible to remove some of the fixed points, by using such symmetries. To simplify the discussion, we will consider in the following only the first generator, D (the action of $\tilde{D}$ can be obtained by T-duality). Modding out by the group generated by D amounts to cutting half of the states in the twisted sector and to a modification of the lattice of momenta and windings: the momenta are restricted to even values and the windings are shifted to half-integer values. In order to realize this projection, we must pair the D-projection on the twisted $c = (4, 4)$ block with a translation in one direction of the untwisted $c = (2, 2)$ block. From (3.12), we see that this translation can itself be considered as a D-projection. The operation therefore amounts to the insertion of a D-projection into two circles belonging to two different complex planes. There are then always two $N = 4$ sectors for which the pair of D-operations acts by reducing the number of fixed points.

In order to account for the various possibilities, we extend the definition of twisted/shifted $c = (2, 2)$ conformal blocks to account also for the D-operations. We therefore define

$$\Gamma^{(i)}_{2,2}\begin{bmatrix} H;\ h_1,\ h_2 \mid H^1,\ H^2 \\ G;\ g_1,\ g_2\ \mid G^1,\ G^2 \end{bmatrix}, \qquad i = 1, 2, 3, \tag{3.14}$$

where the index i indicates the planes (1,2), (3,4) and (5,6) respectively. The pair (H, G) specifies the ordinary twist, the pairs (h_1, g_1), (h_2, g_2) specify the lattice shifts in the two circles of T^2, while the pairs (H^1, G^1), (H^2, G^2) refer to the D-operations. When $(H, G) \neq (0, 0)$, the block is non-zero only if (H^1, G^1), (H^2, G^2) equal $(0, 0)$ or (H, G). In this case, also the shifts (h_1, g_1), (h_2, g_2) are constrained in the same way: (h_1, g_1), $(h_2, g_2) = (0, 0)$ or (H, G) and we set, by definition,

$$\Gamma_{2,2}\begin{bmatrix} H;\ h_1,\ h_2 \mid H^1,\ H^2 \\ G;\ g_1,\ g_2\ \mid G^1,\ G^2 \end{bmatrix}_{(H,G)\neq(0,0)} \equiv \frac{4\,|\eta|^6}{\left|\vartheta\left[{1+H \atop 1+G}\right]\vartheta\left[{1-H \atop 1-G}\right]\right|}\ , \tag{3.15}$$

$$(h_i, g_i),\ (H^j, G^j) = (0, 0) \quad \text{or} \quad = (H, G)\ . \tag{3.16}$$

When $(H, G) = (0, 0)$, the D-action amounts to a shift, which adds to the shifts (h_1, g_1), (h_2, g_2). In this case, (3.14) is a doubly shifted lattice sum; the lattice shifts in the two circles are specified by $(h_1 + H^1, g_1 + G^1)$ and $(h_2 + H^2, g_2 + G^2)$:

$$\Gamma_{2,2}\begin{bmatrix} 0;\ h_1,\ h_2 \mid H^1,\ H^2 \\ 0;\ g_1,\ g_2\ \mid G^1,\ G^2 \end{bmatrix} = \Gamma_{2,2}\begin{bmatrix} h_1 + H^1, h_2 + H^2 \\ g_1 + G^1, g_2 + G^2 \end{bmatrix}\ . \tag{3.17}$$

There is, however, a subtlety: the pair of D-operations was defined as a projection, which mods out states by using a symmetry of the twisted sector. It makes sense only when the $Z_2\,(b)$ projections do not act freely. The recipe is that one first projects with at least one $Z_2\,(b)$, to reduce supersymmetry to $N = 4$ or $N = 2$, then the D-projection can be inserted in the $Z_2(b)$-twisted sectors and related to a shift in the untwisted coordinates. If the $Z_2(b)$ acts freely, i.e. has no fixed points, such an operation cannot be performed. This means that the D-operation is not independent of the shifts (h_i, g_i), $i = 1, 2$, which in turn depend on the two projections introduced by b_1 and b_2. We can however extend the definition of the D-projection to include also the case of freely acting orbifolds, by specifying that, in the absence of fixed points, it acts simply as a shift, i.e. as the natural restriction of (3.12), with the constraint that it must always act in a direction independent of that of $(\vec{h}, \vec{g}) = \{(h_1, g_1), (h_2, g_2)\}$. Its interference with the shift $(\vec{h}, \vec{g})$ then produces the following shifted lattice sum:

$$\frac{1}{2}\Gamma^{w_1}_{2,2}\begin{bmatrix} H \\ G \end{bmatrix} + \frac{1}{2}\Gamma^{w_2}_{2,2}\begin{bmatrix} H \\ G \end{bmatrix}, \qquad (w_1)^2 = (w_2)^2 = w_1 \cdot w_2 = 0\,. \tag{3.18}$$

Once this is pointed out, we can unambiguously express, in full generality, the $c = (6, 6)$ conformal block $Z^{\vec{T},\vec{U}}_{6,6}$ of the orbifold partition function at a generic point in the space of the Kähler and the complex structure moduli, $\vec{T} \equiv (T^1, T^2, T^3)$ and $\vec{U} \equiv (U^1, U^2, U^3)$, as a product of the above defined twisted/shifted characters:

$$Z^{\vec{T},\vec{U}}_{6,6} = \frac{1}{|\eta|^{12}}\Gamma_{6,6}\begin{bmatrix} H_1,\ H_2,\ \vec{h} \mid \vec{H} \\ G_1,\ G_2,\ \vec{g} \mid \vec{G} \end{bmatrix} \equiv Z^{\vec{T},\vec{U}}_{6,6}\begin{bmatrix} H_1,\ H_2,\ \vec{h} \mid \vec{H} \\ G_1,\ G_2,\ \vec{g} \mid \vec{G} \end{bmatrix}, \tag{3.19}$$

where

$$\begin{aligned} \Gamma_{6,6}\begin{bmatrix} H_1,\ H_2,\ \vec{h} \mid \vec{H} \\ G_1,\ G_2,\ \vec{g} \mid \vec{G} \end{bmatrix} &= \Gamma^{(1)}_{2,2}\begin{bmatrix} H_2, \vec{h}_{(1)} \mid H^1,\ H^2 \\ G_2, \vec{g}_{(1)} \mid G^1,\ G^2 \end{bmatrix} \times \Gamma^{(2)}_{2,2}\begin{bmatrix} H_1, \vec{h}_{(2)} \mid H^3,\ H^4 \\ G_1, \vec{g}_{(2)} \mid G^3,\ G^4 \end{bmatrix} \\ &\quad \times \Gamma^{(3)}_{2,2}\begin{bmatrix} H_1 + H_2, \vec{h}_{(3)} \mid H^5,\ H^6 \\ G_1 + G_2, \vec{g}_{(3)} \mid G^5,\ G^6 \end{bmatrix}. \end{aligned} \tag{3.20}$$

In the above expression, the shifts $(\vec{h}, \vec{g})$ depend on the projections $Z_2(b_i)$, and can always be expressed in terms of the twists (H_1, G_1), (H_2, G_2), while $\vec{H} \equiv (H^1, \ldots, H^6)$, $\vec{G} \equiv (G^1, \ldots, G^6)$ refer to the D-projections. At a generic point in the moduli space, the partition function (3.1) becomes:

$$Z^{\rm string}(\vec{T}, \vec{U}) = \frac{1}{{\rm Im}\,\tau|\eta(\tau)|^4}\,\frac{1}{4}\sum_{(H_1,G_1,H_2,G_2)}\left(\frac{1}{2}\right)^{n_D}\sum_{(\vec{H},\vec{G})} C\begin{bmatrix} H_1,\ H_2,\ H^i,\ H^j \\ G_1,\ G_2,\ G^i,\ G^j \end{bmatrix} Z^F_{\rm L}\ Z^F_{\rm R}\ Z^{\vec{T},\vec{U}}_{6,6}\,. \tag{3.21}$$

$Z^F_{L,R}$ are defined as in (3.2), (3.3); n_D indicates the number of D-projections. Notice that even though n_D can be greater than 2, in each $N = 4$ sector the maximal number of projections that effectively act is 2, because there are only two independent directions in which it is possible to pick a modular-invariant shift (see Appendix C of [12]). Further projections superpose and their effect vanish. What remains of the coefficient $C\left[{}^{\gamma, e_i, H_j}_{\delta, d_i, G_j}\right]$ of expression (3.1) is:

$$C\begin{bmatrix} H_1,\ H_2,\ H^i,\ H^j \\ G_1,\ G_2,\ G^i,\ G^j \end{bmatrix} = {\rm e}^{\frac{i\pi}{2}\left(1 - C_{(b_1|b_2)}\right)(H_1G_2 + H_2G_1)}\prod_{ij} C_{ij}\begin{bmatrix} H^i,\ H^j \\ G^i,\ G^j \end{bmatrix}, \tag{3.22}$$

where C_{ij} can be either $+1$ or $(-)^{H^iG^j+H^jG^i}$; (H^i, G^i), (H^j, G^j) refer to the D-operations in the circles i, j. These coefficients are always $+1$ in the $N = 4$ sectors; however they can play a role in the $N = 2$ sector.

As is clear from the definition of twisted/shifted lattice given in (3.14) and its properties, an $N = 4$ sector can provide 16, 8, 4 or 0 supermultiplets, depending on the shift $(\vec{h}, \vec{g})$ and on the D-projections. Thanks to the interpretation of the different choices of modular coefficients in terms of such operations, we understand why it is not possible to obtain models with any number $N_V + N_H$ of supermultiplets between 48 and 0, modulo 4 (with the obvious exception of 44, which would require one $N = 4$ sector with twelve supermultiplets). We saw that, in order to effectively reduce the number of states, the D-operation must always be inserted in at least two circles belonging to two different complex planes, which implies that there are always at least two $N = 4$ sectors in which eight supermultiplets become massive. As a consequence, the maximal number $N_V + N_H$ of supermultiplets originating from the twisted sectors, below the 48, which is reached only when all the shifts $\vec{h}, \vec{g}$ and the D-projections are turned off (all the modular coefficients are $+1$), is 32, and is obtained when $n_D = 1$ and the D-projection involves only two different complex planes. This explains why, in the classification of Table D.1, there are no models with $N_V + N_H$ between 48 and 32. For an analogous reason, also $N_V + N_H = 28$ is forbidden: this would require one more D-projection, reducing further the number of states in only one $N = 4$ sector, but it is easy to realize that in order to reduce the number of supermultiplets to four, the two D-projections must be inserted in at least four different circles, two per projection. There are therefore always at least two $N = 4$ sectors with 4 supermultiplets, so that, below 32, the maximal number of supermultiplets is 24. Below this, all the numbers modulo 4 are allowed: it is possible to construct models with $N_V + N_H = 20, 16, 12, 8, 4, 0$. One can check that these are precisely the numbers that appear in the list of Table D.1. In Section 4.2 we will also see how the operations described above can be interpreted in terms of stringy (super-)Higgs mechanisms.

4. Helicity supertraces and (super-)Higgs phenomena

When specified for a certain model, formula (3.21) encodes, in principle, all the perturbative information about it. It can be used to investigate the BPS spectrum and to compute one-loop threshold corrections. In each model, all the non-trivial moduli dependence is contained in the three $N = 4$ sectors, so that essentially the classification of $Z_2 \times Z_2$ symmetric orbifolds amounts to assigning the three $N = 4$ sectors for each one of the massless spectra appearing in Table D.1. According to the analysis of Section 3, this is equivalent to specifying the form of the lattice sum, $\Gamma^{(i)}_{2,2}$, for each one of the three untwisted tori. The set $(N_V, N_H, \Gamma^1_{2,2}, \Gamma^2_{2,2}, \Gamma^3_{2,2})$ then fixes unambiguously the entire partition function. Actually, as far as we are interested only in the $N = 4$ sectors, the notation (3.14) is highly redundant, because the shifts and the D-projections are constrained, in each (H, G)-twisted sector, to be either $(0, 0)$ or equal to (H, G). It is therefore sufficient to specify the direction and the nature of the translations through a pair of lattice vectors, w_1 and w_2 (see Appendix E). We can then account for the various situations by introducing the following notation for the (H, G)-shifted lattice sums:

$$
\begin{aligned}
O &= \Gamma_{2,2}\begin{bmatrix}0\\0\end{bmatrix}, && (4.1)\\
F &= \Gamma^{w_1}_{2,2}\begin{bmatrix}H\\G\end{bmatrix}, && (4.2)\\
FD &= \frac{1}{2}\Gamma^{w_1}_{2,2}\begin{bmatrix}H\\G\end{bmatrix} + \frac{1}{2}\Gamma^{w_2}_{2,2}\begin{bmatrix}H\\G\end{bmatrix}, && (4.3)\\
D &= \frac{1}{2}\Gamma_{2,2}\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{2}\Gamma^{w_1}_{2,2}\begin{bmatrix}H\\G\end{bmatrix}, && (4.4)\\
DD &= \frac{1}{4}\Gamma_{2,2}\begin{bmatrix}0\\0\end{bmatrix} + \frac{1}{4}\Gamma^{w_1}_{2,2}\begin{bmatrix}H\\G\end{bmatrix} + \\
&\quad + \frac{1}{4}\Gamma^{w_2}_{2,2}\begin{bmatrix}H\\G\end{bmatrix} + \frac{1}{4}\Gamma^{w_1+w_2}_{2,2}\begin{bmatrix}H\\G\end{bmatrix}. && (4.5)
\end{aligned}
$$

The shift vectors satisfy:

$$(w_1)^2 = (w_2)^2 = w_1 \cdot w_2 = 0 \tag{4.6}$$

and we indicated by $\Gamma_{2,2}\left[{0 \atop 0}\right]$ the ordinary unshifted lattice sum. The result of our analysis, which accounts for all the possible "partition functions", is quoted in Appendix C.

4.1. Helicity supertraces

We are interested in the second and fourth helicity supertraces, B_2 and B_4, through which we control the behaviour of a model under a motion in moduli space. The helicity supertraces are defined, for a given representation R of supersymmetry, as

$$B_{2n}(R) \equiv \mathrm{Str}\lambda^{2n} = \mathrm{Tr}_{\mathrm{R}}[(-)^{2\lambda}\lambda^{2n}], \tag{4.7}$$

where λ stands for the physical four-dimensional helicity. In the framework of string theory, $\lambda = \lambda_{\mathrm{L}} + \lambda_{\mathrm{R}}$, where $\lambda_{\mathrm{L,R}}$ are the contributions to the helicity from the left and right movers. The quantities B_{2n} are computed by taking appropriate derivatives of the generating function

$$Z^{\mathrm{string}}(v, \bar{v}) = \mathrm{Tr}' q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \mathrm{e}^{2\pi i (v\lambda_{\mathrm{L}} - \bar{v}\lambda_{\mathrm{R}})}, \tag{4.8}$$

by defining

$$\lambda_{\mathrm{L}} = Q = \frac{1}{2\pi i}\frac{\partial}{\partial v}, \qquad \lambda_{\mathrm{R}} = \bar{Q} = \frac{1}{2\pi i}\frac{\partial}{\partial \bar{v}}. \tag{4.9}$$

We then have

$$B^{\mathrm{string}}_{2n} = (Q + \bar{Q})^{2n} Z^{\mathrm{string}}(v, \bar{v})|_{(v=\bar{v}=0)}. \tag{4.10}$$

An explicit expression for $Z^{\mathrm{string}}(v, \bar{v})$, in the case of $Z_2 \times Z_2$ symmetric orbifolds, is given by an expression that is similar to the $v = \bar{v} = 0$ case presented in (3.21):

$$Z^{\mathrm{string}}(v, \bar{v}) = \frac{1}{\mathrm{Im}\,\tau|\eta(\tau)|^2}\,\frac{1}{4}\sum_{(H_1,G_1,H_2,G_2)}\left(\frac{1}{2}\right)^{n_D}\sum_{(\vec{H},\vec{G})} Z^F_{\mathrm{L}}(v)\; Z^F_{\mathrm{R}}(\bar{v})\;\xi(v)\bar{\xi}(\bar{v})\; Z^{\vec{T},\vec{U}}_{6,6}, \tag{4.11}$$

where

$$\xi(v) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n \mathrm{e}^{2\pi i v})(1-q^n \mathrm{e}^{-2\pi i v})} = \frac{\sin \pi v}{\pi} \frac{\vartheta_1'(0)}{\vartheta_1(v)} \tag{4.12}$$

is an even function of v ($\xi(v) = \xi(-v)$) that counts the helicity contributions of the space-time bosonic oscillators and $Z_{\mathrm{L}}^F(v)$, $Z_{\mathrm{R}}^F(\bar{v})$ are the contributions of the world-sheet fields ψ_μ^L, χ_I^L, ψ_μ^R, χ_I^R, (3.2) and (3.3), modified by a change in the argument of the theta functions:

$$Z_{\mathrm{L}}^F(v) = \frac{1}{2} \sum_{(a,b)} \frac{\mathrm{e}^{i\pi\varphi_{\mathrm{L}}}}{\eta^4}\, \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(v)\, \vartheta\begin{bmatrix} a+H_1 \\ b+G_1 \end{bmatrix} \vartheta\begin{bmatrix} a+H_2 \\ b+G_2 \end{bmatrix} \vartheta\begin{bmatrix} a-H_1-H_2 \\ b-G_1-G_2 \end{bmatrix}, \tag{4.13}$$

and

$$Z_{\mathrm{R}}^F(\bar{v}) = \frac{1}{2} \sum_{(\bar{a},\bar{b})} \frac{\mathrm{e}^{i\pi\varphi_{\mathrm{R}}}}{\bar{\eta}^4}\, \bar{\vartheta}\begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix}(\bar{v})\, \bar{\vartheta}\begin{bmatrix} \bar{a}+H_1 \\ \bar{b}+G_1 \end{bmatrix} \bar{\vartheta}\begin{bmatrix} \bar{a}+H_2 \\ \bar{b}+G_2 \end{bmatrix} \bar{\vartheta}\begin{bmatrix} \bar{a}-H_1-H_2 \\ \bar{b}-G_1-G_2 \end{bmatrix}, \tag{4.14}$$

In order to compute the quantities B_{2n}, we observe that, by using the Riemann identity, these two terms can be cast in the form

$$\begin{aligned} Z_{\mathrm{L}}^F(v) &= \frac{1}{\eta^4}\, \vartheta\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(\frac{v}{2}\right) \vartheta\begin{bmatrix} 1+H_1 \\ 1+G_1 \end{bmatrix}\left(\frac{v}{2}\right) \vartheta\begin{bmatrix} 1+H_2 \\ 1+G_2 \end{bmatrix}\left(\frac{v}{2}\right) \vartheta\begin{bmatrix} 1-H_1-H_2 \\ 1-G_1-G_2 \end{bmatrix}\left(\frac{v}{2}\right), \\ Z_{\mathrm{R}}^F(\bar{v}) &= \frac{1}{\bar{\eta}^4}\, \bar{\vartheta}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\left(\frac{\bar{v}}{2}\right) \bar{\vartheta}\begin{bmatrix} 1+H_1 \\ 1+G_1 \end{bmatrix}\left(\frac{\bar{v}}{2}\right) \bar{\vartheta}\begin{bmatrix} 1+H_2 \\ 1+G_2 \end{bmatrix}\left(\frac{\bar{v}}{2}\right) \bar{\vartheta}\begin{bmatrix} 1-H_1-H_2 \\ 1-G_1-G_2 \end{bmatrix}\left(\frac{\bar{v}}{2}\right). \end{aligned} \tag{4.15}$$

Then, to evaluate the various derivative terms, we use the properties that $\vartheta\left[{1 \atop 1}\right]$ and its even derivatives with respect to v are odd under $v \to -v$ and vanish at $v = 0$.

Taking this into account, it is easy to see that the only non-zero contribution to the second helicity supertrace, B_2:

$$B_2 = (Q + \bar{Q})^2 \; Z^{\text{string}}(v, \bar{v})|_{v=\bar{v}=0}\,, \tag{4.16}$$

comes from the $N = 2$ sector. This is easily computed to be a constant:

$$\begin{aligned} B_2 &= \frac{1}{2|\eta|^{12}} \,\mathrm{Re} \sum_{H_1,G_1,H_2,G_2} \left(\frac{1}{2}\right)^{n_D} \sum_{\vec{H},\vec{G}} C\begin{bmatrix} H_1, & H_2, & H^i, & H^j \\ G_1, & G_2, & G^i, & G^j \end{bmatrix} Z_{6,6}^{\vec{T},\vec{U}} \\ &= 8 \sum_{H_1,G_1,H_2,G_2} \left(\frac{1}{2}\right)^{n_D} \sum_{\vec{H},\vec{G}} C\begin{bmatrix} H_1, & H_2, & H^i, & H^j \\ G_1, & G_2, & G^i, & G^j \end{bmatrix} \\ &= N_V - N_H, \end{aligned} \tag{4.17}$$

in agreement with the supergravity computation, for which the gravity multiplet, as well as a vector multiplet, contribute $+1$, while a hypermultiplet and a hypertensor multiplet (the multiplet that contains the dilaton) contribute -1. In this way the contribution of the untwisted sector cancels, leaving precisely the difference between the number of vector and hypermultiplets coming from the twisted sectors. The result (4.17) also shows that, for the symmetric constructions we are considering, the contribution of the massive short multiplets sums up to zero[6].

It is also easy to see that B_4, which counts the number of short multiplets,

$$B_4 = (Q+\bar{Q})^4 \; Z^{\rm string}(v,\bar{v})|_{v=\bar{v}=0} \;, \tag{4.18}$$

receives contributions from both the $N=2$ and the $N=4$ sectors. The contribution of the $N=2$ sector is due to the term $4(Q^3\bar{Q}+Q\bar{Q}^3)\; Z^{\rm string}(v,\bar{v})|_{v=\bar{v}=0}$ in the expansion of (4.18), and in the models we are considering it turns out to be equal to B_2, while the contribution of the $N=4$ sectors is due to the term $6Q^2\bar{Q}^2\; Z^{\rm string}(v,\bar{v})|_{v=\bar{v}=0}$. The contribution of each $N=4$ sector is easily computed:

$$B_4^{(i)} = 6|\eta|^4 \sum_{(H,G)}{}' \, Z_{2,2}^{(i)}(X)\begin{bmatrix} H \\ G \end{bmatrix} \;, \tag{4.19}$$

where the prime on the summation means that the value $(H,G)=(0,0)$ is excluded and $Z_{2,2}^{(i)}(X)\left[{H \atop G}\right]$ is the conformal block that encodes the contribution of the i-th unshifted plane:

$$Z_{2,2}^{(i)}(X)\begin{bmatrix} H \\ G \end{bmatrix} = \frac{X}{|\eta|^4}, \tag{4.20}$$

where X stands for one of the expressions (4.1)–(4.5). The massless limit of (4.20) is:

$$B_4^{(i)} \underset{{\rm Im}\,\tau\to\infty}{\longrightarrow} 6+\frac{3}{4}\left(N_V^{(i)}+N_H^{(i)}\right) \;, \tag{4.21}$$

where $N_V^{(i)}$ and $N_H^{(i)}$ are respectively the number of vector multiplets and hypermultiplets originating from the i-th twisted sector. Summing over the three sectors and adding the contribution of the $N=2$ sector, we obtain the expected massless contribution, in agreement with supergravity:

$$B_4|_{\rm massless} = 18 + \frac{7N_V - N_H}{4} \;. \tag{4.22}$$

Since the threshold corrections in general are expressed in terms of integrals over the fundamental domain of torus partition functions, for later convenience we quote in Appendix E also the integrals of the various $Z_{2,2}(X)$.

4.2. Higgs and super-Higgs phenomena

It is a general property of shifted lattices, $\Gamma_{2,2}^{w}\left[{H \atop G}\right]$, that there is always at least one corner in moduli space in which, for $(H,G)=(1,0)$ or $(1,1)$, the lattice sum vanishes (for a detailed account, see for instance [22]). The particular limit(s) at which this happens depends on the shift vector w, and, once specified the modular properties of the lattice sum, the various situations, which differ in the choice of w, are mapped into one another by $SL(2,Z)$

[6]The contribution of a short massive multiplet S^j of spin j is $B_2(S^j)=(-)^{2j+1}(2j+1)$, so that (4.17) puts constraints on the ratios of the numbers of massive short multiplets of integer and half-integer spin.

transformations performed on the toroidal moduli T and U (see for instance [22]). The vanishing of the lattice sum at a particular limit means that the states originating from the (H,G)-twisted sector become infinitely massive and decouple from the spectrum. In this limit the unprojected theory is recovered.

This phenomenon is reflected in the behaviour of B_4, as it appears from eq. (4.19). The contribution of a given $N=4$ sector vanishes completely in an appropriate limit in the (T,U) space when X is given by expression (4.2) or (4.3), namely when the Z_2 projection which breaks supersymmetry acts freely in that sector. In the cases (4.4) and (4.5), instead, there is a decoupling of respectively one-half and three-quarters of the states of the twisted sector. In some cases, in the limit in which the states with shifted mass decouple from the spectrum, there is also an associated effective restoration of a certain number of supersymmetries [11, 23]. In this case, the decoupling of some states is accompanied by the appearance of new massless states, which fit into multiplets of the enlarged supersymmetry. This happens in the limits in which one or both of the $Z_2(b)$ projections effectively vanish. We can therefore restore four or even eight supersymmetries. A necessary condition for the existence of a limit of restoration of $N=4$ supersymmetry is the vanishing of B_2. In this limit, B_4 receives a non-zero contribution only from one $N=4$ sector of the orbifold. When there is a restoration of $N=8$, also the massless contribution to B_4 must vanish. As is clear from our formulae, however, this implies the full vanishing of this helicity supertrace.

5. Perturbative and non-perturbative dualities

The knowledge of the partition function allows us to analyse many properties of the string constructions. In this section we consider perturbative and non-perturbative string–string dualities.

5.1. Mirror symmetry from the partition function

In Appendix B.2 we illustrate how to pass from type IIA to type IIB in the framework of the fermionic construction. Here we want to show how mirror symmetry, namely the statement that the type IIA string compactified on the Calabi–Yau manifold M is equivalent to the type IIB string compactified on the mirror manifold $\tilde{M}$ [7], can be easily read off at the orbifold points we are considering. In order to see this, we start by going to the fermionic point of the moduli space of the orbifolds. At such a point, as we saw, the operation of passing from a space M to the mirror $\tilde{M}$ is implemented by a change in the modular coefficient $C_{(b_1|b_2)}$, which is responsible for the sign of $B_2 = N_V - N_H = -\chi/2$. On the other hand, passing from IIA to IIB requires the changes quoted in Appendix B.2, which involve also a change of sign of B_2. When combined, the two operations of exchanging IIA with IIB and M with $\tilde{M}$ leave B_2 invariant and exchange T^i with U^i in B_4. However, at the level of the partition function, such an exchange simply amounts to a different choice of the vectors w, which specify, for each plane, the lattice shifts. The initial choice of w is arbitrary. In particular, we can

[7] See for instance [24].

choose w in such a way that the quantities (4.1)–(4.5), which encode all the non-trivial moduli dependence of the models, are invariant under the exchange of T with U: all the other choices are related to that by $SL(2, Z)$ transformations in T and/or U. We therefore see that, modulo $SL(2, Z)$ transformations, B_2 and B_4 are invariant under mirror symmetry. To conclude that this is a perturbative symmetry of the theory, it is then sufficient to observe that the pair (B_2, B_4) is in a one-to-one correspondence with the partition function. This means that this pair uniquely determines the model, encoding all the perturbative physics at any order of perturbation.

5.2. String–string U-dualities

The non-perturbative dualities we consider here are string–string U-dualities, which relate the type IIA orbifolds to heterotic or type II duals. The type II duals are constructed as asymmetric orbifolds in which the $N = 2$ supersymmetry is realized only among left-movers. As for the heterotic constructions, also in these type II orbifolds the dilaton–axion field belongs to the vector manifold (see Appendix B.3); it is exchanged by U-duality with one of the moduli of the vector manifold of the type IIA duals. This kind of duality is therefore much similar to the duality between type IIA and heterotic strings. In the case of the heterotic/type IIA duality, a necessary condition for the identification of the moduli is the compactification of the type IIA string on a $K3$ fibration [16]. When the conformal field theory can be explicitly solved, as in our $Z_2 \times Z_2$ orbifolds, this requirement translates into the property of spontaneous breaking of the $N = 4$ supersymmetry, in the sense we described above[8].

Therefore the models that are orbifold limits of $K3$ fibrations are the following:

$$\begin{array}{ll} (8,8) & (O, F, F) \\ (4,4) & (D, FD, F^*) \\ (2,2) & (DD, FD, F^*) \\ (0,0) & (F, F, F) \\ & (FD, FD, F^*) \end{array} \tag{5.1}$$

(here F^* stands for F as well as for FD). The heterotic duals of the first three models were considered in [17, 18, 25]. The model $(0, 0)$ is special, possessing also a spontaneously broken $N = 8$ supersymmetry. The detailed study of this last model has been considered in [19].

[8]The connection relies on the fact that in the limit of large volume of the compact space, any $K3$ fibration looks locally like $\mathbf{C} \times K3$. This means that locally, an observer sitting on a point of the base sees 16 supercharges, as in the $T^2 \times K3$ compactification, instead of 8. The extra 8 supercharges are projected out by global, not local, projection. In the case of $Z_2 \times Z_2$ orbifolds, the $K3$ is described by the orbifold limit T^4/Z_2: the $Z_2 \times Z_2$ orbifolds which correspond to $K3$ fibrations are those for which, in some corner of the moduli T, U, which always corresponds to the decompactification of some dimensions, one of the two Z_2 projections can be made to effectively vanish, thereby recovering a $T^2 \times T^4/Z_2$, $N = 4$ orbifold. In all such models, the $N = 2$ and $N = 4$ phases are continuously related by a change of size, or shape, of the compact space. The $N = 4$ phase is reached when at least one dimension is decompactified.

In the case of type IIA/type II asymmetric orbifold duality, on which we will concentrate in the following, the recipe is not the compactification on a $K3$ fibration (or some of its orbifold limits). However, the type IIA/heterotic U-duality can be tested by looking at the renormalization of certain terms in the effective action. In [18], a particular linear combination of R^2 and $F_{\mu\nu}F^{\mu\nu}$ terms, smooth and analytic in the full space of moduli T and U, was shown to be appropriate for a comparison of type IIA and heterotic constructions. The same combination of gravitational and gauge field-strengths can be used here as a guideline in the search of type IIA/type II asymmetric dual pairs. Actually, since in the type II constructions all the gauge bosons are Ramond–Ramond states, the above amplitude turns out to coincide with the R^2 term alone. As it happens for the type IIA, also in the type II asymmetric orbifold constructions the genus zero contribution to this term vanishes. In the type II effective action there is therefore no bare coupling constant, and the dilaton contribution is only non-perturbative and exponentially suppressed. Such a behaviour is reproduced on type IIA by the moduli T^i of the planes denoted, in our convention, by F or FD. Their contribution to the renormalization of the R^2 term is therefore

$$\log \mathrm{Im}\, T \left|\vartheta_4\left(T\right)\right|^4 \longrightarrow \log \mathrm{Im}\, T \quad (\sim\, 0) \;+\; O\left(\mathrm{e}^{-iT}\right) \qquad (|T| \to \infty)\,. \tag{5.2}$$

The mild logarithmic divergence is an infrared artefact and can be lifted by switching on an appropriate cut-off (these planes behave in fact like the planes shifted by the projection that spontaneously breaks the supersymmetry in the models of Refs. [12, 18, 19].)

Once the plane whose Kähler class modulus T is mapped into the dilaton field of the asymmetric orbifold has been identified, the contribution of the moduli of the remaining planes has to match the contribution of the perturbative vector multiplet moduli in the type II asymmetric orbifold. It turns out that the models that possess such an asymmetric dual construction are

$$\begin{array}{ll} (16,16) & (F,O,O) \\ (8,8) & (F^*,D,D) \\ (4,4) & (F^*,DD,DD) \\ (0,0) & (F,F,F) \\ & (F^*,FD,FD)\,. \end{array} \tag{5.3}$$

Among these, a special role is still played by the model $(0,0)$, which therefore possesses both a heterotic and a type II asymmetric dual. In this model, one of the moduli T is mapped in the dilaton of the asymmetric construction and in the inverse of the dilaton of the heterotic dual (in the limit $T \to 0$, (5.2) shows up a linear behaviour in $\tilde{T} \equiv -1/T$, which matches the tree level $\frac{1}{g^2} \sim \mathrm{Im}\, S$ correction on the heterotic side [19]). Since this modulus plays the role of a Higgs field for the spontaneous breaking of some of the supersymmetries in the type IIA orbifold, we learn through the duality map that there is, on the heterotic side, a non-perturbative spontaneous breaking of an $N = 8$ supersymmetry [19]. Similar arguments can be applied to the first three models of (5.3), namely the constructions with $(N_V, N_H) =$ $(16,16)$, $(8,8)$ and $(4,4)$, which we analyse here in detail. It is possible to show that also in such models there is a non-perturbative super-Higgs phenomenon. The dependence on the dilaton $S^{\rm II}$, in these cases, can be obtained by looking at the asymmetric duals. The dual of $S^{\rm II}$ is in fact a perturbative modulus belonging to a hypermultiplet, whose dependence is explicit in the asymmetric constructions; it is not difficult to identify the latter with one of the super-Higgs fields responsible for the spontaneous breaking of some of the supersymmetries. Through the duality between symmetric and asymmetric orbifolds, we therefore learn that, in the strong coupling limit, these type IIA orbifolds have an approximate restoration of a $N=4$ supersymmetry.

In order to see the above issues in detail, we start by discussing the type IIA orbifolds. In these specific cases, Eq. (3.21) reads:

$$\begin{aligned} Z_{\rm II}^{(1,1)} &= \frac{1}{{\rm Im}\,\tau|\eta|^{24}}\frac{1}{4}\sum_{H^{\rm o},G^{\rm o}}\sum_{H^{\rm f},G^{\rm f}}\Gamma_{6,6}^{N_V}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix} \\ &\times\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\ \vartheta\begin{bmatrix}a\\ b\end{bmatrix}\vartheta\begin{bmatrix}a+H^{\rm o}\\ b+G^{\rm o}\end{bmatrix}\vartheta\begin{bmatrix}a+H^{\rm f}\\ b+G^{\rm f}\end{bmatrix}\vartheta\begin{bmatrix}a-H^{\rm o}-H^{\rm f}\\ b-G^{\rm o}-G^{\rm f}\end{bmatrix} \\ &\times\frac{1}{2}\sum_{\bar a,\bar b}(-)^{\bar a+\bar b+\bar a\bar b}\bar\vartheta\begin{bmatrix}\bar a\\ \bar b\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a+H^{\rm o}\\ \bar b+G^{\rm o}\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a+H^{\rm f}\\ \bar b+G^{\rm f}\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a-H^{\rm o}-H^{\rm f}\\ \bar b-G^{\rm o}-G^{\rm f}\end{bmatrix}\,. \end{aligned} \tag{5.4}$$

The characters $\Gamma_{6,6}^{N_V}\left[{}^{H^{\rm o},H^{\rm f}}_{G^{\rm o},G^{\rm f}}\right]\equiv|\eta|^{12}Z_{6,6}$, are given by:

$$\Gamma_{6,6}^{16}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix} = \Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\rm o}\mid H^{\rm f}\\ G^{\rm o}\mid G^{\rm f}\end{bmatrix}\Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\rm f}\mid 0\\ G^{\rm f}\mid 0\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}H^{\rm o}+H^{\rm f}\mid 0\\ G^{\rm o}+G^{\rm f}\mid 0\end{bmatrix}\,, \tag{5.5}$$

$$\begin{aligned} \Gamma_{6,6}^{8}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix} &= \frac{1}{2}\sum_{H^{D_1},G^{D_1}}\Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\rm o}\mid H^{\rm f}\ ;\ 0\\ G^{\rm o}\mid G^{\rm f}\ ;\ 0\end{bmatrix}\Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\rm f}\mid H^{D_1}\\ G^{\rm f}\mid G^{D_1}\end{bmatrix} \\ &\times\ \Gamma_{2,2}^{(3)}\begin{bmatrix}H^{\rm o}+H^{\rm f}\mid H^{D_1}\\ G^{\rm o}+G^{\rm f}\mid G^{D_1}\end{bmatrix}\,, \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \Gamma_{6,6}^{4}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix} &= \frac{1}{4}\sum_{H^{D_1},G^{D_1}}\ \sum_{H^{D_2},G^{D_2}}\Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\rm o}\mid H^{\rm f}\ ;\ 0\\ G^{\rm o}\mid G^{\rm f}\ ;\ 0\end{bmatrix} \\ &\times\ \Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\rm f}\mid 0\ ;\ H^{D_1},\ H^{D_2}\\ G^{\rm f}\mid 0\ ;\ G^{D_1},\ G^{D_2}\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}H^{\rm o}+H^{\rm f}\mid 0\ ;\ H^{D_1},\ H^{D_2}\\ G^{\rm o}+G^{\rm f}\mid 0\ ;\ G^{D_1},\ G^{D_2}\end{bmatrix}\,, \end{aligned} \tag{5.7}$$

where $(H^{\rm o},G^{\rm o})$ refer to the boundary conditions introduced by the projection $Z_2(b_2)$ (see Section 2) and $(H^{\rm f},G^{\rm f})$ refer to the projection $Z_2(b_1)$, which acts freely, as a rotation in the complex planes 2, 3 and a translation, ${\rm e}^{i\pi m_2 G^{\rm f}}$, in the lattice of the first complex plane. In (5.6), (5.7) we used the generalized characters that include the action of the D-projections as they were defined in the Eq. (3.14) and the following. The corresponding helicity supertraces

B_4 are[9]:

$$B_4^{N_V=16} = 6 \sum_{(h,g)}{}' \, \Gamma_{2,2}^{(1)} \begin{bmatrix} 0|h \\ 0|g \end{bmatrix} + 18 \sum_{i=2,3} \Gamma_{2,2}^{(i)} \, , \tag{5.8}$$

$$\begin{aligned} B_4^{N_V=8} &= 3 \sum_{(h,g)}{}' \left(\Gamma_{2,2}^{(1)} \begin{bmatrix} 0|h,0 \\ 0|g,0 \end{bmatrix} + \Gamma_{2,2}^{(1)} \begin{bmatrix} 0|h,h \\ 0|g,g \end{bmatrix} \right) \\ &+ 9\Gamma_{2,2}^{(2)} + 3 \sum_{(h,g)}{}' \, \Gamma_{2,2}^{(2)} \begin{bmatrix} 0|h \\ 0|g \end{bmatrix} + 9\Gamma_{2,2}^{(2)} + 3 \sum_{(h,g)}{}' \, \Gamma_{2,2}^{(3)} \begin{bmatrix} 0|h \\ 0|g \end{bmatrix} \, , \end{aligned} \tag{5.9}$$

$$\begin{aligned} B_4^{N_V=4} &= 3 \sum_{(h,g)}{}' \left(\Gamma_{2,2}^{(1)} \begin{bmatrix} 0|h,0 \\ 0|g,0 \end{bmatrix} + \Gamma_{2,2}^{(1)} \begin{bmatrix} 0|h,h \\ 0|g,g \end{bmatrix} \right) \\ &+ \frac{9}{2}\Gamma_{2,2}^{(2)} + \frac{3}{2} \sum_{(h,g)}{}' \left(\Gamma_{2,2}^{(2)} \begin{bmatrix} 0|h,0 \\ 0|g,0 \end{bmatrix} + \Gamma_{2,2}^{(2)} \begin{bmatrix} 0|0,h \\ 0|0,g \end{bmatrix} + \Gamma_{2,2}^{(2)} \begin{bmatrix} 0|h,h \\ 0|g,g \end{bmatrix} \right) \\ &+ \frac{9}{2}\Gamma_{2,2}^{(3)} + \frac{3}{2} \sum_{(h,g)}{}' \left(\Gamma_{2,2}^{(3)} \begin{bmatrix} 0|h,0 \\ 0|g,0 \end{bmatrix} + \Gamma_{2,2}^{(3)} \begin{bmatrix} 0|0,h \\ 0|0,g \end{bmatrix} + \Gamma_{2,2}^{(3)} \begin{bmatrix} 0|h,h \\ 0|g,g \end{bmatrix} \right) \, . \end{aligned} \tag{5.10}$$

In order to obtain the gravitational corrections, we proceed as in [18, 19]: the four derivative gravitational corrections we will consider are precisely those that were analysed in the framework of $N=4$ ground states of Ref. [12] and the $N=2$ ground states of Refs. [18, 19]. There is no tree-level contribution to these operators, and the R^2 correction is related to the insertion of the two-dimensional operator $2\lambda^2\bar{\lambda}^2$ in the one-loop partition function. In the models at hand, since supersymmetry is realized symmetrically and $N_V = N_H$, the contribution of the $N=2$ sector to B_4 vanishes, and $\left\langle 2\lambda^2\bar{\lambda}^2 \right\rangle$ is identified with $B_4/3$. The massless contributions of the latter give rise to an infrared logarithmic behaviour $2b_{\rm II} \log[M^{({\rm IIA})\,2}/\mu^{({\rm IIA})\,2}]$ [26, 28], where $M^{({\rm IIA})} \equiv \frac{1}{\sqrt{\alpha'_{\rm IIA}}}$ is the type IIA string scale and $\mu^{({\rm IIA})}$ is the type IIA infrared cut-off. Besides this running, the one-loop correction contains the thresholds $\Delta_{\rm IIA}$, which account for the infinite tower of string modes.

The one-loop corrections of the R^2-term are then related to the infrared-regularized genus-one integral of $B_4/3$. In the type IIA string, these R^2 corrections depend on the Kähler moduli (spanning the vector manifold), and are independent of the complex-structure moduli (spanning the scalar manifold):

$$\partial_{T^i} \Delta_{\rm IIA} = \frac{1}{3} \int_{\mathcal{F}} \frac{d^2\tau}{{\rm Im}\,\tau} \partial_{T^i} B_4 \ , \quad \partial_{U^i} \Delta_{\rm IIA} = 0 \, . \tag{5.11}$$

In the type IIB string, the roles of T^i and U^i are interchanged. We obtain the following one-loop correction to the coupling constant:

$$\frac{16\,\pi^2}{g^2_{\rm grav}\left(\mu^{({\rm IIA})}\right)} = -2 \log \, {\rm Im}\, T^1 \left| \vartheta_4 \left(T^1\right) \right|^4 \ + \ \Delta^{N_V}\left(T^2, T^3\right) \ + \left(6 + \frac{N_V}{2}\right) \log \frac{M^{({\rm IIA})}}{\mu^{({\rm IIA})}} \ , \tag{5.12}$$

[9] We recall that the prime summation over (h,g) stands for $(h,g) = \{(0,1),(1,0),(1,1)\}$.

where the various "thresholds" $\Delta^{N_V}(T^2,T^3)$ read:

$$\Delta^{16}\left(T^2,T^3\right) = -6\log\,\mathrm{Im}\,T^2\left|\eta\left(T^2\right)\right|^4 - 6\log\,\mathrm{Im}\,T^3\left|\eta\left(T^3\right)\right|^4 \ , \tag{5.13}$$

$$\begin{aligned}\Delta^{8}\left(T^2,T^3\right) &= -3\log\,\mathrm{Im}\,T^2\left|\eta\left(T^2\right)\right|^4 - \log\,\mathrm{Im}\,T^2\left|\vartheta_4\left(T^2\right)\right|^4 \\ &\quad -3\log\,\mathrm{Im}\,T^3\left|\eta\left(T^3\right)\right|^4 - \log\,\mathrm{Im}\,T^3\left|\vartheta_4\left(T^3\right)\right|^4 \ ,\end{aligned} \tag{5.14}$$

$$\begin{aligned}\Delta^{4}\left(T^2,T^3\right) &= -\frac{3}{2}\log\,\mathrm{Im}\,T^2\left|\eta\left(T^2\right)\right|^4 - \frac{3}{2}\log\,\mathrm{Im}\,T^2\left|\vartheta_4\left(T^2\right)\right|^4 \\ &\quad -\frac{3}{2}\log\,\mathrm{Im}\,T^3\left|\eta\left(T^3\right)\right|^4 - \frac{3}{2}\log\,\mathrm{Im}\,T^3\left|\vartheta_4\left(T^3\right)\right|^4 \ .\end{aligned} \tag{5.15}$$

In the last model, $N_V = N_H = 4$, if the semi-freely acting projection on the third complex plane is a product of $\tilde{D}$- instead of D-operations we obtain:

$$\begin{aligned}\Delta^{4}\left(T^2,T^3\right) &= -\frac{3}{2}\log\,\mathrm{Im}\,T^2\left|\eta\left(T^2\right)\right|^4 - \frac{3}{2}\log\,\mathrm{Im}\,T^2\left|\vartheta_4\left(T^2\right)\right|^4 \\ &\quad -3\log\,\mathrm{Im}\,T^3\left|\eta\left(T^3\right)\right|^4 \ .\end{aligned} \tag{5.16}$$

Notice that, except for the planes 2 and 3 of model $N_V = 16$, the shifts on the $\Gamma_{2,2}^{(i)}$ lattices break the $SL(2,Z)_{T^i}$ duality groups. As in [18, 19], the actual subgroup left unbroken depends on the kind of shifts performed (see Refs. [12, 22, 23]). All the above corrections diverge linearly, in both the large and small T^2 and T^3 limits. On the other hand, the contribution of T^1 diverges only logarithmically in the large-$\mathrm{Im}\,T^1$ limit, and linearly in the inverse modulus $\tilde{T} = -1/T^1$, for small T^1. As we previously discussed, the logarithmic divergence is an infrared artefact and can be removed by switching on an appropriate cut-off.

5.3. The type II asymmetric duals

We now discuss the type II asymmetric dual orbifolds. The model $N_V = N_H = 16$ is constructed starting from the $N = 8$ IIA superstring compactified on T^6 and applying two projections: $Z_2^{(F)}$ and $Z_2^{(\mathrm{o})}$. $Z_2^{(F)}$ acts freely, as $(-)^{F_{\mathrm{R}}}$ together with a translation on T^6, and projects out all the left-moving supersymmetries. $Z_2^{(\mathrm{o})}$, instead, acts as a rotation that reduces symmetrically the number of supersymmetries by 1/2. The properties of the $N = 4$ models obtained by applying only $Z_2^{(F)}$ were already analysed in [12]. The orbifold obtained by the further application of $Z_2^{(\mathrm{o})}$ has an $N = 2$ supersymmetry, which is realized only among left-movers. The partition function of the model reads

$$\begin{aligned}Z_{\mathrm{II}}^{(2,0)} &= \frac{1}{\mathrm{Im}\,\tau|\eta|^{24}}\frac{1}{4}\sum_{H^{\mathrm{F}},G^{\mathrm{F}}}\sum_{H^{\mathrm{o}},G^{\mathrm{o}}}\Gamma_{6,6}\begin{bmatrix}H^{\mathrm{F}},H^{\mathrm{o}}\\G^{\mathrm{F}},G^{\mathrm{o}}\end{bmatrix} \\ &\quad\times\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\vartheta^2\begin{bmatrix}a\\b\end{bmatrix}\vartheta\begin{bmatrix}a+H^{\mathrm{o}}\\b+G^{\mathrm{o}}\end{bmatrix}\vartheta\begin{bmatrix}a-H^{\mathrm{o}}\\b-G^{\mathrm{o}}\end{bmatrix} \\ &\quad\times\frac{1}{2}\sum_{\bar{a},\bar{b}}(-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}(-)^{\bar{a}G^{\mathrm{F}}+\bar{b}H^{\mathrm{F}}+H^{\mathrm{F}}G^{\mathrm{F}}}\bar{\vartheta}^2\begin{bmatrix}\bar{a}\\\bar{b}\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\bar{a}+H^{\mathrm{o}}\\\bar{b}+G^{\mathrm{o}}\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\bar{a}-H^{\mathrm{o}}\\\bar{b}-G^{\mathrm{o}}\end{bmatrix} \ ,\end{aligned} \tag{5.17}$$

where now

$$\Gamma_{6,6}\begin{bmatrix} H^{\rm F}, H^{\rm o} \\ G^{\rm F}, G^{\rm o} \end{bmatrix} = \Gamma^{(1)}_{2,2}\begin{bmatrix} H^{\rm o} \mid H^{\rm F} \\ G^{\rm o} \mid G^{\rm F} \end{bmatrix}\Gamma^{(2)}_{2,2}\begin{bmatrix} H^{\rm o} \mid 0 \\ G^{\rm o} \mid 0 \end{bmatrix}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0 \\ 0 \mid 0 \end{bmatrix} . \qquad (5.18)$$

Notice that neither $Z_2^{(\rm F)}$ nor $Z_2^{(\rm o)}$ act on the third plane. The massless spectrum can be easily analysed by computing the helicity supertraces B_2 and B_4. B_2 turns out to be zero (for the details of this computation, see the Appendix F). This tells us that $N_V = N_H$. The supertrace B_4 is (cf. [19]):

$$\begin{aligned} B_4 \;&=\; \frac{3}{16}\,\frac{1}{\bar\eta^{12}}\sum_{\bar a,\bar b}\sum_{(H^{\rm F},G^{\rm F})}(-)^{\bar a+\bar b+\bar a\bar b}(-)^{\bar a G^{\rm F}+\bar b H^{\rm F}+H^{\rm F}G^{\rm F}}\bar\vartheta^4\begin{bmatrix}\bar a\\ \bar b\end{bmatrix}\Gamma_{6,6}\begin{bmatrix} H^{\rm F},0 \\ G^{\rm F},0\end{bmatrix} \\ &\quad + 36\,\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0 \\ 0 \mid 0 \end{bmatrix} \end{aligned} \qquad (5.19)$$

The massless contribution, which coincides with the supergravity result, is

$$B_4|_{\rm massless} = 18 + \frac{7N_V - N_H}{4} = 42 \,, \qquad (5.20)$$

from which we derive $N_V = N_H = 16$. The massless spectrum therefore contains, besides the supergravity multiplet, 3+16 vector multiplets and 4+16 hypermultiplets: it is therefore the same as that of the type IIA model $N_V = 16$. However, here the dilaton belongs to a vector multiplet. This is a general property of all the $N = 2$ string compactifications in which all the supersymmetries are realized either between only left or between only right movers, such as the heterotic strings or type II asymmetric orbifolds as the ones we consider. The reason is that the dilaton, in such cases, is uncharged under the $SU(2)$ operators that rotate the supercharges of the $N = 2$ supergravity[10].

The $N_V = N_H = 8$ orbifold is constructed by modding out the previous model with a further $Z_2^{(D)}$ projection, which acts semi-freely. The partition function of this orbifold is given as in (5.17), but with (5.18) replaced by

$$\Gamma_{6,6}\begin{bmatrix} H^{\rm F}, H^{\rm o} \\ G^{\rm F}, G^{\rm o} \end{bmatrix} = \frac{1}{2}\sum_{(H^{D_1},G^{D_1})}\Gamma^{(1)}_{2,2}\begin{bmatrix} H^{\rm o} \mid H^{\rm F}\,;\,0 \\ G^{\rm o} \mid G^{\rm F}\,;\,0 \end{bmatrix}\Gamma^{(2)}_{2,2}\begin{bmatrix} H^{\rm o} \mid 0\,;\,H^{D_1} \\ G^{\rm o} \mid 0\,;\,G^{D_1} \end{bmatrix}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0\,;\,H^{D_1} \\ 0 \mid 0\,;\,G^{D_1} \end{bmatrix} . \qquad (5.21)$$

The helicity supertrace B_4 is now given by an expression similar to (5.19), but the second term, instead of being $36\Gamma^{(3)}_{2,2}\left[{}^{0|0}_{0|0}\right]$, is now

$$12\sum_{(h,g)}{}' \left(\frac{1}{2}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0 \\ 0 \mid 0 \end{bmatrix} + \frac{1}{2}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid h \\ 0 \mid g \end{bmatrix}\right) \,, \qquad (5.22)$$

[10] One can construct the vertex operators, which represent the states of the string theory, and then construct explicitly the generators of supersymmetry in this representation (see Appendix B.3). One then finds that the spinor vertex operator, which corresponds to the two transverse space-time coordinates $e^{-\frac{i}{2}H_0}$, is common to both the supercharges (in type II symmetric orbifolds, we have instead $Q \sim e^{-\frac{i}{2}H_0}$, $\bar Q \sim e^{-\frac{i}{2}\bar H_0}$). As a consequence, the generators of the $SU(2)$ symmetry of the $N = 2$ do not act on the space-time degrees of freedom, and therefore leave invariant the dilaton, whose vertex operator actually contains only space-time degrees of freedom.

which gives

$$B_4|_{\rm massless} = 30 \tag{5.23}$$

consistently with $N_V = 8$ ($B_2 = N_V - N_H$ is zero in all these models).

Finally, the model $N_V = N_H = 4$ is obtained by applying two $Z_2^{(D)}$ projections, which act on the compact space, producing the following character:

$$\begin{aligned}\Gamma_{6,6}\begin{bmatrix}H^{\rm F}, H^{\rm o}\\ G^{\rm F}, G^{\rm o}\end{bmatrix} &= \frac{1}{4}\sum_{(H^{D_1},G^{D_1})}\sum_{(H^{D_2},G^{D_2})}\Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\rm o}\mid H^{\rm F}\ ; H^{D_1}\\ G^{\rm o}\mid G^{\rm F}\ ;\ G^{D_1}\end{bmatrix}\\ &\times\Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\rm o}\mid 0\ ;\ H^{D_2}\\ G^{\rm o}\mid 0\ ;\ G^{D_2}\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}0\mid 0\ ;\ H^{D_1},H^{D_2}\\ 0\mid 0\ ;\ G^{D_1},G^{D_2}\end{bmatrix}.\end{aligned} \tag{5.24}$$

In this case, the second term in B_4 is

$$12\sum_{(h,g)}{}'\left(\frac{1}{4}\Gamma_{2,2}^{(3)}\begin{bmatrix}0\mid 0,0\\ 0\mid 0,0\end{bmatrix}+\frac{1}{4}\Gamma_{2,2}^{(3)}\begin{bmatrix}0\mid h,0\\ 0\mid g,0\end{bmatrix}+\frac{1}{4}\Gamma_{2,2}^{(3)}\begin{bmatrix}0\mid 0,h\\ 0\mid 0,g\end{bmatrix}+\frac{1}{4}\Gamma_{2,2}^{(3)}\begin{bmatrix}0\mid h,h\\ 0\mid g,g\end{bmatrix}\right), \tag{5.25}$$

which gives $B_4|_{\rm massless} = 24$ and $N_V = 4$.

Owing to the free action of $Z_2^{F_{\rm R}}$, all these models possess a spontaneously broken $N = 4 = (2,2)$ supersymmetry, restored in the limit in which the projection $Z_2^{F_{\rm R}}$ becomes irrelevant: this phenomenon takes place for special values of the moduli belonging to hypermultiplets. We remark that in all these models $B_2 \equiv 0$. As a consequence there are no points in the moduli space in which there appear “$N = 2$”, $\Delta N_V \neq \Delta N_H$ singularities.

As in the case of the type IIA orbifolds, also in the asymmetric duals the R^2 gravitational corrections receive a contribution only from one loop, and are obtained by the insertion of the operator $\lambda_{\rm L}^2\lambda_{\rm R}^2$ (see [19]). The only non-zero contribution is provided by the sectors with $(H^{\rm o}, G^{\rm o}) \neq (0,0)$. We obtain:

$$\frac{16\,\pi^2}{g^2_{\rm grav}\left(\mu^{(\rm As)}\right)} = \left(4+\frac{N_V}{2}\right)\log\frac{M^{(\rm As)}}{\mu^{(\rm As)}} + \Delta^{N_V}\left(T^{\rm As}, U^{\rm As}\right)\ , \tag{5.26}$$

where we introduced the type II asymmetric mass scale and infrared cut-off, $M^{(\rm As)}$ and $\mu^{(\rm As)}$ respectively, and

$$\Delta^{16}\left(T^{\rm As}, U^{\rm As}\right) = -6\log{\rm Im}\,T^{\rm As}\left|\eta\left(T^{\rm As}\right)\right|^4 - 6\log{\rm Im}\,U^{\rm As}\left|\eta\left(U^{\rm As}\right)\right|^4\ ; \tag{5.27}$$

$$\begin{aligned}\Delta^{8}\left(T^{\rm As}, U^{\rm As}\right) &= -3\log{\rm Im}\,T^{\rm As}\left|\eta\left(T^{\rm As}\right)\right|^4 - 3\log{\rm Im}\,U^{\rm As}\left|\eta\left(U^{\rm As}\right)\right|^4\\ &\quad-\log{\rm Im}\,T^{\rm As}\left|\vartheta_i\left(T^{\rm As}\right)\right|^4 - \log{\rm Im}\,U^{\rm As}\left|\vartheta_j\left(U^{\rm As}\right)\right|^4\ ;\end{aligned} \tag{5.28}$$

$$\begin{aligned}\Delta^{4}\left(T^{\rm As}, U^{\rm As}\right) &= -3\log{\rm Im}\,T^{\rm As}\left|\eta\left(T^{\rm As}\right)\right|^4 - \frac{3}{2}\log{\rm Im}\,U^{\rm As}\left|\eta\left(U^{\rm As}\right)\right|^4\\ &\quad-\frac{3}{2}\log{\rm Im}\,T^{\rm As}\left|\vartheta_4\left(T^{\rm As}\right)\right|^4\ .\end{aligned} \tag{5.29}$$

5.4. Comparison of symmetric and asymmetric orbifolds

We now come to the comparison of the type IIA symmetric and the type II asymmetric orbifolds. As is clear from expressions (5.13)–(5.16) and the analogous (5.27)–(5.29) for the asymmetric orbifolds, for any type IIA symmetric orbifold it is possible to choose actions of the D-projections that can be reproduced in the type II asymmetric orbifolds, leading to the same corrections Δ^{N_V}: $\Delta^{N_V}(T^2,T^3) = \Delta^{N_V}(T^{\rm As},U^{\rm As})$. A comparison of the corrections with the R^2 term therefore leads to the following identification of the moduli in the vector manifold: $T^2 = T^{\rm As}$, $T^3 = U^{\rm As}$. These moduli are perturbative in both the constructions. On the other hand, the contribution of the modulus T^1 in (5.12), in the limit $T^1 \to \infty$, diverges only logarithmically. This is consistent with the vanishing of the perturbative, genus-zero ($O(S^{\rm As})$) contribution to this term in the type II asymmetric orbifolds. We are therefore led to identify this modulus with the dilaton–axion field $\tau_S^{\rm As} \equiv 4\pi S^{\rm As}$ of the asymmetric orbifolds [12, 19]. The logarithmic behaviour is then interpreted as a non-perturbative effect.

We remark that the identification of the perturbative moduli is possible only for special choices of the translations introduced by the D-projections. The reason is that it is not always possible to reconstruct the properties of the Kähler class moduli of a product of two tori, whose lattice of momenta and windings is shifted by the action of the D-projections, with the Kähler class and complex structure moduli of a single torus with shifted lattice: different translations correspond to different cuts in the moduli space of the model, and not all the cuts correspond to dual constructions. However, we learned that a correct cut exists, and the identification of (T^2,T^3) with $(T^{\rm As},U^{\rm As})$ provides a test of the duality. The corrections computed in the type IIA symmetric orbifolds therefore provide the full, perturbative and non-perturbative, corrections for the asymmetric orbifolds. These are given by the expression (5.12), in which

i) the moduli T^1, T^2 are replaced by $T^{\rm As}, U^{\rm As}$,

ii) the modulus T^1 is replaced by the dilaton $4\pi S^{\rm As}$,

iii) the type IIA string mass and cut-off have to be replaced by those of the type II asymmetric theory. Indeed, we are using a regularization scheme for which the ratio of the mass and the cut-off is duality-independent, and can be expressed in terms of the ratio of the Plank mass and a physical cut-off μ:

$$\frac{M^{\rm (IIA)}}{\mu^{\rm (IIA)}} = \frac{M^{\rm (As)}}{\mu^{\rm (As)}} = \frac{M_{\rm Planck}}{\mu} \, . \tag{5.30}$$

This duality also provides, on the other hand, a window on the non-perturbative physics of the type IIA orbifolds. As we already pointed out, in the above type IIA orbifolds supersymmetry is broken in a "rigid" way; namely, it is not possible to restore some of the broken supersymmetries by taking some special limit in the moduli of the compact space. This has to be contrasted with the situation of the type II asymmetric dual orbifolds, in which instead the asymmetric projection $Z_2^{\rm F_R}$ acts freely, and it can be made to effectively vanish by taking an appropriate limit in the moduli of the first plane. These are hypermultiplet moduli. In these orbifolds, the first and the second plane provide four such moduli, that correspond to the three hypermultiplet moduli U^1, U^2, U^3 and the dilaton $S^{\rm II}$ of the type IIA orbifolds. Three moduli are therefore perturbative in both the theories, and the mismatch in the perturbative mechanism of spontaneous breaking of supersymmetry forces us to identify the modulus of the asymmetric orbifolds, which plays the role of super-Higgs field as the dual of the type IIA dilaton. From the point of view of type IIA, the mechanism of spontaneous breaking of supersymmetry is then entirely non-perturbative.

6. Conclusions

In this work we constructed and classified all the four-dimensional $N = 2$, $Z_2 \times Z_2$ symmetric orbifolds of the type IIA/B superstring. After having constructed the models at the fermionic point, according to the rules of the "fermionic construction", we established the equivalence, for this class of orbifolds, of world-sheet fermions and bosons, and we derived the partition function of each model at a generic point in the space of the moduli of the three tori of $T^6 = T^2 \times T^2 \times T^2$. Through an analysis of some helicity supertraces, easily computable from the partition function, we investigated the appearance of stringy Higgs and super-Higgs phenomena, which served as a guideline in the search for heterotic and/or type II duals. We devoted particular attention to the study of the latter, for which we provided an analysis analogous to that of the symmetric orbifolds. These pairs are related by a map that exchanges the dilaton–axion field of the type II asymmetric construction for a perturbative modulus of the type IIA dual, associated to a vector multiplet. Conversely, the type IIA dilaton is mapped into a perturbative modulus of the type II asymmetric dual, associated to a hypermultiplet. Through the comparison of the corrections to the R^2 term, we then provided a test of this duality, obtaining also a prediction on the non-perturbative physics of the type II asymmetric models. On the other hand, we observed a perturbative super-Higgs mechanism on the type II asymmetric models, unobservable on the perturbative type IIA duals, because it involves as super-Higgs field a modulus dual to the type IIA dilaton.

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Appendix A: The type II string in the free fermionic formulation

In the free fermionic formulation of string theory [2]–[4] the string degrees of freedom are expressed in terms of free world-sheet fermions. For the four-dimensional type II string and in the light cone-gauge theory these fermions are [5]:

- the space-time degrees of freedom

$$\begin{array}{l} \text{left}: \partial_z X_\mu(z,\bar{z}),\ \psi^{\rm L}_\mu(z) \\ \text{right}: \partial_{\bar{z}} X_\mu(z,\bar{z}),\ \psi^{\rm R}_\mu(z) \end{array} \quad , \ \mu = 1,2$$

- the internal degrees of freedom

$$\begin{array}{l} \text{left}: \chi^{\rm L}_I(z),\ y^{\rm L}_I(z),\ \omega^{\rm L}_I(z) \\ \text{right}: \chi^{\rm R}_I(z),\ y^{\rm R}_I(z),\ \omega^{\rm R}_I(z) \end{array} \quad , \ I = 1,\dots 6.$$

The world-sheet supersymmetry, necessary for a consistent theory, is realized in the usual way among the space-time coordinates and non-linearly among the internal coordinates [29]

$$\delta f^A = \eta^{ABC} f^B f^C \epsilon \ , \qquad f \in \{\chi^{\rm L}_I, y^{\rm L}_I, \omega^{\rm L}_I\,, I = 1\dots 6\}\ ,$$

where ϵ is a Grassmann field and η^{ABC} are the properly normalised structure constants of a Lie algebra $G = G^L_W$ and similarly for right movers $G = G^R_W$.

It is known that the transportation properties of fermionic fields on surfaces with non-trivial topology, such as the string world-sheet, are not completely determined by the two-dimensional metric and an extra information must be supplied. This information is known as spin structure and describes the phases emerging when each fermionic field moves around a non-contractible circle of the surface. Spin structures are in principle arbitrary, but a consistent string model should satisfy a set of physical requirements:

(i) multiloop modular invariance,

(ii) factorisation of physical amplitudes,

(iii) global existence of left and right supercurrents.

After imposing these constraints much freedom is left in choosing spin structures and a very big number of consistent string models can be obtained.

If we restrict ourselves to periodic–antiperiodic fermionic fields, and demand space-time supersymmetry to emerge symmetrically from left and right movers (which means that left and right space-time fermions will be treated symmetrically), the choice of the internal fermion gauge groups (G^L_W, G^R_W) is essentially unique:

$$G^L_W = G^R_W = \prod_{I=1}^{6} SO(3)^I$$

and the left and right supercurrents take the form

$$T_F(z) = i\partial_z X^\mu \psi_\mu^{\rm L} + \sum_{I=1}^{6} \chi_I^{\rm L} y_I^{\rm L} \omega_I^{\rm L} \,, \qquad \overline{T}_F(z) = i\partial_{\bar z} X^\mu \psi_\mu^{\rm R} + \sum_{I=1}^{6} \chi_I^{\rm R} y_I^{\rm R} \omega_I^{\rm R}, \tag{A.1}$$

Then the solution of the consistency constraints can be expressed in a simple set of rules for constructing any type II string model. A specific model is defined by

1) A set of boundary condition basis vectors $\{b_1 = 1, \dots, b_N\}$, which generate a set of 2^N boundary conditions vectors Ξ: $\xi \in \Xi \leftrightarrow \xi = \sum_{i=1}^N m_i b_i, m_i = 0, 1$. Each boundary condition vector b_i has 40 entries[11]:

$$b_i \equiv \{b_i(\psi^\mu), b_i(\chi^1), \dots, b_i(y^1) \dots, b_i(\omega^1) \dots; b_i(\bar\psi^\mu), b_i(\chi^1), \dots, b_i(\bar y^1) \dots, b_i(\bar\omega^1) \dots\}$$

where $b_i(f) = 0, 1$ correspond to the transportation properties of each fermion $f \to -\mathrm{e}^{i\pi b_i(f)}$. The basis vectors are subject to some restrictions, namely

   1. $b_i \cdot b_i = \bmod 8 \,, \forall\ i = 0, \dots, N,$
   2. $b_i \cdot b_j = \bmod 4 \,, \forall\ i \neq 0 = 1, \dots, N,$
   3. $\prod_f b_i(f) b_j(f) b_k(f) b_\ell(f) = 0 \bmod 2 \,, \forall\ i{\neq}j{\neq}k{\neq}\ell = 0, \dots, N,$
   4. $|\{b_i(\chi_I), b_i(y_I), b_i(\omega_I)\}|^2 \ = b_i(\psi^\mu) \bmod 2 \ \ ,$
   $|\{b_i(\bar\chi_I), b_i(\bar y_I), b_i(\bar\omega_I)\}|^2 = b_i(\bar\psi^\mu) \bmod 2 \,, \quad \forall\ a = 1, \dots, 6 \text{ and } \forall\ i = 0, \dots, N.$

2) A set of $\frac{N(N-1)}{2} + 1$ phases $c\left[{1 \atop 1}\right] = \pm 1 \,, c\left[{b_i \atop b_j}\right] = \pm 1$, $i > j$ (we will use also the notation $C_{(b_i|b_j)} \equiv c\left[{b_i \atop b_j}\right]$), which determine the weight of each spin structure to the string partition function.

The model's partition function is then given by

$$Z = \frac{1}{2^N} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3} \frac{1}{|\eta|^4} \sum_{\alpha,\beta \in \Xi} c\begin{bmatrix}\alpha\\ \beta\end{bmatrix} Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix},$$

where

$$Z\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \prod_{f=\text{left}} \left(\frac{\vartheta\left[{\alpha(f) \atop \beta(f)}\right]}{\eta} \right)^{\frac{1}{2}} \prod_{f=\text{right}} \left(\frac{\bar\vartheta\left[{\alpha(f) \atop \beta(f)}\right]}{\bar\eta} \right)^{\frac{1}{2}}$$

and $c\left[{\alpha \atop \beta}\right]$ can be expressed in terms of $c\left[{b_i \atop b_j}\right]$, $i > j$ and $c\left[{b_1 \atop b_1}\right] \equiv c\left[{1 \atop 1}\right]$ using

$$c\begin{bmatrix}0\\ \alpha\end{bmatrix} \ = \ \delta_\alpha \tag{A.2}$$

[11] Along the paper we employed a slightly different notation, that is to present a basis vector as the set of periodic fermions. For example, $b = \{1, 1, 1, 1, 1, 1, 1, 0, \dots, 0\}$ is written as $b = \{\psi_\mu^{\rm L}, \chi_1^{\rm L}, \chi_2^{\rm L}, \chi_3^{\rm L}, \chi_4^{\rm L}, \chi_5^{\rm L}, \chi_6^{\rm L}\}$ in this notation.

$$c\begin{bmatrix}\alpha\\ \beta+\gamma\end{bmatrix} = \delta_\alpha c\begin{bmatrix}\alpha\\ \beta\end{bmatrix} c\begin{bmatrix}\alpha\\ \gamma\end{bmatrix} \tag{A.3}$$

$$c\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \mathrm{e}^{\frac{i\pi}{2}\alpha\cdot\beta} c\begin{bmatrix}\beta\\ \alpha\end{bmatrix} \tag{A.4}$$

$$c\begin{bmatrix}\alpha\\ \alpha\end{bmatrix} = \mathrm{e}^{\frac{i\pi}{4}\alpha^2} c\begin{bmatrix}\alpha\\ b_1\end{bmatrix} \tag{A.5}$$

$$\tag{A.6}$$

with $\delta_\alpha = \mathrm{e}^{i\pi\left(\alpha(\psi^\mu)+\alpha(\bar{\psi}^\mu)\right)}$.

Along the paper, we have used indifferently two equivalent notations for the operation of composition of fermion sets: the sum, as here, which is more convenient when we specify the fermion sets through the boundary conditions they assign, and the "symmetric difference" (see [3]), indicated as a product of sets, which contains the union of two fermion sets minus their intersection, and is more convenient when, as in many sections of the paper, we specify the periodic fermions contained in the various sets.

Appendix B: Massless spectrum and vertex operators

B.1. $Z_2 \times Z_2$ orbifolds in the fermionic construction

We write here the vertex operator representation of the massless states of the $Z_2 \times Z_2$ orbifold constructed in Section 2. The massless states of the untwisted sector come from the NS-NS, R-NS, NS-R and RR sectors, which are given, in our convention, by the sectors $\oslash$ (the "vacuum"), S, $\bar{S}$ and $S\bar{S}$ respectively. In order to describe the physical states, we introduce the $SU(2)$, R-parity currents $\mathrm{e}^{\pm\frac{i}{2}H_i}$, ∂H_i, $\mathrm{e}^{\pm\frac{i}{2}\overline{H}_i}$, $\overline{\partial H}_i$, $i=0,1,2,3$ [12], where:

$$\partial H_0 = \psi_1^{\mathrm{L}}\psi_2^{\mathrm{L}}\ , \quad \partial H_1 = \chi_1^{\mathrm{L}}\chi_2^{\mathrm{L}}\ , \quad \partial H_2 = \chi_3^{\mathrm{L}}\chi_4^{\mathrm{L}}\ , \quad \partial H_3 = \chi_5^{\mathrm{L}}\chi_6^{\mathrm{L}}\ ,$$

$$\overline{\partial H}_0 = \psi_1^{\mathrm{R}}\psi_2^{\mathrm{R}}\ , \quad \overline{\partial H}_1 = \chi_1^{\mathrm{R}}\chi_2^{\mathrm{R}}\ , \quad \overline{\partial H}_2 = \chi_3^{\mathrm{R}}\chi_4^{\mathrm{R}}\ , \quad \overline{\partial H}_3 = \chi_5^{\mathrm{R}}\chi_6^{\mathrm{R}} \tag{B.1}$$

(the eigenvalues of ∂H_0 and $\overline{\partial H}_0$ give the space-time spin). The fermions originate from the S and $\bar{S}$ sectors. They are two gravitinos, represented, in the $-1/2$-picture, by

$$\mathrm{e}^{\pm\frac{i}{2}(2H_0+\bar{H}_0+\bar{H}_1+\bar{H}_2+\bar{H}_3)}\ , \qquad \mathrm{e}^{\pm\frac{i}{2}(2\bar{H}_0+H_0+H_1+H_2+H_3)}\ ; \tag{B.2}$$

two dilatinos:

$$\mathrm{e}^{\pm\frac{i}{2}(2H_0-\bar{H}_0-\bar{H}_1-\bar{H}_2-\bar{H}_3)}\ , \qquad \mathrm{e}^{\pm\frac{i}{2}(2\bar{H}_0-H_0-H_1-H_2-H_3)}\ ; \tag{B.3}$$

and twelve spin-1/2 fermions:

$$\mathrm{e}^{\pm iH_1}\mathrm{e}^{\pm\frac{i}{2}(\bar{H}_0+\bar{H}_1-\bar{H}_2-\bar{H}_3)}, \qquad \mathrm{e}^{\pm i\bar{H}_1}\mathrm{e}^{\pm\frac{i}{2}(H_0+H_1-H_2-H_3)}, \tag{B.4}$$

[12] In this notation, $\partial H \equiv \partial_z H$, $\overline{\partial H} \equiv \partial_{\bar{z}}\overline{H}$, where $H \equiv X^{\mathrm{L}}(z)$, $\overline{H} \equiv X^{\mathrm{R}}(\bar{z})$ are respectively the holomorphic (left-moving) and the antiholomorphic (right-moving) part into which a world-sheet boson X is decomposed: $X = H + \overline{H}$.

$$\mathrm{e}^{\pm iH_2}\mathrm{e}^{\pm\frac{i}{2}(\bar{H}_0-\bar{H}_1+\bar{H}_2-\bar{H}_3)}, \qquad \mathrm{e}^{\pm i\bar{H}_2}\mathrm{e}^{\pm\frac{i}{2}(H_0-H_1+H_2-H_3)}, \tag{B.5}$$

$$\mathrm{e}^{\pm iH_3}\mathrm{e}^{\pm\frac{i}{2}(\bar{H}_0-\bar{H}_1-\bar{H}_2+\bar{H}_3)}, \qquad \mathrm{e}^{\pm i\bar{H}_3}\mathrm{e}^{\pm\frac{i}{2}(H_0-H_1-H_2+H_3)}, \tag{B.6}$$

The bosons originate from the sectors $\oslash$ and $S\bar{S}$. The vacuum sector $\oslash$ contains the graviton $\mathrm{e}^{\pm i(H_0+\bar{H}_0)}$, the pair dilaton–pseudoscalar $\mathrm{e}^{\pm i(H_0-\bar{H}_0)}$, and the six complex scalars $\mathrm{e}^{\pm i(H_i\pm\bar{H}_i)}$. From the vertex operator representation of the graviton and the gravitinos we read off the generators of the $N=2$ supersymmetry:

$$Q = \mathrm{e}^{-\frac{i}{2}(H_0-H_1-H_2-H_3)} \quad \text{and} \quad \bar{Q} = \mathrm{e}^{-\frac{i}{2}(\bar{H}_0-\bar{H}_1-\bar{H}_2-\bar{H}_3)}\,, \tag{B.7}$$

from which we derive also the expressions of the generators of the $N=2$, $SU(2)$ algebra: J^+, J^- $(\equiv J^{+*})$ and J^0, whose charge operator I^0 is given by:

$$I^0 = \frac{1}{2}\oint(\partial H_0 - \partial H_1 - \partial H_2 - \partial H_3) - \frac{1}{2}\oint(\overline{\partial H_0} - \overline{\partial H_1} - \overline{\partial H_2} - \overline{\partial H_3}). \tag{B.8}$$

It is then easy to see that the dilaton-pseudoscalar

$$\mathrm{e}^{\pm i(H_0-\bar{H}_0)}\,, \tag{B.9}$$

and the three complex scalars

$$\mathrm{e}^{\pm i(H_i-\bar{H}_i)} \qquad i=1,2,3, \tag{B.10}$$

carry a non-zero I^0 charge: they therefore belong to hypermultiplets. Since we are considering a type IIA orbifold, the three complex scalars (B.10) correspond to the complex structure moduli U^i of the three tori of the internal space. The three complex scalars

$$\mathrm{e}^{\pm i(H_i+\bar{H}_i)} \qquad i=1,2,3, \tag{B.11}$$

have no I^0 charge and are superpartners of the vectors: they therefore correspond to the three Kähler class moduli T^i. The $S\bar{S}$ sector contains four complex scalars, which carry I^0 charge:

$$\begin{aligned}
&\mathrm{e}^{\pm\frac{i}{2}(H_0+H_1+H_2+H_3-\bar{H}_0-\bar{H}_1-\bar{H}_2-\bar{H}_3)}\,,\\
&\mathrm{e}^{\pm\frac{i}{2}(H_0-H_1-H_2+H_3-\bar{H}_0+\bar{H}_1+\bar{H}_2-\bar{H}_3)}\,,\\
&\mathrm{e}^{\pm\frac{i}{2}(H_0-H_1+H_2-H_3-\bar{H}_0+\bar{H}_1-\bar{H}_2+\bar{H}_3)}\,,\\
&\mathrm{e}^{\pm\frac{i}{2}(H_0+H_1-H_2-H_3-\bar{H}_0-\bar{H}_1+\bar{H}_2+\bar{H}_3)}\,,
\end{aligned} \tag{B.12}$$

and four vectors:

$$\begin{aligned}
&\mathrm{e}^{\pm\frac{i}{2}(H_0+H_1+H_2+H_3+\bar{H}_0+\bar{H}_1+\bar{H}_2+\bar{H}_3)}\,,\\
&\mathrm{e}^{\pm\frac{i}{2}(H_0-H_1-H_2+H_3+\bar{H}_0-\bar{H}_1-\bar{H}_2+\bar{H}_3)}\,,\\
&\mathrm{e}^{\pm\frac{i}{2}(H_0-H_1+H_2-H_3+\bar{H}_0-\bar{H}_1+\bar{H}_2-\bar{H}_3)}\,,\\
&\mathrm{e}^{\pm\frac{i}{2}(H_0+H_1-H_2-H_3+\bar{H}_0+\bar{H}_1-\bar{H}_2-\bar{H}_3)}\,.
\end{aligned} \tag{B.13}$$

It is easy to recognise that the first vector belongs to the gravity multiplet, being obtained by applying twice the supersymmetry generators to the graviton.

B.2. IIA versus IIB in the fermionic language

The type IIA↔B exchange is realized by changing the chirality of, say, the right-moving spinors. In the fermionic construction this is implemented by the following changes with respect to the type IIA choice of Table 2.1: $C_{(\bar{S}|\bar{S})} \rightarrow +1$, $C_{(\bar{S}|F)} \rightarrow -1$, $C_{(\bar{S}|b_{11})} \rightarrow -1$, $C_{(\bar{S}|b_{22})} \rightarrow -1$. Under this exchange, the right-moving supersymmetry generator becomes:

$$\bar{Q} \rightarrow \mathrm{e}^{-\frac{i}{2}(\bar{H}_0+\bar{H}_1+\bar{H}_2+\bar{H}_3)} \,. \tag{B.14}$$

As a consequence, the $SU(2)$ Cartan charge operator I^0 is now:

$$I^0 = \frac{1}{2}\oint (\partial H_0 - \partial H_1 - \partial H_2 - \partial H_3) - \frac{1}{2}\oint (\overline{\partial H_0} + \overline{\partial H_1} + \overline{\partial H_2} + \overline{\partial H_3}) \,. \tag{B.15}$$

In this case the states that are associated, in the type IIA construction, to the complex scalars T^1, T^2, T^3, given in (B.11), now carry I^0 charge, while those associated to the three complex scalars U^1, U^2, U^3, given in (B.10), do not. The role of T^i and U^i is therefore exchanged, as expected. A change in the sign of $C_{(b_{11}|b_{22})}$ exchanges N_V and N_H. This then changes the sign of the Euler characteristic of the compact space, $\chi = 2(N_H - N_V)$.

B.3. Massless states of type II asymmetric orbifolds

The analysis of the massless spectrum of the asymmetric orbifolds discussed in Section 5 can be performed in a similar way, by going to the fermionic point. We quote here the vertices for the bosonic massless states of the untwisted sectors: $(H^{\mathrm{F}}, G^{\mathrm{F}})$, $(H^{\mathrm{o}}, G^{\mathrm{o}})$ equal to $(0,0)$ or $(0,1)$. They are:

$$\begin{array}{lll} \mathrm{e}^{\pm i(H_0+\bar{H}_0)} & & \text{(graviton)} \\ \mathrm{e}^{\pm i(H_0-\bar{H}_0)} & & \text{(dilaton, pseudoscalar)} \\ \mathrm{e}^{\pm i(H_i-\bar{H}_j)}, & i,j=1,2 & \text{(hyperscalars)} \\ \mathrm{e}^{\pm iH_0}\mathrm{e}^{\pm\frac{i}{2}\bar{H}_3}, & \mathrm{e}^{\pm iH_3}\mathrm{e}^{\pm i\bar{H}_0} & \text{(graviphoton, vectors)} \\ \mathrm{e}^{\pm i(H_3-\bar{H}_3)} & & \text{(vectorscalars)} \end{array} \tag{B.16}$$

In particular, we notice that, although the dilaton is represented by the same vertex operator as the dilaton of the type IIA orbifolds, Eq. (B.9), in this case it is uncharged under the $SU(2)$ symmetry of the $N=2$ superalgebra. The analogous of the operators Q and $\bar{Q}$, Eq. (B.7), are in fact in this case:

$$Q_+,\, Q_- = \mathrm{e}^{-\frac{i}{2}(H_0+\epsilon H_1 \pm H_2 \pm H_3)} \,, \tag{B.17}$$

where ϵ takes the values ± 1 and depends on the (immaterial) choice of the chirality of the spinors. The analogous of the operator I^0 is therefore

$$I^0_{\mathrm{As}} = \oint (\partial H_2 + \partial H_3) \,. \tag{B.18}$$

Appendix C: GSO projections on the twisted sectors

We quote here the contribution to the quantity $4(N_V - N_H)$, as a function of the modular coefficients, of each one of the 48 twisted supersectors. By supersector we mean a twisted sector and the sectors derived from it by addition of the sets S, $\bar{S}$, $S\bar{S}$, which provide the supersymmetric partners. For simplicity, we omit to indicate the latter, so that b_1 is a short-hand notation for the sets b_1, Sb_1, $\bar{S}b_1$, $S\bar{S}b_1$. The quantity $4(N_V + N_H)$ is given by the product of the two square brackets.

$\underline{b_1}$:

$$[1 + C_{(e_1|b_1)}][1 + C_{(e_2|b_1)}] \cdot \alpha, \tag{C.1}$$

$\underline{b_1 + e_3}$:

$$[1 + C_{(e_1|b_1)}C_{(e_1|e_3)}][1 + C_{(e_2|b_1)}C_{(e_2|e_3)}] \cdot \alpha C_{(e_3|e_4)}C_{(e_3|b_2)}, \tag{C.2}$$

$\underline{b_1 + e_4}$:

$$[1 + C_{(e_1|b_1)}C_{(e_4|e_1)}][1 + C_{(e_2|b_1)}C_{(e_2|e_4)}] \cdot \alpha C_{(e_3|e_4)}C_{(e_4|b_2)}, \tag{C.3}$$

$\underline{b_1 + e_5}$:

$$[1 + C_{(e_1|b_1)}C_{(e_1|e_5)}][1 + C_{(b_1|e_2)}C_{(b_2|e_5)}] \cdot \alpha\gamma C_{(e_3|e_5)}C_{(e_4|e_5)}, \tag{C.4}$$

$\underline{b_1 + e_6}$:

$$[1 + C_{(b_1|e_1)}C_{(e_1|e_6)}][1 + C_{(b_1|e_2)}C_{(e_2|e_6)}] \cdot \gamma\beta C_{(e_5|e_6)}C_{(b_2|e_3)}C_{(b_2|e_4)}, \tag{C.5}$$

$\underline{b_1 + e_3 + e_4}$:

$$[1 + C_{(b_1|e_1)}C_{(e_1|e_3)}C_{(e_1|e_4)}][1 + C_{(b_1|e_2)}C_{(e_2|e_3)}C_{(e_2|e_4)}] \cdot \alpha C_{(b_2|e_3)}C_{(b_2|e_4)}, \tag{C.6}$$

$\underline{b_1 + e_3 + e_5}$:

$$\begin{aligned} &[1 + C_{(b_1|e_1)}C_{(e_1|e_3)}C_{(e_1|e_5)}]\times \\ &\quad\times[1 + C_{(b_1|e_2)}C_{(e_2|e_3)}C_{(e_2|e_5)}] \cdot \alpha\gamma C_{(e_3|e_4)}C_{(e_4|e_5)}C_{(b_2|e_3)}, \end{aligned} \tag{C.7}$$

$\underline{b_1 + e_3 + e_6}$:

$$\begin{aligned} &[1 + C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_1|e_4)}C_{(e_1|e_5)}]\times \\ &\quad\times[1 + C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_2|e_4)}C_{(e_2|e_5)}] \cdot \beta\gamma C_{(b_2|e_4)}C_{(e_1|e_5)}C_{(e_2|e_5)}C_{(e_4|e_5)}, \end{aligned} \tag{C.8}$$

$\underline{b_1 + e_4 + e_5}$:

$$\begin{aligned} &[1 + C_{(b_1|e_1)}C_{(e_1|e_4)}C_{(e_1|e_5)}]\times \\ &\quad\times[1 + C_{(b_1|e_2)}C_{(e_2|e_4)}C_{(e_2|e_5)}] \cdot \alpha\gamma C_{(b_2|e_4)}C_{(e_3|e_4)}C_{(e_3|e_5)}, \end{aligned} \tag{C.9}$$

$\underline{b_1 + e_4 + e_6}$:

$$\begin{aligned} &[1 + C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_1|e_3)}C_{(e_1|e_5)}]\times \\ &\quad\times[1 + C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_2|e_3)}C_{(e_2|e_5)}] \cdot \beta\gamma C_{(b_2|e_3)}C_{(e_1|e_5)}C_{(e_2|e_5)}C_{(e_3|e_5)}, \end{aligned} \tag{C.10}$$

$\underline{b_1+e_5+e_6}$:

$$[1+C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_1|e_3)}C_{(e_1|e_4)}]\times \\ \times[1+C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_2|e_3)}C_{(e_2|e_4)}]\cdot\beta C_{(b_2|e_3)}C_{(b_2|e_4)}, \tag{C.11}$$

$\underline{b_1+e_4+e_5+e_6}$:

$$[1+C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_1|e_3)}][1+C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_2|e_3)}]\cdot\beta C_{(b_2|e_3)}, \tag{C.12}$$

$\underline{b_1+e_3+e_4+e_6}$:

$$[1+C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_1|e_5)}][1+C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_2|e_5)}]\cdot\beta\gamma C_{(e_1|e_5)}C_{(e_2|e_5)}, \tag{C.13}$$

$\underline{b_1+e_3+e_5+e_6}$:

$$[1+C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_1|e_4)}][1+C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_2|e_4)}]\cdot\beta C_{(b_2|e_4)}, \tag{C.14}$$

$\underline{b_1+e_3+e_4+e_5}$:

$$[1+C_{(b_1|e_1)}C_{(e_1|e_3)}C_{(e_1|e_4)}C_{(e_1|e_5)}]\times \\ \times[1+C_{(b_1|e_2)}C_{(e_2|e_3)}C_{(e_2|e_4)}C_{(e_2|e_5)}]\cdot\alpha\gamma C_{(b_2|e_3)}C_{(b_2|e_4)}, \tag{C.15}$$

$\underline{b_1+e_3+e_4+e_5+e_6}$:

$$[1+C_{(b_1|e_1)}C_{(e_1|e_2)}][1+C_{(b_1|e_2)}C_{(e_1|e_2)}]\cdot\beta, \tag{C.16}$$

$\underline{b_2}$:

$$[1+C_{(b_2|e_3)}][1+C_{(b_2|e_4)}]\cdot\beta, \tag{C.17}$$

$\underline{b_2+e_1}$:

$$[1+C_{(b_2|e_3)}C_{(e_1|e_3)}][1+C_{(b_2|e_4)}C_{(e_1|e_4)}]\cdot\beta C_{(b_1|e_1)}C_{(e_1|e_2)}, \tag{C.18}$$

$\underline{b_2+e_2}$:

$$[1+C_{(b_2|e_3)}C_{(e_2|e_3)}][1+C_{(b_2|e_4)}C_{(e_2|e_4)}]\cdot\beta C_{(b_1|e_2)}C_{(e_1|e_2)}, \tag{C.19}$$

$\underline{b_2+e_5}$:

$$[1+C_{(b_2|e_3)}C_{(e_3|e_5)}][1+C_{(b_2|e_4)}C_{(e_4|e_5)}]\cdot\beta\gamma C_{(e_1|e_5)}C_{(e_2|e_5)}, \tag{C.20}$$

$\underline{b_2+e_6}$:

$$[1+C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_2|e_3)}C_{(e_4|e_3)}C_{(e_5|e_3)}][1+C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_2|e_4)}\cdot \\ \cdot C_{(e_3|e_4)}C_{(e_5|e_4)}]\cdot\alpha\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(e_1|e_5)}C_{(e_2|e_5)}C_{(e_3|e_5)}C_{(e_4|e_5)}, \tag{C.21}$$

$\underline{b_2+e_1+e_2}$:

$$[1+C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_2|e_3)}][1+C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_2|e_4)}]\cdot\beta C_{(b_1|e_1)}C_{(b_1|e_2)}, \tag{C.22}$$

$\underline{b_2+e_1+e_5}$:

$$[1+C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_5|e_3)}]\times \\ \times[1+C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_5|e_4)}]\cdot\beta\gamma C_{(b_1|e_1)}C_{(e_1|e_2)}C_{(e_5|e_2)}, \tag{C.23}$$

$\underline{b_2 + e_2 + e_5}$:

$$[1 + C_{(b_2|e_3)}C_{(e_2|e_3)}C_{(e_5|e_3)}]\times$$
$$\times[1 + C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_5|e_4)}] \cdot \beta\gamma C_{(b_1|e_2)}C_{(e_1|e_2)}C_{(e_1|e_5)}, \tag{C.24}$$

$\underline{b_2 + e_2 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_4|e_3)}C_{(e_5|e_3)}]\times$$
$$\times[1 + C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_3|e_4)}C_{(e_5|e_4)}] \cdot \alpha\gamma C_{(b_1|e_1)}C_{(e_1|e_5)}C_{(e_3|e_5)}C_{(e_4|e_5)}, \tag{C.25}$$

$\underline{b_2 + e_1 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_2|e_3)}C_{(e_4|e_3)}C_{(e_5|e_3)}]\times$$
$$\times[1 + C_{(b_2|e_4)}C_{(e_2|e_4)}C_{(e_3|e_4)}C_{(e_5|e_4)}] \cdot \alpha\gamma C_{(b_1|e_2)}C_{(e_2|e_5)}C_{(e_3|e_5)}C_{(e_4|e_5)}, \tag{C.26}$$

$\underline{b_2 + e_5 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_2|e_3)}C_{(e_4|e_3)}]\times$$
$$\times[1 + C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_2|e_4)}C_{(e_3|e_4)}] \cdot \alpha C_{(b_1|e_1)}C_{(b_1|e_2)}, \tag{C.27}$$

$\underline{b_2 + e_1 + e_2 + e_5}$:

$$[1 + C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_2|e_3)}C_{(e_5|e_3)}]\times$$
$$\times[1 + C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_2|e_4)}C_{(e_5|e_4)}] \cdot \beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}, \tag{C.28}$$

$\underline{b_2 + e_1 + e_2 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_4|e_3)}C_{(e_5|e_3)}][1 + C_{(b_2|e_4)}C_{(e_3|e_4)}C_{(e_5|e_4)}] \cdot \alpha\gamma C_{(e_3|e_5)}C_{(e_4|e_5)}, \tag{C.29}$$

$\underline{b_2 + e_1 + e_5 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_2|e_3)}C_{(e_4|e_3)}][1 + C_{(b_2|e_4)}C_{(e_2|e_4)}C_{(e_3|e_4)}] \cdot \alpha C_{(b_1|e_2)}, \tag{C.30}$$

$\underline{b_2 + e_2 + e_5 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_1|e_3)}C_{(e_4|e_3)}][1 + C_{(b_2|e_4)}C_{(e_1|e_4)}C_{(e_3|e_4)}] \cdot \alpha C_{(b_1|e_1)}, \tag{C.31}$$

$\underline{b_2 + e_1 + e_2 + e_5 + e_6}$:

$$[1 + C_{(b_2|e_3)}C_{(e_4|e_3)}][1 + C_{(b_2|e_4)}C_{(e_3|e_4)}] \cdot \alpha, \tag{C.32}$$

$\underline{b_3}$:

$$[1 + \gamma][1 + \alpha\beta C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}] \cdot \beta C_{(b_1|e_1)}C_{(b_1|e_2)}, \tag{C.33}$$

$\underline{b_3 + e_1}$:

$$[1 + \gamma C_{(e_1|e_5)}][1 + \alpha\beta C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}\cdot$$
$$\cdot C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_1|e_3)}C_{(e_1|e_4)}] \cdot \beta C_{(b_1|e_2)}C_{(e_1|e_2)}, \tag{C.34}$$

$\underline{b_3 + e_2}$:

$$[1+\gamma C_{(e_2|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_1|e_2)}\cdot \\ \cdot C_{(e_2|e_3)}C_{(e_2|e_4)}C_{(e_2|e_5)}]\cdot\beta C_{(b_1|e_1)}C_{(e_1|e_2)}, \tag{C.35}$$

$\underline{b_3 + e_3}$:

$$[1+\gamma C_{(e_3|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_3|e_1)}\cdot \\ \cdot C_{(e_3|e_2)}C_{(e_3|e_4)}C_{(e_3|e_5)}]\cdot\alpha C_{(b_2|e_4)}C_{(e_3|e_4)}, \tag{C.36}$$

$\underline{b_3 + e_4}$:

$$[1+\gamma C_{(e_4|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_4|e_1)}\cdot \\ \cdot C_{(e_4|e_2)}C_{(e_4|e_3)}C_{(e_4|e_5)}]\cdot\alpha C_{(b_2|e_3)}C_{(e_4|e_3)}, \tag{C.37}$$

$\underline{b_3 + e_1 + e_2}$:

$$[1+\gamma C_{(e_1|e_5)}C_{(e_2|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}\cdot \\ \cdot C_{(b_2|e_4)}C_{(e_1|e_3)}C_{(e_1|e_4)}C_{(e_1|e_5)}C_{(e_2|e_3)}C_{(e_2|e_4)}C_{(e_2|e_5)}]\cdot\beta, \tag{C.38}$$

$\underline{b_3 + e_1 + e_3}$:

$$[1+\gamma C_{(e_1|e_5)}C_{(e_3|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_1|e_4)}\cdot \\ \cdot C_{(e_1|e_5)}C_{(e_2|e_3)}C_{(e_3|e_4)}C_{(e_3|e_5)}]\cdot\beta C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(e_1|e_2)}C_{(e_2|e_3)}, \tag{C.39}$$

$\underline{b_3 + e_1 + e_4}$:

$$[1+\gamma C_{(e_1|e_5)}C_{(e_4|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_1|e_3)}\cdot \\ \cdot C_{(e_1|e_5)}C_{(e_2|e_4)}C_{(e_3|e_4)}C_{(e_4|e_5)}]\cdot\beta C_{(b_1|e_2)}C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_2|e_4)}, \tag{C.40}$$

$\underline{b_3 + e_2 + e_3}$:

$$[1+\gamma C_{(e_2|e_5)}C_{(e_3|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_2|e_4)}\cdot \\ \cdot C_{(e_2|e_5)}C_{(e_1|e_3)}C_{(e_3|e_4)}C_{(e_3|e_5)}]\cdot\beta C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(e_1|e_2)}C_{(e_1|e_3)}, \tag{C.41}$$

$\underline{b_3 + e_2 + e_4}$:

$$[1+\gamma C_{(e_2|e_5)}C_{(e_4|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_2|e_3)}\cdot \\ \cdot C_{(e_2|e_5)}C_{(e_1|e_4)}C_{(e_3|e_4)}C_{(e_4|e_5)}]\cdot\beta C_{(b_1|e_2)}C_{(b_2|e_4)}C_{(e_1|e_2)}C_{(e_1|e_4)}, \tag{C.42}$$

$\underline{b_3 + e_3 + e_4}$:

$$[1+\gamma C_{(e_3|e_5)}C_{(e_4|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}\cdot \\ \cdot C_{(b_2|e_4)}C_{(e_3|e_1)}C_{(e_3|e_2)}C_{(e_3|e_5)}C_{(e_4|e_1)}C_{(e_4|e_2)}C_{(e_4|e_5)}]\cdot\alpha, \tag{C.43}$$

$\underline{b_3 + e_1 + e_2 + e_3}$:

$$[1+\gamma C_{(e_1|e_5)}C_{(e_2|e_5)}C_{(e_3|e_5)}][1+\alpha\beta\gamma C_{(b_1|e_1)}C_{(b_1|e_2)}C_{(b_2|e_3)}C_{(b_2|e_4)}\cdot \\ \cdot C_{(e_1|e_4)}C_{(e_1|e_5)}C_{(e_2|e_4)}C_{(e_2|e_5)}C_{(e_3|e_4)}C_{(e_3|e_5)}]\cdot\beta C_{(b_2|e_3)}, \tag{C.44}$$

$\underline{b_3 + e_1 + e_3 + e_4}$:

$$[1 + \gamma C_{(e_1|e_5)} C_{(e_3|e_5)} C_{(e_4|e_5)}][1 + \alpha\beta\gamma C_{(b_1|e_1)} C_{(b_1|e_2)} C_{(b_2|e_3)} C_{(b_2|e_4)} \cdot \\ \cdot C_{(e_1|e_2)} C_{(e_1|e_5)} C_{(e_3|e_2)} C_{(e_3|e_5)} C_{(e_4|e_2)} C_{(e_4|e_5)}] \cdot \alpha C_{(b_1|e_1)}, \tag{C.45}$$

$\underline{b_3 + e_1 + e_2 + e_4}$:

$$[1 + \gamma C_{(e_1|e_5)} C_{(e_2|e_5)} C_{(e_4|e_5)}][1 + \alpha\beta\gamma C_{(b_1|e_1)} C_{(b_1|e_2)} C_{(b_2|e_3)} C_{(b_2|e_4)} \cdot \\ \cdot C_{(e_1|e_3)} C_{(e_1|e_5)} C_{(e_2|e_3)} C_{(e_2|e_5)} C_{(e_4|e_3)} C_{(e_4|e_5)}] \cdot \beta C_{(b_2|e_4)}, \tag{C.46}$$

$\underline{b_3 + e_2 + e_3 + e_4}$:

$$[1 + \gamma C_{(e_2|e_5)} C_{(e_3|e_5)} C_{(e_4|e_5)}][1 + \alpha\beta\gamma C_{(b_1|e_1)} C_{(b_1|e_2)} C_{(b_2|e_3)} C_{(b_2|e_4)} \cdot \\ \cdot C_{(e_1|e_2)} C_{(e_2|e_5)} C_{(e_3|e_1)} C_{(e_3|e_5)} C_{(e_4|e_1)} C_{(e_4|e_5)}] \cdot \alpha C_{(b_1|e_2)}, \tag{C.47}$$

$\underline{b_3 + e_1 + e_2 + e_3 + e_4}$:

$$[1 + \gamma C_{(e_1|e_5)} C_{(e_2|e_5)} C_{(e_3|e_5)} C_{(e_4|e_5)}][1 + \alpha\beta\gamma C_{(b_1|e_1)} C_{(b_1|e_2)} C_{(b_2|e_3)} \cdot \\ \cdot C_{(b_2|e_4)} C_{(e_1|e_5)} C_{(e_2|e_5)} C_{(e_3|e_5)} C_{(e_4|e_5)}] \cdot \alpha C_{b_1|e_1)} C_{(b_1|e_2)}, \tag{C.48}$$

In the previous expressions, all the modular coefficients are symmetric under the exchange of the arguments: $C_{(\alpha|\beta)} = C_{(\beta|\alpha)}$. We defined also:

$$\begin{aligned} \alpha &= C_{(b_1|F)} C_{(b_1|e_3)} C_{(b_1|e_4)}, \\ \beta &= C_{(b_2|F)} C_{(b_2|e_1)} C_{(b_2|e_2)}, \\ \gamma &= C_{(b_1|e_5)} C_{(b_2|e_5)}. \end{aligned} \tag{C.49}$$

By running the above formulae with a computer program, we find the following pairs of (N_V, N_H): $(0,48)$, $(48,0)$, $(28,4)$, $(4,28)$, $(16,16)$, $(0,24)$, $(24,0)$, $(6,18)$, $(18,6)$, $(12,12)$, $(4,16)$, $(16,4)$, $(2,14)$, $(14,2)$, $(8,8)$, $(0,12)$, $(12,0)$, $(6,6)$, $(3,9)$, $(9,3)$, $(4,4)$, $(2,2)$, $(0,0)$.

The formulae above simplify if, instead of factorizing the boundary conditions of the six circles with the sets e_i, $i = 1, ..., 5$, we consider only the factorization of the three tori $(1,2)$, $(3,4)$ and $(5,6)$ with the sets T_i, $i = 1,2,3$, given by:

$$\begin{aligned} T_1 &= \left\{ y_1^L, y_2^L, \omega_1^L, \omega_2^L \mid y_1^R, y_2^R, \omega_1^R, \omega_2^R \right\} , \\ T_2 &= \left\{ y_3^L, y_4^L, \omega_3^L, \omega_4^L \mid y_3^R, y_4^R, \omega_3^R, \omega_4^R \right\} , \\ T_3 &= \left\{ y_5^L, y_6^L, \omega_5^L, \omega_6^L \mid y_5^R, y_6^R, \omega_5^R, \omega_6^R \right\} . \end{aligned} \tag{C.50}$$

In this case, we can write a compact expression for the formulae, which give the sum and the difference of the total number of vector and hypermultiplets provided by the twisted sectors:

$$\frac{N_V + N_H}{4} = 6 + (1 + C_{(T_1|T_2)}) \times \\ \times (C_{(b_1|T_1)} + C_{(b_2|T_2)} + \alpha\beta C_{(b_1|T_1)} C_{(b_2|T_2)}), \tag{C.51}$$

$$\frac{N_V - N_H}{4} = \frac{3}{2} C_{(b_1|b_2)} (\alpha + \beta) \times \\ \times (1 + C_{(b_1|T_1)} + C_{(b_2|T_2)} + C_{(b_1|T_1)} C_{(b_2|T_2)} C_{(T_1|T_2)}), \tag{C.52}$$

where α and β are given by:

$$\begin{aligned} \alpha &= C_{(b_1|F)} C_{(b_1|T_2)}, \\ \beta &= C_{(b_2|F)} C_{(b_2|T_1)}. \end{aligned} \tag{C.53}$$

In this case, we can only find a subset of models, namely those with $(N_V, N_H) = (48, 0)$, $(0, 48)$, $(24, 0)$, $(0, 24)$, $(16, 16)$, $(12, 12)$, $(8, 8)$ and $(0, 0)$.

C.1. A choice of modular coefficients for each model

We give here a choice of modular coefficients per each one of the constructions of Section 5:

1) (48,0). All coefficients $= +1$.

2) (28,4). (D, D, O): $C_{(e_1|e_3)}$, $C_{(e_1|e_4)}$, $C_{(e_2|e_3)}$, $C_{(e_2|e_4)} = -1$.

3) (16,16). (O, O, F): $C_{(b_1|F)} = -1$.

4) (24,0). (D, D, D): $C_{(e_2|e_5)}$, $C_{(e_4|e_5)} = -1$.

5) (18,6). (DD, DD, O): $C_{(e_1|e_2)}$, $C_{(e_1|e_4)}$, $C_{(e_2|e_3)}$, $C_{(e_3|e_4)} = -1$.

6) (12,12). (R, O, FD): $C_{(e_1|e_2)}$, $\gamma = -1$.
(D, D, D): $C_{(e_1|e_2)}$, $C_{(e_3|e_4)} = -1$.

7) (16,4). (DD, D, D): $C_{(e_1|e_2)}$, $C_{(e_2|e_4)} = -1$.

8) (14,2). (DD, DD, D): $C_{(e_1|e_4)}$, $C_{(e_2|e_3)} = -1$.

9) (8,8). (DD, DD, D): $C_{(e_1|e_2)}$, $C_{(e_2|e_4)}$, $C_{(e_3|e_4)} = -1$.
(D, D, F): $C_{(e_1|e_3)}$, $C_{(e_1|e_4)}$, $C_{(e_2|e_3)}$, $C_{(e_3|e_4)}$, $\gamma = -1$. (D, D, FD): $C_{(e_1|e_3)}$, $\gamma = -1$.
(O, F, F): $C_{(b_2|e_3)}$. (O, FD, FD): $C_{(b_2|e_3)}$, $C_{(e_4|e_5)} = -1$.

10) (12,0). (DD, DD, DD): $C_{(e_1|e_5)}$, $C_{(e_2|e_4)}$, $C_{(e_3|e_5)} = -1$.

11) (6,6). (DD, DD, DD): $C_{(e_1|e_2)}$, $C_{(e_2|e_5)}$, $C_{(e_3|e_4)}$, $C_{(e_4|e_5)} = -1$.
(DD, D, FD): $C_{(e_1|e_2)}$, $C_{(e_2|e_4)}$, $\gamma = -1$.

12) (9,3). (RR, RR, RR): $C_{(e_1|e_2)}$, $C_{(e_2|e_4)}$, $C_{(e_3|e_4)}$, $C_{(e_3|e_5)}$, α, $\beta = -1$.

13) (4,4). (D, FD, F): $C_{(b_2|e_3)}$, $C_{(e_1|e_3)}$, $C_{(e_1|e_4)}$, $C_{(e_2|e_3)}$, $C_{(e_2|e_4)} = -1$. (D, FD, FD): $C_{(b_2|e_3)}$, $C_{(e_2|e_5)}$, $C_{(e_4|e_5)}$, $\alpha = -1$.

(DD, DD, F): $C_{(e_1|e_2)}$, $C_{(e_1|e_4)}$, $C_{(e_2|e_3)}$, $C_{(e_3|e_4)}$, $\gamma = -1$. (DD, DD, FD): $C_{(e_1|e_3)}$, $C_{(e_2|e_4)}$, $\gamma = -1$.

14) (2,2). (DD, FD, FD): $C_{(b_2|e_4)}$, $C_{(e_1|e_2)}$, $C_{(e_2|e_3)}$, $\gamma = -1$.

15) (0,0). (F, F, F): $C_{(b_1|e_1)}$, $C_{(b_2|e_3)}$, $\gamma = -1$. (FD, FD, F): $C_{(b_1|e_1)}$, $C_{(b_2|e_3)}$, $C_{(e_1|e_3)}$, $C_{(e_1|e_4)}$, $C_{(e_2|e_3)}$, $C_{(e_2|e_4)}$, $\gamma = -1$. (FD, FD, FD): $C_{(b_1|e_1)}$, $C_{(b_2|e_3)}$, $C_{(e_2|e_5)}$, $C_{(e_4|e_5)}$, $\gamma = -1$.

C.2. Reading the (super-)Higgs mechanism directly from (C.1)–(C.48)

In Section 3 we saw how it is possible to interpret the GSO projections of the fermionic construction in terms of lattice shifts and twists and, in the light of the previous analysis, we are also able to understand them in terms of Higgs and super-Higgs mechanisms. The Higgs mechanism is present whenever there are shifts due to modular coefficients $C_{(e_i|e_j)} = -1$: they translate in fact into D projections. There is a super-Higgs mechanism when there is a shift due to modular coefficients $C_{(b_i|e_j)} = -1$ and/or $C_{(b_i|F)} = -1$ (by symmetric difference, the set F can be seen to assign the boundary conditions of the sixth circle of $\mathcal{T}^6$, which in the notation of Section 3, Eq. (3.5), are (γ, δ)). According to this interpretation, we see that the missing massless states still belong to the string spectrum, and there are corners in moduli space in which some or all of them become massless. On each $N = 4$ sector, besides the GSO projection that reduces the number of states, starting from a maximum of sixteen, there is also in action a GSO projection on the world-sheet chiralities, which determines whether such states are hyper- or vector multiplets. By looking at formulae (C.1)–(C.48), which express the quantity $N_V \pm N_H$ for each one of the 48 twisted (super)sectors, we can see that in each sector the GSO projection splits into a product of three factors: the first two factors, which determine whether a given twisted supersector provides massless states or not, can be translated in terms of lattice shifts. The modular coefficients entering the first factor in square brackets determine the shift in the first circle of the corresponding untwisted torus, while the shift in the second circle is determined by the coefficients entering the second square brackets. The product of coefficients after the square brackets translates into a shift in the twisted T^4. A shift on a twisted lattice is directly related to the sign of $N_V - N_H$. The coefficients inside the square brackets therefore determine whether such states are massless or massive, while the coefficients out of the square brackets determine whether the massless states are hyper- or vector multiplets.

Appendix D: Classification of partition functions

The classification of the "partition functions" can be easily carried out by observing that the situations listed in (4.1), (4.2), (4.4) and (4.5) are in a one-to-one correspondence with the number of massless multiplets, no matter whether they are hypermultiplets or vector-multiplets, of the corresponding twisted sector. The classification of "partitions functions" therefore amounts essentially to a complete account of these numbers. There is, however, a subtlety, because this method does not allow a distinction between (4.2) and (4.3). In fact, "D" can always be superposed to "F", but not all the combinations are allowed, because the insertion of "D" makes sense only when it involves at least two circles belonging to two different tori. The result is shown in Table D.1. In this table we did not quote the constructions that differ from the above by an exchange of N_V and N_H and/or by a permutation of the three planes.

From Table D.1, it easy to readread the number of supersymmetries that are spontaneously broken. When the free action of a SUSY-breaking projection appears in at most one plane (indicated by F or FD), there is no spontaneous breaking of supersymmetry. When the free action involves two planes, there is a spontaneous breaking of $N = 4$. When finally the free action involves all the three planes, there is spontaneous breaking of $N = 8$.

(N_V, N_H)	N_V^1	N_V^2	N_V^3	N_H^1	N_H^2	N_H^3	plane 1	plane 2	plane 3
(48,0)	16	16	16	0	0	0	O	O	O
(28,4)	8	8	12	0	0	4	D	D	O
(16,16)	8	8	0	8	8	0	O	O	F
(24,0)	8	8	8	0	0	0	D	D	D
(18,6)	4	4	10	0	0	6	DD	DD	O
(12,12)	4	8	0	4	8	0	D	O	FD
	4	4	4	4	4	4	D	D	D
(16,4)	4	6	6	0	2	2	DD	D	D
(14,2)	4	4	6	0	0	2	DD	DD	D
(8,8)	2	2	4	2	2	4	DD	DD	D
	4	4	0	4	4	0	D	D	F
							D	D	FD
	8	0	0	8	0	0	O	F	F
							O	FD	FD
(12,0)	4	4	4	0	0	0	DD	DD	DD
(6,6)	2	2	2	2	2	2	DD	DD	DD
	2	4	0	2	4	0	DD	D	FD
(9,3)	3	3	3	1	1	1	DD	DD	DD
(4,4)	4	0	0	4	0	0	D	FD	F
							D	FD	FD
	2	2	0	2	2	0	DD	DD	F
							DD	DD	FD
(2,2)	2	0	0	2	0	0	DD	FD	FD
(0,0)	0	0	0	0	0	0	F	F	F
							FD	FD	F
							FD	FD	FD

Table D.1: The contribution to the massless spectrum and lattice sums in the $N = 4$ sectors of the models.

Appendix E: Lattice integrals and threshold corrections

We give below our notation and conventions for the usual (2,2) and (2,2)-shifted lattice sums used in the text. The Z_2-shifted (2,2) lattice sums are

$$\Gamma^w_{2,2}(T,U)\begin{bmatrix}h\\g\end{bmatrix} = \sum_{\{p_{\rm L},p_{\rm R}\}\in\Gamma_{2,2}+w\frac{h}{2}} {\rm e}^{-\pi i g \ell\cdot w} q^{\frac{1}{2}p_{\rm L}^2}\bar{q}^{\frac{1}{2}p_{\rm R}^2} \ , \tag{E.1}$$

where the shifts h and projections g take the values 0 or 1. Here, w denotes the shift vector with components (a_1, a_2, b^1, b^2) and $\ell \equiv (m_1, m_2, n^1, n^2)$. We have also introduced the inner product[13]

$$\ell\cdot w = \vec{m}\vec{b} + \vec{a}\vec{n} \ , \quad w^2 = 2\vec{a}\vec{b} \ , \tag{E.2}$$

so that a_I generates a winding shift in the I direction, whereas b^I shifts the Ith momentum. The vector ℓ is associated to the $\Gamma_{2,2}$ lattice and therefore the vector associated to the shifted lattice will be

$$p \equiv \ell + w\frac{h}{2} \ . \tag{E.3}$$

With these conventions, left and right momenta read:

$$p_{\rm L}^2 = \frac{\left|U\left(m_1 + a_1\frac{h}{2}\right) - \left(m_2 + a_2\frac{h}{2}\right) + T\left(n^1 + b^1\frac{h}{2}\right) + TU\left(n^2 + b^2\frac{h}{2}\right)\right|^2}{2T_2U_2} \ , \tag{E.4a}$$

$$p_{\rm L}^2 - p_{\rm R}^2 = 2\left(m_I + a_I\frac{h}{2}\right)\left(n^I + b^I\frac{h}{2}\right) \ . \tag{E.4b}$$

It is easy to check the periodicity properties (h, g integers)

$$Z^w_{2,2}\begin{bmatrix}h\\g\end{bmatrix} = Z^w_{2,2}\begin{bmatrix}h+2\\g\end{bmatrix} = Z^w_{2,2}\begin{bmatrix}h\\g+2\end{bmatrix} = Z^w_{2,2}\begin{bmatrix}-h\\-g\end{bmatrix} \tag{E.5}$$

as well as the modular transformations that the expression

$$Z^w_{2,2}\begin{bmatrix}h\\g\end{bmatrix} = \frac{\Gamma^w_{2,2}\left[{h \atop g}\right]}{|\eta|^4} \tag{E.6}$$

obeys:

$$\tau \to \tau+1 : \quad Z^w_{2,2}\begin{bmatrix}h\\g\end{bmatrix} \to {\rm e}^{\pi i \frac{w^2}{2}\frac{h^2}{2}} Z^w_{2,2}\begin{bmatrix}h\\h+g\end{bmatrix} \tag{E.7}$$

$$\tau \to -\frac{1}{\tau} : \quad Z^w_{2,2}\begin{bmatrix}h\\g\end{bmatrix} \to {\rm e}^{-\pi i \frac{w^2}{2}hg} Z^w_{2,2}\begin{bmatrix}g\\-h\end{bmatrix} . \tag{E.8}$$

The relevant parameter for these transformations is

$$\lambda \equiv \frac{w^2}{2} = \vec{a}\vec{b}. \tag{E.9}$$

[13]For $w_1 = \left(\vec{a}_1, \vec{b}_1\right)$ and $w_2 = \left(\vec{a}_2, \vec{b}_2\right)$, the inner product is defined as $w_1\cdot w_2 = \vec{a}_1\vec{b}_2 + \vec{a}_2\vec{b}_1$.

From expressions (E.1) we learn that the integers a_I and b^I are defined modulo 2, in the sense that adding 2 to any one of them amounts at most to a change of sign in $Z^w_{2,2}\left[{}^1_1\right]$. Such a modification is necessarily compensated by an appropriate one in the rest of the partition function, in order to ensure modular invariance; we are thus left with the same model. On the other hand, adding 2 to a_I or b^I translates into adding a multiple of 2 to λ. Therefore, although λ can be any integer, only $\lambda = 0$ and $\lambda = 1$ correspond to truly different situations. Only when $\lambda = 0$, is the (2,2) block modular invariant by itself, when the sum over (h, g) is taken.

In the case where there are two independent shifts Z_2, as in the cases we indicate by RR, modular invariance requires also orthogonality of the two shift vectors: $w_1 \cdot w_2 = 0$. The lattice sum, which we denote by $\Gamma^{w_1,w_2}_{2,2}\left[{}^{h_1,\ h_2}_{g_1,\ g_2}\right]$, satisfies the following equalities:

$$\Gamma^{w_1,w_2}_{2,2}\begin{bmatrix}h,\ 0\\ g,\ 0\end{bmatrix} = \Gamma^{w_1}_{2,2}\begin{bmatrix}h\\ g\end{bmatrix}\ ,\quad \Gamma^{w_1,w_2}_{2,2}\begin{bmatrix}0,\ h\\ 0,\ g\end{bmatrix} = \Gamma^{w_2}_{2,2}\begin{bmatrix}h\\ g\end{bmatrix}\ ,\quad \Gamma^{w_1,w_2}_{2,2}\begin{bmatrix}h,\ h\\ g,\ g\end{bmatrix} = \Gamma^{w_{12}}_{2,2}\begin{bmatrix}h\\ g\end{bmatrix}\ , \tag{E.10}$$

where $w_{12} \equiv w_1 + w_2$ reflects the action of the diagonal Z_2. We refer to Appendix C of [12] for a detailed discussion of target space duality. One of the issues, valid when $\lambda = 0$, our case of interest, is that a change in the lattice vector w, which preserves modular invariance (i.e. $w^2/2 = 0$ mod2), amounts to an $SL(2, Z)$ transformation performed on T and/or U, and vice versa. We can therefore fix the lattice shift vectors and then derive the general result by $SL(2, Z)$ transformations. If we choose $w_1 = (0, 0, 1, 0)$, $w_2 = (0, 0, 0, 1)$, the first Z_2 translates the momenta of the first circle (insertion of $(-1)^{m_1}$), the second Z_2 translates the momenta of the second (insertion of $(-1)^{m_2}$) [14]. In this case the lattice sum reads:

$$\begin{aligned}\Gamma_{2,2}\begin{bmatrix}h_1, h_2\\ g_1, g_2\end{bmatrix} \ =\ & \sum_{\vec m \vec n\in\mathbb{Z}} (-1)^{m_1 g_1 + m_2 g_2} \exp\left\{2\pi i\bar\tau\left(m_1\left(n^1 + \frac{h_1}{2}\right) + m_2\left(n^2 + \frac{h_2}{2}\right)\right)\right.\\ & \left. - \frac{\pi\tau_2}{T_2 U_2}\left|T\left(n^1 + \frac{h_1}{2}\right) + TU\left(n^2 + \frac{h_2}{2}\right) + U m_1 - m_2\right|^2\right\}, \end{aligned}\tag{E.11}$$

By performing a Poisson resummation over the momenta (m_1, m_2), we can express the shifted lattice in the Lagrangian formulation. When $w = (w_1, w_2) = ((0, 0, 1, 0), (0, 0, 0, 1))$, we have

$$Z_{2,2}\begin{bmatrix}h_1,\ h_2\\ g_1,\ g_2\end{bmatrix} = \frac{1}{|\eta|^4} \sum_{(m_1,n_1)}\sum_{(m_2,n_2)} Z_{2,2}\begin{bmatrix}n_1(h_1),\ n_2(h_2)\\ m_1(g_1),\ m_2(g_2)\end{bmatrix}\ , \tag{E.12}$$

where

$$Z_{2,2}\begin{bmatrix}n_1(h_1),\ n_2(h_2)\\ m_1(g_1),\ m_2(g_2)\end{bmatrix} = (\mathrm{Im}\tau)^{-1}\sqrt{\det G_{ij}}\,\exp\left[-\pi T_{ij}\ \frac{(m_i + n_i\tau)(m_j + n_j\bar\tau)}{\mathrm{Im}\tau}\right]\ . \tag{E.13}$$

In the above expression, the tensor T_{ij} is defined as

$$T_{ij} = G_{ij} + B_{ij}\ , \tag{E.14}$$

[14] Notice that these are the same translations as were introduced when projecting with D (3.12).

where

$$B_{ij} = \begin{pmatrix} 0 & -\mathrm{Im}T \\ \mathrm{Im}T & 0 \end{pmatrix} \ , \qquad G_{ij} = \frac{\mathrm{Im}T}{\mathrm{Im}U} \begin{pmatrix} 1 & \mathrm{Re}U \\ \mathrm{Re}U & |U|^2 \end{pmatrix} \ , \tag{E.15}$$

and

$$m_i \in \mathbb{Z} + \frac{g_i}{2} \ , \qquad n_j \in \mathbb{Z} + \frac{h_j}{2} \ . \tag{E.16}$$

Relations (E.15) can be inverted giving T and U as functions of B_{ij}, G_{ij}:

$$T = -B_{12} + i\sqrt{detG_{ij}} \ , \qquad U = \frac{G_{12}}{G_{11}} + i\frac{\sqrt{detG_{ij}}}{G_{11}} \ . \tag{E.17}$$

In terms of the metric G_{ij} and the antisymmetric tensor B_{ij}, the argument in the exponential of (E.13) becomes

$$-\pi G_{ij} \, \frac{(m_i + n_i\tau)(m_j + n_j\bar{\tau})}{\mathrm{Im}\tau} + 2\pi i B_{ij} \, m_i n_j \ . \tag{E.18}$$

By using the identity:

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix} \bar{\vartheta}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2\mathrm{Im}\tau}} \sum_{(m,n)} \mathrm{e}^{-\pi \ \frac{|m+n\tau|^2}{2\mathrm{Im}\tau}} \mathrm{e}^{i\pi(am+bn+mn)} \ , \tag{E.19}$$

it is easy to prove that the equality (3.11):

$$\Gamma^w_{2,2}\begin{bmatrix} h_1, \ h_2 \\ g_1, \ g_2 \end{bmatrix} = \sum_{a_1,b_1,a_2,b_2} \mathrm{e}^{i\pi(a_1g_1+b_1h_1+h_1g_1)} \mathrm{e}^{i\pi(a_2g_2+b_2h_2+h_2g_2)} \left| \vartheta\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \vartheta\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \right|^2 \ , \tag{E.20}$$

holds for $w = (w_1, w_2) = ((0,0,1,0),(0,0,0,1))$ at the particular value of moduli

$$T_0 = i \ , \qquad U_0 = i \ . \tag{E.21}$$

Notice also that this is the self-dual point.

⋆ ⋆ ⋆

We recall here the integrals of shifted lattice sums. If there is no shift, we have [30]:

$$\int_{\mathcal{F}} (\Gamma_{2,2}(T,U) - 1) = -\log\left(T_2 \, |\eta(T)|^4 \, U_2 \, |\eta(U)|^4\right) - \log \frac{8\pi \mathrm{e}^{1-\gamma}}{\sqrt{27}} \ . \tag{E.22}$$

In the case where there is only one Z_2 shift, as in expressions (4.2), i.e. when $w \equiv (w_1, 0)$, we have:

$$\int_{\mathcal{F}} \left({\sum_{(h,g)}}' \Gamma^w_{2,2}\begin{bmatrix} h \\ g \end{bmatrix}(T,U) - 1 \right) = -\log\left(T_2 \, |\vartheta_i(T)|^4 \, U_2 \, |\vartheta_j(U)|^4\right) - \log \frac{\pi \mathrm{e}^{1-\gamma}}{6\sqrt{3}} \ , \tag{E.23}$$

where the relation between the shift vector $w_1 = (\vec{a}, \vec{b})$ and the pairs (i,j) is given in Table E.1:

Case	$\vec{a}$	$\vec{b}$	i	j
I	$(0,0)$	$(1,0)$	4	2
II	$(0,0)$	$(0,1)$	4	4
III	$(0,0)$	$(1,1)$	4	3
IV	$(1,0)$	$(0,0)$	2	4
V	$(0,1)$	$(0,0)$	2	2
VI	$(1,1)$	$(0,0)$	2	3
VII	$(1,0)$	$(0,1)$	3	4
VIII	$(0,1)$	$(1,0)$	3	2
IX	$(1,-1)$	$(1,1)$	3	3

Table E.1: The nine physically distinct models with $\lambda = 0$.

For the other cases, given in Eqs. (4.3)-(4.5), the integral is obtained by taking the proper combination of (E.22) and (E.23). We collect here the results, including also the infrared running. Modulo an integration constant, the result is:

$$I(O) = 3\log M_S^2/\mu^2 - 3\log(|\eta(T)|^4|\eta(U)|^4 T_2U_2) \ , \tag{E.24}$$

$$I^w(F) = \log M_S^2/\mu^2 - \log(|\vartheta_i(T)|^4|\vartheta_j(U)|^4 T_2U_2) \ , \tag{E.25}$$

$$\begin{aligned} I^w(FD) &= \log M_S^2/\mu^2 - 1/2\ \log(|\vartheta_i(T)|^4|\vartheta_j(U)|^4 T_2U_2) + \\ &\quad -1/2\ \log(|\vartheta_k(T)|^4|\vartheta_\ell(U)|^4 T_2U_2) \end{aligned} \tag{E.26}$$

$$\begin{aligned} I^w(D) &= 2\log M_S^2/\mu^2 - 3/2\ \log(|\eta(T)|^4|\eta(U)|^4 T_2U_2) + \\ &\quad -1/2\ \log(|\vartheta_i(T)|^4|\vartheta_j(U)|^4 T_2U_2) \ , \end{aligned} \tag{E.27}$$

$$\begin{aligned} I^w(DD) &= 3/2\ \log M_S^2/\mu^2 - 3/4\ \log(|\eta(T)|^4|\eta(U)|^4 T_2U_2) \\ &\quad -1/4 \sum_{a=1}^{3} \log(|\vartheta_{i_a}(T)|^4|\vartheta_{j_a}(U)|^4 T_2U_2) \ . \end{aligned} \tag{E.28}$$

In expression (E.26), the first term is due to the integration of a lattice with only one Z_2 shift, $w = (w_1, 0)$, and the dependence of the pairs (i,j) on w_1 is given in Table E.1. The second term is due to the integration of a lattice with a shift specified by $w' = w_1 + w_2$, where w_1 is the same as in the first term and w_2 is a vector in an independent direction. Modular invariance requires $w_1^2 = (w_1 + w_2)^2 = 0$. The pair (k,ℓ) can be anyone of Table E.1, with the only constraint $(k,\ell) \neq (i,j)$.

In the case of two Z_2 shifts, as in (E.28), there is a sum of three terms, $a = 1,2,3$, which refer respectively to the shifts given by the vectors $(w_1, w_2, w_1 + w_2)$. The requirement of modular invariance reduces the number of possibilities to the following six:

Case	w_1	w_2
(i)	$(0,0,1,0)$	$(0,0,0,1)$
(ii)	$(1,0,0,0)$	$(0,1,0,0)$
(iii)	$(1,0,0,1)$	$(0,-1,1,0)$
(iv)	$(1,0,0,0)$	$(0,0,0,1)$
(v)	$(0,0,1,0)$	$(0,1,0,0)$
(vi)	$(0,0,1,1)$	$(1,-1,0,0)$

Table E.2: The six physically distinct models with $w_i \cdot w_j = 0\ \forall i, j = 1, 2$.

Appendix F: The helicity supertraces in the type II asymmetric orbifolds

As we discussed in Section 4, the helicity supertraces are defined in terms of the four-dimensional helicity λ as

$$B_{2n} \equiv \text{Str}\ \lambda^{2n}\ . \tag{F. 1}$$

In the framework of string theory, the physical four-dimensional helicity is the sum of the contributions of the left and right movers: $\lambda = \lambda_{\rm L} + \bar\lambda_{\rm R}$. The supertraces are computed by acting on the helicity-generating partition function $Z(v, \bar v)$ with the differential operators that represent $\lambda_{\rm L}$ $\left(\bar\lambda_{\rm R}\right)$:

$$\lambda_{\rm L} = \frac{1}{2\pi i}\partial_v\ , \qquad \bar\lambda_{\rm R} = -\frac{1}{2\pi i}\partial_{\bar v}\ . \tag{F. 2}$$

In the type II asymmetric orbifolds of Section 5.3, the contribution of the right-moving fermions cannot be cast directly in the form (4.14) (with $H^2 = G^2 = 0$). In order to compute the helicity supertraces we must start from the expression

$$Z_{\rm R}^{\rm F}\begin{bmatrix} a, H^{\rm o} \\ b, G^{\rm o}\end{bmatrix}(\bar v) = \frac{(-)^{\bar a + \bar b + \bar a \bar b}}{\bar\eta^4}\,\bar\vartheta\begin{bmatrix}\bar a \\ \bar b\end{bmatrix}(\bar v)\,\bar\vartheta\begin{bmatrix}\bar a \\ \bar b\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a + H^{\rm o} \\ \bar b + G^{\rm o}\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a - H^{\rm o} \\ \bar b - G^{\rm o}\end{bmatrix}\ . \tag{F. 3}$$

The helicity-generating partition function reads

$$\begin{aligned} Z(v, \bar v) \;&=\; \frac{\xi(v)\bar\xi(\bar v)}{|\eta|^4}\,\frac{1}{4}\sum_{H^{\rm F}, G^{\rm F}}\ \sum_{H^{\rm o}, G^{\rm o}} \frac{\Gamma_{6,6}\left[{}^{H^{\rm F}, H^{\rm o}}_{G^{\rm F}, G^{\rm o}}\right]}{|\eta|^{12}} \\ &\times\ \frac{1}{2}\sum_{\bar a, \bar b}(-)^{\bar a G^{\rm F} + \bar b H^{\rm F} + H^{\rm F} G^{\rm F}}\, Z_{\rm L}^{\rm F}\begin{bmatrix} H^{\rm o} \\ G^{\rm o}\end{bmatrix}(v)\, Z_{\rm R}^{\rm F}\begin{bmatrix}\bar a, H^{\rm o} \\ \bar b, G^{\rm o}\end{bmatrix}(\bar v)\ , \end{aligned} \tag{F. 4}$$

where $Z_{\rm L}^{\rm F}\left[{}^{H^{\rm o}}_{G^{\rm o}}\right](v)$ is the same as expression (4.13), with $H^2 = G^2 = 0$ and arguments $(H^{\rm o}, G^{\rm o})$ instead of (H^1, G^1).

The helicity supertrace B_2

The only non-vanishing contribution to B_2 can originate from the sectors of the orbifold for which $(H^{\rm o}, G^{\rm o}) \neq (0, 0)$. In these sectors, there is a constraint, coming from the twisted boson character, which is non-vanishing only for $(H^{\rm F}, G^{\rm F}) = (0,0)$ or $(H^{\rm F}, G^{\rm F}) = (H^{\rm o}, G^{\rm o})$. When $(H^{\rm F}, G^{\rm F}) = (0,0)$, we get 1/2 of the contribution of one $N = (2,2)$ sector of the corresponding type IIA symmetric orbifolds, which is zero because of the non-complete saturation of the fermion zero modes. When $(H^{\rm F}, G^{\rm F}) = (H^{\rm o}, G^{\rm o})$, we get an identical contribution. In order to see this, we redefine the arguments in (F. 3) as:

$$\bar a + H^{\rm o} \to A \ , \qquad \bar b + G^{\rm o} \to B \ . \tag{F. 5}$$

After this substitution, we use the Riemann identity and recast the right-moving fermion contribution

$$\frac{1}{2} \sum_{\bar a, \bar b} (-)^{\bar a + \bar b + \bar a \bar b + \bar a G^{\rm o} + \bar b H^{\rm o} + H^{\rm o} G^{\rm o}} \, \bar\vartheta \begin{bmatrix} \bar a \\ \bar b \end{bmatrix} (\bar v) \ \vartheta \begin{bmatrix} \bar a \\ \bar b \end{bmatrix} \bar\vartheta \begin{bmatrix} \bar a + H^{\rm o} \\ \bar b + G^{\rm o} \end{bmatrix} \bar\vartheta \begin{bmatrix} \bar a - H^{\rm o} \\ \bar b - G^{\rm o} \end{bmatrix} \tag{F. 6}$$

as

$$- \ \bar\vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{\bar v}{2}\right) \ \bar\vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(-\frac{\bar v}{2}\right) \ \bar\vartheta \begin{bmatrix} 1 + H^{\rm o} \\ 1 + G^{\rm o} \end{bmatrix} \left(\frac{\bar v}{2}\right) \ \bar\vartheta \begin{bmatrix} 1 - H^{\rm o} \\ 1 - G^{\rm o} \end{bmatrix} \left(\frac{\bar v}{2}\right) \ . \tag{F. 7}$$

Also the contribution of this term therefore vanishes, due to the non-complete saturation of the fermion zero modes.

The helicity supertrace B_4

In these orbifolds, all the $N = 4$ sectors, namely:

1. the $N = (4,0)$ sector with $(H^{\rm o}, G^{\rm o}) = (0,0)$,
2. the $N = (2,2)$ sector with $(H^{\rm o}, G^{\rm o}) \neq (0,0)$ and $(H^{\rm F}, G^{\rm F}) = (0,0)$,
3. the $N = (2,2)$ sector with $(H^{\rm o}, G^{\rm o}) \neq (0,0)$ and $(H^{\rm F}, G^{\rm F}) = (H^{\rm o}, G^{\rm o})$,

contribute to B_4. In the first sector the contribution is given by $\langle \lambda_{\rm L}^4 \rangle$. We obtain

$$\langle \lambda_{\rm L}^4 \rangle = \frac{3}{16} \, \frac{1}{\bar\eta^{12}} \sum_{\bar a, \bar b} \ \sum_{(H^{\rm F}, G^{\rm F})} (-)^{\bar a + \bar b + \bar a \bar b + \bar a G^{\rm F} + \bar b G^{\rm F} + H^{\rm F} G^{\rm F}} \, \bar\vartheta^4 \begin{bmatrix} \bar a \\ \bar b \end{bmatrix} \frac{1}{2^{n_D}} \sum_{\vec H^D, \vec G^D} \Gamma_{6,6} \begin{bmatrix} H^{\rm F}, & \vec H^D \\ G^{\rm F}, & \vec G^D \end{bmatrix} \tag{F. 8}$$

(n_D indicates the number of D-projections.) This expression is a series that starts with a square-root pole:

$$a_{-1} \, q^{-\frac{1}{2}} + a_0 + \dots \tag{F. 9}$$

The term a_0 gives the massless contribution, which turns out to be constant in the full space of $T^{\rm As}$, $U^{\rm As}$, the moduli in the vector multiplets, but not in the space of the moduli belonging to hypermultiplets. At a generic point in moduli space, we have $a_0 = 6$. When $(H^{\rm o}, G^{\rm o}) \neq (0,0)$, B_4 amounts to $6 \langle \lambda_{\rm L}^2 \lambda_{\rm R}^2 \rangle$. In this case, the arguments $(H^{\rm F}, G^{\rm F})$, as we saw, are constrained. One therefore proceeds as for the computation of B_2, by splitting the sum over $(H^{\rm F}, G^{\rm F})$ into the two terms $(H^{\rm F}, G^{\rm F}) = (0,0)$ and $(H^{\rm F}, G^{\rm F}) = (H^{\rm o}, G^{\rm o})$. After the same substitution of variables as in (F. 5), one obtains that the contribution of each one of the two terms is equal to the contribution of one complex plane of the symmetric orbifold. In these sectors, the arguments $(\vec{H}^D, \vec{G}^D)$ are constrained as well: $(H^D, G^D) = (0,0)$ or $(H^D, G^D) = (H^{\rm o}, G^{\rm o})$. One therefore gets the various expressions:

$$6\langle \lambda^2 \bar\lambda^2 \rangle = 36\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0 \\ 0 \mid 0 \end{bmatrix} \qquad N_V = 16\ ; \qquad \text{(F. 10)}$$

$$= 12 \sum_{(h,g)}{}' \left(\frac{1}{2}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0 \\ 0 \mid 0 \end{bmatrix} + \frac{1}{2}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid h \\ 0 \mid g \end{bmatrix} \right) \qquad N_V = 8\ ; \qquad \text{(F. 11)}$$

$$= 12 \sum_{(h,g)}{}' \left(\frac{1}{4}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0,0 \\ 0 \mid 0,0 \end{bmatrix} + \frac{1}{4}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid h,0 \\ 0 \mid g,0 \end{bmatrix} \right.$$
$$\left. + \frac{1}{4}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid 0,h \\ 0 \mid 0,g \end{bmatrix} + \frac{1}{4}\Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid h,h \\ 0 \mid g,g \end{bmatrix} \right) \qquad N_V = 4\ ; \qquad \text{(F. 12)}$$

$$= 12 \sum_{(h,g)}{}' \Gamma^{(3)}_{2,2}\begin{bmatrix} 0 \mid h \\ 0 \mid g \end{bmatrix} \qquad N_V = 0\ . \qquad \text{(F. 13)}$$

References

[1] C. Vafa and E. Witten, Nucl. Phys. **B447** (1995) 261.

[2] H. Kawai, D. C. Lewellen and S. H. Tye, Nucl. Phys. **B288** (1987) 25.

[3] I. Antoniadis, C. P. Bachas and C. Kounnas, Nucl. Phys. **B289** (1987) 87.

[4] I. Antoniadis and C. P. Bachas, Nucl. Phys. **B298** (1988) 586.

[5] S. Ferrara and C. Kounnas, Nucl. Phys. **B328** (1989) 406.

[6] R. Gopakumar and S. Mukhi, Nucl. Phys. **B479** (1996) 260.

[7] D. R. Morrison and C. Vafa, Nucl. Phys. **B476** (1996) 437.

[8] G. Papadopoulos and P. K. Townsend, Phys. Lett. **B357** (1995) 300.

[9] A. Sen and C. Vafa, Nucl. Phys. **B455** (1995) 165.

[10] C. Bachas and E. Kiritsis, Nucl. Phys. **B55** [Proc. Suppl.] (1997) 194.

[11] C. Kounnas, Nucl. Phys. **B58** [Proc. Suppl.] (1997) 57;
E. Kiritsis and C. Kounnas, Nucl. Phys. **B503** (1997) 117.

[12] A. Gregori, E. Kiritsis, C. Kounnas, N.A. Obers, P.M. Petropoulos and B. Pioline, Nucl. Phys. **B510** (1998) 423.

[13] C. Kounnas and A. Kumar, Nucl. Phys. **B511** (1998) 216.

[14] R. D'Auria, S. Ferrara and C. Kounnas, Phys. Lett. **B420** (1998) 289.

[15] S. Kachru and C. Vafa, Nucl. Phys. **B450** (1995) 69; A. Klemm, W. Lerche and P. Mayr, Phys. Lett. **B357** (1995) 313; S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. **B459** (1996) 313.

[16] P. S. Aspinwall and J. Louis, Phys. Lett. **B369** (1996) 233.

[17] S. Ferrara, J. Harvey, A. Strominger and C. Vafa, Phys. Lett. **B361** (1995) 69, hep-th/9505162.

[18] A. Gregori, C. Kounnas and M. P. Petropoulos, hep-th/9808024, to appear in Nucl. Phys. **B**;
A. Gregori, hep-th/9811096, to appear in the Proceedings of the *6th Hellenic School and Workshop on Elementary Particle Physics*, Corfu, Greece, 6–26 September 1998.

[19] A. Gregori. C. Kounnas and P.M. Petropoulos, hep-th/9901117.

[20] E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. **B250** (1985) 385; B. de Wit, P.G. Lauwers and A. Van Proeyen, Nucl. Phys. **B255** (1985) 569.

[21] D. Dijkgraaf, E. Verlinde and H. Verlinde, Commun. Math. Phys. **115** (1988) 649.

[22] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, hep-th/9807067, to appear in Nucl. Phys. **B**.

[23] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, Phys. Lett. **B385** (1996) 87.

[24] A. Klemm and S. Theisen, Nucl. Phys. **B389** (1993) 153.

[25] J.A. Harvey and G. Moore, Phys. Rev. **D57** (1998) 2329.

[26] E. Kiritsis and C. Kounnas, Nucl. Phys. **B442** (1995) 472;

[27] J. A. Harvey and G. Moore, Nucl. Phys. **B463** (1996) 315.
P.M. Petropoulos and J. Rizos, Phys. Lett. **B374** (1996) 49.

[28] E. Kiritsis, C. Kounnas, P.M. Petropoulos and J. Rizos, Nucl. Phys. **B483** (1997) 141.

[29] I. Antoniadis, C. P. Bachas, C. Kounnas and P. Windey, Phys. Lett. **B171** (1986) 51.

[30] J. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. **B355** (1991) 649