EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CERN — PS DIVISION

CERN/PS 98-061 (DI)

THICK SCATTERERS SEEN THROUGH THE TWISS FUNCTIONS

Andrew Maier*

Abstract

A treatment of multiple scattering for thick and thin scatterers is derived using the σ-formalism which is then used to give a statistical description of the Twiss functions with the scattering included. Full account is taken of the geometric correlation which occurs between the scattering angle and displacement of the scattered particle. The formalism is first derived for uncoupled beams, but is then extended to take coupling into account. Excellent agreement is demonstrated with Monte-Carlo data. The Twiss-Scatterer relations presented make it possible to include arbitrarily, thick scatterers as elements in accelerator codes.

Geneva, Switzerland
31st October1998

*A. Maier is currently a Ph. D student in the EP Division, CERN, working on the **P**roton-**I**on **M**edical **M**achine **S**tudy (PIMMS) hosted by the PS Division, CERN

Introduction

Multiple coulomb scattering can be an important source of emittance blowup and betatron mismatch when considering stripping foils, semi-destructive monitors, beam scrapers and specialist applications such as hadron therapy. The thick-scatterer formulae derived in the present paper give a description of the beam envelope and the emittances after scattering by a uniform, isotropic absorber of arbitrary thickness. The derivation exploits the statistical description of the beam given by the well-known σ-matrix and the final expressions are written in terms of Twiss beam parameters. The modified Twiss parameters apply to the beam envelope and emittances and cannot be used for single particle tracking, since the concept of betatron phase advance for an individual particle is lost when crossing the scatterer. The more usual thin-scatterer formulae are shown to be consistent with the thick scatterer results and the thick scatterer formulae are shown to be in excellent agreement with Monte-Carlo tracking. Indications are also given for finding the energy loss and calculating emittance growth from the betatron mismatch caused by scattering.

The σ-formalism

The σ-matrix formalism is briefly reviewed following Buon [1], where the σ-matrix of the particles in a beam with transverse phase-space co-ordinates $\vec{y}$ is defined as:

$$\sigma = \langle\vec{y}\rangle\vec{y}^{\mathsf{T}} = \left\langle \begin{pmatrix} x \\ x' \\ z \\ z' \end{pmatrix} \begin{pmatrix} x & x' & z & z' \end{pmatrix} \right\rangle = \begin{pmatrix} \langle xx\rangle & \langle x'x\rangle & \langle zx\rangle & \langle z'x\rangle \\ \langle xx'\rangle & \langle x'x'\rangle & \langle zx'\rangle & \langle z'x'\rangle \\ \langle xz\rangle & \langle x'z\rangle & \langle zz\rangle & \langle z'z\rangle \\ \langle xz'\rangle & \langle x'z'\rangle & \langle zz'\rangle & \langle z'z'\rangle \end{pmatrix} \quad (1)$$

where the $\langle yy\rangle$ and $\langle yy'\rangle$ denotes the variance and the co-variance of the phase-space co-ordinates of the particle beam.

A linear transformation $\mathbf{M}$ which transfers an initial vector $\vec{y}_1$, so that,

$$\vec{y}_2 = \mathbf{M}\vec{y}_1$$

has the following effect on the σ-matrix:

$$\sigma_2 = \langle\vec{y}_2\vec{y}_2^{\mathsf{T}}\rangle = \langle\mathbf{M}\vec{y}_1(\mathbf{M}\vec{y}_1)^{\mathsf{T}}\rangle = \mathbf{M}\underbrace{\langle\vec{y}_1\vec{y}_1^{\mathsf{T}}\rangle}_{\sigma_1}\mathbf{M}^{\mathsf{T}}$$

and reveals that the σ-matrix transfers via

$$\sigma_2 = \mathbf{M}\sigma_1\mathbf{M}^{\mathsf{T}}. \quad (2)$$

Thus, the knowledge of the σ-matrix at one point in a lattice is sufficient to calculate the σ-matrix at any other point, provided $\mathbf{M}$ is known.

For an uncoupled beam, the co-variance terms between the planes disappear, so that

$$\sigma_{uc} = \begin{pmatrix} \langle xx \rangle & \langle x'x \rangle & 0 & 0 \\ \langle xx' \rangle & \langle x'x' \rangle & 0 & 0 \\ 0 & 0 & \langle zz \rangle & \langle z'z \rangle \\ 0 & 0 & \langle zz' \rangle & \langle z'z' \rangle \end{pmatrix}.$$

It is possible to link the elements of the σ-matrix with the Courant and Snyder invariants. Using the definition of the statistical emittance†,

$$E_x = \pi\sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2};\ E_z = \pi\sqrt{\langle z^2 \rangle \langle z'^2 \rangle - \langle zz' \rangle^2} \tag{3}$$

where α and β have their usual meanings as Twiss parameters and $\gamma = (1+\alpha^2)/\beta$, σ_{uc} can be written as

$$\sigma_{uc} = \begin{pmatrix} \langle xx \rangle & \langle x'x \rangle & 0 & 0 \\ \langle xx' \rangle & \langle x'x' \rangle & 0 & 0 \\ 0 & 0 & \langle zz \rangle & \langle zz' \rangle \\ 0 & 0 & \langle z'z \rangle & \langle z'z' \rangle \end{pmatrix} = \frac{1}{\pi}\begin{pmatrix} E_x\beta_x & -E_x\alpha_x & 0 & 0 \\ -E_x\alpha_x & E_x\gamma_x & 0 & 0 \\ 0 & 0 & E_z\beta_z & -E_z\alpha_z \\ 0 & 0 & -E_z\alpha_z & E_z\gamma_z \end{pmatrix}. \tag{4}$$

Thick–scatterer transformations for an uncoupled beam

Consider a particle with co-ordinates $\vec{y}_0 = (y_0, y'_0)$ at the entry to a scatterer (see Figure 1). Multiple scattering will cause the particle to deviate from its original path. Instead of exiting the scatterer of thickness L at co-ordinates (y_1, y'_1) the path will be perturbed and the particle will exit at the new co-ordinates (y_2, y'_2). Compared to the unperturbed path the particle will have received an additional angular deflection θ_s and will have undergone a displacement δy.

It is intuitively obvious that there is a correlation between θ_s and δy, since a particle which has received a large kick has also a higher probability of exiting the absorber with a larger displacement. This correlation is entirely geometric in origin [2] and can be calculated to be,

$$\langle \theta_s \delta y \rangle = \frac{L}{2}\theta_s^2. \tag{5}$$

†Emittance is here defined as an area in phase space, i. e. π is included inside E_x and E_z

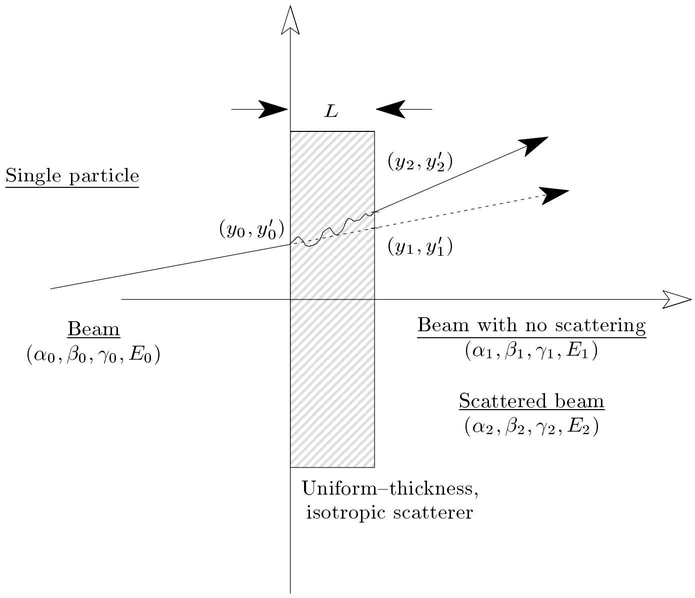

Figure 1: Effect of a thick scatterer

Furthermore, the variance of the displacement δy can be calculated to be

$$\langle \delta y^2 \rangle = \left(\frac{L}{\sqrt{3}} \theta_0 \right)^2, \tag{6}$$

where $\theta_0 = \sqrt{\langle \theta_s^2 \rangle}$, the characteristic scattering angle, which can be calculated from Highland's equation (19) as described in the Appendix A.

Three relationships for the statistical effect of scattering can now be established. Firstly, the change in divergence,

$$y_2' = y_1' + \theta_s. \tag{7}$$

By squaring (7) and then averaging over the whole beam, the increase in divergence can be related to the characteristic scattering angle θ_0.

$$\langle {y'}_2^2 \rangle = \langle {y_1'}^2 \rangle + 2\langle y_1' \theta_s \rangle + \langle \theta_s^2 \rangle$$

where $2\langle y_1' \theta_s \rangle = 0$ because y_1' and θ_s are uncorrelated since scattering is isotropic. Thus the new divergence is given by,

$$\langle {y'}_2^2 \rangle = \langle {y'}_1^2 \rangle + \theta_0^2. \tag{8}$$

Similarly with the use of (6),

$$\langle y_2^2 \rangle = \langle y_1^2 \rangle + 2\langle y_1 \delta y \rangle + \langle \delta y^2 \rangle$$

where $\langle y_1, \delta y \rangle = 0$ and

$$\langle y_2^2 \rangle = \langle y_1^2 \rangle + \left(\frac{L}{\sqrt{3}} \theta_0 \right)^2 . \tag{9}$$

Finally, the change in the cross-term yy' is given by,

$$\langle y_2 y_2' \rangle = \langle (y_1 + \delta y)(y_1' + \theta_s) \rangle$$

which can be evaluated to be

$$\langle y_2 y_2' \rangle = \langle y_1 y_1' \rangle + \langle \theta_s \delta y \rangle + \langle y_1 \theta_s \rangle + \langle y_1' \delta y \rangle . \tag{10}$$

Since scattering is isotropic and has no correlations with y_1 or y_1', the last two terms in (10) are zero. The second term, however, makes a finite contribution, since there is a correlation between the displacement δy and the scattering angle θ_s, which is described by (5). Inserting (5) into (10) therefore leads to

$$\langle y_2 y_2' \rangle = \langle y_1 y_1' \rangle + \frac{L}{2} \theta_0^2 . \tag{11}$$

The changes due to scattering are described statistically in equations (8), (9) and (11) and provide the three relationships needed to solve the Twiss functions after scattering. Thus, using the correspondences in the matrix equation (4),

$$\begin{aligned} E_2 \gamma_2 &= E_1 \gamma_1 + \pi \theta_0^2 \\ E_2 \beta_2 &= E_1 \beta_1 + \pi \left(\frac{L}{\sqrt{3}} \theta_0 \right)^2 \\ -E_2 \alpha_2 &= -E_1 \alpha_1 + \pi \left(\frac{L}{2} \theta_0^2 \right) . \end{aligned} \tag{12}$$

The right-hand side of each equation in (12) is fully evaluated by conditions at the exit to the scatterer without scattering, and by the characteristics of the scatterer. Re-writing (12) with three constants

$$\begin{aligned} E_2 \gamma_2 &= A \\ E_2 \beta_2 &= B \\ E_2 \alpha_2 &= C \end{aligned}$$

and substituting into the Twiss relation below leads to the solution of the three equations by

$$\frac{1+\alpha^2}{\beta} = \gamma \quad \text{so that} \quad \frac{1+\left(\frac{C}{E_2}\right)^2}{\frac{B}{E_2}} = \frac{A}{E_2} \quad \text{and} \quad E_2 = \sqrt{AB - C^2}. \tag{13}$$

Consequently E_2 can be evaluated as

$$E_2 = E_1 \sqrt{\left(\gamma_1 + \pi \frac{\theta_0^2}{E_1}\right)\left(\beta_1 + \pi \frac{L^2 \theta_0^2}{3E_1}\right) - \left(\alpha_1 - \pi \frac{L}{2E_1}\theta_0^2\right)^2}. \tag{14}$$

Backsubstitution into (12) now yields the modified Twiss functions after the scatterer.

Extension to a coupled beam

The approach given in the previous section can now be generalised to coupled beams. The σ-matrix for a coupled beam contains additional elements:

$$\sigma_{\text{coupled}} = \begin{pmatrix} \langle xx \rangle & \langle x'x \rangle & \langle zx \rangle & \langle z'x \rangle \\ \langle xx' \rangle & \langle x'x' \rangle & \langle zx' \rangle & \langle z'x' \rangle \\ \langle xz \rangle & \langle x'z \rangle & \langle zz \rangle & \langle z'z \rangle \\ \langle xz' \rangle & \langle x'z' \rangle & \langle zz' \rangle & \langle z'z' \rangle \end{pmatrix}. \tag{15}$$

By using the same technique as for the uncoupled beam — adding the scattering and then squaring and averaging over the beam — the overall effect of scattering can be evaluated. Since scattering is isotropic, the elements of the σ-matrix which do not describe the coupling remain the same. The only additional work to be done is to deal with the three coupled terms in the matrix.

Starting with

$$\langle x_2 z_2 \rangle = \langle (x_1 + \delta x)(z_1 + \delta z) \rangle$$

which can be transformed to

$$\begin{aligned} \langle x_2 z_2 \rangle &= \langle x_1 z_1 \rangle + \langle x_1 \delta z \rangle + \langle z_1 \delta x \rangle + \langle \delta x \delta z \rangle \\ &= \langle x_1 z_1 \rangle \end{aligned}$$

resulting in no extra change, since scattering does not introduce any coupling between the x and z planes and all coupling terms cancel.

Similarly,

$$
\begin{aligned}
\langle x_2 z_2' \rangle &= \langle (x_1 + \delta x)(z_1' + \theta_{zs}) \rangle \\
&= \langle x_1 z_1' \rangle + \langle x_1 \theta_{zs} \rangle + \langle z_1' \delta x \rangle + \langle \theta_{zs} \delta x \rangle \\
&= \langle x_1 z_1' \rangle.
\end{aligned}
$$

Finally the last term,

$$
\begin{aligned}
\langle x_2' z_2' \rangle &= \langle (x_1' + \theta_{xs})(z_1' + \theta_{zs} \rangle \\
&= \langle x_1' z_1' \rangle + \langle x_1' \theta_{zs} \rangle + \langle z_1' \theta_{xs} \rangle + \langle \theta_{xs} \theta_{zs} \rangle \\
&= \langle x_1' z_1' \rangle.
\end{aligned}
$$

All coupled terms are unchanged after scattering and only the uncoupled elements change. Thus, it is possible to treat a coupled beam passing through a scatterer in the same way as an uncoupled beam, which is a welcome simplification, although at first, a little surprising.

Approximation for a thin scatterer

In practical applications it is often necessary to deal with thin scatterers, such as stripping foils. For such an application, it is possible to make a thin scatterer approximation. In a thin-scatterer, it is assumed, that the position of the particle remains unchanged when passing through the scatterer, and only the change in divergence has to be taken into account. Thus neglecting the terms depending on L in (12) leads to,

$$
\begin{aligned}
E_2 \gamma_2 &= E_1 \gamma_1 + \pi \theta_0^2 \\
E_2 \beta_2 &= E_1 \beta_1 \\
E_2 \alpha_2 &= E_1 \alpha_1
\end{aligned}
\tag{16}
$$

Equation (14) is still valid and yields,

$$
E_2 = E_1 \sqrt{1 + \pi \frac{\beta_1 \theta_0^2}{E_1}}. \tag{17}
$$

Be sure that $\pi \beta_1 \theta_0^2 / E_1 \ll 1$ if the square root is to be expanded.

Emittance dilution from betatron mismatch

As an application of the Twiss-Scatterer relations, consider the following example. For a luminescent screen introduced into a transfer line, the multiple scattering in the monitor will cause a betatron mismatch. If the beam then enters a ring, the mismatch will cause filamentation and hence an increase in emittance according to

$$E_{\text{diluted}} = \frac{1}{2}\left[\frac{\beta_s}{\beta_n} + \left(\alpha_s - \alpha_n\frac{\beta_s}{\beta_n}\right)^2\frac{\beta_n}{\beta_s} + \frac{\beta_n}{\beta_s}\right]E_s, \qquad (18)$$

which can be found in [3]. Here the index s denotes the Twiss parameters after scattering, while the index n denotes the non-scattered values, i. e. the values expected at the entry to the ring for the beam without beam monitors.

By isolating α_2 and β_2 by dividing by E_2 in equation (12) and inserting the new scattered values into equation (18) using the matched values for the line for α_1 and β_1 and using equation (14) to obtain the intermediate emittance blown-up by scattering (E_s in (18)), it is possible to evaluate the emittance dilution of the mismatched beam due to scattering.

Comparison with a Monte-Carlo simulation

	Input beam		Beam after scattering	
	Parameters used to generate distribution	Statistical estimation from generated distribution	Calculated by Twiss-scatter relation	Estimated statistically from tracking results
$E_{\text{RMS},x}$ [π mm mrad]	2.30	2.30	77.3	76.5
$E_{\text{RMS},z}$ [π mm mrad]	2.30	2.30	77.3	76.6
β_x[m]	6.25	6.27	0.19	0.19
β_z[m]	6.25	6.25	0.19	0.19
α_x[m]	0.00	0.00	0.00	0.00
α_z[m]	0.00	0.00	0.00	0.00

Table 1: Comparison of calculated beam parameters and those obtained by tracking

A simple comparison with tracking shows the validity of (12). Using a code for designing passive spreading systems in hadron therapy [4], an uncorrelated Gaussian beam of 100 000 protons at 180 MeV with a momentum spread of 0.1% and a spatial cut-off at 2σ was generated and tracked through a copper foil of 3.67 mm thickness. The absorber adds an average angle $\theta_0 = 20.4$ mrad using Highland's equation (19). The numerical data is collected in Table 1. The estimated Twiss parameters of the scattered beam obtained by statistical analysis of the distribution calculated by tracking agree extremely well with those calculated using the Twiss-Scatterer relations of equation (12).

Appendix A. Multiple scattering

Multiple coulomb scattering of charged heavy particles is probably best described by the theory of Molière [5] which has been extended and corrected by Bethe [6], Fano [7] and Scott [8]. This theory describes the width and distribution of the projected scattering angle, which is not quite Gaussian. However, Molière's theory is somewhat complicated and difficult to apply, it is fortunate that it is often possible to use a Gaussian approximation of the scattering distribution. Highland [9, 10] has developed such an approximation and he calculates the width by an astute observation that certain parameters in Molière's theory occur in the same way as in the equation of the radiation length:

$$\theta_0 = \frac{14.1[\mathrm{MeV}]}{p\beta c[\mathrm{MeV}]} Z_{inc} \sqrt{\frac{L}{L_R}} \left[1 + \frac{1}{9} \log_{10} \left(\frac{L}{L_R} \right) \right]. \tag{19}$$

This equation is said to be accurate to 5% in the range $10^{-3} \leq L/L_R \leq 100$, except for very light elements, where the accuracy is supposed to be better than 11% [11]. Note however, that there are a variety of similar equations quoted in the literature, which all deviate at the one to two percent level. Especially for low particle energies, such as those used in hadron therapy, the validity of (19) has to be questioned and it may be necessary to modify Highland's equation.

The Highland formula is a fit that takes into account the energy loss of the beam. Thus equations (12) and (14) give the Twiss functions and emittance at the new energy of the beam. To find the energy loss the well known Bethe-Bloch Formula can be used which can be found in [11]. It should be remembered that if the energy loss is not accounted for by a reduction in the focusing strength of the line, additional betatron mismatch will be created.

The formulation of scattering in terms of Twiss functions is, however, independent of the method chosen to calculate the characteristic scattering angle. It is sufficient to be aware that Highland's formula is an approximation and should be replaced by measured data when possible.

Conclusions

- An analytically rigorous approach has been developed for evaluating scattering in the context of the Twiss functions.
- In particular, full account is taken for the geometrical correlation occurring between the scattering angle and the displacement of the scattered particle.
- The resultant Twiss-Scatterer relations allow the Twiss functions to be calculated in a continuous manner through a lattice, which opens the way to incorporating scatterers into lattice programs and even using them for matching beam envelopes and emittances. This has already been implemented in `WINAGILE`[12].

- Excellent agreement with the Monte-Carlo data is achieved and demonstrates the accuracy achievable.

Acknowledgements

This work was performed at CERN, within the framework of the Proton-Ion Medical Machine Study. The author would like to thank the PS Division for its hospitality in hosting this study and to acknowledge all the friendly help and advice given by PS staff members. In particular he would like to thank P. Bryant for the continuous help in the study of the subject and for checking the calculations.

References

[1] J. Buon. A statistical description of a partice beam subjected to a linear coupled betatron motion. Technical report, LAL/RT, April 1996.

[2] R. K. Bock, K. Bos, S. Brandt, J. Myrheim, and M. Regler, editors. *Formulae and Methods in Experimental Data Evaluation.* The European Physical Society, 1983.

[3] P. J. Bryant. Beam transfer lines. In Turner [13], pages 219–238.

[4] A. Maier. PhD thesis, University of Vienna, 1998.

[5] G. Molière. Theorie der Streuung schneller geladener Teilchen II. *Zeitschrift für Naturforschung*, 3a:78–97, 1948.

[6] H. A. Bethe. Molière's theory of multiple scattering. *Physical Review*, 89(6):1256–1266, 1952.

[7] U. Fano. Inelastic collisions and the molière theory of multiple scattering. *Physical Review*, 93(1):117–120, 1953.

[8] W. T. Scott. The theory of small-angle multiple scattering of fast charged particles. *Review of Modern Physics*, 35:231–313, 1963.

[9] V. L. Highland. Some practical remarks on multile scattering. *Nuclear Instruments and Methods*, 129:497–499, 1975.

[10] V. L. Highland. Erratum. *Nuclear Instruments and Methods*, 161:171, 1979.

[11] C. Caso et al. Review of particle properties. *The European Physical Journal*, C3:1, 1998.

[12] P. Bryant. Private communication. `WINAGILE` – lattice calculation program, 1998.

[13] S. Turner, editor. *CERN Accelerator School – Fifth General Accelerator Physics Course.* CERN, 1994.