Study of the non-linear behaviour of the LHC using normal forms

Y. Papaphilippou and F. Schmidt, CERN

Abstract

Normal form tools are used in view of understanding the causes of dynamic aperture limitations in the LHC optics versions 4 and 5. To this end, maps up to 11th order have been produced and analysed by means of Lie algebraic methods, for all models and 60 different realisations of the accelerator, at injection energy. The resonances up to 12th order are systematically analysed and correlated with the dynamic aperture. It has been of particular interest to establish the link between the individual multipole coefficients and the higher order contributions to a specific resonance.

1 INTRODUCTION

High order perturbation theory and normal form approaches have been proved to be very beneficial for the analysis of the non-linear dynamics of particle accelerator models [1]. The usual procedure necessitates the computation of a Poincaré map, through a Taylor expansion around the 1-periodic orbit. The construction of normal forms consists in transforming the variables of the original map to a new set, in order for the resulting map to have a simpler form. The generating function, through which this symplectic transformation is performed, provides valuable information about the influence of resonances on the dynamics of the system. However, one usually needs an indication about the strength of resonances for specific initial conditions, especially close to the parts of the phase space where the beam is lost. Even though these areas are chaotic and the usual representation given by normal forms will diverge from the real dynamics of the system, one can try to evaluate the strengths of resonances for these types of initial conditions. This is a crucial point for coping with the non-linear dynamics of an accelerator, as these strengths are related with the multipolar magnet errors. Through a resonance analysis of this type, one can associate the physical characteristics of the accelerator to the dynamic aperture (DA) of the model, find the limits and provide the necessary cures, ensuring the long-term stability of the beam. In this respect, we developed a simple numerical tool, the Graphical Representation of Resonances (GRR), which uses as input the output of standard Lie algebraic numerical codes [2] and allows the evaluation and graphical representation of the resonance strengths and detuning, up to a desired order (see also [3]).

The aim of this work is to present the impact of this tool in the ongoing studies with respect to the dynamics of the LHC optics versions 4 and 5. The article is organised as follows: in Sect. 2, we outline the basic ideas regarding normal form analysis and the GRR. In the next section, we present some examples of the application of the tool in LHC models. The last section is devoted to summarising the principal results.

2 GRAPHICAL REPRESENTATION OF RESONANCES

In order to construct the normal form $\mathcal{U}$ of a Poincaré map $\mathcal{M}$ representing the successive intersections of a kicked accelerator beam at a fixed longitudinal position s, one performs a symplectic transformation of the form $\mathcal{U} = \Phi^{-1} \circ \mathcal{M} \circ \Phi$, where Φ can be expressed as a Lie operator:

$$\Phi = e^{:F:} \; . \tag{1}$$

Neglecting the weak coupling between the vertical and horizontal motion with the longitudinal one, the generating function F is written as a sum

$$F = \sum_{jklm} f_{jklm} {\zeta_x^+}^j {\zeta_x^-}^k {\zeta_y^+}^l {\zeta_y^-}^m \tag{2}$$

of homogeneous polynomials of the new set of "action-angle" variables $\zeta_{x,y}^{\pm} = \sqrt{\epsilon_{x,y}}\; e^{\mp i(\psi_{x,y}+\psi_{x0,y0})}$, with $f_{jklm} \in \mathbb{C}$ and $j,k,l,m \in \mathbb{N}$. The emittances are $\epsilon_x = n_\sigma(\epsilon_n/\gamma)^{1/2}\cos\phi$ and $\epsilon_y = n_\sigma(\epsilon_n/\gamma)^{1/2}\sin\phi$, where the factor n_σ determines the amplitude in terms of the rms beam size σ, ϵ_n is the normalised emittance, γ the energy factor and ϕ is the emittance ratio ($\tan\phi = \epsilon_y/\epsilon_x$). Inserting the expressions of the new variables in the series (2), one obtains:

$$F = \sum_{jklm} f_{jklm} (\epsilon_x)^{\frac{j+k}{2}} (\epsilon_y)^{\frac{l+m}{2}} e^{-i\psi_{jklm}} \; , \tag{3}$$

where $\psi_{jklm} = (j-k)(\psi_x + \psi_{x0}) + (l-m)(\psi_y + \psi_{y0})$. In practice, this infinite series has to be truncated, as the number of terms grows very sharply with the order n. We usually find sufficient to carry out the calculation up to 12th order (11th order of the map).

The series (3) is formed by terms corresponding to resonances of the form $(a,b) = (j-k, l-m)$ which are related to the non-linear dynamical behaviour of the system. The norm of the coefficients f_{jklm} provides an indication about the resonance strength associated with the multipolar magnet errors. In fact, multipole errors of a certain order n_m will have a contribution to the coefficients f_{jklm} of order $n \geq n_m$. Hence, one could imagine that the importance of a specific resonance can be revealed by checking the amplitude of the corresponding coefficients. However, the series terms associated with the same resonance (a,b) of order n will appear in higher orders as well, i.e. $n+2, n+4, n+6, \ldots$. Thus, a more precise estimation of the resonance strength in a given position of the phase space can be efficiently computed only by including the contribution of these higher order terms. Without loss of generality, the phase can be fixed at an arbitrary value, e.g. $\psi_{jklm} = 0$. Thus, the strength of a specific resonance (a, b) is given by:

$$F_{(a,b)} \approx \sum_{\substack{jklm \\ j+k+l+m \leq n \\ j-k=a\ ,\ l-m=b}} f_{jklm}\, (\epsilon_x)^{\frac{j+k}{2}}\, (\epsilon_y)^{\frac{l+m}{2}}\ . \qquad (4)$$

In the same way, for a certain amplitude and emittance ratio, one can calculate the non-linear detuning, by adding up to an order the contribution of the detuning terms corresponding to the partial derivatives of the associated Hamiltonian function with respect to ϵ_x and ϵ_y. The validity of the approximation can be checked by comparing the tune calculated through the normal form approach and the one provided by a direct application of a Laskar type frequency map analysis [4] (using the SUSSIX code [5]) of tracking data generated by SIXTRACK [6] for the same orbit (8σ and $\phi = 15°$), for the "nominal" error table of LHC optics version 5. In Fig. 1, we present graphically the average difference, over 60 random realisations of the magnet errors ("seeds"), between the vertical and horizontal tune computed with the two different methods, versus the detuning order. It is apparent that after adding the contribution of the 4th order, the two approaches converge to the same value, with a precision of 10^{-4}.

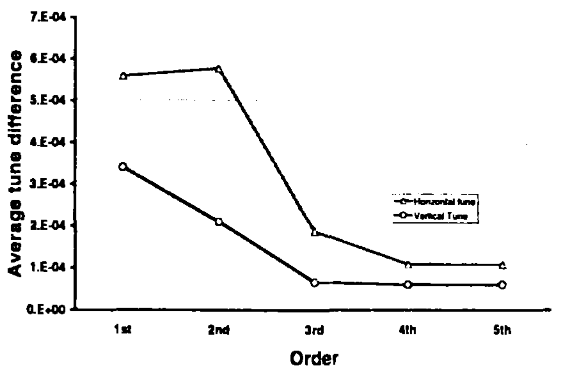

Figure 1: *Difference between the tune values calculated with two methods, for LHC optics version 5 nominal error table at 8σ and $\phi = 15°$.*

3 APPLICATION TO THE LHC

3.1 *Limitations in the dynamic aperture of LHC optics versions 4 and 5*

The function F can be computed by means of Lie algebraic numerical tools [2]. A global picture of the resonances strength given by F is represented graphically in the 3D plot of Fig. 2. This graph represents the 7th order resonance coefficients f_{jklm} for LHC optics version 5, using the "nominal" error table, for 60 seeds. Each number on the horizontal axis corresponds to a different 7th order resonance (60 in total). The first 7 series of spikes represent the amplitudes of the normal resonances and the next 23 the normal sub-resonances. The remaining 30 correspond to the 7th order skew (sub)resonances.

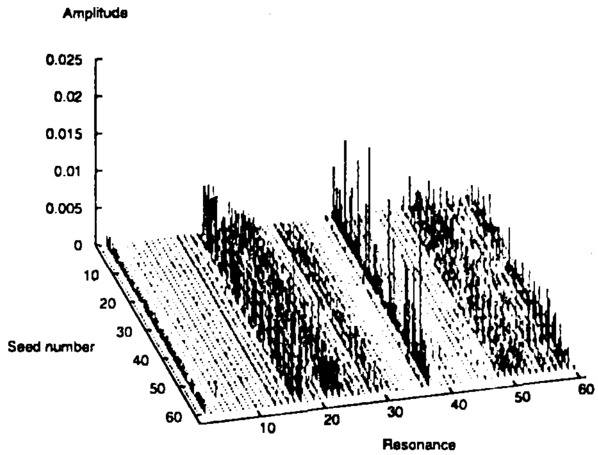

Figure 2: *Norm of the 7th order resonance coefficients f_{jklm} of the generating function (3), for 60 seeds of LHC optics versions 5 "nominal" error table.*

The importance of the (7,0) resonance for predominantly horizontal motion was reported in previous studies [7]. Nevertheless, this is not at all visible in this picture, i.e. the first series of spikes is quite small as compared with certain sub-resonances and skew resonances, in the middle of the graph. This is indeed due to the fact that the generating function F gives an indication of the strength of resonances close to the 1-periodic orbit (at "zero amplitude") and, further, the higher order series terms contributing to a specific resonance are not included in the coefficients f_{jklm}.

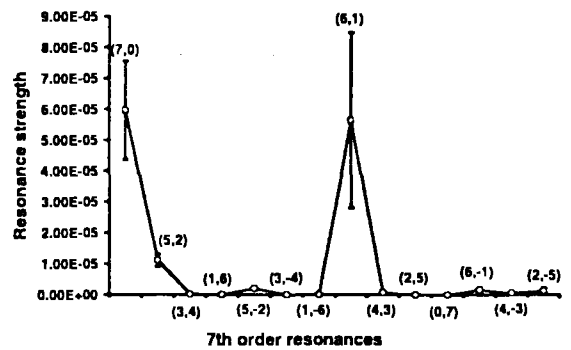

Figure 3: *Average value and standard deviation of the 7th order resonance strength at 8σ and $\phi = 15°$ for LHC optics version 5 "nominal" error table.*

Through eq.(4), we were able to evaluate the strength of the 7th order resonances up to 12th order, for an emittance ratio of 15° and an amplitude of 8σ (close to the minimum DA of this model). A graphical representation of these resonances (14 in total) is given in Fig. 3. The points correspond to the average value of the resonance driving terms over the 60 seeds and the error bars are equal to one standard deviation. It is now clear that the (7,0) resonance is quite important. On the other hand, this graph also shows that the strength of the (6,1) resonance is indeed big. A further analysis of the detuning of the system reveals that at this part of the phase space, the particle tunes are close to this resonance.

By using GRR, we were able to understand the reason for this strong excitation of the (7,0) resonance, limiting the DA of LHC optics version 5 with respect to version 4, at least for motion close to the horizontal plane. It was found that the large field errors corresponding to some special type of quadrupoles ("warm" quads.) on the two high-beta insertions (I.P.3 and I.P.7) of the LHC optics version 5 were in the heart of the problem [8]. In Fig. 4, we present the average (over 60 seeds) absolute value of the relative difference (ARD) between the 12 most prominent resonances of LHC optics version 4 and 5. There is a 50% difference between the amplitudes of the (7,0) resonance in the two lattices. By switching off the errors in the "warm" quadrupoles the two lattices become approximately identical. This fact guided us in the correction of the LHC optics version 5 with an important average gain of 2σ for the DA.

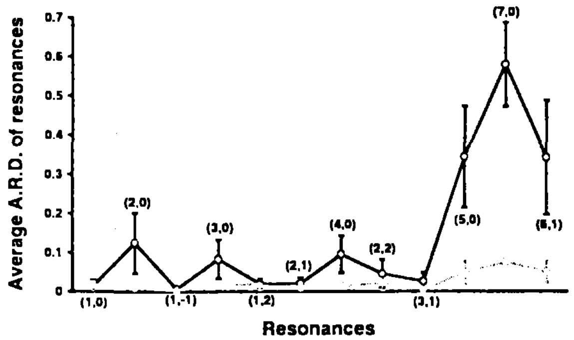

Figure 4: *Average absolute value of the resonance amplitude relative difference at 8σ and $15°$ between LHC optics version 4 and 5 with and without the errors on the "warm" quadrupoles.*

3.2 Correction of the effect of the octupole error bias on the LHC dipoles

A similar resonance analysis was followed in order to understand the drop of the DA in LHC optics version 5 with the "target" error table after the inclusion of a large bias of the systematic per arc octupoles in the dipoles [9]. This bias can usually be cancelled by erect and skew octupole spool pieces each located at one end of the dipoles and powered in series. The correction of the bias of the erect octupole component with erect octupole spool pieces in half of the machine fully restored the DA of the "target" error table. Nevertheless, it was not sufficient to suppress the bias of the skew octupole component with skew octupole spool pieces of this kind. A resonance analysis with GRR has revealed that the skew octupoles mainly drive the $(1,-1)$ resonance. A minimisation in first order of these resonances with skew octupole spool pieces powered in series was very beneficial for the DA (2σ gain on average).

This can be easily seen in Fig. 5 where we present the strength difference of some important resonances between the "target" error table and the tables with the large skew octupole bias before and after the correction. The resonances in this graph correspond to an amplitude of 10σ and an emittance ratio of $15°$. All points represent the average values over 60 seeds and the error bars correspond to the standard deviation. The effect of the correction of the $(1,-1)$ resonance is indeed very visible. Both, the average resonance difference (and the resonance strength as well) are considerably reduced after the correction with the skew octupole spool pieces. In particular, the $(1,-1)$ resonance strength decreases by a factor of three so that there is approximately no difference left with the one of the "target" error table.

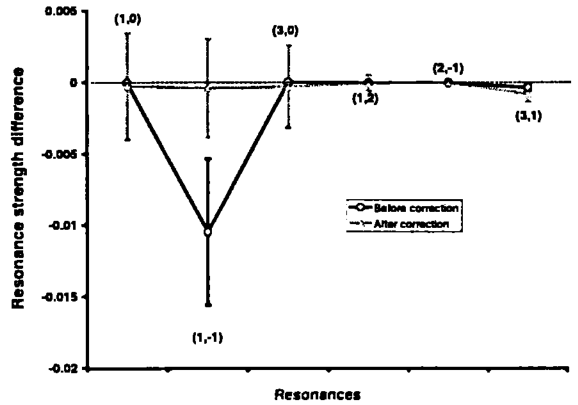

Figure 5: *Strength of resonances for LHC optics version 5 due to skew octupole errors before and after correction.*

4 CONCLUSIONS

Resonance analysis can be very effective in order to understand the reasons which limit the DA of an accelerator model. The use of a simple numerical tool enabled us to understand the effect of the large errors in some special quadrupoles of LHC optics version 5. Moreover, we were able to identify and minimise the resonance which was correlated with the drop of the DA due to the large skew octupole bias introduced in the LHC optics version 5 "target" error table. As shown by subsequent tracking studies, this correction lead to a considerable improvement of the DA of the studied LHC models. We can thus be confident that GRR is a quite efficient numerical tool which can be used as a guide for correcting and improving the DA of accelerator lattices.

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