Glueball calculations in large-N_c gauge theory

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We use the light-front Hamiltonian of transverse lattice gauge theory to compute from first principles the glueball spectrum and light-front wavefunctions in the leading order of the $1/N_c$ colour expansion. We find 0^{++}, 2^{++}, and 1^{+-} glueballs having masses consistent with $N_c = 3$ data available from Euclidean lattice path integral methods. The wavefunctions exhibit a light-front constituent gluon structure.

PACS numbers: 11.15.Tk, 11.10.Jj, 12.38.Bx, 13.85.Ni

I. INTRODUCTION

There is growing experimental evidence that glueballs, boundstates of gluons in the $SU(3)$ gauge theory Quantum Chromodynamics (QCD), have been discovered in the mass range $1.5-1.7$ GeV [1–3]. But the confinement feature of QCD — interactions grow stronger as the energy of a process decreases — complicates any first principles calculation of the boundstate problem. To date, the most successful boundstate calculations use a Euclidean spacetime lattice and simulate the path integral by Monte Carlo methods. These difficult calculations are now roughly consistent with the experimental signatures of glueballs [2,4], although much detail remains to be clarified. Therefore, it is important to have some independent method of calculating the properties of glueballs from first principles in gauge theory. In this letter, we present our first results on this problem using an effective light-front Hamiltonian quantisation (canonical quantisation on a null-plane in spacetime). This is the transverse lattice method, suggested originally by Bardeen and Pearson [5,6], which we have developed to the extent that quantitative calculations are now feasible [7,8].

Although this work is ostensibly about glueballs, we have a more general motivation for developing the light-front Hamiltonian formulation of gauge theory. A light-front Hamiltonian has Lorentz-frame-independent wavefunctions. Together with the simplicity of the light-front vacuum, this leads to a field-theoretic realisation of the parton model for hadrons on which so much understanding is based. The rich phenomenology in hadronic and nuclear physics that would follow from knowledge of the light-front wavefunctions is surveyed in the lectures of Brodsky [9]. We will use the glueball problem in QCD, which is especially difficult computationally, as a quantitative test of our light-front formalism.

A detailed account of our methods and various quantitative tests that we have performed, mostly in $2+1$ dimensions, can be found in Refs. [7,8] but we will briefly review the salient points below. We will work in the leading order of the $1/N_c$-expansion of $SU(N_c)$ gauge theory, which omits the $1/N_c^2$-suppressed glue configurations. It also removes "sea" quarks from our theory, an approximation also used in the Euclidean lattice calculations. We work on a coarse transverse lattice, using an effective potential tuned to minimize discretisation errors. The groundstate $\mathcal{J}^{\mathcal{PC}} = 0^{++}$ glueball mass is $\mathcal{M} = 3.3 \pm 0.2\sqrt{\sigma}$, where $\sigma \approx 0.1936\mathrm{GeV}^2$ is the string tension, and is consistent with $SU(2)$ and $SU(3)$ results available from the Euclidean lattice. Components of the 2^{++} and 1^{+-} glueballs which are behaving covariantly are also consistent. The glueball wavefunctions that we obtain are new.

II. TRANSVERSE LATTICE FORMULATION

In $3+1$ spacetime dimensions we use a square lattice of spacing a in the 'transverse' directions $\mathbf{x} = \{x^1, x^2\}$, and a continuum in the $\{x^0, x^3\}$ directions. Our action is

$$S = \int dx^0 dx^3 \sum_{\mathbf{x}} \Bigg(\mathrm{Tr}\Big\{ \overline{D}_\alpha M_r(\mathbf{x}) \left(\overline{D}^\alpha M_r(\mathbf{x})\right)^\dagger \Big\} - \frac{1}{2G^2}\,\mathrm{Tr}\big\{F_{\alpha\beta}F^{\alpha\beta}\big\} - V_{\mathbf{x}}[M] \Bigg) \ , \qquad (2.1)$$

where $r, s \in \{1,2\}$; $\alpha, \beta \in \{0,3\}$; and

$$\overline{D}_\alpha M_r(\mathbf{x}) = \left(\partial_\alpha + iA_\alpha(\mathbf{x})\right) M_r(\mathbf{x}) - iM_r(\mathbf{x})A_\alpha(\mathbf{x}+a\hat{\mathbf{r}}) \ . \qquad (2.2)$$

$A_\alpha(\mathbf{x})$ is the gauge potential at transverse site $\mathbf{x}$, while $M_r(\mathbf{x}) \in GL(N_c, C)$ is a link variable from $\mathbf{x}$ to $\mathbf{x}+a\hat{\mathbf{r}}$ which carries colour flux between sites. Our use of non-compact variables at finite a is the basis of the colour-dielectric formulation of lattice gauge theory [10]. The action (2.1) is gauge invariant at each transverse lattice site under

$$A_\alpha(\mathbf{x}) \to U(\mathbf{x})A_\alpha(\mathbf{x})U^\dagger(\mathbf{x}) + \mathrm{i}\left(\partial_\alpha U(\mathbf{x})\right)U^\dagger(\mathbf{x}) \qquad (2.3)$$

$$M_r(\mathbf{x}) \to U(\mathbf{x})M_r(\mathbf{x})U^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) \qquad (2.4)$$

for $U \in SU(N_c)$. The dimensionful coupling $G^2(a)$ is such that $a^2G^2 \to g^2$ in the classical continuum limit

$a \to 0$, where g^2 is the usual continuum gauge coupling. We include a gauge-invariant effective potential $V_{\mathbf{x}}[M]$ consisting of Wilson flux loops on the transverse lattice. The exact form we use is given below. Part of its job is to ensure that M is forced to be a unitary matrix as $a \to 0$, so that continuum QCD is recovered à la Wilson [11]. More generally, we choose it to minimize violations of Lorentz covariance at finite a. The idea is that we tune the couplings of V along an approximation to the Lorentz covariant scaling trajectory which leads to continuum QCD as $a \to 0$. Then for low energy observables, such as boundstate masses, one can work at finite a.

We perform light-front quantisation, treating $x^+ = (x^0+x^3)/\sqrt{2}$ as time, and pick the light-front gauge $A_- = (A_0 - A_3)/\sqrt{2} = 0$. In this gauge $A_+ = (A_0 + A_3)/\sqrt{2}$ is a constrained field which we eliminate by solving its equation of motion classically. Working in the leading $1/N_c$ approximation has some simplifying consequences. In the transverse rest frame $\mathbf{P} = 0$, Eguchi-Kawai reduction occurs at large a, so that the argument $\mathbf{x}$ may be dropped [7]. This results in a light-front Hamiltonian of the form

$$
\begin{aligned}
P^- = \int dx^- &- \frac{G^2}{4} \mathrm{Tr}\Big\{ J^+ \frac{1}{\partial_-^2} J^+ \Big\} \\
&- \frac{\beta}{N_c a^2} \mathrm{Tr}\Big\{ M_2^\dagger M_1^\dagger M_2 M_1 + M_1 M_2 M_1^\dagger M_2^\dagger \Big\} \\
&+ \mu^2 \sum_r \mathrm{Tr}\{M_r M_r^\dagger\} + \frac{\lambda_3}{a^2 N_c^2} \sum_r \left(\mathrm{Tr}\{M_r M_r^\dagger\}\right)^2 \\
&+ \frac{\lambda_2}{a^2 N_c} \sum_r \mathrm{Tr}\{M_r M_r M_r^\dagger M_r^\dagger\} \\
&+ \frac{\lambda_1}{a^2 N_c} \sum_r \mathrm{Tr}\{M_r M_r^\dagger M_r M_r^\dagger\} \\
&+ \frac{\lambda_4}{a^2 N_c} \sum_{\sigma=\pm 2, \sigma'=\pm 1} \mathrm{Tr}\Big\{ M_\sigma^\dagger M_\sigma M_{\sigma'}^\dagger M_{\sigma'} \Big\} \\
&+ \frac{4\lambda_5}{a^2 N_c^2} \mathrm{Tr}\Big\{ M_1 M_1^\dagger \Big\} \mathrm{Tr}\Big\{ M_2 M_2^\dagger \Big\} \ , \qquad (2.5)
\end{aligned}
$$

$$
J^+ = \mathrm{i}(M_r \overleftrightarrow{\partial}_- M_r^\dagger + M_r^\dagger \overleftrightarrow{\partial}_- M_r) \ . \qquad (2.6)
$$

This Hamiltonian includes all Eguchi-Kawai-reduced Wilson loops on the transverse lattice up to fourth order in M; this is our approximation to the effective potential V. Generalisation to higher orders is straightforward. The Hamiltonian is diagonalised in a parton Fock space of M_r, at fixed total momentum P^+, built from its Fourier modes at $x^+ = 0$

$$
M_r = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} \left(a_{-r}\, e^{-ik^+x^-} + a_r^\dagger\, e^{ik^+x^-} \right) \ . \qquad (2.7)
$$

However, only closed flux loops, invariant under residual x^--independent gauge transformations U, have finite energy. These loops propagate without splitting or joining at leading $1/N_c$ order.

Even with a finite lattice spacing a, the Hamiltonian (2.5) is not yet completely regulated. We impose periodic boundary conditions in x^- and a cut-off on the maximum number of link partons a_r in the Fock space. For each choice of couplings in the Hamiltonian, we remove these latter cut-offs by extrapolation, following a continuum improvement procedure.

III. TUNING THE EFFECTIVE POTENTIAL

To test for Lorentz covariance, we measure the dispersion relations of glueball boundstates as a function of couplings in V. This requires us to work at $\mathbf{P} \neq 0$, which can be achieved by adding appropriate phase factors to matrix elements of P^- above [8]. Using $G^2 N_c$ to set the dimensionful scale, the dispersion relation of each glueball can be written

$$
\begin{aligned}
2P^+P^- = G^2 N_c \big(&\mathcal{M}_0^2 + \mathcal{M}_1^2\, a^2 |\mathbf{P}|^2 \\
&+ 2\overline{\mathcal{M}}_1^2\, a^2 P^1 P^2 + O(a^4 |\mathbf{P}|^4) \big) \ . \qquad (3.1)
\end{aligned}
$$

For each glueball in the light spectrum and for fixed μ^2, we tune the couplings λ_i and β so that, as far as possible,

$$
a^2 G^2 N_c \mathcal{M}_1^2 \equiv c_{\rm on}^2 = 1 \qquad (3.2)
$$

$$
a^2 G^2 N_c (\mathcal{M}_1^2 + \overline{\mathcal{M}}_1^2) \equiv c_{\rm off}^2 = 1 \ . \qquad (3.3)
$$

$c_{\rm on}$ is the speed of light in direction $\mathbf{x} = (1,0)$; $c_{\rm off}$ is the speed of light in the direction $\mathbf{x} = (1,1)$. In practice, we apply a χ^2-test over a range of light glueballs to optimize isotropy of the speed of light. In this way we search for a Lorentz covariant scaling trajectory parameterized by μ^2, which may later be related to a.

It is necessary to know the dimensionless combination $a^2 G^2 N_c$ for this procedure. This can be deduced from two measurements of the string tension σ. The first measures the mass squared of winding strings in the transverse lattice directions $\mathbf{x}$, fit to $n^2 a^2 \sigma_T^2$ for winding number n. The second measures the heavy quark potential in the continuous x^3 direction [12], fit to $\sigma_L R$ for separation R. Demanding $\sigma_T = \sigma_L \equiv \sigma$ and expressing eigenvalues in units of $G^2 N_c$, we may eliminate σ and deduce $a^2 G^2 N_c$. We may also deduce $G^2 N_c$, and hence all dimensionful quantities, in units of σ.

To further help optimize lattice discretisation errors, we include in the χ^2 test a measure of deviations from rotational invariance of the heavy-quark potential, including deviations from $\sigma_T = \sigma_L$ and deviations from Lorentz multiplet structure in the glueball spectrum. $\sigma_L / G^2 N_c$ is the only basic quantity which is poorly determined by our current approximations, and our results were particularly sensitive to the variance assignment in the χ^2 test for this datum (though still compatible with the systematic error estimate we make below).

According to the colour-dielectric picture, large μ will correspond to large a where discretisation errors grow, while small μ corresponds to intermediate spacing a. Sufficiently small a produces a non-trivial light-front vacuum, signalled by the appearance of tachyons; it would be difficult to construct an effective potential in this regime. Instead, we use an effective potential at intermediate a on a trivial light-front vacuum. In $2+1$ dimensions the above procedure accurately identified a Lorentz covariant scaling trajectory at intermediate a [8]. The results we present below are based on a similar trajectory which we have found in $3+1$ dimensions, though further work is needed to reduce systematic errors.

IV. RESULTS

In Table I we show results for various components of the lightest glueballs at a transverse lattice spacing $a = 1.1/\sqrt{\sigma} \approx 0.49\,\text{fm}$, at the overall χ^2 minimum for this a. In order to establish the scaling trajectory, we have performed calculations at other lattice spacings: At smaller lattice spacings tachyons appear, indicating breakdown of the trivial vacuum regime. For larger lattice spacings the couplings evolve gradually, and the minimum χ^2 value grows. For the groundstate 0^{++} we find approximate scaling and maintenance of Lorentz covariance $(a/\sqrt{\sigma}, \mathcal{M}/\sqrt{\sigma}, c) = \{(1.1, 3.3, 1.13), (1.2, 3.1, 1.18), (1.3, 3.5, 1.19)\}$. The errors shown on glueball masses in Table I are estimates of the error in removing the periodic boundary condition on x^- and the cut-off on the maximum number link variables M_r in a wavefunction; these errors are under control. Only for the 0^{++} mass are we confident enough to estimate the finite-a systematic error. The Lorentz covariant scaling trajectory will need to be determined more accurately before similar statements can be made for the higher states. In $2+1$ dimensions, we found that when the deduced speed of light becomes isotropic to within about 10%, the corresponding glueball mass is also accurate at this level; this also seems to be the case in 3+1 dimensions.

$\mathcal{J}^{\mathcal{PC}}$	$\lvert\mathcal{J}_z\rvert^{\mathcal{P}_1}$	c_{on}	c_{off}	$\frac{\mathcal{M}}{\sqrt{\sigma}}$
0^{++}	0^+	1.13	1.13	3.3(1)
2^{++}	2^+	0.85	0.86	4.4(1)
	2^-	Im	–	5.9(2)
	$1^\pm$	0.89, –	0.92, Im	5.4(2)
	0^+	0.84	0.83	4.5(1)
1^{+-}	$1^\pm$	0.75, 0.88	0.74, 0.89	5.0(1)
	0^-	0.99	0.99	5.9(2)

TABLE I. The glueball multiplet components showing masses $\mathcal{M}^2 = 2P^+P^-(\mathbf{P}=0)$ and c from the dispersion relation (3.1). $\mathcal{P}_1$ is the symmetry under reflections $x^1 \to -x^1$; "Im" indicates $c^2 < 0$ and "–" indicates the quantity has not been measured. The $\mathcal{J}_z = \pm 1$ states are exactly degenerate.

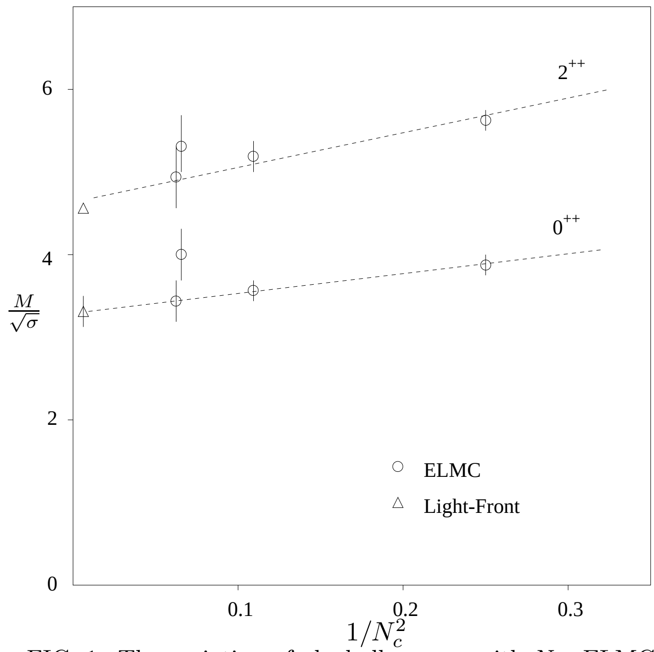

FIG. 1. The variation of glueball masses with N_c. ELMC predictions are continuum ones for $N_c = 2, 3$, and fixed lattice spacings (higher points are $\beta = 11.1$ and lower are $\beta = 10.9$) for $N_c = 4$ [15]. $N_c = \infty$ light-front predictions are from components behaving covariantly in Table I, including the total error estimate for the 0^{++}. The dotted lines are to guide the eye, and correspond to the expectation of leading linear dependence on $1/N_c^2$.

We now compare our results with spectra from conventional Euclidean lattice Monte Carlo (ELMC) simulations. See Fig. 1. (Recently, large N_c glueball mass ratios have been calculated using a conjectured duality with supergravity [13]; we will not discuss these results since their errors are not quantified.)

<u>0^{++}</u>
Based on violations of Lorentz covariance and variation of $\mathcal{M}$ along the scaling trajectory, we include an estimate of systematic error for this state, $\mathcal{M} = 3.3 \pm 0.2\sqrt{\sigma}$. A continuum extrapolation of $SU(3)$ ELMC data has been made in Ref. [14], $\mathcal{M} = 3.65(15)\sqrt{\sigma}$ (statistical error). Furthermore, measurements for $SU(2)$ and $SU(4)$ [15] indicate a slight downward drift with decreasing $1/N_c^2$.

<u>2^{++}</u>
Not all components of this state are behaving covariantly. The $\mathcal{P}_1$-odd combinations of $\mathcal{J}_z = \pm 1$ and $\mathcal{J}_z = \pm 2$ have large lattice artifacts and cannot be trusted. As a result, we are unsure of the correct parity assignment for the $\mathcal{J}_z = \pm 1$ but assume here that it belongs to the 2^{++} multiplet. The $\mathcal{P}_1$-even $\mathcal{J}_z = 0$ and $\mathcal{J}_z = \pm 2$ combination are nearly degenerate and isotropic, pointing to a tensor mass near $\mathcal{M} \approx 4.5\sqrt{\sigma}$. The $SU(3)$ ELMC extrapolation of Ref. [14] gives 5.15(23), consistent with more recent coarse anisotropic ELMC simulations [16]. Again, there is a slight drift with $1/N_c^2$.

<u>1^{+-}</u>
Here the $\mathcal{P}_1$-odd components fair better. The $\mathcal{J}_z = 0$

component of the lightest spin 1 is behaving covariantly, and points to $\mathcal{M} \approx 5.9\sqrt{\sigma}$. The $\mathcal{J}_z = \pm 1$ component is split away as a consequence of its anisotropy. An $SU(3)$ value $\mathcal{M} = 6.6(6)\sqrt{\sigma}$ is given in Ref. [4], while the more accurate measurement of Ref. [16] implies $\mathcal{M} = 6.30(4)\sqrt{\sigma}$.

$\underline{0^{-+}}$

Our lightest candidate for this state appears around $7.4\sqrt{\sigma}$ though we have not measured its covariance. $SU(3)$ ELMC work suggests a lighter mass [4]. Barring an unusual large-N_c extrapolation, it seems likely that this state (which posed problems for early ELMC studies also) is poorly described with our current approximations.

As an example of what may be deduced from wavefunctions, in Figure 2 we plot the distribution of longitudinal momentum fraction $x = k^+/P^+$ in the 0^{++} glueball at $a = 1.1/\sqrt{\sigma}$. $G_d(x)$ is the probability distribution for a link parton $a_r(k^+)$ and becomes the usual gluon distribution function in the continuum limit $a \to 0$ [8]. At finite a, G_d may be interpreted as a light-front constituent gluon distribution that evolves with a. We find that our 0^{++} light-front wavefunction consists mostly of four such constituents; it does *not* obey a gluonium picture (two constituent gluons).

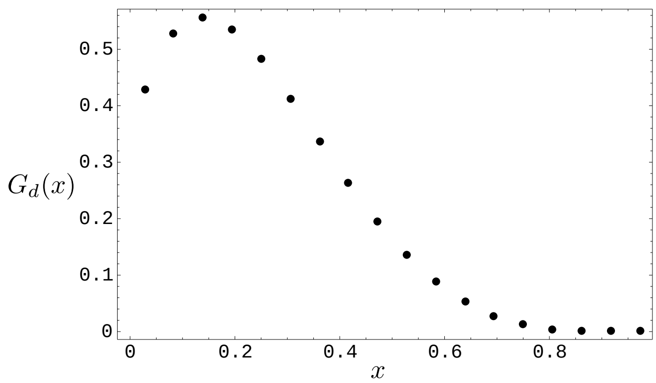

FIG. 2. The 0^{++} momentum distribution function.

V. CONCLUSIONS

We have measured glueball masses in QCD, using the $1/N_c$ expansion and different quantisation, field variables, regulators, and gauge fixing from the traditional Euclidean lattice path integral simulations. The results from both approaches are consistent for QCD without quarks, indicating a 0^{++} glueball near 1.6 GeV, and $1/N_c^2$ corrections appear to be small for the 0^{++} and 2^{++} states. (In this latter regard, it would be interesting to see better data at $SU(4)$.) We found that light-front wavefunctions exhibit a constituent gluon structure, although not of the form one might naively guess. In order to improve the accuracy of the Lorentz covariant scaling trajectory, future work includes choosing other observables to set the scale and incorporating higher order effective interactions. Finally, we note that the reintroduction of quarks will be necessary for a detailed comparison with experimental glueball candidates, which mix with nearby scalar quarkonia [1,3].

ACKNOWLEDGMENTS

The work of SD is supported by CERN. We thank M. Teper for helpful interactions. A portion of the computations were performed at Argonne's Center for Computational Science and Technology.

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