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Improvement of LHC Dynamic Aperture via Octupole Spool Pieces for the Nominal Tunes

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Abstract

The dynamic aperture of the LHC optics version 5 at injection energy has been calculated for an optimistic error table, the so called *target error table*, in which erect and/or skew octupolar components were increased up to values close to realistic estimates. Correction strategies, using octupole spool pieces or the lattice octupoles, have been tested so as to recover, as much as possible, the loss in dynamic aperture.

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1 Introduction

Since 1991 (LHC optics version 1 [1]), the effect of octupolar components on the dynamic aperture has been studied [2]. At that time the systematic b_4 component was considered to be about 3 times as large as the value of the uncertainty of the present *target error table* (Table 1), while the systematic a_4 component was about as large as the a_4 uncertainty of the present *target error table*. According to symmetry considerations [1], the b_4 changed sign from octant to octant. As a result the dynamic aperture was not influenced by the erect octupolar component so that the correction with erect octupole spool pieces was discontinued. No considerable effect of the skew octupoles were reported.

The problem of the systematic octupolar components has been studied again for LHC optics version 4.1 [3]. The sevenfold larger value of a_4 in the latter optics version as compared to version 1 led to a drastic decrease of the dynamic aperture. The b_4, on the other hand, while very dangerous when considered as one systematic value around the machine, had practically no “offensive” effect when the sign of b_4 was changed from octant to octant.

Version	**Error Table**	**Octupolar Component**			
		Mean		**Uncertainty**	
		b_4	a_4	b_4	a_4
1	–	±0.246	0.147	0.0	0.0
4.1	–	±0.983	0.983	0.0	0.0
4.2	–	0.0	0.0	0.983	0.983
4.3	–	0.0	0.0	0.491	0.349
5.0	9607	0.0	0.0	0.266	0.349
5.0	Target Error Table	0.0	0.0	0.069	0.138
5.0	9712	0.0	0.0	0.35	0.555

Table 1: Evolution of the Octupole Components at injection, in units of 10^{-4} relative field errors and at a reference radius of 17mm

As of LHC optics version 4.2, a new model of field errors including a *systematic per arc* has been introduced [4]. When studied at initially very large values (third row in Table 1), the dynamic aperture was clearly too small, but could be corrected with erect and skew spool pieces added at every dipole and powered in series so as to correct the average around the machine for each type of octupoles [5].

Using exact scaling laws [6], the estimate for the erect and skew octupoles have been reduced by a factor of 2 and 3 respectively [7]. Tracking runs confirmed that the dynamic aperture was enlarged due to this reduction of the systematic octupoles per arc [8]. However, these components were still dominant, so that the b_4 component had been further reduced by a factor of 2 in the error table **9607**.

A new evaluation [9] of the uncertainties of the dynamic aperture led to an increase of the target dynamic aperture from 10 to 12σ so as to have a safety factor of 2 [8] with respect to the position of the collimators at 6σ. In the light of this increased requirement for the dynamic aperture, a *target error table* has been proposed [10] to reach the target dynamic aperture in part due to largely reduced octupolar components.

It should be mentioned that for the LHC optics version 4, a correlation of the dynamic aperture with the fourth order resonances has been found and in particular with the (1, –1) subresonance driven by a_4 [11]. All the progress made in the prediction and requirement of the dynamic aperture has thus emphasised the detrimental role of b_4 and a_4.

The aim of this report is to evaluate how much the dynamic aperture would be reduced considering realistic erect and skew octupoles (last row in Table 1) and whether a spool piece correction system is suitable to recover this loss. For the LHC optics version 5, it seems feasible to include one type of extra octupolar spool pieces at the location of the decapole spool pieces already foreseen at one of the ends of each dipole. Following the twin–aperture design of the dipoles one type of octupole spool pieces will be located in the inner and the other in the outer channel. As the beam crosses from one channel to the other in IP1, IP2, IP5, IP8 it will encounter one type of octupolar spool pieces in sectors 1, 5, 6, 7 and the other type in sectors 2, 3, 4, 8. It has also been studied if the lattice octupoles can be used instead of erect octupole spool piece correctors.

After describing (Sec. 2) the error table to be studied, the tracking results are discussed in detail (Sec. 3). Lastly, a thorough analysis of the nonlinear detuning and resonance driving terms is presented in Sec. 4.

2 Target Error Table

As a reference case we have used the *target error table*. Table 2) shows the mean, systematic and random part of this error table, whereby the persistent and geometric contributions are added quadratically. Moreover, it also gives the more realistic $\mathbf{b_4}$ and $\mathbf{a_4}$ systematic per arc components from the latest official error table 9712, thereby referred as $\mathbf{b_4}^{\mathbf{9712}}$ and $\mathbf{a_4}^{\mathbf{9712}}$, respectively. In that case, the values are the quadratic sum of the persistent, geometric and decay contributions. With respect to the *target error table*, these $\mathbf{b_4}^{\mathbf{9712}}$ and $\mathbf{a_4}^{\mathbf{9712}}$ values are larger by a factor of 5.1 and 4.0, respectively.

3 Tracking Results

The tracking for LHC optics version 5 has been performed following the procedure described in Ref. [5]. We have used the tracking code SIXTRACK [12] which is linked to the LHC data–base in MAD 8 language [13] via the well tested MADTOSIX converter [14]. For this systematic study, we made full use of the extended NAP DEC cluster [15] which now consists of two times ten stations with 350MHz EV5 and 500MHz EV56 alpha chips, respectively.

The tracking has been performed using the following parameters:

- 60 different realizations of the magnetic field errors (seeds)
- Six–dimensional phase space with initial relative momentum deviation of $\frac{\delta p}{p_0} = 0.00075$
- The dynamic aperture is defined as the smallest amplitude with a particle loss that occurs before 100’000 turns
- 30 particles were tracked for each σ of beam amplitude
- The amplitude is varied for 2 ratios of the linear invariants $\phi = \arctan(\sqrt{\frac{I_y}{I_x}})$: for predominantly horizontal motion ($\phi = 15°$) and for equal linear invariants ($\phi = 45°$). At the nominal working point ($\omega_x = 0.28$, $\omega_x = 0.31$), it is needed to study at least these two cases, since the dynamic aperture has its minimum at $\phi = 15°$, whilst the case of $\phi = 45°$ represents roughly the dynamic aperture averaged over all ratios of linear invariants (see Ref. [16]).

Besides the reference case (*Case 0*), which is the LHC optics version 5 tracked with the *target error table*, three cases have been studied including various correction schemes:

- *Case 1*: *Target error table* plus the erect octupole component $\mathbf{b_4}^{\mathbf{9712}}$

Target Error Table				Taken from Error Table 9712
	Persistent & Geometric			Persistent & Geometric & Decay
b_n	**Mean**	**Uncertainty**	**Random**	**Uncertainty**
3	–11.07	2.38	1.45	
4		0.069	0.49	0.35
5	0.376	0.125	0.65	
6		0.057	0.28	
7	–0.097	0.024	0.24	
8			0.21	
9	0.349	0.084	0.22	
10			0.24	
11	0.58		0.20	
a_n				
3		0.29	0.43	
4		0.138	0.49	0.555
5		0.376	0.334	
6		0.057	0.14	
7			0.24	
8			0.22	
9			0.29	
10			0.24	
11			0.20	

Table 2: Multipole Components of *Target Error Table* and more realistic values of b_4 and a_4 taken from Error Table 9712 (at injection in units of 10^{-4} relative field errors and at a reference radius of 17mm)

- *Case 1a*: Correction of the erect octupole average with spool pieces in the outer channel ${b_4}^{sp1-3-}$ being powered in series
- *Case 1b*: Least square minimisation of the three first order detuning terms using the lattice octupoles ${B_4}^{lo}$ with the OD and OF type each being powered in series

– *Case 2*: *Target error table* plus the skew octupole component ${a_4}^{9712}$

- *Case 2a*: Local correction of the skew octupole component in each sector ${a_4}^{spa}$
- *Case 2b*: Correction of the skew octupole average with spool pieces in the inner channel ${a_4}^{sp-3-1}$ being powered in series
- *Case 2c*: Correction of the large driving term of the subresonance (1, –1) excited by the skew octupole component via spool pieces in the inner channel ${a_4}^{spc-3-1}$ being powered in series (the spool pieces strength in the 3rd octant has opposite sign, see below for details)

– *Case 3*: *Target error table* plus both erect and skew octupole component ${b_4}^{9712}$, ${a_4}^{9712}$

- *Case 3a*: Correction of the erect and skew octupole average with spool pieces in the outer channel $\mathbf{b_4}^{\mathbf{sp1-3-}}$ and inner channel $\mathbf{a_4}^{\mathbf{sp-3-1}}$ being powered in series, respectively
- *Case 3b*: Correction of the erect octupole average with spool pieces in the outer channel $\mathbf{b_4}^{\mathbf{sp1-3-}}$ and correction of the large driving terms of the subresonance (1, –1) excited by the skew octupole component via spool pieces in the inner channel $\mathbf{a_4}^{\mathbf{spc-3-1}}$, each being powered in series
- *Case 3c*: Local correction of the erect and skew octupole component in each sector with spool pieces, $\mathbf{b_4}^{\mathbf{spa}}$, $\mathbf{a_4}^{\mathbf{spa}}$

Case		**Errors and Correctors**	**Linear Invariant Ratio [°]**			
			15		**45**	
			Dynamic Aperture [σ]			
			Minimum	**Average**	**Minimum**	**Average**
0		**Target error table**	11.3	12.4	12.3	13.8
1		$\mathbf{b_4}^{\mathbf{9712}}$	9.6	12.2	10.8	13.4
1a	+	$\mathbf{b_4}^{\mathbf{sp1-3-}}$	11.8	12.6	12.0	13.7
1b	+	$\mathbf{B_4}^{\mathbf{lo}}$	11.2	12.6	11.4	13.6
2		$\mathbf{a_4}^{\mathbf{9712}}$	10.4	12.1	10.0	12.9
2a	+	$\mathbf{a_4}^{\mathbf{spa}}$	11.4	12.4	12.1	14.0
2b	+	$\mathbf{a_4}^{\mathbf{sp-3-1}}$	10.2	12.1	9.5	12.5
2c	+	$\mathbf{a_4}^{\mathbf{spc-3-1}}$	11.3	12.5	11.7	13.8
3		$\mathbf{b_4}^{\mathbf{9712}}$, $\mathbf{a_4}^{\mathbf{9712}}$	10.1	12.0	10.0	12.6
3a	+	$\mathbf{b_4}^{\mathbf{sp1-3-}}$, $\mathbf{a_4}^{\mathbf{sp-3-1}}$	9.7	12.0	9.5	12.5
3b	+	$\mathbf{b_4}^{\mathbf{sp1-3-}}$, $\mathbf{a_4}^{\mathbf{spc-3-1}}$	11.2	12.6	11.8	13.6
3c	+	$\mathbf{b_4}^{\mathbf{spa}}$, $\mathbf{a_4}^{\mathbf{spa}}$	11.6	12.4	11.7	13.9

Table 3: The effect of realistic erect and skew octupolar errors and their corrections on the dynamic aperture of LHC optics version 5 at injection

Table 3 holds the tracking results for all cases:

- *Case 1*: The correction of the average of the erect octupole component with erect octupole spool pieces in half of the machine corrected in series, fully restores the dynamic aperture. The lattice octupoles are also effective although the minimum dynamic aperture is reduced for the case of $\phi = 45°$.
- *Case 2*: A local correction of the skew octupole component in each sector (*Case 2a*) is very effective to recover the loss in dynamic aperture. On the other hand, it is not sufficient to suppress the average of the skew octupole component with skew octupole spool pieces correctors, contrary to our experience from LHC optics version 4 (*Case 2b*). This fact can be explained, as follows: Let us first mention that, similar to what has been found for LHC optics version 4 [11], the skew octupoles mainly drive the (1, –1) subresonance (see next chapter). The interplay between this resonance and the fact that the LHC optics

version 5 has an integer tune split of 4 (63 versus 59), is crucial. As a result of this tune split, the real and imaginary part of the (1,–1) resonance coefficients change sign from octant to octant. For an efficient correction, one has therefore to respect this sign flipping. In fact, the skew octupole correctors can have all the same absolute value but one of them (octant 3) should have opposite sign as compared to the others. A minimisation in first order of the (1, –1) subresonance with skew octupole spool pieces being powered in series, following the above mentioned sign flipping in their strengths, is equally beneficial for the dynamic aperture (*Case 2c*), as compared to the *Case 2a*. The latter method is the preferred solution as it allows the correction of erect and skew octupole components in only half of the machine.

– *Case 3*: As expected from the previous case it is not sufficient to correct the erect and the skew octupolar average with alternating erect and skew octupole spool pieces (*Case 3a*). However, a combined correction of the erect octupole average and the (1, –1) skew octupole subresonance (*Case 3b*) allows to restore the dynamic aperture almost completely. Not even the ultimate solution, which is a local correction of erect and skew octupolar components sector by sector (*Case 3c*), leads to any significant improvement. In fact, one even finds a small drop of the minimum value at $\phi = 45°$ (see next chapter for details).

Table 4 summarises the average, standard deviation and maximum absolute values of the integrated octupolar strengths needed to do the various corrections at injection energy for 60 random seeds. Notice that the total strength of all lattice octupoles combined is about 1.5 times larger than what is needed for the erect octupole spool piece correction. This implies that the correction with spool pieces is slightly more efficient.

Case	**Type**	**Number**	$B_4^{(r)}$ $[Tm]$ @ r_0=17mm		
			Average	σ	**Maximum**
			$[10^{-4}]$	$[10^{-4}]$	$[10^{-4}]$
1b	**Lattice Octupoles OD**	96	1.9	1.5	5.4
	Lattice Octupoles OF	88	2.2	1.6	6.9
1a, 3a, 3b	**Spool Pieces** $b_4{}^{sp1-3-}$	616	0.79	0.64	2.3
2b, 3a	**Spool Pieces** $a_4{}^{sp-3-1}$	616	1.4	0.79	3.1
2c, 3b	**Spool Pieces** $a_4{}^{spc-3-1}$	616	1.3	0.98	3.6

Table 4: Strength requirements for the octupole correctors at injection.

4 Analysis

In parallel to the systematic tracking studies the same cases are also analysed with respect to the nonlinear detuning with amplitude and the driving terms of resonances.

4.1 Detuning

Figure 1 shows the effect of the erect and skew octupoles on the horizontal and vertical amplitude detuning before and after correction at predominantly horizontal motion ($\phi = 15°$). All graphs represent, order by order, the difference of the detuning strength with respect to the one of the *target error table*.

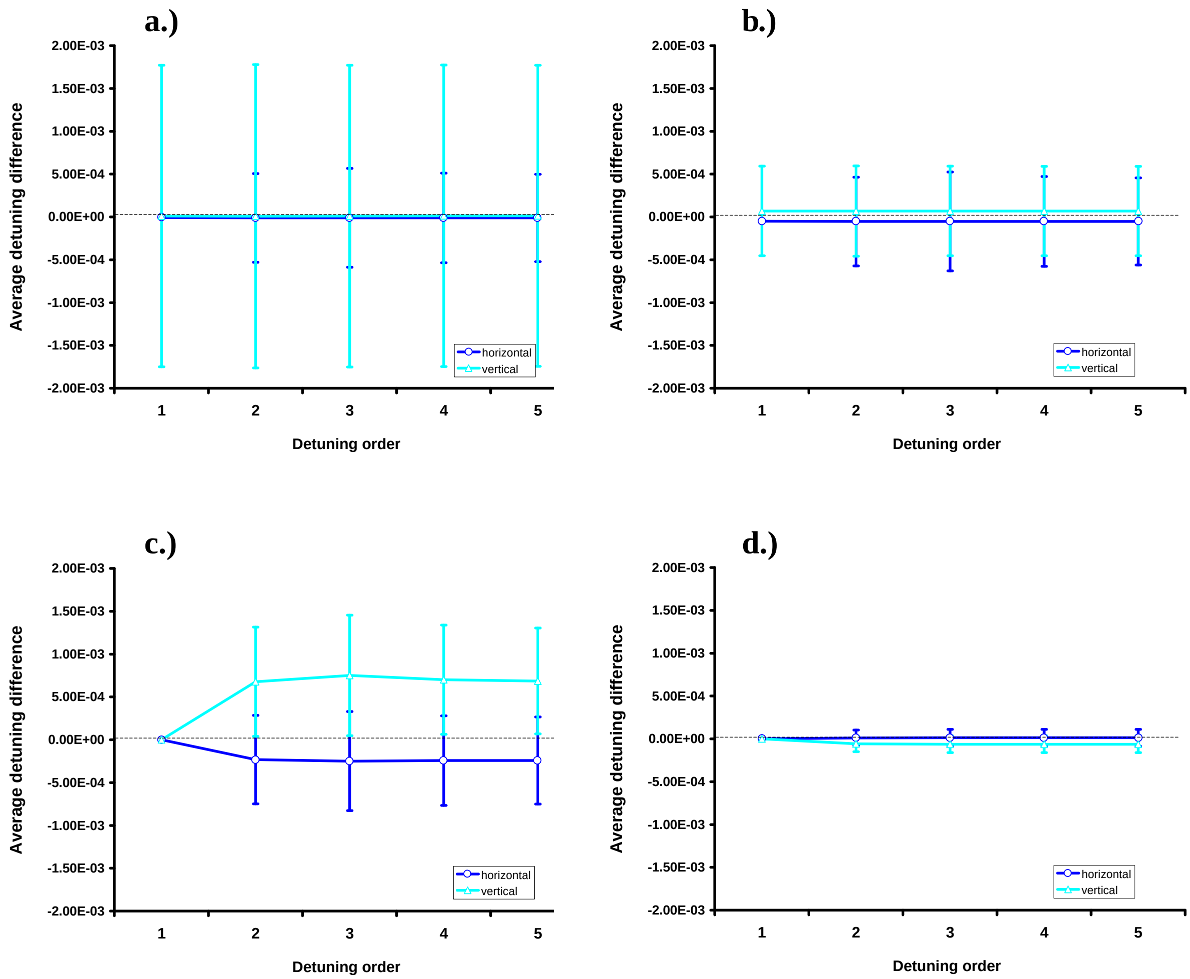

Figure 1: *Change of amplitude detuning with respect to the target error table due to octupole errors before and after correction at* $\phi = 15°$

In all graphs we show the average detuning and the error bars corresponding to one standard deviation. The amplitude detuning is evaluated using Normal Forms up to a certain order and at 8σ.

Part **a.)** erect octupoles

Part **b.)** erect octupoles after correction

Part **c.)** skew octupoles

Part **d.)** skew octupoles after correction

In *Case 1* (erect octupoles, Figure 1 a.), the average over 60 seeds is close to zero but the spread is wide, i.e. certain seeds have large detuning values as compared to the values of the *target error table*. After the correction of the erect octupole average (*Case 1a*), the detuning spread is largely reduced (Figure 1 b.). In *Case 2* (skew octupoles, Figure 1 c.), the difference in the detuning terms can only arise from orders larger than one. In this case an average change in the detuning is found both in the horizontal and vertical plane and the spread is large as well. Once the (1, –1) skew subresonance is compensated with spool pieces (*Case 2c*, Figure 1 d.), this large change in the detuning is almost completely suppressed and we find the *target error table* detuning values. The fact that the compensation of a first order resonance leads to a good second order detuning correction shows that the spool piece correction is quasi–local even if it is implemented in half of the machine only. This interconnection of the effects may therefore allow a correction of the large erect or skew octupole components by choosing observables which are most easily accessible during operation.

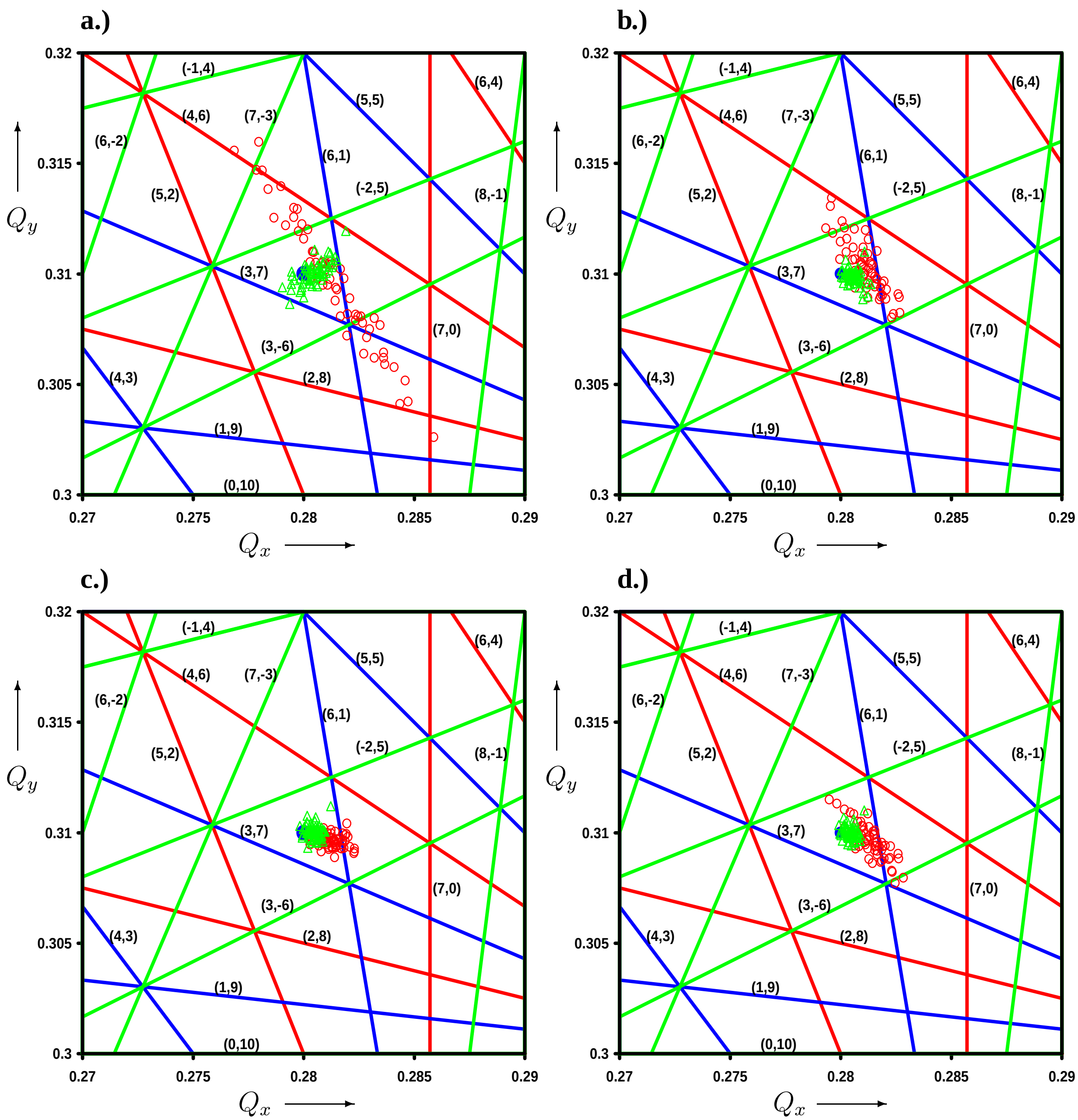

Figure 2: *Tune diagrams evaluated by tracking at 8σ for 60 seeds of LHC optics version 5*
The open circles are calculated for predominantly horizontal motion ($\phi = 15°$) while the open triangles represent the tunes of round beam motion ($\phi = 45°$)
Part **a.)** erect octupoles
Part **b.)** skew octupoles
Part **c.)** erect and skew octupoles after correction
Part **d.)** *target error table* only

These theoretical predictions of the detuning have been checked with a refined FFT analysis [17] of short–term tracking data. Figure 2 shows the tunes at 8σ of 60 seeds for the two ratios of linear invariants $\phi = 15°, 45°$: in particular at $\phi = 15°$ the detuning is dominated by b_4 but a_4 is significant as well (see Figure 2 a. and b., respectively). Considering both erect and skew octupolar errors including their correction (*Case 3b*) the detuning reduces to smaller values (see Figure 2 c.). In fact, the detuning is even slightly better compared to what has been found for the *target error table* (Figure 2 d.). This implies that the 4–5 times smaller values of the octupole errors, that are left in the *target error table*, still create some detuning. The residual detuning of about $\pm 1 \times 10^{-3}$ is mostly due to sextupoles in second order.

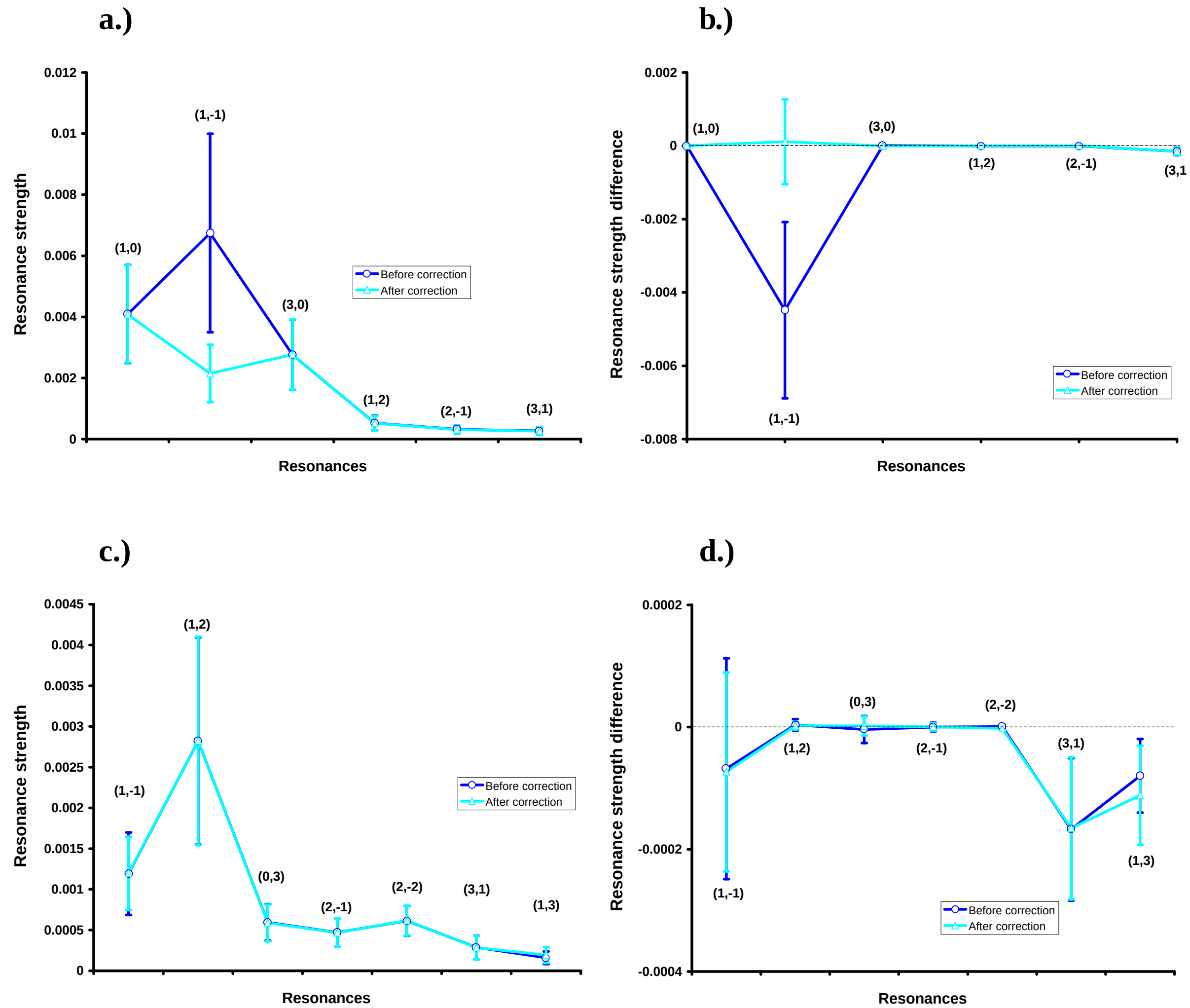

Figure 3: *Strength of resonances for LHC optics version 5 due to skew octupole errors before and after correction*

The two left hand graphs show the resonance strength including skew octupolar errors and the graphs on the right hand side show the difference to the resonance strengths of the *target error table*. The average over 60 seeds and the error bars corresponding to one standard deviation are depicted before and after skew octupole spool piece correction, respectively.

Part **a.)** $\phi = 15^\circ$ resonance strength

Part **b.)** $\phi = 15^\circ$ resonance strength difference

Part **c.)** $\phi = 45^\circ$ resonance strength

Part **d.)** $\phi = 45^\circ$ resonance strength difference

4.2 Resonances

A study of resonances for LHC optics version 4 (see Ref. [11, 18]) has shown that a multi–variant fit of the driving terms of three octupole resonances with the dynamic aperture yielded a large correlation coefficient. The (1, –1) subresonance, although having the largest driving term, did not correlate very well by itself with the dynamic aperture.

In order to proceed to an analysis of the resonances affecting the dynamics of LHC optics version 5, we developed a numerical tool GRR (Graphical Resonance Representation) [19, 20]. This tool uses as input the resonance terms of the generating function computed by standard Lie transformation procedures [21] and evaluates them up to a desired order for a specific amplitude close to the minimum dynamic aperture and for linear invariant ratios of interest. We usually evaluate the resonances up to 12th order.

The real and imaginary coefficients of the two (1, –1) subresonances ($(1,0,1,2)$ and $(2,1,0,1)$) are evaluated and the a_4 spool correctors are powered such as to minimise these four terms with a least square method. It should be mentioned that the higher order contributions to these subresonances are insignificant. The upper part of Figure 3 shows the $\phi = 15°$ case (before and after correction). The $45°$ case is shown in the lower part of Figure 3. The resonance strength and difference to the resonance strength of the *target error table* is shown in Figure 3 a., c. and Figure 3 b., d., respectively. All graphs show the average values over 60 seeds and the error bars corresponding to one standard deviation. The resonances are chosen such that the strengths as well as differences of strengths are at their largest values.

For $15°$, the effect of the correction of the (1, –1) subresonance is indeed very visible. Both, the average resonance strengths (Figure 3 a.) and the average resonance differences (Figure 3 b.) are considerably reduced after the correction with the skew octupole spool pieces. In particular the (1, –1) subresonance strength decreases by a factor of three so that there is practically no difference to that of the *target error table*.

For $45°$, on the other hand, the situation is less clear. The effect of the correction is visible neither in the average nor in the standard deviation value of the resonance strength (Figure 3 c.). The difference to the resonance strength of the *target error table* (Figure 3 d.), however, shows some improvement although by far less pronounced than for the $15°$ case. One may argue that this difference is due to the fact that the coupling subresonance (1, –1) is less effective at $45°$ but this leaves unexplained why the minimum dynamic aperture has improved also in this case (compare *Case 1* and *Case 1a* in Table 3). In fact, this finding seems to be in line with the above stated observation of the poor correlation of a single resonance strength with dynamic aperture for LHC optics version 4. Lastly, one can also find the skew resonances (3, 1) and (1, 3) which are somewhat increased after correction in the $45°$ case. This is of no importance, however, given that their driving terms are by far smaller compared to those of the (1, –1) subresonance.

The coupling between the motion planes due to the (1, –1) subresonance, excited by the skew octupole components, can be directly observed in phase space. In Figure 4 a. and b. the horizontal and vertical phase space projections are shown for tracking of the LHC optics version 5 with the *target error table* at 10σ and for the linear invariant ratio $\phi = 15°$. When the skew octupole errors are included (Figure 4 c. and d.) the exchange of energy between the planes of motion due to the coupling subresonance leads to larger variations of the amplitude in both phase plane projections. A compensation of the subresonance with the skew octupolar spool pieces (Figure 4 e. and f.) restores the phase space images back to those of the *target error table*.

Lastly, it should be mentioned that the peculiar drop of the minimum dynamic aperture in the case of the sector–by–sector local correction with erect and skew octupole spool pieces (*Case 3c*) could be explained by a strong excitation of the (2, –2) resonance. A correction of this resonance led to an improvement of the dynamic aperture in this particular case only. A general correction of this resonance seems therefore unnecessary.

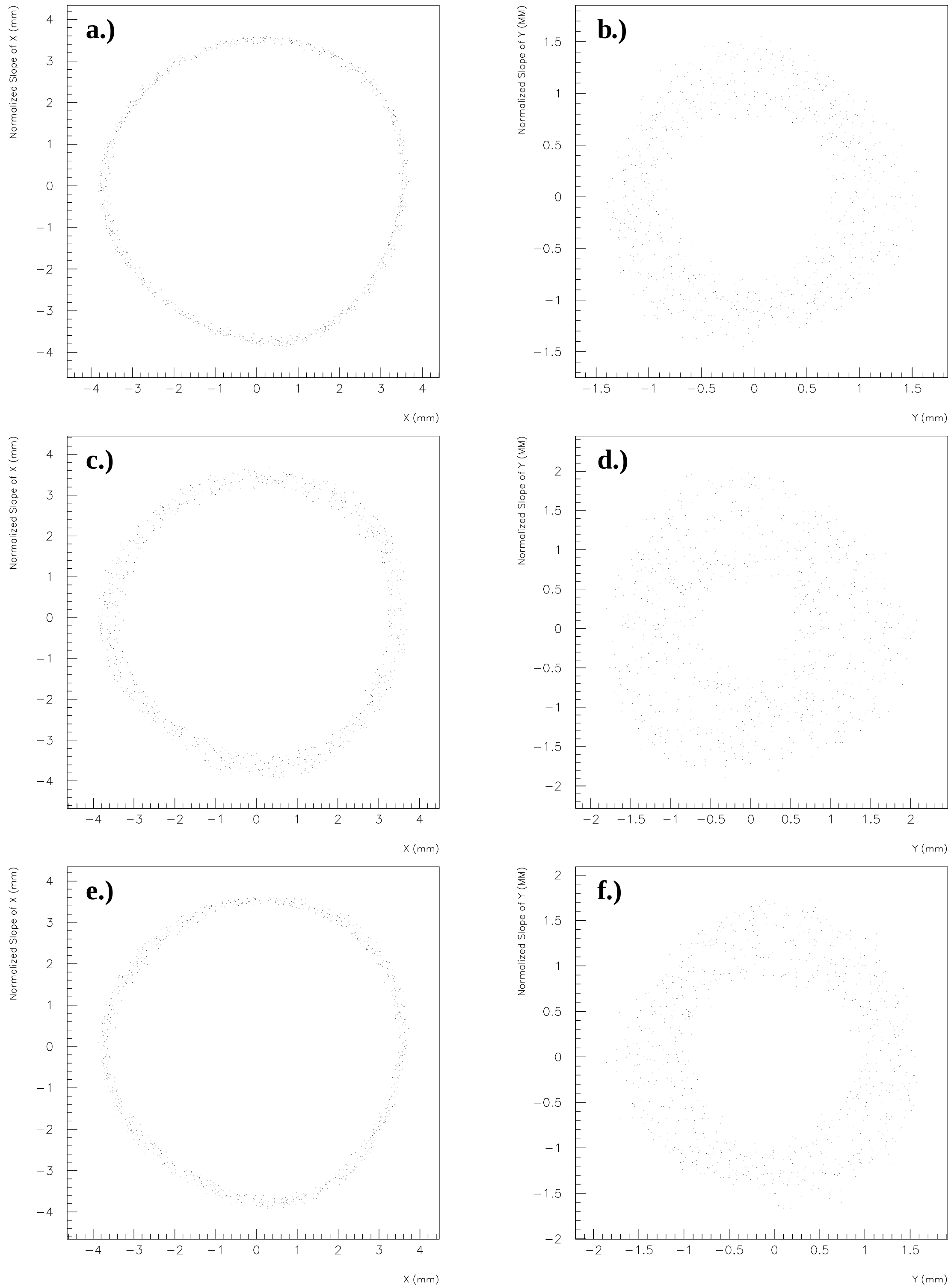

Figure 4: *Influence of skew octupoles on particle trajectories in transverse phase space*
Shown are the horizontal and vertical phase space projections for one seed of LHC optics version 5
Part **a.)** and **b.)** *target error table* only
Part **c.)** and **d.)** skew octupoles
Part **e.)** and **f.)** skew octupoles after correction

5 Conclusion

The expected values of the erect and skew octupolar components of the main dipoles lead to a significant reduction of the dynamic aperture compared to the case tracked with the *target error table*. The dynamic aperture can be fully restored by erect and skew octupole spool pieces which are used to cancel the erect octupole average of the machine and the skew subresonance (1, –1), respectively. Even though, each type of spool piece is present in only half of the machine the correction works almost as well as a truly local scheme. As a result there may be alternative ways to implement the correction during machine operation at least at the studied working point.

Minimisation of the detuning using lattice octupoles also proved useful to improve the dynamic aperture, although less effective. Moreover, the reduction by a factor of two in the number of lattice octupoles in LHC optics version 6 will further limit their effectiveness.

The preferred solution, assuming that the field quality of the dipoles and the alignment tolerances of the b_5 spool pieces cannot be improved, is therefore the correction scheme with both spool piece types, in particular as there is no alternative for the skew octupole correction. It remains to be seen if this semi–local correction works equally well for a working point close to fourth order resonances.

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