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Temperature in Fermion Systems and the Chiral Fermion Determinant

Shoichi ICHINOSE [1]

Physics Department, Brookhaven National Laboratory,
Upton, NY 11973, USA

Abstract

We give an interpretation to the issue of the chiral determinant in the heat-kernel approach. The extra dimension (5-th dimension) is interpreted as (inverse) temperature. The 1+4 dim Dirac equation is naturally derived by the Wick rotation for the temperature. In order to define a "good" temperature, we choose those solutions of the Dirac equation which propagate in a *fixed direction* in the extra coordinate. This choice fixes the regularization of the fermion determinant. The 1+4 dimensional Dirac mass (M) is naturally introduced and the relation: |4 dim electron momentum| $\ll |M| \ll$ ultraviolet cut-off, naturally appears. The chiral anomaly is explicitly derived for

[1]On leave of absence from Department of Physics, University of Shizuoka, Yada 52-1, Shizuoka 422-8526, Japan (Address after Nov.6, 1998).
E-mail address: ichinose@u-shizuoka-ken.ac.jp

the 2 dim Abelian model. Typically two different regularizations appear depending on the choice of propagators. One corresponds to the chiral theory, the other to the non-chiral (hermitian) theory.

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1 Introduction

Since Dirac introduced the first-order differential operator (Dirac operator) to describe the fermion propagation, we have been provided by a tool to describe rich fermion physics. The wonderful success of QED shows its deep importance. It is, however, true that interesting ambiguities related to the subtlety in its vacuum structure remain with the chiral anomaly problem. In order to define the quantum field theory, we must generally regularize the space-time. Equivalently, we must define the measure for quantum fluctuations. One of the fundamental regularizations is the lattice. Since its birth, the lattice approach has suffered from the fermion problem known as species doubling. In the necessity of simulating fermions on a lattice, a new regularization was found by [1, 2] using an idea of the higher dimensional theory[3]. This approach is called "Domain wall fermions," and has been well developed by some people [4, 5, 6, 7, 8, 9]. Especially Ref.[4] made seminal development in the early stage. The most characteristic point of the regularization is the introduction of one extra dimension. In accord with this, a new regularization mass parameter, that is, the 5 dimensional (dim) fermion mass M, appears. It controls the selection of chirality and plays the central role in this regularization scheme.

A continuum analysis has also been done by Narayanan-Neuberger and Randjbar-Daemi-Strathdee[10, 11, 12, 13, 14] in the form of the "Overlap formalism". Here we present another continuum formalism based on the heat-kernel. In this approach, a "temperature" parameter t is used. Previously for the elliptic operator, t is naturally introduced because of the positivity of its eigenvalues. For the Dirac operator, however, the direct application is difficult because of the negative eigenvalues. (So far, in order to avoid the difficulty, a roundabout method has customarily been taken, that is, we transform the problem of $\det \not{D}$ into that of $\det \not{D}^2$. See, for example, Chapter IV of [15].) We solve this difficulty by an idea based on the fact: when the temperature is defined in a system, the system evolves in a *fixed direction*.

The present formalism is a continuum approach based on the heat-kernel method. This is, of course, different from the lattice discrete regularization approach. Both approaches, however, have a common property: they are coordinate-based regularizations and are characterized by length parameters, t (inverse temperature or proper time) and a (lattice constant) respectively.

Because of this we can expect to find some common qualitative behaviours. In fact the analysis presented in the following has been developed by looking at the lattice formulation. Conversely some qualitative features (such as the "wall" configulation and the restriction to the regularization parameter M), which are rather complicated to show in lattice, can be simply shown in the present formalism.

In the (continuum) field theory, we must generally treat the space-time in some regularized way in order to control divergent quantities. So far many regularization methods have been invented. They are, with their characteristic parameters, Pauli-Vilars (M: heavy field mass), dimensional ($1/(n-4)$, n: space-time dimension), heat-kernel (t: inverse temperature), etc.. In this general standpoint of regularization, what status the domain wall regularization has ? Undoubtedly it has an advantageous point in regularizing chiral quantities. But how does it essentially work ? It has the 1+4 dim fermion mass M as the characteristic regularization parameter. In the lattice numerical simulation, we know the choice of M is rather delicate. It should be most appropriately chosen for a best-fit output. Why is the domain wall parameter M so different from that in other familiar regularizations ? In order to understand these things qualitatively, we are urged to have a new formalism which can be compared with known regularizations.

The main points found in this formalism are as follows.

1. Different regularizations appear depending on the choice of solutions in the 1+4 dim Dirac equation. Using these regularizations, the Adler-Bell-Jackiw (U(1) chiral) anomaly is obtained for 2 dim QED. Typically two regularizations appear. One gives the anomaly of QED, the other gives that of the chiral QED. The ABJ anomaly, in the original overlap formalism, was obtained in [10] .

2. As the extra axis, it should be a half line (not a line) like the temperature. (Otherwise the infra-red cut-off appears in the chiral anomaly, which should be avoided.) This is in close analogy to the domain wall and overlap formalisms[4].

3. The (+)- and (−)-domains are introduced simply as the result of the property of the continuum Dirac operator $\hat{D}$: $\hat{D}\gamma_5+\gamma_5\hat{D}=0$. This is contrasted with the original formulation of the overlap where the sign of 5 dim fermion masses distinguishes the two domains.

4. The reason why the "overlap" of $|+>$ and $|->$ should be taken in the anomaly calculation is manifest. The "overlap" in the partition function corresponds to a "difference" in the effective action.

5. The characteristic limit of the present regularization : $|k^\mu| \ll |M| \ll 1/t$, is explained carefully.

In Sec.2 we review the heat-kernel approach taking the 2 dim quantum gravity as an example. It is the ordinary treatment and should be compared with the proposed new one in the remaining sections. We present the new treatment for fermion systems in Sec.3. It is regarded as new regularization (for the chiral determinant) and is composed of three stages. We apply it to the chiral anomaly calculation of 2 dim QED in Sec.4. We conclude in Sec.5.

2 Heat-Kernel Approach

The heat-kernel method has a long history as a tool to define $\det \hat{D}$ for a differential operator $\hat{D}$. Here the cap symbol ($\hat{}$) denotes the operator nature of $\hat{D}$. Let us briefly review the properties necessary for the next section. We take, as an explicit model, the 2 dim (Euclidean) quantum scalar theory ϕ on the background curved space $g_{\mu\nu}$:

$$\mathcal{L}[g_{\mu\nu},\phi] = \sqrt{g}\phi(-\frac{1}{2}\nabla^2)\phi \equiv \frac{1}{2}\tilde{\phi}\hat{D}\tilde{\phi}(x) \ , \ \hat{D} \equiv -\sqrt[4]{g}\nabla_x^2\frac{1}{\sqrt[4]{g}} \ , \ \tilde{\phi} \equiv \sqrt[4]{g}\ \phi \ . \quad (1)$$

We focus on the partition function and the Weyl anomaly of this theory. $\hat{D}$ is a positive (semi)definite operator. The partition function is expressed as

$$Z[g] = \int \mathcal{D}\tilde{\phi}\, \mathrm{e}^{-\int d^2x\mathcal{L}} = (\det \hat{D})^{-\frac{1}{2}}$$
$$= \exp\{-\frac{1}{2}\mathrm{Tr}\ \ln \hat{D}\} = \exp[\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{\mathrm{e}^{-t\hat{D}}}{t}dt + \mathrm{const}\,] \quad (2)$$

Here we have used a useful formula for a matrix A: $\int_0^\infty \frac{\mathrm{e}^{-t}-\mathrm{e}^{-tA}}{t}dt = \ln A$. This is well-defined when all eigenvalues of A are positive[16]. The heat-kernel is defined by

$$G(x,y;t) \equiv < x|e^{-t\hat{D}}|y> \quad ,$$
$$(\frac{\partial}{\partial t} + \hat{D}_x)G(x,y;t) = 0 \quad , \quad \lim_{t\to+0} G(x,y;t) = \delta^2(x-y) \quad , \quad (3)$$

where $< x|$ and $|y>$ are in the x-representation of $\hat{D}$ (Dirac's bra- and ket-vectors respectively). (This formalism is familiar in statistical mechanics as the "density matrix"[17].) The final equation is the boundary condition we take. The parameter t has two properties: 1) Dimension of t is (length)^2 ($\hat{D}$ is the quadratic differential operator.); 2) t is restricted to be positive. The first property shows that the distribution in x-space at t is localized within a distance $\sqrt{t}$. The second one reflects the property of heat: heat propagates from the high temperature to the low temperature (the second law of the thermodynamics). For the positive definite operator $\hat{D}$, like the present case, we can naturally introduce the temperature as $1/t$ using this heat-kernel.

$G(x,y;t)$ is solved by the (weak-field) perturbation: $g_{\mu\nu} = \delta_{\mu\nu}+h_{\mu\nu}, |h_{\mu\nu}| \ll 1, \hat{D} = -\partial^2 - \vec{V}, \vec{V} = -h_{\mu\nu}\partial_\mu\partial_\nu - \partial_\nu h_{\mu\nu}\partial_\mu - \frac{1}{4}\partial^2 h + O(h^2)$ (see Ref.[18]) using two ingredients.

(i) Heat Equation without source(Free Equation)

$$
\begin{gathered}
(\frac{\partial}{\partial t} - \partial^2)G_0(x-y;t) = 0 \quad , \quad t > 0 \quad , \lim_{t\rightarrow +0} G_0(x-y;t) = \delta^2(x-y) \quad , \\
G_0(x-y;t) = \int \frac{d^2k}{(2\pi)^2} exp\{-k^2 t + ik\cdot(x-y)\} = \frac{1}{4\pi t} e^{-\frac{(x-y)^2}{4t}} \quad . \quad (4)
\end{gathered}
$$

(ii) Heat Propagator

$$
\begin{gathered}
(\frac{\partial}{\partial t} - \partial^2)S(x-y;t-s) = \delta(t-s)\delta^2(x-y) \quad , \lim_{t\rightarrow +0} S(x-y;t) = \delta^2(x-y) \quad , \\
S(x-y;t) = \int \frac{d^2k}{(2\pi)^2}\frac{dk^0}{2\pi}\frac{exp\{-ik^0 t + ik\cdot(x-y)\}}{-ik^0 + k^2} = \theta(t)G_0(x-y;t) \quad . \quad (5)
\end{gathered}
$$

We have imposed an appropriate boundary condition above, corresponding to the one in (3). The perturbative solution is given by iterating

$$
G(x,y;t) = G_0(x-y;t) + \int d^2z \int_{-\infty}^{\infty} ds\ S(x-z;t-s)\hat{V}(z)G(z,y;s) \quad , \quad (6)
$$

where $\hat{D} = -\partial^2 - \hat{V}$. The lowest order solution of the present 2 dim scalar-gravity theory is given by

$$
G(x,x;t) \ = \ G_0(0;t) + G_1(x,x;t) + O(h^2)
$$

$$= \frac{1}{4\pi t}(1+\frac{h}{2}) - \frac{1}{24\pi}(\partial^2 h - \partial_\mu\partial_\nu h_{\mu\nu}) + O(h^2) + O(t) \quad (7)$$
$$= \frac{1}{4\pi t}(\sqrt{g} + O(h^2)) - \frac{\sqrt{g}}{24\pi}(R + O(h^2)) + O(t) \quad , \quad (8)$$

where the Riemann scalar curvature R is expanded as $R = \partial^2 h_{\mu\mu} - \partial_\mu\partial_\nu h_{\mu\nu} + O(h^2)$. Introducing an (inverse) ultra-violet cut-off $\epsilon(\rightarrow +0)$ for t-integral in (2), we can regularize $\Gamma[g] \equiv \ln Z[g]$ as

$$\Gamma^{\mathrm{reg}}[g] = \frac{1}{2}\int_\epsilon^\infty \frac{dt}{t}\int d^2x < x|\mathrm{e}^{-t\hat{D}}|x > +\mathrm{const} \quad . \quad (9)$$

Its variation under the scale transformation $\delta\epsilon = \Delta \cdot \epsilon$ gives us the Weyl anomaly.

$$\delta\Gamma^{\mathrm{reg}} = \frac{1}{2}\left[\int d^2x\, G(x,x;\epsilon)\right]\cdot\Delta$$
$$= \Delta \cdot \int d^2x \left[-\frac{1}{48\pi}\sqrt{g}R + \text{cosmological term}\right] \quad . \quad (10)$$

The same result is obtained by the measure change[19, 20, 18].

$$\delta \ln J = \delta(\ln \det \frac{\partial\tilde{\phi}'}{\partial\tilde{\phi}}) = -\mathrm{Tr}\,\alpha(x)\delta^2(x-y) = -\lim_{t\rightarrow+0}\mathrm{Tr}\;\alpha(x)G(x,y;t) \quad ,(11)$$

where $\tilde{\phi}' = \mathrm{e}^{-\alpha(x)}\tilde{\phi}$. This approach will be taken in the evaluation of the chiral anomaly in Sec.4. The result (10) was exploited in the 2 dim quantum gravity[21].

We pointed out (in eq.(5)) that the heat propagator $S(x-y;t)$ has the characteristic form : the theta function, $\theta(t)$, times the free solution $G_0(x,y;t)$. The theta function guarantees that the propagation is going in the fixed direction, that is, forward in the t-axis. We will utilize this fact in the choice of boundary conditions.

3 Chiral Fermion Determinant

Let us start with 4 dim Euclidean *massless Dirac* fermion ψ in the external Abelian gauge field A_μ.[22]

$$\mathcal{L} = \bar{\psi}\hat{D}\psi \quad , \quad \hat{D} = i\gamma_\mu(\partial_\mu + i\mathrm{e}A_\mu) \quad , \quad \hat{D}^\dagger = \hat{D} \quad . \quad (12)$$

We can formally express the determinant as in the previous section.

$$Z[A]=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi \mathrm{e}^{-\int d^4x\mathcal{L}}=(\det\hat{D})^{+1}$$
$$=\exp\{\mathrm{Tr}\ \ln\hat{D}\}=\exp[-\mathrm{Tr}\int_0^\infty\frac{\mathrm{e}^{-t\hat{D}}}{t}dt+\mathrm{const}\ ] \tag{13}$$

Clearly the following things are different from Sec.2: 1) Dimension of t is length ($\hat{D}$ is the *linear* differential operator); 2) $\hat{D}$ has both positive and negative solutions. Due to 2), the t-integral is divergent and further regularization is necessary to define $\det\hat{D}$. The purpose of this paper is to define the integrand inside the t-integral above, using the operator $\hat{D}$ in such a way that the t-integral is convergent and the property as the determinant is preserved (regularization of the fermion determinant). We will do it in three stages.

(1) First Stage

Continuing formally a little longer, the heat-kernel $G(x,y;t)=<x|e^{-t\hat{D}}|y>$ satisfies (3). The operator $\exp\{-t\hat{D}\}$ satisfies, using $(\gamma_5)^2=1$, the relation

$$\exp\{-t\hat{D}\}=\frac{1}{2}(1+i\gamma_5)\exp\{+it\gamma_5\hat{D}\}+\frac{1}{2}(1-i\gamma_5)\exp\{-it\gamma_5\hat{D}\}\quad . \tag{14}$$

This is valid for any operators which satisfy

$$\gamma_5\hat{D}+\hat{D}\gamma_5=0\quad . \tag{15}$$

In eq.(14), $\frac{1}{2}(1\pm i\gamma_5)$ are Wick-rotated variations of the chiral projection operators $P_\pm=\frac{1}{2}(1\pm\gamma_5)$. This hints at the 5 dimensionality behind the discussion.

Here we introduce two new heat-kernels corresponding to the two terms in RHS of (14).

$$G^5_\pm(x,y;t)\equiv<x|\exp\{\pm it\gamma_5\hat{D}\}|y>\quad ,\quad (\frac{\partial}{\partial t}\mp i\gamma_5\hat{D})G^5_\pm(x,y;t)=0\quad . \tag{16}$$

Very interestingly, the above heat equations satisfy the 1+4 dim Minkowski Dirac equation after appropriate Wick rotations for t.

$$(i\partial\!\!\!/-M)G^{5M}_+=ie A\!\!\!/ G^{5M}_+\quad ,\quad (X^a)=(-it,x^\mu)\quad ,$$

$$
\begin{aligned}
(i\partial\!\!\!/ - M')G_-^{5M'} = ie A\!\!\!/ G_-^{5M'} \quad &, \quad (X^a) = (+it, x^\mu) \quad , \\
G_+^{5M}(x,y;t) = < x|\exp\{+it\gamma_5(\hat{D}+iM)\}|y > \quad &, \\
G_-^{5M'}(x,y;t) = < x|\exp\{-it\gamma_5(\hat{D}+iM')\}|y > \quad &.
\end{aligned} \tag{17}
$$

where $A\!\!\!/ \equiv \gamma_\mu A_\mu(x)$, $\partial\!\!\!/ \equiv \Gamma^a \frac{\partial}{\partial X^a}$, and we have introduced new regularization mass parameters $M, M'(\to 0)$ for the next step[23]. Note that the Wick-rotation is taken differently for G_+^{5M} and $G_-^{5M'}$. The sign convention for the 1+4 dim fermion masses formally follows the textbook by Bjorken and Drell[24]. However we do *not* fix their signs here. In the above Dirac equations, $A\!\!\!/$ in the right-hand sides is purely a 4 dim object[25]. Others are 1+4 dim ones. Hence we obtain a regularized partition function, at the present stage, as

$$
\begin{aligned}
\ln Z = -\int_0^\infty \frac{dt}{t}\mathrm{Tr}\left[\frac{1}{2}(1+i\gamma_5)\exp\{it\gamma_5\hat{D}\} + \frac{1}{2}(1-i\gamma_5)\exp\{-it\gamma_5\hat{D}\}\right] \\
= -\lim_{M\to 0}\int_0^\infty \frac{dt}{t}\frac{1}{2}(1-i\frac{\partial}{\partial(tM)})\mathrm{Tr}\, G_+^{5M}(x,y;t) \\
-\lim_{M'\to 0}\int_0^\infty \frac{dt}{t}\frac{1}{2}(1-i\frac{\partial}{\partial(tM')})\mathrm{Tr}\, G_-^{5M'}(x,y;t) \quad .
\end{aligned} \tag{18}
$$

Here we understand G_+^{5M} and $G_-^{5M'}$ are, first calculated in the 1+4 dim(X^a), and then go back to (t, x^μ) by the Wick rotations. Note "Tr" means the 4 dimensional trace (not over the 5-th dimension). The rôle of M, M' looks just like a technical trick at present. From the usage above, the limiting condition for the parameters $(M, M' \to 0)$ should be taken as

$$
t|M| \ll 1 \quad , \quad t|M'| \ll 1 \quad . \tag{19}
$$

In the following we take $M = M'$ for simplicity.

(2) Second Stage

Next stage of the regularization program is to give solutions to G_+^{5M} and $G_-^{5M'}$, taking into account the boundary conditions. We do the calculation in 1+4 dim. The perturbative solution of the 1+4 dim Dirac equation is again given by Ref.[24].

$$
\begin{aligned}
(i\partial\!\!\!/ - M)G_M^5 = ie A\!\!\!/ G_M^5 \quad &, \\
G_M^5(X,Y) = G_0(X,Y) + \int d^5Z\, S(X,Z) ie A\!\!\!/(z) G_M^5(Z,Y) \quad &,
\end{aligned} \tag{20}
$$

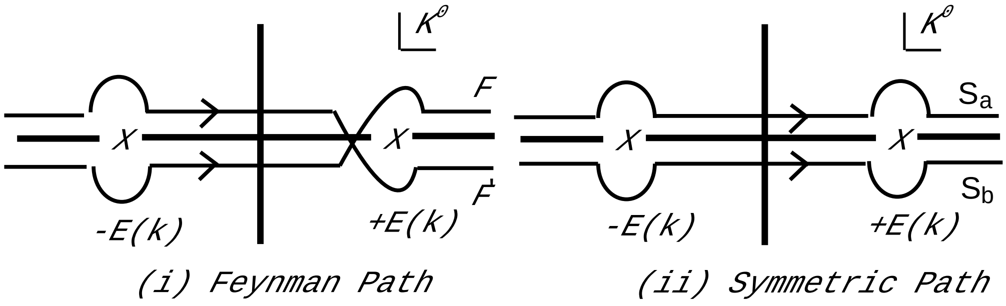

Fig.1 Four possible pathes for the 1+4 dim Dirac Fermion propagator. $E(k) = \sqrt{k^2 + M^2} > 0$.

where $(X^a) = (X^0, X^\mu = x^\mu)$ and $G_0(X,Y)$ is the free solution and $S(X,Z)$ is the propagator.

$$(i\partial\!\!\!/ - M)G_0(X,Y) = 0 \quad , \quad (i\partial\!\!\!/ - M)S(X,Y) = \delta^5(X-Y) \quad . \tag{21}$$

Solutions depend on boundary conditions on $X^0 - Y^0$. There are four possible ways to go around the positive and negative energy poles in the K^0-integral in its complex plane. (See Fig.1.) 1)[F-path] The familiar Feynman propagator takes the path: "below" for the negative pole and "above" for the positive one. This treats the positive and negative poles discriminatively and gives the retarded propagator and the advanced propagator. 2)[F′-path] Its opposite choice ("above" for negative and "below" for positive) can also be considered. The other pair of choices treats both poles equally. 3)[Sa-path] The path taken as "above" for the positive and negative poles gives a retarded propagator. 4)[Sb-path] Its counter choice, "below" for both poles, gives an advanced propagator. Let us discuss them separately.

i) Feynman Propagator (F-path,F′-path)
First we consider the Feynman propagator.

$$S_F(X,Y) = \int \frac{d^5K}{(2\pi)^5} \frac{\mathrm{e}^{-iK(X-Y)}}{K\!\!\!/ - M + i\epsilon} \quad , \quad K\!\!\!/ = \Gamma^a K_a \quad , \tag{22}$$

where $\epsilon(\to +0)$ is introduced to define the boundary condition. (Note that ϵ is only for specifying the path of the K^0-integral and should not be regarded as a present regularization parameter.) After the K^0-integral, we see $S_F(X,Y)$ is composed of the retarded part and the advanced one in X^0-axis.

$$
\begin{aligned}
S_F(X,Y) &= \theta(X^0-Y^0)G_0^p(X,Y)+\theta(Y^0-X^0)G_0^n(X,Y) \quad , \\
G_0^p(X,Y) &\equiv -i\int\frac{d^4k}{(2\pi)^4}\Omega_+(k)\mathrm{e}^{-i\tilde{K}(X-Y)} \quad , \ \Omega_+(k)\equiv\frac{M+\tilde{K}\!\!\!/}{2E(k)} \quad , \\
G_0^n(X,Y) &\equiv -i\int\frac{d^4k}{(2\pi)^4}\Omega_-(k)\mathrm{e}^{+i\bar{K}(X-Y)} \quad , \ \Omega_-(k)\equiv\frac{M-\bar{K}\!\!\!/}{2E(k)} \quad ,
\end{aligned}
\tag{23}
$$

where
$E(k)=\sqrt{k^2+M^2}$, $(\tilde{K}^a)=(E(k),K^\mu=-k^\mu)$, $(\bar{K}^a)=(E(k),-K^\mu=k^\mu)$. k^μ is the momentum in the 4 dim Euclidean space.[26] $\tilde{K}$ and $\bar{K}$ are on-shell momenta($\tilde{K}^2=\bar{K}^2=M^2$), which correspond to the positive and negative energy states respectively.[27] Hence $G_0^p(X,Y)$ ($G_0^n(X,Y)$) is constructed by the positive (negative) energy eigenstates (see Ref.[24]).

In analogy to Sec.2, we should take a solution in such a way that there exists a fixed direction in time. Taking into account the t-integral convergence, we are uniquely led to the following solution of G_+^{5M} and G_-^{5M}.

$$
\text{Retarded solution for} \quad G_+^{5M}:
$$

$$
G_0(X,Y)=G_0^p(X,Y) \quad , \quad S(X,Y)=\theta(X^0-Y^0)G_0^p(X,Y) \tag{24}
$$

$$
\text{Advanced solution for} \quad G_-^{5M}:
$$

$$
G_0(X,Y)=G_0^n(X,Y) \quad , \quad S(X,Y)=\theta(Y^0-X^0)G_0^n(X,Y) \tag{25}
$$

This is the first choice of the 2nd stage regularization. The configuration where the positive energy states propagate only in the forward direction of X^0 constitutes the $(+)$-domain, while the configuration where the negative enegy states propagate only in the backward direction constitutes the $(-)$-domain.

The similar regularization is obtained by the opposite choice of path to Feynman propagator, that is, 2)[F′-path].

ii) Symmetric Propagators (Sa-path,Sb-path)
We have another choices of the K^0-integral path which is symmetric with

respect to positive and negative energy states.

$$
\begin{aligned}
S_{\mathrm{sym}}^{\mathrm{ret}}(X,Y) &= \theta(X^0-Y^0)G_0^{\text{p-n}}(X,Y) \quad , \\
G_0^{\text{p-n}}(X,Y) \equiv -i\int\frac{d^4k}{(2\pi)^4}&\{\Omega_+(k)\mathrm{e}^{-i\tilde{K}(X-Y)}-\Omega_-(k)\mathrm{e}^{i\bar{K}(X-Y)}\} \\
&= G_0^p(X,Y)-G_0^n(X,Y) \quad , \\
S_{\mathrm{sym}}^{\mathrm{adv}}(X,Y) &= \theta(Y^0-X^0)G_0^{\text{n-p}}(X,Y) \quad , \\
G_0^{\text{n-p}}(X,Y) &\equiv -G_0^{\text{p-n}}(X,Y) \quad , \qquad (26)
\end{aligned}
$$

where $S_{\mathrm{sym}}^{\mathrm{ret}}$ and $S_{\mathrm{sym}}^{\mathrm{adv}}$ are obtained by taking Sa-path and Sb-path respectively. Using these solutions we obtain the second choice for the 2nd stage regularization.

$$
\text{Symmetric retarded solution for} \quad G_+^{5M}:
$$

$$
G_0(X,Y)=G_0^{\text{p-n}}(X,Y) \quad , \quad S(X,Y)=\theta(X^0-Y^0)G_0^{\text{p-n}}(X,Y) \qquad (27)
$$

$$
\text{Symmetric advanced solution for} \quad G_-^{5M}:
$$

$$
G_0(X,Y)=G_0^{\text{n-p}}(X,Y) \quad , \quad S(X,Y)=\theta(Y^0-X^0)G_0^{\text{n-p}}(X,Y) \qquad (28)
$$

In this second choice, both positive and negative states propagate in the forward direction in + domain, while in the backward direction in - domain.

Differences between the cases i) and ii) will be explained later. We consider the two cases as different regularizations generally.

(3) Third Stage

As the final stage of the regularization, we give the condition:

$$
|\frac{k^\mu}{M}| \ll 1 \quad . \qquad (29)
$$

This condition corresponds to "fermion zero mode limit". This limit plays the role of selecting a definite chirality. In fact in this limit ($M>0$ is taken here), the Fourier transform of G_0^p and G_0^n is

$$
iG_0^p(k)\sim\Omega_+(k)=\frac{M+\tilde{K}\!\!\!/}{2E(k)}\to\frac{1+\gamma_5}{2} \ , \ iG_0^n(k)\sim\Omega_-(k)=\frac{M-\bar{K}\!\!\!/}{2E(k)}\to\frac{1-\gamma_5}{2} \ ,(30)
$$

which indicates the present regularization clearly selects both chiralities in the first choice (of the 2nd stage) and treats them equally in the second choice.

Let us reconsider another condition (19). The phases of G_0^p and G_0^n are given by

$$\mathrm{e}^{-i\tilde{K}X}=\mathrm{e}^{-iE(k)X^0-ikx}\quad,\quad \mathrm{e}^{i\bar{K}X}=\mathrm{e}^{iE(k)X^0-ikx}\quad. \tag{31}$$

As far as the condition (29) is valid, the condition (19) means $|E(k)X^0|=|E(k)t|\ll 1$. The coordinate X^0 becomes irrelevant and the system dominantly works in the 4 dim x^μ-space (Dimensional reduction[28]). Furthermore, in the case of the first choice (Feynman, F-path), both phases above are proportional to $\mathrm{e}^{-E(k)t}\sim \mathrm{e}^{-|M|t}$ in the original coordinate t ($X^0=-it$ for the former, $X^0=it$ for the latter). It says the fermion is localized within a distance $1/|M|$ around the origin of the extra axis of t("wall" structure). As for the second choice (symmetric), both $\mathrm{e}^{-|M|t}$ and $\mathrm{e}^{+|M|t}$ modes coexist. Therefore the configuration looks like that one wall exists around the origin and the other exists near the boundary($t=\infty$). For both choices we may say the two conditions (19) and (29) combined give the reduction to the 4 dim theory. Considering these two conditions (19) and (29), we obtain

$$|k^\mu|\ll|M|\ll\frac{1}{t}\quad. \tag{32}$$

This relation shows the delicacy in taking the limit in the 1+4 dimensional regularization scheme. It restricts the configuration to the ultra-violet region ($t\ll|M|^{-1}$) in the extra space, whereas to the infra-red (surface) region in the real 4 dim space ($|k^\mu|\ll|M|$) [29]. In the present case we must note that both k^μ and t are integration variables. In the concrete calculation below, first the condition $|k^\mu|\ll|M|$ is realized by suppressing the large $|k^\mu|$ region, compared to $|M|$, in the k^μ-integral. (See [32] for the practical situation.) After the k^μ-integral, $|M|t\to+0$ limit is taken before the t-integral is performed. This relation (32) implies, in lattice, $|M|$ should be appropriately chosen depending on the regularization scale and the considered momentum-region of 4 dim fermions. (In fact, in the lattice simulation, the best-fit value of M ($\sim$ a few Gev for the hadron simulation) looks to depend on the simulation "environment" such as the size of the ordinary-space-axes, the size of the extra-axis and the quark mass[9, 30].)

Using (30), we can easily read off the boundary conditions for the free solutions in (24),(25),(27) and (28). They are equal to corresponding full solutions by its construction. Then we obtain the following table of the boundary conditions.

	G_+^{5M}(Retarded)	G_-^{5M}(Advanced)
Limit	$\lvert M\rvert(X^0 - Y^0) \to +0$	$\lvert M\rvert(X^0 - Y^0) \to -0$
Feynman	$-i\int \frac{d^4k}{(2\pi)^4}\Omega_+(k)\mathrm{e}^{-ik(x-y)}$	$-i\int \frac{d^4k}{(2\pi)^4}\Omega_-(k)\mathrm{e}^{-ik(x-y)}$
Symmetric	$-i\gamma_5\delta^4(x-y)$	$+i\gamma_5\delta^4(x-y)$

Table 1 List of boundary conditions.

For the Feynman path, the boundary condition above says

$$i\gamma_5(G_+^{5M}(X,Y) - G_-^{5M}(X,Y)) \to \delta^4(x-y) \quad , \quad |M|\cdot|X^0 - Y^0| \to +0 \quad ,(33)$$

which will be used later.

4 Chiral Anomaly in 2 Dim QED

Let us evaluate the chiral anomaly to check that the present regularization works correctly. For simplicity, a 2-dim Abelian gauge model is taken. The previous formulae are all valid by the replacement: μ runs from 1 to 2, $\int d^4k/(2\pi)^4 \to \int d^2k/(2\pi)^2$. It is known that the chiral anomaly is obtained by the measure change due to the chiral transformation of ψ[19, 20].

$$\delta\psi = i\alpha(x)\gamma_5\psi \quad , \quad \delta\bar{\psi} = \bar{\psi}(i\alpha(x)\gamma_5) \quad , \quad |\alpha(x)| \ll 1 \quad . \tag{34}$$

The variation of the Jacobian is formally given by

$$\delta\ \ln\ J \equiv \delta\ \ln\ \det\ \frac{\partial(\bar{\psi}',\psi')}{\partial(\bar{\psi},\psi)} = \mathrm{Tr}\left\{i\alpha(x)\gamma_5\delta^2(x-y) + i\alpha(x)\gamma_5\bar{\delta}^2(x-y)\right\} \quad ,(35)$$

where $\delta^2(x-y)$ and $\bar{\delta}^2(x-y)$ are two delta functions which are not necessarily regularized in the same way. Now we regularize $\gamma_5\delta^2(x-y)$ by replacing it by the heat-kernels obtained in Sec.3. We have two different choices corresponding to i) and ii) of Sec.3. In this section we take $M>0$.

i) Feynman path

First we consider the case of Feynman propagator. Taking into account the boundary condition (33), we should take

$$\frac{1}{2}\delta \ln J = \lim_{M\cdot|X^0-Y^0|\to+0} \mathrm{Tr}\ i\alpha(x) i(G_+^{5M}(X,Y) - G_-^{5M}(X,Y)) \quad . \tag{36}$$

Only the 1-st order (with respect to A_μ) perturbation contributes. The first term is evaluated as

$$\begin{aligned} G_+^{5M}|_A &= \int_0^{X^0} dZ^0 \int d^2Z G_0^p(X,Z) ie\not{A}(z) G_0^p(Z,Y) \\ &= \int_0^{X^0} dZ^0 \int d^2z \int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2} \\ &\times(-i)\frac{M+\tilde{\not{K}}}{2E(k)} ie\not{A}(z)(-i)\frac{M+\tilde{\not{L}}}{2E(l)} \mathrm{e}^{-i\tilde{K}(X-Z)}\mathrm{e}^{-i\tilde{L}(Z-Y)} \quad . \end{aligned} \tag{37}$$

Here we have made an important assumption about the extra axis: the axis is a half line (not a (straight) line) like the temperature (t) axis of Sec.2.[31] Instead of z, we take a shifted variable z' (we do not change Z^0), and expand $A_\mu(z)$ around the "center" $(x+y)/2$.

$$z = z' + \frac{x+y}{2}\ ,\ A_\mu(z) = A_\mu(\frac{x+y}{2}) + \partial_\alpha A_\mu|_{\frac{x+y}{2}}\cdot z'^\alpha + O(z'^2) \quad . \tag{38}$$

The second term, $\partial_\alpha A_\mu$, contributes to the anomaly. The final relevant term, after the momentum integrals taking X^0-coordinate($Y^0=0$), turns out to be

$$\begin{aligned} \mathrm{Tr}\ \alpha(x) G_+^{5M}|_A &\sim \int d^2x\, \alpha(x)(-\frac{i}{8\pi})e\epsilon_{\mu\nu}\partial_\mu A_\nu \\ \times MX^0 \int_0^\infty ds \frac{s(s^2+2)}{(s^2+1)^{3/2}} \sin(MX^0\sqrt{s^2+1}) &\to \int d^2x\, \alpha(x)\left[-\frac{i}{8\pi}e\epsilon_{\mu\nu}\partial_\mu A_\nu\right] \end{aligned} \quad ,(39)$$

as $MX^0 \to +0$.[32] In this case, we can also obtain the same result in the original t-coordinate($X^0 = -it, Y^0 = 0$).

$$
\begin{aligned}
\mathrm{Tr}\ \alpha(x) G_+^{5M}|_A \sim \int d^2x\, \alpha(x)(-\frac{i}{8\pi}) e\epsilon_{\mu\nu}\partial_\mu A_\nu \times Mt \int_0^\infty ds \frac{s(s^2+2)}{(s^2+1)^{3/2}} \mathrm{e}^{-Mt\sqrt{s^2+1}} \\
\to \int d^2x\, \alpha(x) \left[-\frac{i}{8\pi} e\epsilon_{\mu\nu}\partial_\mu A_\nu \right] \quad (Mt \to +0) \quad .(40)
\end{aligned}
$$

G_-^{5M} is evaluated similarly and gives the same result except the sign. The final chiral anomaly is

$$
\frac{1}{2}\frac{\delta}{\delta\alpha(x)} \ln J = \frac{1}{2}\frac{1}{J}\frac{\delta J}{\delta\alpha(x)} = +\frac{i}{4\pi} e\epsilon_{\mu\nu}\partial_\mu A_\nu \quad . \tag{41}
$$

which is the half of the well-known ABJ (U(1) chiral) anomaly. This result will be commented after the next case.

ii) Symmetric path

The second choice at the stage 2 of Sec.3 gives us another regularization. From the boundary condition of the symmetric path, in Table 1, we obtain,

$$
\begin{aligned}
\delta \ln J = & \lim_{M(X^0-Y^0)\to+0} \mathrm{Tr}\ i\alpha(x) i G_+^{5M}(X,Y) \\
& + \lim_{M(X^0-Y^0)\to-0} \mathrm{Tr}\ i\alpha(x)(-i) G_-^{5M}(X,Y) \\
= & \lim_{M\cdot|X^0-Y^0|\to+0} \mathrm{Tr}\ i^2\alpha(x)(G_+^{5M}(X,Y) - G_-^{5M}(X,Y)) \quad .
\end{aligned} \tag{42}
$$

The $(+)$-domain term is evaluated from

$$
\begin{aligned}
G_+^{5M}|_A = \int_0^{X^0} dZ^0 \int d^2Z (G_0^p(X,Z) - G_0^n(X,Z)) ie\not{A}(z)(G_0^p(Z,Y) - G_0^n(Z,Y)) \\
= \int_0^{X^0} dZ^0 \int d^2z \int \frac{d^2k}{(2\pi)^2}\frac{d^2l}{(2\pi)^2} \\
\times(-i)(\frac{M+\tilde{\not{K}}}{2E(k)} \mathrm{e}^{-iE(k)(X^0-Z^0)} - \frac{M-\bar{\not{K}}}{2E(k)} \mathrm{e}^{iE(k)(X^0-Z^0)}) ie\not{A}(z) \\
\times(-i)(\frac{M+\tilde{\not{L}}}{2E(l)} \mathrm{e}^{-iE(l)(Z^0-Y^0)} - \frac{M-\bar{\not{L}}}{2E(l)} \mathrm{e}^{iE(l)(Z^0-Y^0)}) \mathrm{e}^{-ik(x-z)-il(z-y)} \quad .(43)
\end{aligned}
$$

As in i), after expanding $A(z) = A(z' + (x+y)/2)$, we are led to the following one, as the relevant part for the anomaly,

$$
\begin{aligned}
&\mathrm{Tr}\ \alpha(x) G_+^{5M}|_A \sim \\
&= \int_0^{X^0} dZ^0 \int \frac{d^2k}{(2\pi)^2} (-i)^2 \cdot ie \cdot (-\frac{1}{i}) \int d^2x \alpha(x) \partial_\mu A_\nu \\
&\times \mathrm{tr}\,[\frac{i}{4}\gamma_5(\partial_\mu \frac{\not{k}}{E})\gamma_\nu(-2i \sin EX^0) \\
&-i\partial_\mu E \cdot (X^0 - Z^0) \cdot \frac{i\gamma_5 \not{k}}{2E} \cdot \gamma_\nu \cdot 2\cos E(X^0 - 2Z^0)] \\
&= \int d^2x\, \alpha(x)(-\frac{i}{4\pi}) e\epsilon_{\mu\nu}\partial_\mu A_\nu \\
&\times MX^0 \int_0^\infty ds [\frac{s(s^2+2)}{(s^2+1)^{3/2}} + \frac{s^3}{(s^2+1)^{3/2}}] \sin(MX^0\sqrt{s^2+1}) \\
&\to \int d^2x\, \alpha(x) \left[-\frac{i}{2\pi} e\epsilon_{\mu\nu}\partial_\mu A_\nu \right] \quad (MX^0 \to +0) \quad ,
\end{aligned} \tag{44}
$$

where we have done the k^μ-integral in X^0-coordinate. In comparison with the previous one, we cannot do the same calculation using the t-coordinate because of the apperance of $\mathrm{e}^{+tE(k)}$ factor. Adding the G_-^{5M} contribution, we obtain

$$
\frac{1}{J}\frac{\delta J}{\delta\alpha(x)} = +\frac{i}{\pi} e\epsilon_{\mu\nu}\partial_\mu A_\nu \quad . \tag{45}
$$

which is the known ABJ anomaly (eqs.(4.156),(4.181), and (12.195) of [15]) and is two times of the previous result (41).

The symmetric path gives the correct value, whereas the Feynman path gives half of it. The discrepancy comes from the fact that we have taken , for the latter case, the "chirally-divided" propagators, $S_F^+(X,Y) \equiv \theta(X^0 - Y^0) G_0^p(X,Y)$, $S_F^-(X,Y) \equiv \theta(Y^0 - X^0) G_0^n(X,Y)$. They do not satisfy (21), but satisfy (2D version of)

$$
\begin{aligned}
(i\not{\partial} - M) S_F^+(X,Y) &= P_+ \delta^5(X-Y) + O(\frac{1}{M}) \quad , \\
(i\not{\partial} - M) S_F^-(X,Y) &= P_- \delta^5(X-Y) + O(\frac{1}{M}) \quad .
\end{aligned} \tag{46}
$$

Therefore the chiral anomaly computed in i) Feynman path is not that of the (2D)QED (12), but is effectively that of the chiral (2D)QED:

$$\mathcal{L}_{chiral} = \bar{\psi}(\gamma_\mu \partial_\mu + i\mathrm{e} P_\pm \not{A})\psi \quad . \tag{47}$$

In fact the Feynman result (41) is consistent with the minimal case of (14.12) of [15]. The practical advantage of the Feynman path is its simplicity in the evaluation.

5 Conclusion

The main motivation for the present work is to clarify the real meaning or the rôle of the extra dimension in the lattice formulation. So far the extra axis has been given a rather obscure meaning such as "a sophisticated flavour space"[4]. In the present formalism, the 5-th (extra) dimension is interpreted as the (inverse) temperature. For the (Euclidean) boson system which has an elliptic differential operator, such as the case of Sec.2, it is easy to treat its canonical ensemble using the heat equation (3) and the temperature can be introduced without any difficulty. For the fermion system, however, the direct use has been considered hard because of the appearance of the negative eigenvalues. We have solved the difficulty by *generalizing the concept of the temperature* as the parameter along which the system evolves in a *fixed direction*. It is based on the analogy to the thermo-dynamical system. By choosing the "directed" Dirac fermion propagation in the 1+4 Minkowski space, we define the temperature. Some different definitions of temperature appear depending on the choice of propagators. We understand they correspond to different regularizations. To "sort" the fermion propagation with respect to "forward" and "backward" in the extra (time) axis controls chiral properties in the fermion determinant evaluation.

Two kinds of regularization naturally appear depending on the choice of solutions: Feynman path and symmetric path. Although the present treatment for the Feynman path could look a little "artificial" in the kinematical viewpoint, we stress the decomposition into the advanced and the retarded parts is quite natural from the requirement of *fixed direction* in the system movement. Furthermore we also point out its practical usefulness of calculational simplicity. The relation between Feynman and symmetric tempts us to identify it with the relation between the consistent and covariant anomalies[19] in the chiral gauge theories. In the latter case, the chiral anomaly caused by a non-hermitian operator $\hat{D}_{chi}$ is the central concern, and, in the evaluation, typically two types regularization appear. They can be prescribed by two operators: $\hat{D}_{cons} = \hat{D}_{chi}\hat{D}_{chi} + \hat{D}^{\dagger}_{chi}\hat{D}^{\dagger}_{chi}$ and $\hat{D}_{cov} = \hat{D}_{chi}\hat{D}^{\dagger}_{chi} + \hat{D}^{\dagger}_{chi}\hat{D}_{chi}$. $\hat{D}_{cons}$ leads to the consistent anomaly and $\hat{D}_{cov}$ to the covariant one[33, 34]. $\hat{D}_{cons}$ is composed of two non-hermitian operators (which are hermite conjugate each other), while $\hat{D}_{cov}$ are of two hermitian operators. We understand that the regularization ambiguity produces two different anomalies from one chiral theory. In the present case, things go somewhat contrastively. The initial concern is the chiral anomaly caused by the hermitian (QED) operator (12). We have found typically two types of path in the evaluation: Feynman and symmetric. The latter path leads to the chiral anomaly of QED (the starting theory) which is hermitian, while the former path leads to that of the chiral QED (47) which is non-hermitian. It shows one can analyze not only an initial non-chiral (hermitian) theory (by choosing the symmetric path) but also the *chiral version* of the initial theory (by choosing Feynman path). The Feynman (F′-path) path gives the chiral (anti-chiral) determinant, whereas the symmetric one gives the non-chiral (hermitian) determinant. In spite of the difference in the above two cases, we still regard Feynman and symmetric as two different regularizations of a same result. This is because both results are simply related (just a factor of 2 in the present model) and essentially the same as far as the chiral anomaly is concerned.

It looks that the usual lattice approach corresponds to the symmetric path, not to the Feynman path. The configuration image of the former is two "walls", one around the origin($t = 0$) and the other near the boundary ($t = \infty$). This is noted below (31) and fits the image in lattice. The present formalism suggests the possible usefulness of the other domain wall configuration appearing in the Feynman path : one "wall" at the origin. (If we take F′-path, the configuration is one "wall" at the boundary ($t = \infty$).) In this respect, the present approach looks to extend other ones known so far.

In the Feynman path, positive energy 5D "electrons" propagate to the future (in 1+4 space-time) for G^{5M}_{+}, and negative energy ones propagate to the past for G^{5M}_{-}. As for the symmetric path, both positive and negative 5D "electrons" propagate to the future for G^{5M}_{+}, to the past for G^{5M}_{-}. This

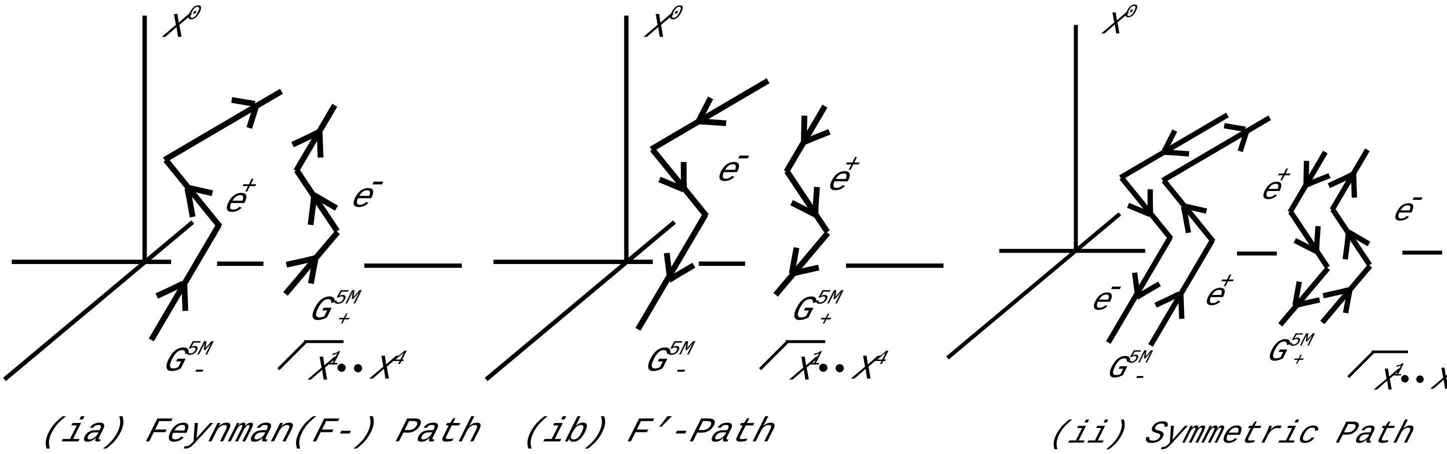

Fig.2 (+)-domain (G^{5M}_+) and (−)-domain (G^{5M}_-) in three different regularizations : (ia) Feynman (F-) path, (ib) F′-path and (ii) Symmetric path. e^- represents the positive energy 1+4 dim "electron" and e^+ represents the positive energy 1+4 dim "positron" which is interpreted as the negative energy "electron" propagating in the opposite direction in time X^0.

situation is schematically drawn in Fig.2 where the negative energy 5D "electrons" are expressed as the positive energy 5D "positrons" propagating in the opposite direction in time (the Dirac's hole theory). In Fig.2 we show the F′-path solution (ib) besides F-path one (ia). Note that the present formalism of the chiral determinant ($\mathrm{Tr}\, G^{5M}_+$, $\mathrm{Tr}\, G^{5M}_-$) is based not on the vacuum structure but on the solutions of the (1+4)-dim Dirac equation. This point is different from the original ones[10, 11, 12, 13, 14].

As for the correspondence to the overlap formalism we point out some comments.

1. G^{5M}_+ and $G^{5M'}_-$ correspond to the "$|+>$ domain" and "$|->$ domain" respectively. In the Feynman case, they describe the right, $(1+\gamma_5)/2$, and the left, $(1-\gamma_5)/2$, chiral fermion contributions, respectively, in the limit of (32). If we change the sign of the masses, their chiralities change each other. This corresponds to the definition of $|\pm>$ in the overlap formalism.

2. The relations (36) and (42) clearly say that the "$|+>$-domain" and "$|->$-domain" are both necessary to regularize $-i\gamma_5\delta^4(x-y)$. This is the present understanding why the overlap of $|+>$ and $|->$ is necessary to give the correct anomaly.

The results based on our interpretation are consistent with others known so far. We hope the present approach helps to further development of chiral fermions on lattice.

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[16] When we apply this formula in (2), we must pay care to zero modes of $\hat{D}$. Generally this point is related to the fixing of local (gauge) symmetry and the regularization of the infrared behaviour. In the present case, there exists no problem in this respect. See Chapter IV of [15].

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[22] We adopt the convention of Ref.[11] for the gamma matrices (γ_μ) and the metric in 2 and 4 dim Euclidean space and for those (Γ_a) in 1+4 dim Minkowski space.

[23] Here we introduce two independent regularization parameters , M and M', in order to stress that the two domains are defined not by (the sign of) the 1+4 dim fermion mass but by the relation (14). This point distinguishes the present formalism from the original overlap formalism. We will soon take $M = M'$ for simplicity.

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[25] A slash '/' is used for 4 dim gamma matrix (γ_μ) contraction, whereas a backslash '\' for 1+4 dim one (Γ_a).

[26] The relation between 4 dim quantities(x^μ and k^μ) and 1+4 dim ones(X^a and K^a): $(X^a) = (X^0, X^\mu = x^\mu), (K^a) = (K^0, K^\mu = -k^\mu), K_a X^a = K_0 X^0 - K^\mu X^\mu = K_0 X^0 + k^\mu x^\mu$.

[27] Useful relations: $-i\tilde{K}X = -iE(k)X^0 - ikx,\ i\bar{K}X = iE(k)X^0 - ikx,\ M + \tilde{\not{K}} = M + E(k)\gamma_5 + i\not{k},\ M - \bar{\not{K}} = M - E(k)\gamma_5 + i\not{k}$.

[28] In the Kaluza-Klein reduction, the extra dimension is compactified as S^1 (circle). In the present case, however, the extra space can not be compactified since it is known, in the lattice formalism, that the fermion doubler appears in the case when the periodic boundary condition is imposed on the extra space[4].

[29] In the standpoint of the 4 dim momentum integral, the limit $|M|t \to +0$ plays the role of the infra-red regularization, whereas the limit $|M|/|k^\mu| \to +\infty$ that of the ultra-violet regularization. In the sense that both the ultra-violet and the infra-red cut-offs exist, the present regularization is similar to the lattice regularization.

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[31] If we take the extra axis as a (straight) line, we have to introduce an infrared cut-off,L, in the Z^0-integral in (37), and the final anomaly result depends on it.

[32] Here the momentum ($s = \sqrt{k^2}/M$) integral is regularized by the analytic continuation. It is essentially equivelent to introduce the cut-off M in the s-integral as $s = \sqrt{k^2}/M \leq 1$. In this sense, the third

stage condition (29) is used in a loosened way. We regard it as a part of regularization. The same thing can be said for the evaluation of (44).

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