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NON-PERTURBATIVE TRIALITY IN HETEROTIC AND TYPE II $N = 2$ STRINGS [†]

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Abstract

The non-perturbative equivalence of four-dimensional $N = 2$ superstrings with three vector multiplets and four hypermultiplets is analysed. These models are obtained through freely acting orbifold compactifications from the heterotic, the symmetric and the asymmetric type II strings. The heterotic scalar manifolds are $(SU(1,1)/U(1))^3$ for the S, T, U moduli sitting in the vector multiplets and $SO(4,4)/(SO(4) \times SO(4))$ for those in the hypermultiplets. The type II symmetric duals correspond to a self-mirror Calabi–Yau threefold compactification with Hodge numbers $h^{1,1} = h^{2,1} = 3$, while the type II asymmetric construction corresponds to a spontaneous breaking of the $N = (4,4)$ supersymmetry to $N = (2,0)$. Both have already been considered in the literature. The heterotic construction instead is new and we show that there is a weak/strong coupling S-duality relation between the heterotic and the asymmetric type IIA ground state with $4\pi S_{\text{Het}} = -(4\pi S_{\text{As}})^{-1}$; we also show that there is a partial restoration of $N = 8$ supersymmetry in the heterotic strong-coupling regime. We compute the full (non-)perturbative R^2 and F^2 corrections and determine the prepotential.

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1 Introduction

In four space-time dimensions, string solutions with $N \leq 4$ supersymmetries can be constructed, through appropriate compactifications of the six-dimensional internal manifold, from any of the perturbative ten-dimensional strings: heterotic, type I, type IIA or type IIB. Although these constructions appear different at the string perturbative level, they might be non-perturbatively equivalent, provided the massless spectrum and the number of supersymmetries is the same [1]–[3]. As far as $N = 4$ supersymmetry is concerned, several tests in favour of the non-perturbative duality equivalence have been presented in the literature, not only in the case of the heterotic string compactified on T^6 versus the type IIA, B compactified on $K3 \times T^2$[1], but also for theories with a lower number of massless vector multiplets [4], including the type IIA, B $N = 4$ asymmetric, freely acting, orbifold constructions [5].

In the $N = 4$ theories the space of the moduli fields is restricted by supersymmetry to be the coset [1, 6]:

$$\left(\frac{SL(2,R)}{U(1)}\right)_S \times \left(\frac{SO(6,6+r)}{SO(6)\times SO(6+r)}\right)_T , \tag{1.1}$$

where $r = 16$ in the heterotic and type IIA, B compactified on T^6 and $K3 \times T^2$, respectively. Models with lower rank, $r < 16$, can be constructed, via freely acting (asymmetric) orbifold compactification, from any of the perturbative superstring theories in ten dimensions [4, 6, 5]. On the heterotic side, the dilaton $S_{\rm Het}$ is always in the gravitational multiplet, while on the type II side it is either one of the moduli of the vector multiplets ($S_{\rm II} = T^1$), when the compactification is left–right-symmetric, with $N = (2, 2)$ supersymmetry, or in the gravitational multiplet in the asymmetric compactification with $N = (4, 0)$ supersymmetry. The non-perturbative string duality therefore implies interchanges between the moduli fields $S_{\rm Het}$ and T^i of the scalar manifold [1, 4, 5], with the perturbative states of one string theory mapped to non-perturbative states of its dual equivalent and vice versa [4, 5, 7, 8]. The non-perturbative equivalence of dual strings has been verified on several occasions: for instance, the anomaly cancellation of the six-dimensional heterotic string implies that there should be a one-loop correction to the gravitational R^2 term in the type II theory. Such a term was found by direct calculation in [4, 9]. Its one-loop threshold correction, upon compactification to four dimensions, implies instanton corrections on the heterotic side, due to five-branes being wrapped around the six-torus. Several other indications are given for dual $N = 4$ theories with rank $r < 16$ [4].

Heterotic/type II dual pairs with lower supersymmetry, $N = 2$, share properties similar to those of $N = 4$. In general (non-freely-acting) symmetric orbifolds still give rise to $N = 2$ heterotic/type II dual pairs in four dimensions [10]–[12]. On the heterotic side they can be interpreted as $K3$ plus gauge-bundle compactifications, while on the type II side they are Calabi–Yau compactifications of the ten-dimensional type IIA theory. The heterotic dilaton is in a vector-tensor multiplet, dual to a vector, and the vector moduli space receives both perturbative and non-perturbative corrections. The hypermultiplet moduli space, however, does not receive perturbative corrections; if $N = 2$ is assumed to be unbroken, it does not receive non-perturbative corrections either. The dilaton in the type II (symmetric) constructions is in a tensor multiplet, dual to a hypermultiplet, and the prepotential for the vector multiplets receives only tree-level contributions. The tree-level type II prepotential was computed and shown to give the correct one-loop heterotic result. This provides a quantitative test of the duality [10, 12] and allows us to reach the non-perturbative corrections of the heterotic side.

In the quantitative tests of non-perturbative dualities, extended supersymmetry plays an essential role, since it allows for the existence of BPS states (states in short representations of the supersymmetry algebra). These states are (generically) non-perturbatively stable and provide a reliable window into non-perturbative corrections to some terms of the effective action that receive contributions only from those states. The relevant structures for this analysis are the helicity supertrace formulae, which distinguish between various BPS and non-BPS states [4, 13]–[16]. For $N \geq 2$, these supertraces appear in particular in the F^2 and R^2 (two-derivative) terms or in a special class of higher-order terms constructed out of the Riemann tensor and the graviphoton field strength [17]. In the four-dimensional heterotic string, these terms are anomaly-related and it can be shown that they receive only tree- and one-loop corrections. In higher dimensions, they do not receive non-perturbative corrections either [18].

The non-perturbative equivalence of some heterotic/type II dual pairs with $N = 2$ supersymmetry has been investigated in Refs. [5, 11, 19, 20]. In this class of $N = 2$ models the scalar manifolds are coset spaces:

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2+N_V)}{SO(2)\times SO(2+N_V)} \quad \text{and} \quad \frac{SO(4,4+N_H)}{SO(4)\times SO(4+N_H)}, \tag{1.2}$$

describing the moduli space of the $N_V + 3$ moduli in vector multiplets and the $N_H + 4$ in the hypermultiplets, respectively. The type II symmetric duals correspond to self-mirror Calabi–Yau threefold compactifications with Hodge numbers $h^{1,1} = N_V + 3 = h^{2,1} = N_H + 3$, which are $K3$ fibrations, necessary condition for the existence of heterotic duals [21, 22]. The equivalence of heterotic/type II model(s) with $N_V = N_H = 8$ was studied in Refs. [11, 19, 20]; recently, in Ref. [20], this analysis has been extended to type II and heterotic duals with $N_V = N_H = 4$ and 2.

The purpose of this work is to extend the analysis of Ref. [20] and establish the non-perturbative equivalence of three different $N = 2$ constructions with $N_V = N_H = 0$: the heterotic construction with supersymmetry $N = (2, 0)$, the symmetric type II construction with $N = (1, 1)$ and the asymmetric type II with $N = (2, 0)$ supersymmetry. All these constructions are based on six-dimensional freely acting (asymmetric) orbifold compactifications and thus the initial $N = 8$ (in type II) or $N = 4$ (in heterotic) supersymmetry is spontaneously broken to $N = 2$ [14]. The heterotic scalar manifold of this $N = 2$ construction is described by the vector scalar manifold $(SU(1,1)/U(1))^3$ associated to the three moduli S, T and U, and by the hypermultiplet quaternionic one, $SO(4,4)/(SO(4) \times SO(4))$. The type IIA symmetric construction corresponds to a self-mirror Calabi–Yau threefold compactification with Hodge numbers $h^{1,1} = h^{2,1} = 3$ [5]. The type II asymmetric construction [5] corresponds to a spontaneous breaking of $N = (4, 4)$ to $N = (2, 0)$ supersymmetry [14].

In Section 2 we construct the symmetric type II model with $N = (1, 1)$ supersymmetry and calculate the one-loop gravitational corrections associated to the R^2 term. The asymmetric type II $N = (2,0)$ construction is presented in Section 3; in the same section we show that the R^2 corrections in the two type II dual theories are in agreement with their non-perturbative equivalence. The heterotic construction, as well as the corresponding corrections to the gauge and gravitational couplings, are presented in Section 4. In Section 5 we compare the heterotic, the type II symmetric and the type II asymmetric corrections, and show that the non-perturbative equivalence of the three models is verified; we furthermore show that due to this triality equivalence, there exists a weak–strong coupling relation (S-duality) between the heterotic and the asymmetric type II theory ($4\pi S_{\rm Het} = -(4\pi S_{\rm As})^{-1}$). We also claim that the $N = 8$ supersymmetry is restored in the heterotic strong coupling regime. Finally, in Section 6 and in the appendix, we derive the perturbative prepotential, as well as part of its non-perturbative corrections, which are argued to be valid for all three dual theories. Our conclusion and comments are given in Section 7.

2 The $N = (1,1)$ type II symmetric construction

We start by considering a type II symmetric construction with $N = (1,1)$ supersymmetry and no vector multiplets or hypermultiplets in the twisted sector. This model is obtained by compactification of the ten-dimensional type II string on a Calabi–Yau manifold $\mathrm{CY}^{(3,3)}$, with Hodge numbers $h^{1,1} = h^{2,1} = 3$. This compactification reduces the $N = (4,4)$ supersymmetry to the desired $N = (1,1)$. In what follows we will always work at the $Z_2^{(1)} \times Z_2^{(2)}$ (freely acting) orbifold limit of this manifold, where the partition function and the one-loop gravitational and gauge corrections can be computed analytically.

At the orbifold point, where $\mathrm{CY}^{(3,3)} \equiv T^6 \Big/ \left(Z_2^{(1)} \times Z_2^{(2)}\right)$, the partition function of the model can be written easily in terms of the characters of the twisted and shifted compactified left and right coordinates $X^I, \bar{X}^I, I = 1, \dots, 6$, and in terms of twisted fermionic superpartners Ψ^I and $\bar{\Psi}^I$. The remaining contribution to the partition function comes from the left- and right-moving non-compact supercoordinates $X^\mu, \Psi^\mu, \bar{X}^\mu, \bar{\Psi}^\mu$ and the super-reparametrization ghosts b, c, β, γ and $\bar{b}, \bar{c}, \bar{\beta}, \bar{\gamma}$. The resulting partition function reads:

$$\begin{aligned} Z_{\rm II}^{(1,1)} &= \frac{1}{\mathrm{Im}\,\tau |\eta|^{24}} \frac{1}{4} \sum_{H^1,G^1} \sum_{H^2,G^2} \Gamma_{6,6} \begin{bmatrix} H^1, H^2 \\ G^1, G^2 \end{bmatrix} \\ &\times \frac{1}{2} \sum_{a,b} (-)^{a+b+ab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \vartheta \begin{bmatrix} a+H^2 \\ b+G^2 \end{bmatrix} \vartheta \begin{bmatrix} a+H^1 \\ b+G^1 \end{bmatrix} \vartheta \begin{bmatrix} a-H^1-H^2 \\ b-G^1-G^2 \end{bmatrix} \\ &\times \frac{1}{2} \sum_{\bar{a},\bar{b}} (-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}} \bar{\vartheta} \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+H^2 \\ \bar{b}+G^2 \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+H^1 \\ \bar{b}+G^1 \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-H^1-H^2 \\ \bar{b}-G^1-G^2 \end{bmatrix}, \end{aligned} \qquad (2.1)$$

where the contribution of $\beta, \gamma, \Psi^\mu, \Psi^I$ and $\bar{\beta}, \bar{\gamma}, \bar{\Psi}^\mu, \bar{\Psi}^I$ gives rise to the functions ϑ and $\bar{\vartheta}$, while $\Gamma_{6,6}\left[{H^1,H^2 \atop G^1,G^2}\right]$ denotes the contribution of X^I and $\bar{X}^I$; (H^1, G^1) refer to the boundary conditions introduced by the projection $Z_2^{(1)}$ and (H^2, G^2) to the projection $Z_2^{(2)}$:

$$\Gamma_{6,6} \begin{bmatrix} H^1, H^2 \\ G^1, G^2 \end{bmatrix} = \Gamma_{2,2}^{(1)} \begin{bmatrix} H^2 | H^1 \\ G^2 | G^1 \end{bmatrix} \Gamma_{2,2}^{(2)} \begin{bmatrix} H^1 | H^1+H^2 \\ G^1 | G^1+G^2 \end{bmatrix} \Gamma_{2,2}^{(3)} \begin{bmatrix} H^1+H^2 | H^2 \\ G^1+G^2 | G^2 \end{bmatrix}. \qquad (2.2)$$

Here we have introduced the twisted and shifted characters of a $c=(2,2)$ block, $\Gamma_{2,2}\left[{}^{h|h'}_{g|g'}\right]$; the first column refers to the twist, the second to the shift. The non-vanishing components are the following:

$$\begin{aligned}\Gamma_{2,2}\begin{bmatrix}h|h'\\ g|g'\end{bmatrix} &= \frac{4\,|\eta|^6}{\left|\vartheta\left[{}^{1+h}_{1+g}\right]\vartheta\left[{}^{1-h}_{1-g}\right]\right|}\ , \quad \text{for } (h',g')=(0,0) \text{ or } (h',g')=(h,g)\\ &= \Gamma_{2,2}\begin{bmatrix}h'\\ g'\end{bmatrix}\ , \quad \text{for } (h,g)=(0,0)\ ,\end{aligned} \tag{2.3}$$

where $\Gamma_{2,2}\left[{}^{h'}_{g'}\right]$ is the Z_2-shifted $(2,2)$ lattice sum. As usual, the shift has to be specified by the way it acts on the momenta and windings (our conventions are those of Refs. [4, 20, 24]). Since the three complex planes of T^6 are translated, there are no fixed points. Therefore there are no extra massless states coming from the twisted sectors: the massless spectrum of this model contains the $N=2$ supergravity multiplet, 3 vector multiplets, 1 tensor multiplet and 3 hypermultiplets. The tensor multiplet is the type II dilaton multiplet and is equivalent to an extra hypermultiplet.

By using the techniques developed in Refs. [14, 23, 24], it can be shown that this model possesses an $N=8$ supersymmetry spontaneously broken through a super-Higgs phenomenon, due to the free actions of $Z_2^{(1)}\times Z_2^{(2)}$. There exist appropriate limits of the moduli, which depend on the precise shifts in the lattices, in which there is an approximate restoration of 16 or 32 supercharges. In such limits, the supersymmetry restoration is accompanied by a logarithmic instead of a linear blow-up of the various thresholds. The logarithmic blow-up is an infrared artefact, which can be lifted by switching on an infrared cut-off μ larger than the mass of the extra massive gravitinos; the thresholds thus vanish, as expected, in the limit $m_{3/2}/\mu \to 0$ in which supersymmetry is extended to $N=8$.

The relevant quantities for the computation of the string correction to the R^2 term are the helicity supertraces B_{2n}. These are defined as the vacuum expectation value of $\left(Q+\overline{Q}\right)^{2n}$, where Q, $\overline{Q}$ stand for the left- and right-helicity contributions to the four-dimensional physical helicity. For the details of the computation of such quantities in the framework of the above models we refer to previous publications [20, 24]. Here we simply quote the results. A straightforward computation shows that $B_2=0$, as expected in the models with $N_V=N_H$ [20]. This feature is common to all the $N=2$ type II $Z_2\times Z_2$ symmetric orbifolds [25], in which B_2 can receive a non-zero contribution only from the $N=(1,1)$ sectors of the orbifold. The internal coordinates in these sectors are twisted; all corrections are therefore moduli-independent and come from the massless states only. One finds $B_2 = B_2|_{\text{massless}}$, which vanishes in the model we are considering here.

On the other hand, B_4 receives non-zero contributions from the $N=(2,2)$ sectors of the orbifold, and we find[1]:

$$B_4 = 6\sum_{i=1,2,3}\sum_{(h,g)}{}' \,\Gamma_{2,2}^{(i)}\begin{bmatrix}0|h\\ 0|g\end{bmatrix}\ . \tag{2.4}$$

[1]The prime summation over (h,g) stands for $(h,g)=\{(0,1),(1,0),(1,1)\}$.

From (2.4), by applying the techniques developed in [24], one can see that there is a limit in moduli space in which B_4 vanishes; this is the signal of the restoration of the $N = 8$ supersymmetry.

The four-derivative gravitational correction we consider here is similar to those that were analysed in Refs. [4] and [20]; in order to obtain it we proceed as in [20]. There is no tree-level contribution to this operator, and the R^2 correction appears at one loop; it is related to the contribution of the $h^{1,1}$ moduli, obtained through the insertion, in the one-loop partition function, of the two-dimensional operator $2Q^2\overline{Q}^2$. In this class of models the contribution of the $N = (1,1)$ sectors to B_4 vanishes, and therefore $\left\langle 2Q^2\overline{Q}^2 \right\rangle$ is $B_4/3$. The massless contributions of the latter give rise to an infrared logarithmic behaviour $\frac{B_4|_{\text{massless}}}{3} \log \left(M^{(\text{IIA})\,2} \big/ \mu^{(\text{IIA})\,2} \right)$ [26, 27], where $M^{(\text{IIA})} \equiv 1/\sqrt{\alpha'_{\text{IIA}}}$ is the type IIA string scale and $\mu^{(\text{IIA})}$ is the type IIA infrared cut-off. Besides this running, the one-loop correction contains, as usual, the thresholds Δ_{IIA}, which account for the infinite tower of massive string modes. The threshold corrections to the R^2 term are related to the infrared-regularized genus-one integral of $B_4/3$. This relationship can be made more precise by noting that the amplitude $\left\langle 2Q^2\overline{Q}^2 \right\rangle$ contains more than the R^2-term corrections: it also accounts for terms such as F^2 or H^2. However, in the type IIA string, the R^2 corrections depend on the Kähler moduli T^1, T^2 and T^3 (spanning the vector manifold), and are independent of the complex-structure moduli U^1, U^2 and U^3 (spanning the scalar manifold) [4, 20]. We thus have

$$\partial_{T^i} \Delta_{\text{IIA}} = \frac{1}{3} \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\operatorname{Im}\tau} \partial_{T^i} B_4 \ , \quad \partial_{U^i} \Delta_{\text{IIA}} = 0 \, . \tag{2.5}$$

For definiteness we choose the half-unit shifts for $Z_2^{(1)}$ and $Z_2^{(2)}$ as defined by the following insertions: $(-)^{n_2 G^1}$, $(-)^{m_1 (G^1 + G^2)}$, $(-)^{n_2 G^2}$ shifting the lattices of the first, second and third plane, respectively. With this choice of lattice shifts the one-loop-corrected gravitational coupling reads (up to a constant):

$$\frac{16\,\pi^2}{g^2_{\text{grav}}(\mu^{(\text{IIA})})} = -2 \sum_{i=1,3} \log \operatorname{Im} T^i \left| \vartheta_2 \left(T^i \right) \right|^4 - 2 \log \operatorname{Im} T^2 \left| \vartheta_4 \left(T^2 \right) \right|^4 + 6 \log \frac{M^{(\text{IIA})}}{\mu^{(\text{IIA})}} \, . \tag{2.6}$$

The shifts on the $\Gamma_{2,2}^{(i)}$ lattices break the $SL(2,Z)_{T^i}$ duality groups, and the actual subgroup left unbroken depends on the kind of shifts performed (see Refs. [4, 20, 23, 24]). Furthermore, there are three $N = 4$ restoration limits, corresponding to $(\operatorname{Im} T^1, 1/\operatorname{Im} T^2) \to 0$, $(\operatorname{Im} T^1, \operatorname{Im} T^3) \to 0$ or $(1/\operatorname{Im} T^2, \operatorname{Im} T^3) \to 0$. The masses of the three extra pairs of gravitinos are in fact given by

$$\begin{aligned} m^2_{3/2}(1) &= \frac{1}{4} \operatorname{Im} T^1 \operatorname{Im} U^1 + \frac{1}{4} \frac{\operatorname{Im} U^2}{\operatorname{Im} T^2} \\ m^2_{3/2}(2) &= \frac{1}{4} \operatorname{Im} T^1 \operatorname{Im} U^1 + \frac{1}{4} \operatorname{Im} T^3 \operatorname{Im} U^3 \\ m^2_{3/2}(3) &= \frac{1}{4} \frac{\operatorname{Im} U^2}{\operatorname{Im} T^2} + \frac{1}{4} \operatorname{Im} T^3 \operatorname{Im} U^3 \, , \end{aligned} \tag{2.7}$$

and each of them vanishes in one of the above limits. Owing to the effective restoration of the $N=4$ supersymmetry, there is no linear divergence in the volume of the decompactifying manifold. For example, when $(1/\,\mathrm{Im}\,T^2,\,\mathrm{Im}\,T^3)\to 0$, we observe the following leading behaviour:

$$\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{\rm (IIA)})}\to -2\log\,\mathrm{Im}\,T^2+2\log\,\mathrm{Im}\,T^3\,. \tag{2.8}$$

However, the threshold correction blows up linearly with respect to the modulus of the first plane in the limit $\,\mathrm{Im}\,T^1\to\infty$:

$$\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{\rm (IIA)})}\to 8\pi\,\mathrm{Im}\,T^1\,. \tag{2.9}$$

Finally, the $N=8$ supersymmetry is restored when $(\,\mathrm{Im}\,T^1, 1/\,\mathrm{Im}\,T^2,\,\mathrm{Im}\,T^3)\to 0$. In this limit, the correction behaves logarithmically in all three moduli:

$$\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{\rm (IIA)})}\to 2\log\,\mathrm{Im}\,T^1-2\log\,\mathrm{Im}\,T^2+2\log\,\mathrm{Im}\,T^3\,. \tag{2.10}$$

3 The $N=(2,0)$ type II asymmetric construction

We now consider the asymmetric type II orbifold, which is obtained from the $N=8$ IIA superstring compactified on T^6 by applying the freely acting projections $Z_2^{F_{\rm R}}$ and $Z_2^{(1)}$. The latter is the same projection we considered in the previous section: it acts as a combination of rotation and translation, and it reduces symmetrically the number of supersymmetries by one half. Instead, $Z_2^{F_{\rm R}}$ acts as $(-)^{F_{\rm R}}$ together with a translation on T^6, and projects out all the right-moving supersymmetries. The properties of the $N=4$ model obtained by applying only $Z_2^{F_{\rm R}}$ were already analysed in [4]. The orbifold obtained by further application of $Z_2^{(1)}$ has an $N=(2,0)$ supersymmetry realized among the left-movers only.

The partition function of the model reads:

$$\begin{aligned} Z_{\rm II}^{2,0} \;=\; & \frac{1}{\mathrm{Im}\,\tau|\eta|^{24}}\frac{1}{4}\sum_{H^1,G^1}\sum_{H^F,G^F}\Gamma_{6,6}\begin{bmatrix}H^1,H^F\\ G^1,G^F\end{bmatrix}\\ & \times\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\vartheta^2\begin{bmatrix}a\\ b\end{bmatrix}\vartheta\begin{bmatrix}a+H^1\\ b+G^1\end{bmatrix}\vartheta\begin{bmatrix}a-H^1\\ b-G^1\end{bmatrix}\\ & \times\frac{1}{2}\sum_{\bar a,\bar b}(-)^{\bar a+\bar b+\bar a\bar b}(-)^{\bar aG^F+\bar bH^F+H^FG^F}\bar\vartheta^2\begin{bmatrix}\bar a\\ \bar b\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a+H^1\\ \bar b+G^1\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a-H^1\\ \bar b-G^1\end{bmatrix}, \end{aligned} \tag{3.1}$$

where now

$$\Gamma_{6,6}\begin{bmatrix}H^1,H^F\\ G^1,G^F\end{bmatrix}=\Gamma_{2,2}^{(1)}\begin{bmatrix}0|H^1\\ 0|G^1\end{bmatrix}\Gamma_{2,2}^{(2)}\begin{bmatrix}H^1|H^F\\ G^1|G^F\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}H^1|H^1\\ G^1|G^1\end{bmatrix}. \tag{3.2}$$

The massless spectrum contains, besides the supergravity multiplet, 1 vector-tensor multiplet, dual to a vector, 2 vector multiplets and 4 hypermultiplets: it is therefore the same as that of the type II symmetric orbifold. However, there is an important difference in the nature of the fields: in this case the dilaton belongs to a vector multiplet. This is a general property of all $N=(2,0)$ string compactifications, where supersymmetry charges involve left-movers only. The reason is that the dilaton, in such cases, is uncharged under the $SU(2)$ operators that rotate the supercharges of the $N=2$ supergravity.

The three vector moduli are in this case the dilaton $S_{\rm As}$, the Kähler class $T_{\rm As}$, and the complex structure $U_{\rm As}$ of the first torus. When $(H^1,G^1)=(0,0)$, expressions (3.1) and (3.2) give half the partition function of an $N=(4,0)$ asymmetric orbifold with gauge group $U(1)^6$. This model was analysed in detail in Ref. [4].

By using the same techniques as for the type II symmetric orbifold, it can be shown that the model at hand possesses a spontaneously broken $N=8$ supersymmetry, due to the free action of $Z_2^{(1)}$ and $Z_2^{F_{\rm R}}$. This can be seen again from the analysis of the helicity supertraces. We find that B_2 is a constant also in this asymmetric construction. There are therefore no "$N=2$ singularities", i.e. lines in moduli space with enhancement of the massless spectrum such that $\Delta N_V \neq \Delta N_H$. On the other hand, for finite values of the moduli, there are no "$N=4$ singularities" either (lines where $\Delta N_V = \Delta N_H$), because the bosonic vacuum energy is $-1/2$, and there are no points in moduli space in which new massless states can appear.

The helicity supertrace B_4 receives non-zero contributions from three $N=4$ sectors: the $N=(4,0)$ sector with $(H^1,G^1)=(0,0)$, the $N=(2,2)$ sector with $(H^1,G^1)\neq(0,0)$ and $(H^F,G^F)=(0,0)$, and the $N=(2,2)$ sector with $(H^1,G^1)\neq(0,0)$ and $(H^F,G^F)=(H^1,G^1)$. We obtain:

$$\begin{aligned} B_4 &= \frac{3}{8}\frac{1}{\bar\eta^{12}} \sideset{}{'}\sum_{(H^F,G^F)} (-)^{H^F G^F} \bar\vartheta^4\begin{bmatrix}1-H^F\\ 1-G^F\end{bmatrix}\Gamma_{6,6}\begin{bmatrix}0,H^F\\ 0,G^F\end{bmatrix} \\ &\quad +12\sideset{}{'}\sum_{(H^1,G^1)} \Gamma_{2,2}^{(1)}\begin{bmatrix}0|H^1\\ 0|G^1\end{bmatrix}. \end{aligned} \tag{3.3}$$

The contributions of the first line come from the $N=(4,0)$ sector, while those of the second line are due to the $N=(2,2)$ sectors.

Expression (3.3) has to be compared with the analogous for the type II symmetric orbifold, Eq. (2.4). In both cases $B_4|_{\rm massless}=18$, in agreement with the expected $N=2$ supergravity result. As in the type II symmetric orbifold, we can make B_4 vanish by taking appropriate limits in the space of moduli belonging to vector multiplets and hypermultiplets. In the asymmetric type II the vector-multiplet moduli are the moduli of the first complex plane $T_{\rm As}, U_{\rm As}$. The moduli of second and third complex planes belong to the hypermultiplet space. The lattice sum in the second line of (3.3) vanishes in some appropriate limits in $T_{\rm As}$ and $U_{\rm As}$. However, only by taking further limits in some of the moduli belonging to hypermultiplets can we make also $\Gamma_{6,6}\left[{}^{0,H^F}_{0,G^F}\right]$ vanish. This is precisely the limit in which the extra massive gravitinos of the asymmetric construction become massless.

As was already pointed out in the framework of symmetric type II constructions, the R^2 gravitational corrections of the asymmetric case do not receive any contribution beyond one loop. These corrections are related to the insertion of the operator $2Q^2\overline{Q}^2$. Again, this amplitude accounts for more terms (like H^2) and only half of it is relevant to the R^2. Therefore, the only non-zero contribution is provided by one sixth of the $N=(2,2)$ sectors of B_4 (the second term in Eq. (3.3)). The part of B_4 associated with the $N=(4,0)$ sector does not enter in the R^2 correction and thus the moduli dependence comes from the vector multiplets only; there is no dependence at all on the hypermultiplet moduli, as expected from general properties of the $N=2$ theories. Both moduli $T_{\rm As}$ and $U_{\rm As}$ of the first plane belong to vector multiplets and appear in the correction to the R^2 term.

With the specific choice of half-unit shift, $(-)^{m_1 G^1}$, induced by $Z_2^{(1)}$ acting on $\Gamma_{2,2}^{(1)}$, we obtain the corrected gravitational coupling in terms of the moduli $T_{\rm As}$, $U_{\rm As}$, and the appropriate string scale and infrared cut-off:

$$\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{(\rm As)})} = -2\log\,{\rm Im}\,T_{\rm As}\left|\vartheta_4\left(T_{\rm As}\right)\right|^4 - 2\log\,{\rm Im}\,U_{\rm As}\left|\vartheta_2\left(U_{\rm As}\right)\right|^4 + 4\log\frac{M^{(\rm As)}}{\mu^{(\rm As)}} \tag{3.4}$$

up to a constant[2].

This expression deserves some comments, as was for (2.6). We first observe that the $Z_2^{(1)}$-shift in $\Gamma_{2,2}^{(1)}$ breaks the $SL(2,Z)_{T_{\rm As}}\times SL(2,Z)_{U_{\rm As}}\times Z_2^{T_{\rm As}\leftrightarrow U_{\rm As}}$ duality group to a subgroup that depends on the kind of shift performed. In the limit ${\rm Im}\,T_{\rm As}\to\infty$, ${\rm Im}\,U_{\rm As}\to 0$ there is a restoration of an $N=(4,0)$ supersymmetry with no linear behaviour either in ${\rm Im}\,T_{\rm As}$ or in $1/\,{\rm Im}\,U_{\rm As}$; the remaining contribution is logarithmic:

$$\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{(\rm As)})} \to -2\log\,{\rm Im}\,T_{\rm As} + 2\log\,{\rm Im}\,U_{\rm As}\,. \tag{3.5}$$

Finally, comparison of (2.6) and (3.4) suggests that the string duality map implies the following identification of the moduli:

$$T_{\rm As}\leftrightarrow T^2\ ,\quad U_{\rm As}\leftrightarrow T^3\ \text{and}\ \ 4\pi S_{\rm As}\leftrightarrow -1/T^1\,. \tag{3.6}$$

The identification of the asymmetric dilaton $S_{\rm As}$ with the $h^{1,1}$ moduli T^1 of the symmetric type II construction follows from the behaviour in the limit ${\rm Im}\,T^1\to 0$, which corresponds, in the asymmetric construction, to the perturbative limit $S_{\rm As}\to\infty$.

4 The $N=(2,0)$ heterotic construction

4.1 The construction of the model

The heterotic dual to the previous type II constructions is based on $\left(T^2\times T^4\right)/Z_2^{(\rm f)}$ freely acting orbifold heterotic compactification. In order to reduce the gauge group we have to introduce a set of "discrete Wilson lines", which separate the boundary conditions of the 32 right-moving fermions, $\Psi_A, A=1,\ldots,32$, in order to twist all the currents $\Psi_A\Psi_B$ of the

[2]This constant, as well as the one appearing later in the perturbative heterotic coupling (4.25), contains in general $\log(M/\mu)$ terms, which account for extra massless states that have been disregarded in the determination of the R^2 amplitude from the B_4. Field-theoretical arguments can be advocated to fix these terms. We will take care of them in the final expression (5.3).

$c = (0,16)$ conformal block. The resulting characters are described by those of 32 right-moving Ising's . The $Z_2^{(\mathrm{f})}$ projection reduces the initial $N=4$ supersymmetry to $N=2$; it acts as a π rotation in T^4 left and right (super)coordinates and as a translation in T^2. Its action on Ψ_A is non-trivial (see below), and it is chosen in such a way that neither vectors nor hypers are produced from the Ψ_A's. In this way the heterotic massless spectrum is identical to that of the previous type II constructions. Namely, it consists of the $N=2$ supergravity multiplet, 1 vector-tensor multiplet that contains the dilaton S_{Het} and is dual to a vector, 2 vector multiplets associated with the Kähler class T and the complex structure U of the torus[3] T^2, and 4 hypermultiplets, obtained by pairing the left-moving negative eigenvalues of the projection $Z_2^{(\mathrm{f})}$ with the 4 right-moving negative eigenvalues in T^4.

The partition function of the heterotic construction has the following expression:

$$\begin{aligned} Z_{\mathrm{Het}} &= \frac{1}{\mathrm{Im}\,\tau|\eta|^4}\frac{1}{2}\sum_{H^{\mathrm{f}},G^{\mathrm{f}}} Z_{6,22}\begin{bmatrix}H^{\mathrm{f}}\\G^{\mathrm{f}}\end{bmatrix} \\ &\times\frac{1}{2}\sum_{a,b}\frac{1}{\eta^4}(-)^{a+b+ab}\vartheta\begin{bmatrix}a\\b\end{bmatrix}^2\vartheta\begin{bmatrix}a+H^{\mathrm{f}}\\b+G^{\mathrm{f}}\end{bmatrix}\vartheta\begin{bmatrix}a-H^{\mathrm{f}}\\b-G^{\mathrm{f}}\end{bmatrix}, \end{aligned} \tag{4.1}$$

where the second line stands for the contribution of the 10 left-moving world-sheet fermions ψ^μ, Ψ^I and the ghosts β,γ of the super-reparametrization; $Z_{6,22}\left[{H^{\mathrm{f}} \atop G^{\mathrm{f}}}\right]$ accounts for the $(6,6)$ compactified coordinates and the $c=(0,16)$ conformal system, which is described by the 32 right-moving fermions Ψ_A. It takes the following form:

$$Z_{6,22}\begin{bmatrix}H^{\mathrm{f}}\\G^{\mathrm{f}}\end{bmatrix} = \frac{1}{2^5}\sum_{\vec h,\vec g}\frac{1}{\eta^6\bar\eta^6}\Gamma_{2,2}\begin{bmatrix}H^{\mathrm{f}},\vec h\\G^{\mathrm{f}},\vec g\end{bmatrix}\Gamma_{4,4}\begin{bmatrix}H^{\mathrm{f}}|\vec h\\G^{\mathrm{f}}|\vec g\end{bmatrix}\overline{\Phi}\begin{bmatrix}H^{\mathrm{f}},\vec h\\G^{\mathrm{f}},\vec g\end{bmatrix}, \tag{4.2}$$

where $(\vec h,\vec g)$ denote the five projections that are needed in order to separate the boundary conditions of all 32 fermions. The function $\Phi\left[{H^{\mathrm{f}},\vec h \atop G^{\mathrm{f}},\vec g}\right]$ can be written explicitly using the $SO(4)$ twisted characters (see Ref. [20]):

$$\widehat{F}_1\begin{bmatrix}\gamma,h\\\delta,g\end{bmatrix} \equiv \frac{1}{\eta^2}\vartheta^{1/2}\begin{bmatrix}\gamma-h_1-h_2-h_3\\\delta-g_1-g_2-g_3\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_3\\\delta+g_3\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_3-h_1\\\delta+g_3-g_1\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_2-h_3\\\delta+g_2-g_3\end{bmatrix} \tag{4.3}$$

and

$$\widehat{F}_2\begin{bmatrix}\gamma,h\\\delta,g\end{bmatrix} \equiv \frac{1}{\eta^2}\vartheta^{1/2}\begin{bmatrix}\gamma\\\delta\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_1-h_2\\\delta+g_1-g_2\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_1\\\delta+g_1\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_2\\\delta+g_2\end{bmatrix}, \tag{4.4}$$

where we introduced the notation $h\equiv(h_1,h_2,h_3)$ and similarly for g. Under $\tau\to\tau+1$, $\widehat{F}_I$ transform as:

$$\widehat{F}_1\begin{bmatrix}\gamma,h\\\delta,g\end{bmatrix} \to \widehat{F}_1\begin{bmatrix}\gamma,h\\\gamma+\delta+1,h+g\end{bmatrix}$$

[3]For simplicity we will use systematically (T,U) for the heterotic two-torus moduli, instead of the more natural notation, which would have been $(T_{\mathrm{Het}},U_{\mathrm{Het}})$.

$$\times \exp -\frac{i\pi}{4}\left(\frac{2}{3}+2\gamma^2+h_1^2+h_2^2+2h_3^2-2\gamma h_1+h_1h_2-4\gamma+2h_1\right)\ ; \quad (4.5)$$

$$\widehat{F}_2\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \to \widehat{F}_2\begin{bmatrix}\gamma, h\\ \gamma+\delta+1, h+g\end{bmatrix} \times \exp -\frac{i\pi}{4}\left(\frac{2}{3}+2\gamma^2+h_1^2+h_2^2+2\gamma h_1-h_1h_2-4\gamma-2h_1\right). \quad (4.6)$$

Notice that $\widehat{F}_I$ are $c=(2,0)$ conformal characters of 4 different left-moving Isings. In the fermionic language [28] this is a system of 4 left-moving real fermions with different boundary conditions. All currents $J^{IJ}=\Psi^I\Psi^J$ are twisted and therefore the initial $SO(4)$ is broken. We have then, as in [20], two alternative constructions, Φ and $\tilde{\Phi}$, that differ with respect to the embedding of the $Z_2^{(\mathrm{f})}$-shift in the two-torus:

$$\Phi\begin{bmatrix}H^{\mathrm{f}}, \vec{h}\\ G^{\mathrm{f}}, \vec{g}\end{bmatrix} = \frac{1}{2}\sum_{\gamma,\delta} \widehat{F}_1\begin{bmatrix}\gamma+H^{\mathrm{f}}, h\\ \delta+G^{\mathrm{f}}, g\end{bmatrix} \widehat{F}_1\begin{bmatrix}\gamma+h_4+H^{\mathrm{f}}, h\\ \delta+g_4+G^{\mathrm{f}}, g\end{bmatrix} \widehat{F}_1\begin{bmatrix}\gamma+h_5, h\\ \delta+g_5, g\end{bmatrix} \widehat{F}_1\begin{bmatrix}\gamma+h_4+h_5, h\\ \delta+g_4+g_5, g\end{bmatrix} \times \widehat{F}_2\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \widehat{F}_2\begin{bmatrix}\gamma+h_4, h\\ \delta+g_4, g\end{bmatrix} \widehat{F}_2\begin{bmatrix}\gamma+h_5, h\\ \delta+g_5, g\end{bmatrix} \widehat{F}_2\begin{bmatrix}\gamma+h_4+h_5, h\\ \delta+g_4+g_5, g\end{bmatrix} \quad (4.7)$$

and

$$\tilde{\Phi}\begin{bmatrix}H^{\mathrm{f}}, \vec{h}\\ G^{\mathrm{f}}, \vec{g}\end{bmatrix} = \frac{1}{2}\sum_{\gamma,\delta} \widehat{F}_1\begin{bmatrix}\gamma+H^{\mathrm{f}}, h\\ \delta+G^{\mathrm{f}}, g\end{bmatrix} \widehat{F}_1\begin{bmatrix}\gamma+h_4+H^{\mathrm{f}}, h\\ \delta+g_4+G^{\mathrm{f}}, g\end{bmatrix} \widehat{F}_1\begin{bmatrix}\gamma+h_5, h\\ \delta+g_5, g\end{bmatrix} \widehat{F}_1\begin{bmatrix}\gamma+h_4+h_5, h\\ \delta+g_4+g_5, g\end{bmatrix} \times \widehat{F}_2\begin{bmatrix}\gamma+H^{\mathrm{f}}, h\\ \delta+G^{\mathrm{f}}, g\end{bmatrix} \widehat{F}_2\begin{bmatrix}\gamma+h_4, h\\ \delta+g_4, g\end{bmatrix} \widehat{F}_2\begin{bmatrix}\gamma+h_5, h\\ \delta+g_5, g\end{bmatrix} \widehat{F}_2\begin{bmatrix}\gamma+h_4+h_5, h\\ \delta+g_4+g_5, g\end{bmatrix}. \quad (4.8)$$

The $(2,2)$ and $(4,4)$ lattice shifts are dictated by modular invariance and are needed in order to cancel the phases that appear under modular transformations. These shifts are different for the two constructions, based on Φ or $\tilde{\Phi}$. In the case of Φ, modular invariance implies an asymmetric shift on the $\Gamma_{2,2}\left[{H^{\mathrm{f}} \atop G^{\mathrm{f}}}\right]$, which we chose to be $(-)^{(m_1+n_1)G^{\mathrm{f}}}$ (this projection was referred to as "X" in Ref. [24], where the various lattice shifts were discussed in detail); the shift in $\Gamma_{4,4}\left[{H^{\mathrm{f}}|\vec{h} \atop G^{\mathrm{f}}|\vec{g}}\right]$, however, has to be symmetric and is chosen to be $(-)^{M_i\, g_i}$.

In the construction based on $\tilde{\Phi}$, on the other hand, the $(2,2)$ lattice must be doubly shifted, $\Gamma_{2,2}\left[{H^{\mathrm{f}}, h_1-h_2 \atop G^{\mathrm{f}}, g_1-g_2}\right]$. In this case we use the projection $(-)^{m_1G^{\mathrm{f}}+n_1(g_1-g_2)}$.

Constructions Φ and $\tilde{\Phi}$ share the same $N=4$ sector (defined by $(H^{\mathrm{f}}, G^{\mathrm{f}})=(0,0)$). The contribution of this sector to the partition function is one half of the partition function of an $N=4$ model in which the gauge group is $U(1)^6$, and all the vectors originating from the torus T^6. In the $N=2$ sectors, $(H^{\mathrm{f}}, G^{\mathrm{f}})\neq(0,0)$, while (h_i, g_i) are either $(0,0)$ or $(H^{\mathrm{f}}, G^{\mathrm{f}})$. This restriction on the values (h_i, g_i) comes from the $(H^{\mathrm{f}}, G^{\mathrm{f}})$-twisted sector of $\Gamma_{4,4}\left[{H^{\mathrm{f}}|\vec{h} \atop G^{\mathrm{f}}|\vec{g}}\right]$.

A quantity relevant to our purpose is the helicity supertrace B_2, which receives a non-zero contribution from the $N=2$ sector, while the contribution of the $N=4$ sector to this quantity vanishes. For the construction based on Φ we find:

$$B_2\left(\Phi\right) = \frac{1}{\bar\eta^{24}} \sum_{(H^{\rm f},G^{\rm f})}{}' \, \Gamma_{2,2}^{\lambda=1}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix} \overline{\Omega}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix}, \tag{4.9}$$

where $\Omega\left[{H^{\rm f} \atop G^{\rm f}}\right]$ turn out to be the same analytic functions as for the models considered in Ref. [20][4], even though the $N = 2$ constructions are different[5]:

$$\begin{aligned} \Omega\begin{bmatrix}0\\1\end{bmatrix} &= \frac{1}{16}\left(\vartheta_3^8+\vartheta_4^8+14\,\vartheta_3^4\,\vartheta_4^4\right)\vartheta_3^6\,\vartheta_4^6\\ \Omega\begin{bmatrix}1\\0\end{bmatrix} &= -\frac{1}{16}\left(\vartheta_2^8+\vartheta_3^8+14\,\vartheta_2^4\,\vartheta_3^4\right)\vartheta_2^6\,\vartheta_3^6\\ \Omega\begin{bmatrix}1\\1\end{bmatrix} &= \frac{1}{16}\left(\vartheta_2^8+\vartheta_4^8-14\,\vartheta_2^4\,\vartheta_4^4\right)\vartheta_2^6\,\vartheta_4^6\,. \end{aligned} \tag{4.10}$$

The construction based on $\tilde\Phi$ with $N_V = N_H = 0$, under study here, has the same universality properties as the corresponding $N = 2$ models of Ref. [20]. The helicity supertrace B_2 is identical for all such models with $N_V = N_H$, irrespectively of whether the latter vanishes or not:

$$B_2\left(\tilde\Phi\right) = \frac{1}{\bar\eta^{24}} \sum_{(H^{\rm f},G^{\rm f})}{}' \, \frac{1}{2}\left(\Gamma_{2,2}^{\lambda=0}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix}\overline{\Omega}^{(0)}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix}+\Gamma_{2,2}^{\lambda=1}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix}\overline{\Omega}^{(1)}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix}\right), \tag{4.11}$$

where in this case

$$\begin{aligned} \Omega^{(0)}\begin{bmatrix}0\\1\end{bmatrix} &= \frac{1}{2}\left(\vartheta_3^4+\vartheta_4^4\right)\vartheta_3^8\,\vartheta_4^8\\ \Omega^{(0)}\begin{bmatrix}1\\0\end{bmatrix} &= -\frac{1}{2}\left(\vartheta_2^4+\vartheta_3^4\right)\vartheta_2^8\,\vartheta_3^8\\ \Omega^{(0)}\begin{bmatrix}1\\1\end{bmatrix} &= \frac{1}{2}\left(\vartheta_2^4-\vartheta_4^4\right)\vartheta_2^8\,\vartheta_4^8 \end{aligned} \tag{4.12}$$

and

$$\Omega^{(1)}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix} = (-)^{H^{\rm f}}\left(\omega\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix}\right)^{10}, \tag{4.13}$$

with

$$\omega\begin{bmatrix}0\\1\end{bmatrix} = \vartheta_3\,\vartheta_4\;,\quad \omega\begin{bmatrix}1\\0\end{bmatrix} = \vartheta_2\,\vartheta_3\;,\quad \omega\begin{bmatrix}1\\1\end{bmatrix} = \vartheta_2\,\vartheta_4\,. \tag{4.14}$$

[4]The parameter λ, which takes the values 0 or 1, determines the phases appearing in the modular transformations of the shifted lattice sums. These phases are complementary of those coming from the corresponding functions $\Omega\left[{H^{\rm f} \atop G^{\rm f}}\right]$ or $\Omega^{(0)}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ and $\Omega^{(1)}\left[{H^{\rm f} \atop G^{\rm f}}\right]$.

[5]This is not surprising since the $N = 2$ models under consideration are known to fall in a very restricted set of universality classes with respect to their elliptic genus [24].

The lattice sums $\Gamma_{2,2}^{\lambda=0}\left[{H^{\rm f}\atop G^{\rm f}}\right]$ and $\Gamma_{2,2}^{\lambda=1}\left[{H^{\rm f}\atop G^{\rm f}}\right]$ correspond to simply shifted lattices with projections $(-)^{m_1 G^{\rm f}}$ and $(-)^{(m_1+n_1)G^{\rm f}}$, respectively (the cases "I" and "X" of [24]).

In both constructions Φ and $\tilde\Phi$, the massless contribution to the B_2 vanishes for generic values of the moduli T and U, as it should for models where $N_V = N_H$. Owing to the $Z_2^{(\rm f)}$-translation on the two-torus, the $N=4$ supersymmetry is spontaneously broken. The analysis of these constructions is the same as for the models discussed in [20], to which we refer for the details. Here we simply recall that the construction based on Φ is not suitable for a comparison with the type II ground states: the region in the space of (discrete) Wilson lines which allows for an easy identification of the map between the moduli of the dual models is the one that corresponds to the construction based on $\tilde\Phi$. In the latter, and for the particular $Z_2^{(\rm f)}$-shift we have considered, $(-)^{m_1 G^{\rm f}+n_1 g_1}$, the mass of the two extra gravitinos is

$$m_{3/2}^2 = \frac{1}{4}\frac{\operatorname{Im} U}{\operatorname{Im} T}\,. \tag{4.15}$$

The $N=4$ supersymmetry is restored when $R_1 = \sqrt{\operatorname{Im} T/\operatorname{Im} U}$ is large. For large values of $\operatorname{Im} U/\operatorname{Im} T$ we recover instead a genuine $N=2$ non-freely-acting orbifold. For the specific directions of the shifts in the two-torus that we have chosen, there are lines in the (T,U)-plane along which two extra hypermultiplets appear in the massless spectrum together with two extra massless vectors leading to an $SU(2)$ enhancement of one of the $U(1)$'s of the torus.

4.2 The gravitational corrections

In order to obtain heterotic gravitational corrections analogous to those of the type II constructions (see Eqs. (2.6) and (3.4)), we must proceed as follows. Instead of considering the pure R^2 term, we must compute the one-loop corrections for a special combination of gravitational and helicity operators. The ordinary R^2-term correction is given by the genus-one amplitude of

$$Q_{\rm grav}^2 \equiv Q^2\, \overline{P}_{\rm grav}^2\,, \tag{4.16}$$

where Q stands again for the left-helicity operator, and $\overline{P}_{\rm grav}^2$ is the usual gravitational operator: when inserted in the one-loop vacuum amplitude, it acts as $\frac{-1}{2\pi i}\frac{\partial}{\partial\bar\tau}$ on $1/\operatorname{Im}\tau\,\bar\eta^2$; namely, it acts on the contribution of the two right-moving transverse space-time coordinates $\bar X^{\mu=3,4}$, *including their zero-modes.* This latter fact is responsible for the appearance of a non-holomorphic gravity-backreaction contribution, which ensures modular covariance but has no type II counterpart, as was discussed in [20]; the one-loop amplitude of the above operator is[6]

$$F_{\rm grav} \equiv \left\langle Q_{\rm grav}^2\right\rangle_{\rm genus-one} = -\frac{1}{12}\left(\overline{E}_2 - \frac{3}{\pi\operatorname{Im}\tau}\right)B_2\,, \tag{4.17}$$

[6]In general the heterotic one-loop amplitude of an operator of the form $Q^2\,\overline{P}^2$ reads: $\left\langle Q^2\,\overline{P}^2\right\rangle_{\rm genus-one} = \overline{P}^2\,B_2$, where, in the l.h.s., $\overline{P}^2$ acts as a differential operator on some specific factor of B_2.

where, for the model under consideration (constructed with $\tilde{\Phi}$), B_2 is given in Eq. (4.11). The massless contribution to $F_{\rm grav}$ is precisely the gravitational anomaly, $b_{\rm grav} = \frac{24-N_V+N_H}{12}$, which in the case at hand equals 2 ($B_2|_{\rm massless}$ vanishes), at generic points of the (T,U) moduli space.

The operator $\overline{P}^2_{\rm grav}$ is not suitable for comparison with the type II result, because it is not holomorphic and its amplitude is sensitive to the $N=2$ singularities occurring in the (T,U) plane: the corresponding beta-function (i.e. the gravitational anomaly) jumps along rational lines where $\Delta N_V \neq \Delta N_H$. We must therefore replace $\overline{P}^2_{\rm grav}$ with an appropriate holomorphic operator $\overline{P}'^2_{\rm grav}$ whose amplitude is regular everywhere in (T,U), at least in the model constructed with $\tilde{\Phi}$, which is the model we will be analysing in the following. To this purpose, we introduce two operators: $\overline{H}_{\rm tw}$ and $\overline{P}^2_{2,2}$.

The operator $\overline{H}_{\rm tw}$ acts, for any $(H^{\rm f},G^{\rm f})$-twisted sector of the orbifold, as a derivative $\frac{-1}{2\pi i}\frac{\partial}{\partial\bar\tau}$ on the factor $\bar\omega\left[{H^{\rm f}\atop G^{\rm f}}\right]/\operatorname{Im}\tau\,\bar\eta^4$, which contains the contribution of twisted coordinates (see Eq. (4.14)). After some straightforward algebra we obtain the amplitude:

$$\overline{H}_{\rm tw}\,B_2\left(\tilde{\Phi}\right) = -\frac{1}{24}\sum_{\lambda=0,1}\sum_{(H^{\rm f},G^{\rm f})}{}' \Gamma^{\lambda}_{2,2}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\left(\overline{E}_2 - \frac{3}{\pi\operatorname{Im}\tau} + \frac{1}{2}\overline{H}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\right)\frac{\overline{\Omega}^{(\lambda)}\left[{H^{\rm f}\atop G^{\rm f}}\right]}{\bar\eta^{24}}\,, \tag{4.18}$$

where we have introduced the modular-covariant functions

$$H\begin{bmatrix}h\\g\end{bmatrix} = \frac{12}{\pi i}\partial_\tau \log\frac{\vartheta\left[{1-h\atop 1-g}\right]}{\eta} = \begin{cases} \vartheta_3^4+\vartheta_4^4\,, & (h,g)=(0,1)\\ -\vartheta_2^4-\vartheta_3^4\,, & (h,g)=(1,0)\\ \vartheta_2^4-\vartheta_4^4\,, & (h,g)=(1,1)\end{cases} \tag{4.19}$$

of weight 2. In the model constructed with Φ, only a $\lambda=1$ term would appear in (4.18).

From expression (4.18) we observe that the insertion of $\overline{H}_{\rm tw}$ is covariant but not holomorphic. This latter property will allow for cancelling the non-holomorphic terms present when $\overline{P}^2_{\rm grav}$ and $\overline{P}^2_{2,2}$ are inserted in the vacuum amplitude, while keeping modular covariance. The beta-function coefficient of this operator, i.e. the constant term of $\overline{H}_{\rm tw}\,B_2\left(\tilde{\Phi}\right)$, vanishes for generic (T,U) while it jumps accross several special lines: $\Delta b\left(\overline{H}_{\rm tw}\right) = 2\,\Delta b_{\rm grav}$.

On the other hand, after insertion into the one-loop heterotic vacuum amplitude, $\overline{P}^2_{2,2}$ acts as $\frac{-1}{2\pi i}\frac{\partial}{\partial\bar\tau}$ on the modular-covariant factor of weight zero, $\operatorname{Im}\tau\,\Gamma^{\lambda}_{2,2}\left[{H^{\rm f}\atop G^{\rm f}}\right]$. This amounts to inserting the sum of the two right-moving lattice momenta $\bar p_1^2+\bar p_2^2$ of T^2, which correspond to the Cartan of the $U(1)$ factor. The amplitude $\left\langle Q^2\,\overline{P}^2_{2,2}\right\rangle$ therefore generates the corresponding gauge-coupling correction. To be more precise, we must in fact consider the integral $\int_{\mathcal F}\frac{d^2\tau}{\operatorname{Im}\tau}\left\langle Q^2\,\overline{P}^2_{2,2}\right\rangle\gamma(\tau,\bar\tau)$, where $\gamma(\tau,\bar\tau)$ is an appropriate modular-invariant infrared-regularizing function [26]. An integration by parts can be performed, which leads to vanishing boundary terms *all over the* (T,U) *plane*, irrespectively of the specific behaviour of the lattice sums across rational lines. This allows us to recast the above amplitude as:

$$\overline{P}^2_{2,2}\,B_2\left(\tilde{\Phi}\right) = \frac{1}{2}\sum_{\lambda=0,1}\sum_{(H^{\rm f},G^{\rm f})}{}'\left(\Gamma^{\lambda}_{2,2}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\left[\frac{1}{2\pi i}\frac{\partial}{\partial\bar\tau} - \frac{1}{2\pi\operatorname{Im}\tau}\right]\frac{\overline{\Omega}^{(\lambda)}\left[{H^{\rm f}\atop G^{\rm f}}\right]}{\bar\eta^{24}}\right) \tag{4.20}$$

(in the model constructed with Φ, only a $\lambda = 1$ term appears). The differential operator inside the brackets is covariant, since $\overline{\Omega}^{(\lambda)}\left[{}^{H^{\rm f}}_{G^{\rm f}}\right]\Big/\bar\eta^{24}$ has modular weight -2, but non-holomorphic. The beta-function coefficient $b(P_{2,2})$ (constant term in (4.20)) vanishes for generic (T, U) and its discontinuity at special lines turns out to be $\Delta b(P_{2,2}) = -12\,\Delta b_{\rm grav}$.

Given the operators $\overline{P}^2_{\rm grav}$, $\overline{H}_{\rm tw}$ and $\overline{P}^2_{2,2}$, there is a *unique* combination, which is holomorphic and whose beta-function coefficient is equal to the gravitational anomaly $b_{\rm grav} = 2$ *everywhere* in the moduli space (T, U):

$$\overline{P}'^2_{\rm grav} = \overline{P}^2_{\rm grav} - \frac{5}{4}\overline{H}_{\rm tw} - \frac{1}{8}\overline{P}^2_{2,2}\,. \tag{4.21}$$

Here we want to remark that, in some specific cases where $N_V = N_H \neq 0$, as for the models considered in [20], the operator $\overline{H}_{\rm tw}$ can be reexpressed through an integration by parts, as a combination of $\overline{P}^2_{2,2}$ and $\overline{P}^2_{\rm gauge}$, the gauge operator for the higher-level currents (here absent). In such cases, expression (4.21) becomes $\overline{P}^2_{\rm grav} + \frac{1}{12}\overline{P}^2_{2,2} + \frac{5}{3N_V}\overline{P}^2_{\rm gauge}$.

The amplitude $\left\langle Q'^2_{\rm grav}\right\rangle = \left\langle Q^2\,\overline{P}'^2_{\rm grav}\right\rangle$ at genus one reads:

$$\overline{P}'^2_{\rm grav}\, B_2\left(\tilde\Phi\right) = 2\sum_{(H^{\rm f},G^{\rm f})}{}' \,\Gamma^{\lambda=0}_{2,2}\left[{H^{\rm f}\atop G^{\rm f}}\right], \tag{4.22}$$

which is regular everywhere. Notice that the contribution of the $\lambda = 1$ term vanishes identically. This amplitude leads to the thresholds

$$\Delta_{\rm Het} = 2\int_{\mathcal F}\frac{d^2\tau}{{\rm Im}\,\tau}\left(\sum_{(H^{\rm f},G^{\rm f})}{}'\,\Gamma^{\lambda=0}_{2,2}\left[{H^{\rm f}\atop G^{\rm f}}\right] - 1\right), \tag{4.23}$$

which is valid only for the construction based on $\tilde\Phi$.

For the specific case in which the shift in $\Gamma^{\lambda=0}_{2,2}$ is due to a translation of momenta, $(-1)^{m_1 G^{\rm f}}$, we get:

$$\Delta_{\rm Het} = -2\log {\rm Im}\,T\left|\vartheta_4\left(T\right)\right|^4 - 2\log {\rm Im}\,U\left|\vartheta_2\left(U\right)\right|^4 + {\rm const.} \tag{4.24}$$

Finally the running of the coupling is given by

$$\begin{aligned}\frac{16\,\pi^2}{g^2_{\rm grav}\left(\mu^{({\rm Het})}\right)} &= 16\,\pi^2\kappa\,{\rm Im}\,S_{\rm Het} - 2\log {\rm Im}\,T\left|\vartheta_4\left(T\right)\right|^4 - 2\log {\rm Im}\,U\left|\vartheta_2\left(U\right)\right|^4\\ &\quad +4\log\frac{M^{({\rm Het})}}{\mu^{({\rm Het})}} + {\rm const.}\,,\end{aligned} \tag{4.25}$$

where $S_{\rm Het}$ is the heterotic dilaton–axion field and

$$\kappa\ {\rm Im}\,S_{\rm Het} = \frac{1}{g^2_{\rm Het}}\,. \tag{4.26}$$

In the limit where $(1/\operatorname{Im} T,\, \operatorname{Im} U) \to 0$, $N = 4$ supersymmetry is restored with the following behaviour:

$$\frac{16\,\pi^2}{g^2_{\rm grav}\,(\mu^{(\rm Het)})} = 16\,\pi^2 \kappa \operatorname{Im} S_{\rm Het} - 2\log\, \operatorname{Im} T + 2\log\, \operatorname{Im} U\,. \tag{4.27}$$

The role of the normalization factor κ will be specified in the next section where the gravitational corrections of the heterotic $\tilde{\Phi}$ construction and that of the two type II ground states will be compared.

5 Comparison of the three duals

In this section we would like to test the triality relation between all three freely acting orbifolds we have considered, through the analysis of the "gravitational" corrections. First of all we observe that the operator $Q'^2_{\rm grav} = Q^2\,\overline{P}'^2_{\rm grav}$ with $\overline{P}'^2_{\rm grav}$ given in (4.21), coincides, on type II side, with the operator $Q^{\rm II2}_{\rm grav} = 2Q^2\overline{Q}^2$ we considered in Sections 2 and 3. Indeed, owing to the absence of perturbative Ramond–Ramond charges, the contribution of the dual of $\overline{P}^2_{\rm grav}$ vanishes; because of the symmetry between left- and right-movers on the world-sheet, there is no need for us to introduce an operator such as $\overline{H}_{\rm tw}$, the insertion of $Q^{\rm II2}_{\rm grav}$ being automatically holomorphic. The duality among the three orbifolds requires the identification of one of the three perturbative vector multiplet moduli of the type IIA symmetric orbifold, with the dual of the field $S_{\rm Het}$, the dilaton–axion field of the heterotic theory, and with the inverse of $S_{\rm As}$, the dilaton–axion field of the type II asymmetric theory. Modulo $SL(2,Z)$ transformations, such modulus can be indifferently any one of the three T^i, $i = 1,2,3$. For definiteness, we will choose T^1, as was anticipated in (3.6). In order to see what the precise duality map is, we consider all three models in their "$N = 4$ phase". In this limit, the heterotic amplitude $\left\langle Q'^2_{\rm grav}\right\rangle$ is expected to receive contributions from genus zero only, while in the type II $\left\langle Q^{\rm II2}_{\rm grav}\right\rangle$ should vanish in the asymmetric orbifold and depend on one complex modulus only in the type IIA symmetric one. This behaviour can be checked by taking the appropriate $N = 4$ limits in the three models (see Eqs. (2.8), (2.9), (3.5) and (4.15), (4.25)). The other surviving contributions, with a logarithmic dependence on the other moduli, are in fact infrared artefacts due to an accumulation of massless states, which can be lifted, in all the three models, by switching on an infrared cut-off larger than the massive gravitinos, as we previously discussed. By comparing Eq. (2.9) with the genus-zero contribution in Eq. (4.27) $(16\,\pi^2\kappa\, \operatorname{Im} S_{\rm Het})$, we obtain

$$T^1 = \frac{\kappa}{2}\tau_{S_{\rm Het}} = 2\pi\kappa S_{\rm Het}\,. \tag{5.1}$$

Here we have introduced the field $\tau_S \equiv 4\pi S$, the actual "modular" parameter of the Montonen–Olive S-duality transformations. Since, after an $SL(2,Z)_T$ inversion $T \to -1/T$, the model becomes symmetric under permutations of the fields $\tau_{S_{\rm Het}}, T$ and U, the behaviour of the effective coupling constant, for large values of the three moduli, must be symmetric as well; this requirement forces us to fix $\kappa = 2$. This normalization of the coupling (4.26), which differs by a factor 2 from the usual tree-level coupling (corresponding to $\kappa = 1$), is required also for a correct interpretation in terms of instanton contributions (see below). In the opposite limit, $T^1 \to 0$, the T^1-dependent contribution vanishes (up to an irrelevant logarithmic term in $-1/T^1$). This is consistent with the identification of $-1/T^1$ with $\tau_{S_{\rm As}} = 4\pi S_{\rm As}$, and implies in particular that *the type II asymmetric orbifold is the strong coupling limit of the heterotic*:

$$\tau_{S_{\rm As}} = -\frac{1}{\tau_{S_{\rm Het}}}\,. \tag{5.2}$$

In order to test the above duality relations, we consider the "$N=2$ phase" of the various theories under consideration, where the dependence on the other moduli, generating from genus one for all three models, remains. The part of the gravitational amplitude that depends on the perturbative moduli is indeed the same in the three models, provided we identify the moduli $(T_{\rm As}, U_{\rm As})$ with (T^2, T^3) and (T, U). Through the duality map, we therefore learn that, similarly to the symmetric type IIA orbifold, the heterotic model possesses an $N=8$ supersymmetry that is broken spontaneously. This breaking on the heterotic side is non-perturbative: the Higgs field whose vev is the order parameter for the spontaneous breaking of the $N=8$ supersymmetry to $N=4$ is the dilaton $S_{\rm Het}$, and the $N=8$ supersymmetry can be restored at the strong coupling limit. The further breaking of the supersymmetry from $N=4$ to $N=2$ is realized spontaneously in a perturbative way, by using as Higgs fields the moduli T and U. The full, perturbative and non-perturbative, correction to the effective coupling constant of the gravitational term considered here is given by the type IIA result, Eq. (2.6). In heterotic string variables, we have:

$$\begin{aligned}\frac{16\,\pi^2}{g^2_{\rm grav}\,(\mu)} &= -2\log\,{\rm Im}\,\tau_{S_{\rm Het}}\left|\vartheta_2\left(\tau_{S_{\rm Het}}\right)\right|^4\\ &\quad -2\log\,{\rm Im}\,T\left|\vartheta_4\left(T\right)\right|^4 - 2\log\,{\rm Im}\,U\left|\vartheta_2\left(U\right)\right|^4\\ &\quad +6\log\frac{M_{\rm Planck}}{\mu} + {\rm const.}\,,\end{aligned} \tag{5.3}$$

where we have expressed the infrared running in terms of the duality-invariant Planck mass and the physical cut-off μ, related to the various string scales by:

$$\frac{M_{\rm Planck}}{\mu} = \frac{M^{\rm (Het)}}{\mu^{\rm (Het)}} = \frac{M^{\rm (IIA)}}{\mu^{\rm (IIA)}} = \frac{M^{\rm (As)}}{\mu^{\rm (As)}}\,. \tag{5.4}$$

From expression (5.3) we can easily read off the instanton numbers k, given by the powers of $q \equiv \exp 2\pi i \tau_{S_{\rm Het}}$ in the expansion of the first term. We obtain $k \in \frac{\kappa N}{2}$, which for $\kappa = 2$ becomes, as expected, $k \in N$.

6 The prepotential

6.1 The one-loop result

The perturbative prepotential can be easily computed from the heterotic side. Owing to the $N=2$ supersymmetry, it receives no perturbative corrections beyond one loop. The tree-level contribution is determined by the geometric properties of the vector manifold, and is the same for both the constructions based on Φ and on $\tilde{\Phi}$ [33]:

$$h^{(0)} = -\frac{i}{2\pi}\tau_{S_{\rm Het}} T U\,. \tag{6.1}$$

The genus-one correction, on the other hand, is different in the two constructions and, moreover, it depends on the choice of shift vectors in the two-torus. Here, we will concentrate on the choices made previously for these shift vectors. The one-loop corrections to the prepotential turn out to satisfy second-order differential equations. These equations are obtained by properly treating the universal part of the gauge corrections in models with spontaneously broken supersymmetries, thereby generalizing the approach of [27, 30, 31]. For the models based on Φ the correction $h^{(1)}$ solves

$$\mathrm{Re}\left(-\frac{1}{2T_2U_2}\left(1-iU_2\frac{\partial}{\partial U}\right)\left(1-iT_2\frac{\partial}{\partial T}\right)h^{(1)}\right) = \frac{1}{64\pi^3}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{(h,g)}{}' \,\Gamma_{2,2}^{\lambda=1}\begin{bmatrix}h\\g\end{bmatrix}\left(i\frac{d}{d\bar\tau}+\frac{1}{\tau_2}\right)\frac{\overline{\Omega}\left[{}^h_g\right]}{\bar\eta^{24}}\,, \tag{6.2}$$

while for models based on $\tilde\Phi$ it solves

$$\mathrm{Re}\left(-\frac{1}{2T_2U_2}\left(1-iU_2\frac{\partial}{\partial U}\right)\left(1-iT_2\frac{\partial}{\partial T}\right)\tilde h^{(1)}\right) = \frac{1}{64\pi^3}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\frac{1}{2}\sum_{\lambda=0,1}\sum_{(h,g)}{}' \,\Gamma_{2,2}^{\lambda}\begin{bmatrix}h\\g\end{bmatrix}\left(i\frac{d}{d\bar\tau}+\frac{1}{\tau_2}\right)\frac{\overline{\Omega}^{(\lambda)}\left[{}^h_g\right]}{\bar\eta^{24}}\,. \tag{6.3}$$

Notice that, in contrast to the universal corrections [24], the r.h.s. of the above equations has singularities across lines in the moduli space. Integrals of this kind and analysis of the singularities have been performed in several papers. We will not present the general result here but give instead the answer for the prepotential. It is important to observe that the above equations define $h^{(1)}$ up to irrelevant linear and quadratic terms as well as cubic terms such as T^2U or U^2T. These ambiguities can be resolved by looking at the ordinary gravitational threshold corrections [24, 30]:

$$\Delta_{\mathrm{grav}}\left(\Phi\right) = -\frac{1}{12}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\left(\sum_{(h,g)}{}'\,\Gamma_{2,2}^{\lambda=1}\begin{bmatrix}h\\g\end{bmatrix}\widehat{E}_2\frac{\overline{\Omega}\left[{}^h_g\right]}{\bar\eta^{24}}+12\,b_{\mathrm{grav}}\right) \tag{6.4}$$

or

$$\Delta_{\mathrm{grav}}\left(\tilde\Phi\right) = -\frac{1}{12}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\left(\frac{1}{2}\sum_{\lambda=0,1}\sum_{(h,g)}{}'\,\Gamma_{2,2}^{\lambda}\begin{bmatrix}h\\g\end{bmatrix}\widehat{E}_2\frac{\overline{\Omega}^{(\lambda)}\left[{}^h_g\right]}{\bar\eta^{24}}+12\,b_{\mathrm{grav}}\right)\,. \tag{6.5}$$

We would like to stress at this point that Eqs. (6.2) and (6.4) (resp. (6.3) and (6.5)) hold for heterotic constructions of the kind Φ (resp. $\tilde\Phi$) with $N_V = N_H \neq 0$, as those presented in [20]. Therefore, our results for the perturbative prepotential $h^{(1)}$ or $\tilde h^{(1)}$ given below (Eqs. (6.6) and (6.7)) are valid for these more general models. This is again due to the fact that the heterotic ground states under consideration fall into the same elliptic-genus universality class, irrespectively of the value of $N_V = N_H$. Non-perturbative contributions, however, depend on the number of vector multiplets and hypermultiplets (through, for example, the instanton numbers), and only the case $N_V = N_H = 0$ is analysed in the following.

After some lengthy algebra, we can solve the above equations and obtain the one-loop prepotential for models based on the construction Φ:

$$\begin{aligned} h^{(1)}(T,U) &= -\frac{1}{(2\pi)^4}\left(\mathcal{L}_c(T,U)+\mathcal{L}_a(T,U)+\mathcal{L}_b(T,U)\right)-\frac{i}{8\pi}T^2U\ , \quad \text{for } T_2>U_2 \\ &= -\frac{1}{(2\pi)^4}\left(\mathcal{L}_c(U,T)+\mathcal{L}_a(T,U)+\mathcal{L}_b(U,T)\right)-\frac{i}{8\pi}TU^2\ , \quad \text{for } T_2<U_2\,. \end{aligned} \tag{6.6}$$

The functions $\mathcal{L}_{c,a,b}(T,U)$ are given in the appendix; $\mathcal{L}_{c,b}(T,U)$ have a branch along $T=U$, where $\Delta B_2|_{\text{massless}}=\Delta N_V-\Delta N_H=-14$. For models based on the construction $\tilde{\Phi}$, we obtain:

$$\begin{aligned} \tilde{h}^{(1)}(T,U) &= -\frac{1}{(2\pi)^4}\frac{1}{2}\Bigg(-\mathcal{L}_c^{(0)}(T,U)+\mathcal{L}_a^{(0)}\left(\frac{T}{2},2U\right)+\mathcal{L}_b^{(0)}\left(\frac{T}{2},U\right) \\ &\qquad +\mathcal{L}_c^{(1)}(T,U)+\mathcal{L}_a^{(1)}(T,U)+\mathcal{L}_b^{(1)}(T,U)\Bigg)\,, \end{aligned} \tag{6.7}$$

where the functions $\mathcal{L}^{(\lambda)}_{c,a,b}(T,U)$ are as displayed in the appendix. In this case there is no branch at $T=U$, where now $\Delta B_2|_{\text{massless}}=0$, and (6.7) is thus valid for any T and U. This makes the monodromy trivial around $T=U$. Remember, however, that in the models at hand, where the two-torus lattices are shifted, the target-space duality group is only a subgroup of $SL(2,Z)_T\times SL(2,Z)_U\times Z_2^{T\leftrightarrow U}$. In particular $T\to -1/T$ is not a symmetry, and the line $T=U$ is not equivalent to the line $-1/T=U$. The latter, where $\Delta B_2|_{\text{massless}}=2$, is a branch for $\tilde{h}^{(1)}(T,U)$, although this is not straightforward from expression (6.7) – a Poisson resummation is needed. Observe also the absence of cubic terms in $\tilde{h}^{(1)}$. Such terms are present in generic models with spontaneously broken supersymmetry, as those studied in [24], for which the corrections to the prepotential can be computed in a similar way. Cubic terms vanish in general when the intersection form of the underlying Calabi–Yau manifold of the type II dual becomes trivial[7], as in our case. On the other hand, the absence of constant term both in $h^{(1)}$ and $\tilde{h}^{(1)}$ reflects the vanishing of the Euler characteristic $\chi=2\left(h^{1,1}-h^{2,1}\right)$.

6.2 Non-perturbative contributions

Let us now try to go beyond the above perturbative result. We will concentrate on the construction based on $\tilde{\Phi}$, for which we have been able to determine the precise duality map. For the prepotential, however, we have no exact type II result that could be used to obtain the heterotic non-perturbative contributions directly. Nevertheless, the structure of the type II model is useful to infer at least part of these contributions.

The type II $(1,1)$ ground state of Section 2 is symmetric under permutations of the three moduli $\{T^1,-1/T^2,T^3\}$. The heterotic dual should therefore possess the same property with respect to $\{\tau_{S_{\text{Het}}},-1/T,U\}$. In fact, invariance under $-1/T\leftrightarrow U$ is a residual target-space

[7] This implies that the heterotic construction based on Φ, in which such cubic terms are present (see Eq. (6.6)), cannot be dual to a type II, $Z_2\times Z_2$ symmetric orbifold, for which the intersection matrix is trivial.

duality symmetry of the shifted lattices we are considering, and Eq. (6.3) is indeed invariant owing to the covariance property[8] of the perturbative result $\tilde{h}^{(1)}(T,U)$ given in (6.7):

$$\tilde{h}^{(1)}\left(-\frac{1}{U},-\frac{1}{T}\right)=\frac{1}{T^2\,U^2}\tilde{h}^{(1)}(T,U) \tag{6.8}$$

(note, however, that $\tilde{h}^{(1)}(U,T)\neq\tilde{h}^{(1)}(T,U)$, owing to the breakdown of the $T\leftrightarrow U$ symmetry).

In order to promote the above $-1/T\leftrightarrow U$ permutation symmetry to the level of the three moduli, we must demand the following covariance properties for the full prepotential $\tilde{h}\left(\tau_{S_{\rm Het}},T,U\right)$:

$$\begin{aligned}\tilde{h}\left(\tau_{S_{\rm Het}},-\frac{1}{U},-\frac{1}{T}\right) &= \frac{1}{T^2\,U^2}\tilde{h}\left(\tau_{S_{\rm Het}},T,U\right)\\ \tilde{h}\left(-\frac{1}{T},-\frac{1}{\tau_{S_{\rm Het}}},U\right) &= \frac{1}{\tau_{S_{\rm Het}}^2\,T^2}\tilde{h}\left(\tau_{S_{\rm Het}},T,U\right)\\ \tilde{h}\left(U,T,\tau_{S_{\rm Het}}\right) &= \tilde{h}\left(\tau_{S_{\rm Het}},T,U\right)\ ,\end{aligned} \tag{6.9}$$

which are fulfilled by the tree-level contribution (6.1). We must therefore add two more terms to $\tilde{h}^{(1)}(T,U)$:

$$\tilde{h}\left(\tau_{S_{\rm Het}},T,U\right)=h^{(0)}+\tilde{h}^{(1)}(T,U)+\tilde{h}^{(1)}\left(T,\tau_{S_{\rm Het}}\right)+T^2\,U^2\,\tilde{h}^{(1)}\left(-\frac{1}{U},\tau_{S_{\rm Het}}\right)\,. \tag{6.10}$$

These extra terms account for non-perturbative corrections and are exponentially suppressed at large $S_{\rm Het}$.

The above covariant symmetrization, which we have been advocating in order to determine non-perturbative corrections to the prepotential, does not exclude the possibility of having also a series of exponentially suppressed terms with, in the arguments, covariant-symmetric functions of $\tau_{S_{\rm Het}}, T$ and U, in the sense of (6.9). Unfortunately, we have no reason to rule out such non-perturbative contributions, nor a method for computing them from the type II symmetric or asymmetric ground states.

7 Conclusions

In this work we studied $N=2$ superstring ground states obtained from the heterotic and type II ten-dimensional superstrings through freely acting (asymmetric) orbifold compactification. The massless spectrum of all these models is the same. Besides the $N=2$ supergravity multiplet, there are three vector multiplets and four hypermultiplets. The construction presented here extends the work of Ref. [20], where we analysed heterotic/type II duals with $N=2$ supersymmetries that have $3+N_V$ vector multiplets and $4+N_H$ hypermultiplets with $N_V=N_H\neq 0$. Here, we presented three different $N=2$ models with $N_V=N_H=0$

[8]The absence of pure power-like terms in (6.8) reflects again the vanishing of the Euler characteristic and intersection form of the dual symmetric type II construction.

and we verified the non-perturbative duality conjecture between them. The models we have considered are the following: (i) the heterotic construction with $N = (2, 0)$ supersymmetry, based on characters $\tilde{\Phi}$ of the $c = (0, 16)$ conformal block, Eq. (4.8) (this choice is equivalent to a particular choice of discrete Wilson lines, reducing the number of the vectors to three and that of the hypers to four); (ii) the type IIA symmetric construction with $N = (1, 1)$ supersymmetry, which corresponds to a self-mirror Calabi–Yau compactification with Hodge numbers $h^{1,1} = h^{2,1} = 3$; (iii) the type II asymmetric $N = (2, 0)$ freely acting orbifold compactification, where the initial $N = (4, 4)$ supersymmetry is spontaneously broken to $N = (2, 0)$.

The equivalence of the heterotic $N = (2, 0)$, type IIA $N = (1, 1)$ and $N = (2, 0)$ was verified for the corrections of a modified gravitational and gauge combination associated with the operator $\overline{P}'^{2}_{\rm grav}$ introduced in Section 4. This operator has the property of being regular in the entire (T, U) moduli space.

In the duality relations between the constructions described above, the heterotic vector moduli $(\tau_{S_{\rm Het}}, T, U)$ are mapped to the three $h^{1,1}$ moduli (T^1, T^2, T^3) of the symmetric type II, as well as to the moduli of the asymmetric type II $(\tilde{\tau}_{S_{\rm As}}, T_{\rm As}, U_{\rm As})$, where $\tilde{\tau}_{S_{\rm As}} = -1/\tau_{S_{\rm As}}$ is the inverse of the asymmetric type II dilaton. Thus, there is a weak–strong coupling S-duality relation between the heterotic and the asymmetric type II ground state, $\tau_{S_{\rm Het}} = -1/\tau_{S_{\rm As}}$. In all above duals there is a (non-)perturbative restoration of $N = 8$ and $N = 4$ supersymmetry in some specific limits of the three moduli, which is in agreement with the duality maps.

By using these duality maps, we found that *the type II corrections provide the complete, perturbative and non-perturbative, heterotic corrections*, as was also the case for all $N = 2$ models with $N_V = N_H$ constructed in Ref. [20]. This remarkable property is due to the universality of the $N = 2$ sector in the heterotic orbifold and of the corresponding $N = (2, 2)$ sectors in the symmetric and asymmetric type II orbifolds. We obtained in this way the full gravitational heterotic corrections. These contain instanton corrections, $n_k \exp 2k\pi i \tau_{S_{\rm Het}}$, which are due to the Euclidean five-brane wrapped around the six-dimensional internal space; they depend only on $\tau_{S_{\rm Het}}$ and not on the other moduli. The explicit expressions for these corrections are given in Eq. (5.3). The Olive–Montonen duality group is a $\Gamma(2)$ subgroup of $SL(2, Z)_{\tau_{S_{\rm Het}}}$.

Finally, we computed the perturbative and part of the non-perturbative corrections to the prepotential for the heterotic ground state. The perturbative result is actually valid beyond the models presented in this paper, and covers more general situations, with $N_V = N_H \neq 0$ such as those of Ref. [20]. The non-perturbative piece was reached by using the triality symmetry between the three moduli of the type II symmetric model $T^1 \leftrightarrow -1/T^2 \leftrightarrow T^3$. The requirement of (partial) restoration of the $(N = 4)$ $N = 8$ supersymmetry in special limits of the vector moduli turned to be too weak to rule out or determine extra potential non-perturbative terms in the prepotential.

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Appendix: The trilogarithm series

We quote here the explicit solution for the series $\mathcal{L}_{c,a,b}$ and $\mathcal{L}^{(\lambda)}_{c,a,b}(T,U)$ appearing in the expressions of the one-loop prepotential (see Section 6). We have:

$$\begin{aligned}
\mathcal{L}_c(T,U) &= \mathcal{L}i_3\left(e^{2\pi i(T-U)}\right) \\
&+c_0 \sum_{k>0} \left(2\mathcal{L}i_3\left(e^{4\pi i Tk}\right) - \mathcal{L}i_3\left(e^{2\pi i Tk}\right)\right) \\
&+c_0 \sum_{\ell>0} \left(2\mathcal{L}i_3\left(e^{4\pi i U\ell}\right) - \mathcal{L}i_3\left(e^{2\pi i U\ell}\right)\right) \\
&+\sum_{k,\ell>0} \Big(- c_{k\ell}\, \mathcal{L}i_3\left(e^{2\pi i(Tk+U\ell)}\right) \\
&\qquad +2\, c_{4k\ell}\, \mathcal{L}i_3\left(e^{2\pi i(2Tk+2U\ell)}\right) \\
&\qquad +2\, c_{4k\ell-2k-2\ell+1}\mathcal{L}i_3\left(e^{2\pi i(T(2k-1)+U(2\ell-1))}\right)\Big)
\end{aligned} \tag{A. 1}$$

$$\begin{aligned}
\mathcal{L}_a(T,U) &= \sum_{k,\ell>0} \left(a_{4k\ell-3k-3\ell+2}\, \mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}(4k-3)+\frac{U}{2}(4\ell-3)\right)}\right)\right. \\
&\qquad \left. +a_{4k\ell-k-\ell}\, \mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}(4k-1)+\frac{U}{2}(4\ell-1)\right)}\right)\right)
\end{aligned} \tag{A. 2}$$

$$\begin{aligned}
\mathcal{L}_b(T,U) &= b_{-1}\mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}-\frac{U}{2}\right)}\right) \\
&+\sum_{k,\ell>0} \left(b_{4k\ell-k-3\ell}\, \mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}(4k-3)+\frac{U}{2}(4\ell-1)\right)}\right)\right. \\
&\qquad \left. +b_{4k\ell-3k-\ell}\, \mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}(4k-1)+\frac{U}{2}(4\ell-3)\right)}\right)\right),
\end{aligned} \tag{A. 3}$$

where the coefficients in the above expansions are defined through

$$\begin{aligned}
\frac{\Omega\left[{0 \atop 1}\right]}{\eta^{24}} &= \frac{1}{q} + \sum_{n\geq 0} c_n\, q^n \\
\frac{\Omega\left[{1 \atop 0}\right] + \Omega\left[{1 \atop 1}\right]}{\eta^{24}} &= \sum_{n\geq 0} a_n\, q^{n+\frac{1}{4}} \\
\frac{\Omega\left[{1 \atop 0}\right] - \Omega\left[{1 \atop 1}\right]}{\eta^{24}} &= \sum_{n\geq -1} b_n\, q^{n+\frac{3}{4}}\,.
\end{aligned} \tag{A. 4}$$

We also have

$$\begin{aligned}\mathcal{L}_c^{(0)}(T,U) &= \mathcal{L}i_3\left(e^{2\pi i(T-U)}\right)\\ &\quad + c_0^{(0)}\sum_{k>0}\left(\mathcal{L}i_3\left(e^{2\pi iTk}\right)+\mathcal{L}i_3\left(e^{2\pi iUk}\right)\right)\\ &\quad + \sum_{k,\ell>0} c_{k\ell}^{(0)}\,\mathcal{L}i_3\left(e^{2\pi i(Tk+U\ell)}\right) \qquad \text{(A. 5)}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_a^{(0)}\left(\frac{T}{2},2U\right) &= a_0^{(0)}\sum_{k>0}\left(\mathcal{L}i_3\left(e^{\pi iTk}\right)+\mathcal{L}i_3\left(e^{4\pi iUk}\right)\right)\\ &\quad + \sum_{k,\ell>0} a_{k\ell}^{(0)}\,\mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}k+2U\ell\right)}\right) \qquad \text{(A. 6)}\end{aligned}$$

$$\begin{aligned}\mathcal{L}_b^{(0)}\left(\frac{T}{2},U\right) &= 288\sum_{k>0}\left(\mathcal{L}i_3\left(e^{2\pi iTk}\right)+\mathcal{L}i_3\left(e^{4\pi iUk}\right)\right)\\ &\quad + \sum_{k,\ell>0}\left(\left(2\,c_{2k\ell}^{(0)}-a_{2k\ell}^{(0)}\right)\mathcal{L}i_3\left(e^{2\pi i(Tk+2U\ell)}\right)\right.\\ &\qquad\left. + b_{2k\ell-k-\ell}^{(0)}\,\mathcal{L}i_3\left(e^{2\pi i\left(\frac{T}{2}(2k-1)+U(2\ell-1)\right)}\right)\right), \qquad \text{(A. 7)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_c^{(1)}(T,U) &= \mathcal{L}_{c^{(1)}}(T,U) &\text{(A. 8)}\\ \mathcal{L}_a^{(1)}(T,U) &= \mathcal{L}_{a^{(1)}}(T,U) &\text{(A. 9)}\\ \mathcal{L}_b^{(1)}(T,U) &= \mathcal{L}_{b^{(1)}}(T,U)\,, &\text{(A. 10)}\\ & &\text{(A. 11)}\end{aligned}$$

where $\mathcal{L}_{c^{(1)},a^{(1)},b^{(1)}}(T,U)$ are displayed in (A. 1)–(A. 3), and $c_n^{(\lambda)}$, $a_n^{(\lambda)}$ and $b_n^{(\lambda)}$ are given by

$$\begin{aligned}\frac{\Omega^{(\lambda)}\left[{0 \atop 1}\right]}{\eta^{24}} &= \frac{1}{q}+\sum_{n\geq 0} c_n^{(\lambda)}\,q^n\\ \frac{\Omega^{(\lambda)}\left[{1 \atop 0}\right]+\Omega^{(\lambda)}\left[{1 \atop 1}\right]}{\eta^{24}} &= \sum_{n\geq 0} a_n^{(\lambda)}\,q^{n+\frac{\lambda}{4}} \qquad \text{(A. 12)}\\ \frac{\Omega^{(\lambda)}\left[{1 \atop 0}\right]-\Omega^{(\lambda)}\left[{1 \atop 1}\right]}{\eta^{24}} &= \sum_{n\geq 0} b_n^{(\lambda)}\,q^{n+\frac{\lambda+2}{4}}\,.\end{aligned}$$

We finally recall that the non-trivial monodromy properties of the above functions are due to the following connexion formula for the trilogarithm:

$$\mathcal{L}i_3\left(e^x\right)=\mathcal{L}i_3\left(e^{-x}\right)+\frac{\pi^2}{3}x-\frac{i\pi}{2}x^2-\frac{1}{6}x^3\,,\quad \text{for}\quad \mathrm{Re}\,x\geq 0\,.$$

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