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Addendum to Finite-size effects on multibody neutrino exchange

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Abstract

The interaction energy of the neutrons due to massless neutrino exchange in a neutron star has recently been proved, using an effective theory, to be extremely small and infrared-safe. Our comment here is of conceptual order: two approaches to compute the total interaction energy density have recently been proposed. Here, we study the connection between these two approaches. From CP invariance, we argue that the resulting interaction energy has to be even in the parameter $b = -G_F n_n/\sqrt{2}$, which expresses the static neutrino potential created by a neutron medium of density n_n.

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The long-range neutrino-mediated interaction between neutrons in a dense core, such as a neutron star, has recently been studied [1–6] and shown to be extremely small. Following the method of Schwinger [7], the interaction energy W can be computed as

$$W = \langle\hat{0}|H|\hat{0}\rangle - \langle 0|H_0|0\rangle \ , \tag{1}$$

where $|\hat{0}\rangle, |0\rangle$ are the matter and matter-free vacua, respectively, and H (H_0) is the Hamiltonian in the presence (in the absence) of the star medium. For convenience we will call, in this note, "matter vacuum" the neutrinoless ground state in the presence of a star. This first step is crucial: *the interaction energy is equal to the shift of the zero-point energy of a neutrino due to the presence of the star.* We have demonstrated [3,5] that there is a non-vanishing zero-point energy density difference (W) between the inside and the outside of the star, which is due to the refraction index at the stellar boundary and the resulting non-penetrating waves. In Ref. [5], this effect was shown analytically and numerically to be the dominant one and lead to an infrared-safe total energy density. This result is in contradiction with the previous claim in [8] that there must be a lower bound on the neutrino mass to ensure the existence of stars. The latter "catastrophic result" is a consequence of summing up large infrared terms outside the radius of convergence of the perturbative series. The use of an effective Lagrangian (2), which can be exactly solved, allows this non-convergence to be circumvented and to provide the result to a good accuracy. The effective Lagrangian writes:

$$\mathcal{L}_{\rm eff} = i\overline{\nu}_L \partial\!\!\!/ \ \nu_L(r) - b\ \overline{\nu}_L\gamma_0\nu_L\ \theta(R-r) \ , \tag{2}$$

where R is the radius of the neutron star and $b = -G_F n_n/\sqrt{2}$ summarizes the static potential felt by the neutrinos, which is generated by the neutrons through Z_0-exchange. The neutrons are treated as static, and represented by a uniform axial-charge density (being electrically neutral the neutrons interact with the Z_0 only the via axial current) while n_n is the star neutron density. Notice that the potential is attractive, with depth b, for neutrinos, which then condense, and repulsive for antineutrinos. It is important to stress that this effective Lagrangian, Eq. (2), is non-CP-invariant since it describes the neutrino in a non-CP-symmetric background: the neutron star. In fact, it is easy to see that $CP\ \mathcal{L}(b)\ (CP)^{-1} \ = \ \mathcal{L}(-b)$.

As a consequence of the effective theory used, expression (1) is formally an ultraviolet-divergent quantity, which needs to be regularized. This can be written as

$$W = \sum_{i<0} E_i - E_i^0, \tag{3}$$

where i runs over the negative neutrino energy levels. Another method [4,5] consists in the following symmetrization:

$$W_{\rm sym} = -1/2 \left(\sum_{i>0} (E_i - E_i^0) - \sum_{i<0} (E_i - E_i^0) \right) . \tag{4}$$

As a matter of fact, Eqs. (3) and (4) could be considered as alternative definitions of the zero-point energy difference, i.e., of the vacuum energy for the theory [7]. Both definitions are equivalent, as we shall see, *only if* the effective theory is symmetric under CP transformation.

Using the latter expression, the result is found to be even in the parameter b [4,5]. Using the former one, we had found it to be odd [5] and the authors of [4,6] stressed that this last result was incomplete. Their criticism triggered the present study which lead us also, although with different arguments, to the conclusion that the result should be even in b.

To summarize we will mainly discuss two issues: the proper description of the vacuum energy and the UV dependence of the result in order to clarify the differences between the approaches in [2,4–6], which, needless to repeat, all agree on the main issue: massless neutrinos do not imply any catastrophe.

The energy density $w(\vec{x})$, the integral of which is W in Eq. (1), is given in terms of Hamiltonian densities by

$$w(\vec{x}) \equiv \langle \hat{0} | \mathcal{H}(\vec{x}) | \hat{0} \rangle - \langle 0 | \mathcal{H}_0(\vec{x}) | 0 \rangle \ . \tag{5}$$

In order to compute it following Schwinger [7], we write it as:

$$w(\vec{x}) = -i \frac{\partial}{\partial x^0} \mathrm{tr} \left\{ \gamma_0 \left[S_F(x,y) - S_F^{(0)}(x,y) \right] \right\}_{y \to x} \ ; \tag{6}$$

the index (0) refers to the free vacuum and S_F is the usual propagator,

$$\begin{aligned} S_F(x,y)\gamma_0 = {} & \theta(x^0 - y^0) \sum\!\!\!\!\!\!\int_{\kappa, E>0} \psi_{\kappa,E}(\vec{x}) \psi^\dagger_{\kappa,E}(\vec{y}) \, e^{-iE(x^0 - y^0)} \\ & - \theta(y^0 - x^0) \sum\!\!\!\!\!\!\int_{\kappa, E<0} \psi_{\kappa,E}(\vec{x}) \psi^\dagger_{\kappa,E}(\vec{y}) \, e^{-iE(x^0 - y^0)} \ , \end{aligned} \tag{7}$$

where $H\psi_{\kappa,E} = E\psi_{\kappa,E}$, κ summarizing the other quantum numbers, for instance the momenta. From Eq. (7), $w(\vec{x})$ can be rewritten using the notation $\sum\!\!\!\!\!\!\int_{\kappa,E} \equiv \sum_\kappa \int dE$ in the following two ways.

i) The first one consists in taking the limit $y^0 \to x^0$ with $y^0 > x^0$:

$$\begin{aligned} w(\vec{x}) = {} & \sum\!\!\!\!\!\!\int_{\kappa, E<0} E \ \psi^\dagger_{\kappa,E}(\vec{x}) \psi_{\kappa,E}(\vec{x}) \\ & - \sum\!\!\!\!\!\!\int_{\kappa, E<0} E \ \psi^{\dagger(0)}_{\kappa,E}(\vec{x}) \psi^{(0)}_{\kappa,E}(\vec{x}) \ . \end{aligned} \tag{8}$$

Obviously this choice is equivalent to (3).

ii) The second one corresponds to taking the symmetric average of the limits $y^0 \to x^0$ with $y^0 > x^0$ and $y^0 \to x^0$ with $y^0 < x^0$ in Eq. (7), as done in [4] and also in [5], which gives

$$
\begin{aligned}
w_{\rm sym}(\vec{x}) = & \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\kappa,E<0} E\psi^{\dagger}_{\kappa,E}(\vec{x})\psi_{\kappa,E}(\vec{x}) \\
& -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\kappa,E<0} E\psi^{\dagger(0)}_{\kappa,E}(\vec{x})\psi^{(0)}_{\kappa,E}(\vec{x}) \\
& -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\kappa,E>0} E\psi^{\dagger}_{\kappa,E}(\vec{x})\psi_{\kappa,E}(\vec{x}) \\
& +\frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\kappa,E>0} E\psi^{\dagger(0)}_{\kappa,E}(\vec{x})\psi^{(0)}_{\kappa,E}(\vec{x}).
\end{aligned}
\tag{9}
$$

corresponding to (4). The symmetrization over the limit on the time components is equivalent to the symmetrization over positive and negative energies as a consequence of time boundary conditions of the usual Feynman propagator[1]. Moreover, this symmetrization leads to an even behaviour in b: if one takes into account the transformation of the Lagrangian under CP, $CP\mathcal{L}(b)(CP)^{-1} = \mathcal{L}(-b)$, Eq. (9) can be rewritten as

$$
\begin{aligned}
w_{\rm sym}(b,\vec{x}) = & \frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\kappa,E<0} E\left\{\psi^{\dagger}_{\kappa,E}\psi_{\kappa,E}(b,\vec{x}) + \right. \\
& \left. \psi^{\dagger}_{\kappa,E}\psi_{\kappa,E}(-b,\vec{x}) - 2\psi^{\dagger}_{\kappa,E}\psi_{\kappa,E}(0,\vec{x})\right\} \quad ;
\end{aligned}
\tag{10}
$$

where the dependence on b of the eigenfunctions is explicitly written.

In Refs. [4,5] this result has been confirmed, while by using the non-symmetric expression (8) the energy was linear in b [5]. In order to understand this difference, we can subtract Eq. (9) from (8):

$$
\begin{aligned}
w(\vec{x}) - w_{\rm sym}(\vec{x}) = & \left\{\frac{i}{2}\frac{\partial}{\partial x_0}\sum\!\!\!\!\!\!\!\int_{\kappa,E}\left(\psi^{\dagger}_{\kappa,E}(\vec{y})\psi_{\kappa,E}(\vec{x})\right.\right. \\
& \left.\left. -\psi^{\dagger\,(0)}_{\kappa,E}(\vec{y})\psi^{(0)}_{\kappa,E}(\vec{x})\right)\; e^{-iE(x_0-y_0)}\right\}_{y\to x} \\
= & \;\frac{i}{2}\frac{\partial}{\partial x_0}\delta_{\alpha,\beta}\left\{\langle\hat{0}|\left[\Psi^{\dagger}_{\alpha}(y),\Psi_{\beta}(x)\right]_{+}|\hat{0}\rangle\right. \\
& \left. -\langle 0|\left[\Psi^{\dagger\,(0)}_{\alpha}(y),\Psi^{(0)}_{\beta}(x)\right]_{+}|0\rangle\right\}_{y\to x} \quad ,
\end{aligned}
\tag{11}
$$

where $[\Psi^{\dagger},\Psi]_{+}$ stands for the canonical anticommutation relation of the fermionic field:

$$
\begin{aligned}
\Psi(x) = & \sum\!\!\!\!\!\!\!\int_{\kappa,E>0}\; \psi_{\kappa,E}(\vec{x})\; e^{-iEx_0}\; b_{\kappa,E} \\
& + \sum\!\!\!\!\!\!\!\int_{\kappa',E<0}\; \psi_{\kappa',E}(\vec{x})e^{-iEx_0}d^{\dagger}_{\kappa',E}
\end{aligned}
\tag{12}
$$

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[1]It is worth pointing out that the use of momentum-space Feynman propagators to rewrite the energy density, $w(\vec{x})$, implies an implicit symmetrization over positive and negative energies because of the Fourier integration over the time and of the time boundary conditions. Thus, in Feynman's picture the symmetrization is assumed.

the "coefficients" $b_{\kappa,E}$ and $d^\dagger_{\kappa,E}$ being the usual fermionic annihilation and antifermionic creation operators; κ' is obtained from κ by performing the canonical transformation generating the positive-energy antiparticle states from the negative-energy ones.

From the CP transformation: $\Psi(x_0, \vec{x}) \;\to\; i\gamma_2\gamma_0\Psi^*(x_0, -\vec{x})$ [9], it is easy to see that the last r.h.s of (11) changes sign under CP. Therefore (11) vanishes *if the vacua* ($|\hat{0}\rangle, |0\rangle$) *are CP-invariant.* An equivalent way of seeing this is to remark that in a CP-symmetric vacuum, there is an $(E, \kappa \to -E, \kappa')$-symmetry, inducing a cancellation in the sums of the first r.h.s in (11). It results that in a CP-invariant vacuum, (3) and (4) or equivalently (8) and (9) are equal.

In a *non-CP-symmetric vacuum* $|\hat{0}\rangle$, namely a neutronic vacuum leading to the non-CP-invariant effective Lagrangian for the neutrinos (2), which has been used by the authors of Refs. [1–6], Eq. (11) *does not vanish.* Indeed, the axial charge (to which the parameter b is proportional) changes sign under CP transformation [9] and, in fact, our effective Lagrangian is only invariant under the product of CP (which changes neutrinos into antineutrinos) and the operation that changes b into $-b$ (which changes the neutron star into an antineutron star, as mentioned by Kiers et Tytgat in [4]).

Therefore CP invariance[2] imposes the invariance of the "vacuum" energy under the exchange $b \;\to\; -b$: the energy due to massless neutrino multibody exchange inside a neutron star is the same as the one inside an antineutron star (see Fig. 1).

In principle, the problem of matching the effective theory with the underlying one is related to the appropriate choice of the ultraviolet regularization. A detailed and faithful description of the transition from the underlying theory to the effective theory would lead to a UV-regularization, bringing automatically into the effective theory the wanted properties such as the CP symmetry. However, this is a very complex task. The limited goal faced in Refs. [1–6] justifies a naive "*description*" of UV physics; for instance, the use of a simple cut-off in Ref. [5]. We will hence invoke the general symmetry properties of the underlying theory (CP) to constrain the description of the vacuum energy in the effective theory (4), exactly as the chiral symmetry of QCD constrains the effective chiral Lagrangian.

Eq. (10), and hence (9), is clearly invariant under $b \;\to\; -b$ transformation. The latter equation is no longer equivalent to (8) since the difference given by Eq. (11) is non-zero, the Dirac equation obtained from (2) is not symmetric in the exchange $E, \kappa \;\to\; -E, \kappa'$. Thus, the CP symmetry of the underlying theory, i.e. Q.C.D., imposes the choice of (9) in order

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[2]In this note we neglect CP violation in the Standard Model. Still, if the neutron stars are considered at equilibrium, it suffices to invoke CPT to impose an equal mass to neutron and antineutron stars.

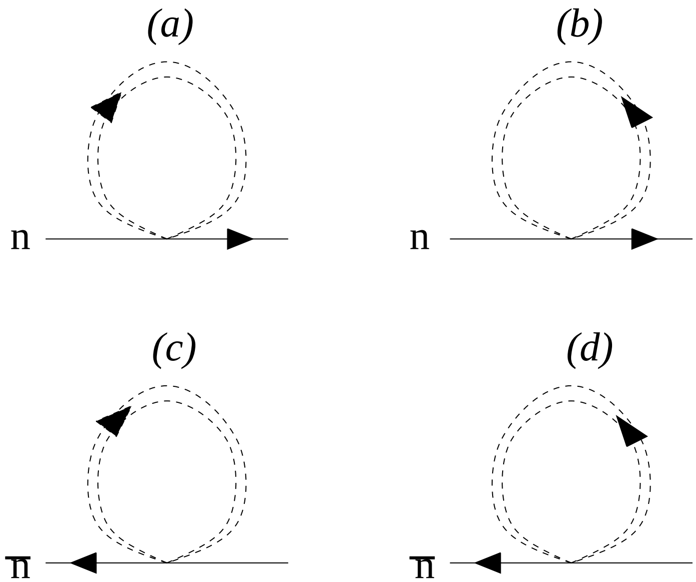

FIG. 1. *The diagram (a) is changed to (d) and (b) to (c) by reversing the time arrow. In the Feynman's picture, reversing the time arrow implies a change of fermionic lines into antifermionic lines. Thus, CP invariance guarantees that the total result will remain the same for both neutron and antineutron stars, (a)+(b)=(d)+(c). The dashed double line represents the neutrino propagator in the medium of the star.*

to compute the interaction energy density.

For practical calculations we need to remember that our effective theory is valid only under the assumption that the neutrons are static and remain so until the energy scale $C \sim 100$ MeV. Moreover, there are other energy scales that limit our theory, as the one related to the confinement of QCD (~ 1 GeV), the one related to the mean free path of neutrinos in the star... The physics related to these UV cut-offs has been discussed in Ref. [5]. The use of a cut-off in the energy integration can be alternatively interpreted as the assumption that a repulsive core prevents the neutrons from "piling-up" in space, as noticed by Fischbach [8]. In our effective theory of Eq. (2), b can be naively interpreted as the coherent and homogeneous amplitude for static neutrons to interact with neutrinos. The energy cut-off expresses the scale at which neutron recoil, repulsive cores, quark and gluon substructures, etc., induce the neutrino interaction with non-static neutrons to become incoherent and inhomogeneous.

In [5], using one of these physical cut-offs, C, we have found for the energies, using respectively Eqs. (3) and (4):

$$W \sim -bC^3R^3. \tag{13}$$

$$W_{\text{sym}} \sim -b^2 C^2 R^3 \ . \tag{14}$$

Eq. (14), which uses (4), *is compatible with the CP symmetry of QCD*. Beside the cut-off explicit regularization, it is probable that the result should also depend on the way of performing the summation over the energy. Indeed, this is the first step in the regularization process [3].

In computing W_{sym}, Kiers and Tytgat [4] counted the energy levels of the Hamiltonian by putting the system in a large box, which was a method different from the one in [5], and their result was

$$W_{\text{sym}}^{\text{KT}} \sim -b^4 R^3. \tag{15}$$

Let us present a few comments to understand the difference between (14) and (15). Both are even in b and respect CP symmetry. Next, the sums leading to (14) and to (15) correspond to different orderings of the same energy levels. Reordering a sum would not change the result if we were not speaking of divergent series [3], which have been regularized in ways that turn out to be different. This can be illustrated simply by applying both summation methods ([5] and [4]) to the (1 + 1)-dimensional toy model we presented in Ref. [3]: we found that $W = 0$ because there was a one-to-one correspondence between the energy levels inside and outside the star; by putting the (1 + 1) star in a box, as done in Ref. [4], we verified that the result was not zero, although it was extremely small. So the way of summing over the energy levels gives a difference in the final result.

Furthermore, the main difference between the two symmetrized results, Eqs. (14) and (15), is the power in the parameter b. The origin of that discrepancy comes from neglecting the two-body contribution to the energy density performed by the authors of [4]. They argue that the two-body interacting energy behaves as R^2, its contribution to the energy density scales as $1/R$ and this becomes negligible in the large R-physical regime. This seems to contradict previous studies that supported the existence of a well-defined neutrino exchange two-body potential [10]

$$\frac{G_F^2}{4\pi^3} \frac{1}{|\vec{r}_1 - \vec{r}_2|^5} \ . \tag{16}$$

The integration of the latter potential over space is UV-divergent. Some regularization procedures lead to a vanishing result, for example, Pauli-Villars in [1,3], and dimensional

[3]The classical example is that of the two possible orderings of the non-convergent series $(-1)^n$: $\sum_n (-1)^n = 1 + (-1+1) + (-1+1) + \cdots$ and $\sum_n (-1)^n = (1-1) + (1-1) + \cdots$, which are both valid to describe the series.

regularization in [4]. This vanishing of the integrated two-neutrino *positive* potential (16) is physically surprising, and is equivalent to assuming a negative distribution located at $r_1 = r_2$. Another regularization, such as a simple UV cut-off: $|\vec{r}_1 - \vec{r}_2| > r_c$ certainly leads to a positive non-vanishing result $\propto 1/r_c^2$.

If one still accepts this vanishing [4], then no b^2-terms remain, and only diagrams with at least four neutrons contribute; those contributions start at b^4. However, in Ref. [4], the authors stated that for a more realistic star, the results should be proportional to b^2.

Finally, in Ref. [5] we use a simple cut-off as regularization procedure, obviously different from both the *Pauli-Villars* and the *dimensional* scheme used, respectively, by us in Refs. [1,3] and by Kiers and Tytgat in Ref. [4]; we then perform a *"crude"* approximation, retaining only the sharp effects of the neutrino potential. Our result is proportional to b^2 (Eq. (14)). We might wonder why this "crude" result does not show larger powers of b. We may conjecture that this is due to the fact the two-body potential, being the shortest range one, is the one that corresponds to the "sharp" effects considered in the "crude" approximation, while the many-body potentials, being long-range, contribute to the corrections to the "crude approximation" studied in [5].

As a conclusion we believe the result (14) to be quite reasonable. However, our main conclusion is that the energy should be even in b because of CP invariance and that the symmetrization given by Eq. (9) is required. That the resulting energy is even in b is also manifest in Schwinger's expansion [8], as it results from the symmetrization (Furry's theorem) implicitly assumed in a Feynman diagram treatment of the problem.

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