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D-particles, Matrix Integrals and KP hierachy

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We derive the determinant representation and Hirota equations for the regularized correlation functions of the light-like coordinate operators $\sim \prod_i \mathrm{Tr}\,(X^+)^{l_i}$ in the reductions to zero dimensions of the matrix models describing D-particles in various dimensions. We investigate in great detail the matrix model originally proposed by J. Hoppe and recently encountered

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in studies of D-particles in four dimensions. We also present a new derivation of the large N and double scaling limits of the one-matrix model with cubic potential.

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1. Introduction

Matrix models describing the behavior of Dp-branes originate in the observation of E. Witten [1] that the massless modes propagating along the worldvolume of N coincident D-branes are those of the supersymmetric Yang-Mills theory, obtained by the dimensional reductions of the $d = 10$ $\mathcal{N} = 1$ theory down to $p + 1$ space-time dimensions.

In various compactifications of string theory one encounters the nearly massless non-perturbative particles, obtained by wrapping the Dp-branes around vanishing p-cycles inside the internal Calabi-Yau manifold. Even in ten dimensional Type IIA string theory there are solitonic particles [2], which are represented by certain black holes in the effective supergravity and are interpreted as Kaluza-Klein modes of the graviton multiplet in the compactification of M-theory on a circle [3][4][5]. Of course, these particles are no longer massless.

Despite the variety of mechanisms by which such objects appear, their internal description at low energies proves to be rather simple. In fact, if N such particles in d space-time dimensions are close to each other then their dynamics is described by the dimensional reduction of $\mathcal{N} = 1$ super-Yang-Mills theory from d down to $0 + 1$ dimensions (first studied a long time ago for different reasons in [6]). The degrees of freedom in such quantum mechanics are represented by $U(N)$ matrices X^i, $i = 1, \ldots, d$, together with the gauge field A_t and their fermionic partners.

Although the exact computations in quantum mechanics of interacting particles are rarely possible, the supersymmetry allows one to get some exact answers. In this paper we are going to concetrate on the correlation functions of the light-like coordinate operator. To state more precisely what we mean by that let us consider the quantum mechanics with periodic time $t \sim t + 2\pi\beta$ and with periodic boundary conditions on fermions. In this case one can show that the observable:

$$\mathcal{O}_R = \mathrm{Tr}_R P \exp \oint dt \left(A_t + X^3 \right)$$

commutes with some of the supercharges (of course the choice of X^3 is arbitrary). In the limit $\beta \to 0$ (and after Wick rotation) the computations in the quantum mechanics reduce to the finite-dimensional integrals, where A_t becomes the 0'th matrix $X_0 = -iX_4$. Then the observables $\mathcal{O}_R$ can be expanded in

$$\mathrm{Tr} \left(X^+ \right)^l, \quad X^+ = X^3 + iX^4$$

The paper is organized as follows. We are going to study the case $d = 4$ in great detail. We derive the determinant representation for the regularized generating function of the correlators of $\mathrm{Tr}\,(X^+)^l$ and show that it obeys Hirota bilinear identities (when working with fixed chemical potential). Then we concetrate on the operators $\left(\mathrm{Tr}\,(X^+)^2\right)^l$ and derive the asymptotics for the generating function in certain limit. We then briefly discuss $d = 6, 10$ cases. Then we proceed with the direct attack on the $d = 4$ integral for fixed but large N using the saddle-point techniques and derive interesting asymptotics in both the strong and weak coupling limit. In the weak coupling limit we get the agreement with the planar graph expansion. In the strong 't Hooft coupling limit we get the agreement with the predictions from KP hierarchy.

In the bulk of the paper we use the notation $\phi \equiv X^+, \bar{\phi} = X^-$. We also denote by Z, $F = \log Z$ the partition function and the free energy at fixed particle number N and by $\mathcal{Z}$, $\mathcal{F} = \log\mathcal{Z}$ the corresponding quantities at the fixed chemical potential μ.

2. Supersymmetric matrix integrals

2.1. Theory with four supercharges

The dimensional reduction of the $\mathcal{N} = 1$ SYM from $d = 4$ dimensions down to zero dimensions would produce a matrix model with 4 bosonic matrices X_μ, $\mu = 1, 2, 3, 4$ and 2 complex fermionic matrices λ_a, $a = 1, 2$. All matrices are in the adjoint representation of the gauge group G, which we will take to be either $U(N)$ or $SU(N)/\mathbb{Z}_N$. The matrix integral has the form:

$$\frac{1}{\mathrm{Vol}(G)} \int dX d\lambda \exp\left(\tfrac{1}{2} \sum_{\mu<\nu} \mathrm{Tr}[X_\mu, X_\nu]^2 + \mathrm{Tr}\bar{\lambda}_{\dot{a}} \sigma_\mu^{a\dot{a}} [X_\mu, \lambda_a]\right) \tag{2.1}$$

where $\sigma_\mu = (1, \sigma_i), i = 1, 2, 3$ are the Pauli matrices. The two complex fermions λ can be viewed as four real fermions which we denote as χ, η, ψ_α, $\alpha = 1, 2$:

$$\lambda_1 = \tfrac{1}{2}\left(\eta - i\chi\right), \quad \lambda_2 = \tfrac{1}{2}\left(\psi_1 + i\psi_2\right)$$

and

$$\bar{\lambda}_{\dot{a}} = \sigma_2^{a\dot{a}} \lambda_a^*.$$

We also redefine the bosonic matrices as:

$$\phi = \frac{1}{\sqrt{2}}\left(X_3 + iX_4\right), \quad \bar{\phi} = \frac{1}{\sqrt{2}}\left(X_3 - iX_4\right) \tag{2.2}$$

and introduce an auxilliary bosonic field H (also in the adjoint). Then integral (2.1) becomes:

$$\begin{aligned}&\int \frac{dX_\alpha d\psi_\alpha d\chi dH d\bar{\phi} d\eta d\phi}{\mathrm{Vol}(G)} \exp(-S) \\ &S = \left(i\mathrm{Tr}Hs + \tfrac{1}{2}\mathrm{Tr}H^2 + \mathrm{Tr}[X_\alpha,\phi][X_\alpha,\bar{\phi}] + \tfrac{1}{2}\mathrm{Tr}[\phi,\bar{\phi}]^2 + \ldots\right)\end{aligned} \tag{2.3}$$

where $s = [X_1, X_2]$ and $\ldots$ represent the fermionic terms which are reconstructed using the following nilpotent symmetry of (2.3):

$$\begin{aligned}\delta X_\alpha = \psi_\alpha, &\qquad \delta\psi_\alpha = [\phi, X_\alpha] \\ \delta\bar{\phi} = \eta, &\qquad \delta\eta = [\phi,\bar{\phi}] \\ \delta\chi = H, &\qquad \delta H = [\phi,\chi] \\ \delta\phi &= 0.\end{aligned} \tag{2.4}$$

The symmetry δ squares to the gauge transformation generated by ϕ, hence it is nilpotent on the gauge-invariant quantities. This symmetry was formally studied in [7] in order to apply it to the model of [8] and was powerfully exploited in [9] in the problem of computing Witten index in certain quantum mechanical systems (first studied in two-particle case by [10][11], see also [12]). The action of the matrix integral (2.3) is δ-exact and in fact it may be written as

$$S = \delta\left(-i\mathrm{Tr}\chi s - \tfrac{1}{2}\mathrm{Tr}\chi H - \sum_\alpha \mathrm{Tr}\psi_\alpha[X_\alpha,\bar{\phi}] - \tfrac{1}{2}\mathrm{Tr}\eta[\phi,\bar{\phi}]\right). \tag{2.5}$$

Now we proceed to reducing the integral (2.3) to an integral with respect of the single matrix variable ϕ. The strategy is known for some time [13] and it consists of two steps.

If the action is perturbed by the expression $\delta(\ldots)$ with nice behavior at infinity then the integral shouldn't change which can be shown by integration by parts. Consider the modification of the action S by the term

$$S \to S + i\delta R$$

with

$$R = \frac{\kappa_1}{2}\varepsilon^{\alpha\beta}\mathrm{Tr}\psi_\alpha X_\beta + \kappa_2\mathrm{Tr}\chi\bar{\phi}. \tag{2.6}$$

This perturbation makes the integral (2.1) localized near the zeroes of $H, \bar{\phi}, \chi, \eta$ is the limit of large κ_2 which can be shown by the saddle-point approximation. It reduces an integral (2.1) to a simpler one

$$\int \frac{d\phi dX_\alpha d\psi_\alpha}{\mathrm{Vol}(G)} \exp(i\kappa_1\mathrm{Tr}\phi[X_1,X_2] + \kappa_1\psi_1\psi_2). \tag{2.7}$$

The behavior of the integrand at large values of ϕ is still not good enough. To make it better behaved we modify the transformation δ. The current δ is designed to respect the ordinary gauge invariance. In particular δ^2 = gauge transformation generated by ϕ. We wish to invoke yet another symmetry of the integral (2.1) which is the global group $U(1)_\epsilon$ acting on the matrices X_α, ψ_α via rotations:

$$e^{i\theta} : X_1 + iX_2 \mapsto e^{i\theta}\left(X_1 + iX_2\right) \tag{2.8}$$

The rest of the fields are invariant under this $U(1)_\epsilon$ group action. Let us denote the generator of this group by ϵ. Then the new supercharge δ acts as follows:

$$\begin{aligned} \delta X_\alpha = \psi_\alpha, \qquad & \delta\psi_\alpha = [\phi, X_\alpha] + i\epsilon \cdot \varepsilon^{\alpha\beta} X_\beta \\ \delta\bar{\phi} = \eta, \qquad & \delta\eta = [\phi, \bar{\phi}] \\ \delta\chi = H, \qquad & \delta H = [\phi, \chi] \\ \delta\phi = 0. & \end{aligned} \tag{2.9}$$

The integral (2.1) has another $U(1)$ symmetry (called the ghost number $U(1)_{\mathrm{gh}}$) under which δ has charge $+1$, the bosons X_α, H have charge 0, the fermions ψ_α have charge $+1$, the fermions χ, η have charge -1, and the bosons ϕ and $\bar{\phi}$ have charges $+2$ and -2 respectively. The measure and the action have the overall charge 0. The modification (2.9) is consistent with the ghost number symmetry iff the generator ϵ is assigned the ghost number two. If we compute the modification of the action (2.7) then we get a better behaved integral

$$\int \frac{d\phi dX_\alpha d\psi_\alpha}{\mathrm{Vol}(G)} \exp \kappa_1 \mathrm{Tr}\left(i\phi[X_1, X_2] - \tfrac{1}{2}\epsilon(X_1^2 + X_2^2) + \psi_1\psi_2\right). \tag{2.10}$$

The factor κ_1 can be now reabsorbed into the X's and ψ's without affecting the measure and then ψ's can be integrated out. Also, the matrices X_1, X_2 can be integrated out, producing the determinant:

$$Z(N, \epsilon, V) = \int \frac{d\phi}{\mathrm{Vol}(G)} \frac{1}{\mathrm{Det}\left(ad(\phi) + \epsilon\right).} \tag{2.11}$$

The supersymmetry δ allows to modify the action by the observables

$$S \to S + \sum_n T_n \mathcal{O}_n$$

where $\delta\mathcal{O}_n = 0$. In our case the operators $\mathcal{O}_n$ are simply the gauge invariant functions of ϕ as they are also $U(1)_\epsilon$ invariant. The simplest operators whose correlation functions may be evaluated are the gauge invariant functionals of ϕ, like $\mathrm{Tr}\phi^l$.

To summarize, we have shown that the computation of the (regularized) correlation functions of the observables $\mathrm{Tr}(\phi^n)$ in the supersymmetric matrix integral (2.1) may be reduced to the computations of the integral over single matrix ϕ of the form $V(z) = -\sum_n T_n z^n$:

$$\int \frac{d\phi}{\mathrm{Vol}(G)} \frac{e^{-\mathrm{Tr}V(\phi)}}{\mathrm{Det}\left(ad(\phi)+\epsilon\right)} \tag{2.12}$$

Remarks.

1. In the paper [9] the similar perturbation has been used in the computations of the Witten index, which can be reduced to the computation of the integral (2.1) for the group $SU(N)/\mathbb{Z}_N$ and without insertions of any observables. In that case the result of the computation was ϵ-independent. Also the integral over the eigenvalues of the matrix ϕ in that case was to be understood as a contour integral, to avoid contribution of flat directions which corresponded to the unbound free particles. In our case the flat directions contribute to the correlation functions as well and the parameter ϵ serves as a regulator as in the computations of [14].
2. One may wonder about the physical meaning of the ϵ-regularized integrals. Here it is:

$$Z(N,\epsilon,V) = \mathrm{Tr}_{\mathcal{H}}(-)^F e^{-\beta H} e^{-\epsilon\cdot J} e^{-\mathrm{Tr}V(\phi)} \tag{2.13}$$

   where the trace is taken over the Hilbert space $\mathcal{H}$ of the quantum mechanical system, H is the Hamiltonian, F is the fermion number, J is the generator of the global symmetry group (which we take to be $SO(d-2)$ for $d=4,6$ and $SO(6)$ for $d=10$, see below). For example, in the case $d=4$, $J = \mathrm{Tr}(\bar{\phi}[X_1,X_2])$. Just like in [11] this expression is related to the matrix integral in the $\beta \to 0$ limit. One can also consider directly the quantum mechanical path integral, i.e. the integral over the space of loops. In this case the rational functions in the formulae (2.12) and the similar formulae below are replaced by their trigonometric counterparts. Also one can consider $1+1$ model (Matrix strings) in which case the ratio of determinants lead to elliptic functions, just like in [15].

2.2. Theory with eight supercharges

This is the model obtained by the dimensional reduction of $\mathcal{N}=1$, $d=6$ theory. In this model the indiex α of the matrices X_α, ψ_α runs from 1 to 4. The symmetry $U(1)_\epsilon$ is extended to $SO(4) \approx SU(2)_L \times SU(2)_R$. The matrices X_α, ψ_α form two copies of the representation $(\frac{1}{2},\frac{1}{2})$ of this group. Also, the fermion χ is promoted to a triplet $\vec{\chi}$ which is in $(\frac{1}{2},0)$. The same metamorphose is experienced by the auxilliary field $H \to \vec{H}$. The action is constructed by same rules, the only difference being:

$$\chi(s-H) \to \vec{\chi}\cdot(\vec{s}-\vec{H})$$

where

$$s_i = [X_4, X_i] + \tfrac{1}{2}\varepsilon_{ijk}[X_j, X_k]$$

The modification of the supercharge (2.9) is achieved by introduction of the generators $\epsilon_L \oplus \epsilon_R = \left(\frac{\epsilon_1+\epsilon_2}{2}\right) \oplus \left(\frac{\epsilon_1-\epsilon_2}{2}\right)$ of the Cartan subalgebra of $SO(4)$. The modified transformations are:

$$\begin{aligned} \delta\psi_1 &= [\phi, X_1] + i\epsilon_1 X_2, & \delta\psi_3 &= [\phi, X_3] + i\epsilon_2 X_4 \\ \delta\psi_2 &= [\phi, X_2] - i\epsilon_1 X_1, & \delta\psi_4 &= [\phi, X_4] - i\epsilon_2 X_3 \\ \delta\chi_i &= H_i & \delta H_3 &= [\phi, \chi_3] \\ \delta H_1 &= [\phi, \chi_1] + 2i\epsilon_L\chi_2 & \delta H_2 &= [\phi, \chi_2] - 2i\epsilon_L\chi_1 \end{aligned} \tag{2.14}$$

Now we get, instead of (2.12), the following one-matrix integral

$$\int \frac{d\phi e^{-\mathrm{Tr}V(\phi)}}{\mathrm{Vol}(G)} \frac{\mathrm{Det}\,(ad(\phi)+\epsilon_1+\epsilon_2)}{\mathrm{Det}\,(ad(\phi)+\epsilon_1)\,\mathrm{Det}\,(ad(\phi)+\epsilon_2)} \tag{2.15}$$

2.3. Theory with sixteen supercharges

It is of great interest to obtain the similar expression for the integrals occuring in the reductions of $d=10$ SYM. In this theory the matrices X_α, ψ_α have index α transforming in the $\mathbf{8}$ of the group $SO(8)$. The antighost $\vec{\chi}$ belongs to $\mathbf{1}\oplus\mathbf{6}$ of $SU(4) \subset SO(8)$. Introduce the notation

$$B_i = X_{2i-1} + iX_{2i}, i = 1,2,3,4.$$

The matrices B_i are in $\mathbf{4}$ of $SU(4)$ and $B_i^\dagger$ are in $\bar{\mathbf{4}}$. The "gauge condition" $\vec{s}$ splits as:

$$\begin{gathered} \vec{s} = \mu \oplus \Phi \qquad \mu = \sum_{i=1}^{4}[B_i, B_i^\dagger] \in \mathbf{1} \\ \Phi_{ij} = [B_i, B_j] + \tfrac{1}{2}\varepsilon_{ijkl}[B_k^\dagger, B_l^\dagger], \\ \Phi_{ij} = \epsilon_{ijkl}\Phi_{kl}^\dagger, \qquad \text{i.e.} \quad \Phi \in \mathbf{6}. \end{gathered} \tag{2.16}$$

The action constructed by the standard rules coincides with that of dimensional reduction of $d = 10$ $\mathcal{N} = 1$ SYM. The gauge field has ten components which become $\phi, \bar{\phi}$ and X_α. Sixteen-component fermion splits as ψ_α, with eight components, $\vec{\chi}$ with seven components and η.

The global group $SU(4)$ (which is not to be confused with the R-symmetry group of $\mathcal{N} = 4$ SYM in four dimensions!) allows to modify the supercharge δ in the manner analogous to (2.9)–(2.14). The generator of Cartan of $SU(4)$ may be written as: $\epsilon_1 \oplus \epsilon_2 \oplus \epsilon_3 \oplus (\epsilon_4 = -\epsilon_1 - \epsilon_2 - \epsilon_3)$. The integrals (2.12)–(2.15) generalize to:

$$\int \frac{d\phi e^{-\mathrm{Tr} V(\phi)}}{\mathrm{Vol}(G)} \frac{\mathrm{Det}\,(ad(\phi) + \epsilon_1 + \epsilon_2)\,\mathrm{Det}\,(ad(\phi) + \epsilon_2 + \epsilon_3)\,\mathrm{Det}\,(ad(\phi) + \epsilon_3 + \epsilon_1)}{\mathrm{Det}\,(ad(\phi) + \epsilon_1)\,\mathrm{Det}\,(ad(\phi) + \epsilon_2)\,\mathrm{Det}\,(ad(\phi) + \epsilon_3)\,\mathrm{Det}\,(ad(\phi) + \epsilon_4)}. \tag{2.17}$$

3. Determinant representation of the correlation functions in $d = 4$ case

In this section we study in details the integral the grand partition function

$$\mathcal{Z}(\mu, \epsilon, V) = \sum_{N=0}^{\infty} e^{\mu N} Z(N, \epsilon, V).$$

We show that $\mathcal{Z}(\mu, \epsilon, V)$ has a determinant representation, very much like the correlation functions in Sine-Gordon and related models are expressed in terms of Fredholm determinants [16].

3.1. Eigenvalue integral

First we write the integral (2.12) in terms of the eigenvalues $i\phi_1, ..., i\phi_N$ of the anti-hermitean matrix ϕ

$$Z(N, \epsilon, V) = \int_{\mathbb{R}^N} \frac{d\phi_1 \dots d\phi_N}{N!(2\pi\epsilon)^N} \prod_{i \neq j} \frac{\phi_i - \phi_j}{\phi_i - \phi_j + i\epsilon} \prod_i e^{-V(\phi_i)} \tag{3.1}$$

where we changed $V(ix)$ to $V(x)$. The integral (3.1) can be rewritten using the Cauchy's formula as

$$Z(N, \epsilon, V) = \sum_{\sigma \in \mathcal{S}_N} (-)^\sigma \int \frac{d\phi_1 \dots d\phi_N}{N!(2\pi i)^N} \prod_k \frac{e^{-V(\phi_k)}}{\phi_k - \phi_{\sigma(k)} + i\epsilon}. \tag{3.2}$$

It turns out that the grand partition function (that is, with fixed chemical potential) can be written as a Fredholm determinant of an integral operator. Let us introduce the notation

$$W_l(\epsilon, V) = \int_{\mathbb{R}^l} \prod_{k=1}^{l} \frac{dx_k}{2\pi} \frac{e^{-V(x_k)}}{\epsilon - i(x_k - x_{k+1})}, \quad x_{l+1} \equiv x_1. \tag{3.3}$$

We may rewrite the sum over all elements of permutation group in (3.2) as the sum over conjugacy classes, which are labelled by the partitions of N:

$$N = \sum_{l=1}^{\infty} l d_l, \quad d_l \geq 0.$$

Every permutation in the conjugacy class, labelled by $\vec{d} = (d_1, d_2, \ldots)$ is similar to the product of cycles of lengths $1, 2, \ldots,$ with the number of times the cycles with length l appear being precisely d_l. The number of permutations in the given conjugacy class $\vec{d}$ is equal to

$$\frac{N!}{\prod_l l^{d_l} d_l!}$$

and the sign of any permutation in this class is $(-)^{\sum_l d_l}(-)^N$. Thus, (3.2) may be represented as

$$Z(N, \epsilon, V) = \sum_{\vec{d}: \sum l d_l = N} \prod_l \frac{1}{d_l!} \left(-\frac{(-)^l W_l}{l} \right)^{d_l} \tag{3.4}$$

and the grand partition function is equal to

$$\mathcal{Z}(\mu, \epsilon, V) = \exp \sum_l (-)^{l-1} e^{l\mu} \frac{W_l(\epsilon, V)}{l} \tag{3.5}$$

The quantity $W_l(\epsilon, V)$ may be represented as $W_l = \mathrm{Tr} K^l$ where K is a linear operator acting in the space of functions of one variable as follows:

$$(Kf)(x) = e^{-V(x)} \int_{\mathbb{R}} \frac{dy}{2\pi} \frac{f(y)}{\epsilon - i(x-y)}. \tag{3.6}$$

Therefore, the grand partition function becomes

$$\mathcal{Z}(\mu, \epsilon, V) = \exp \sum_l \frac{(-)^{l-1}}{l} \mathrm{Tr}(e^{\mu} K)^l = \mathrm{Det}\,(I + e^{\mu} K). \tag{3.7}$$

3.2. Another representation for quadratic V

Let us consider the case of a gaussian potential $V(x) = \frac{1}{2}\left(\frac{\xi x}{\epsilon}\right)^2$ and write the partition function (3.1) again as a matrix integral

$$Z(N,\xi) = \int \frac{d\phi dX}{\mathrm{Vol}(G)} \exp\left(-\tfrac{1}{2}\mathrm{Tr}[\phi, X]^2 + \frac{\xi^2}{\epsilon^2}\mathrm{Tr}\phi^2 - \tfrac{1}{2}\epsilon^2 \mathrm{Tr}X^2\right). \tag{3.8}$$

Considering the matrices X and ϕ as the the hermitean and antihermitean part of the same complex matrix $Z = (\epsilon/\sqrt{\xi})X + (\sqrt{\xi}/\epsilon)\phi$, we rewrite the matrix integral as

$$Z(N,\xi) = \frac{1}{\mathrm{Vol}(G)} \int dZ dZ^\dagger \, e^{-S}, \quad S = \tfrac{1}{2}\mathrm{Tr}[Z, Z^\dagger]^2 + \frac{\xi}{2}\mathrm{Tr}ZZ^\dagger. \tag{3.9}$$

In polar coordinates

$$Z = UH^{1/2}$$

where U is unitary and H is hermitian matrix with *positive* eigenvalues $y_1, \ldots, y_N$, the measure and the action read

$$dZdZ^\dagger = dUdH, \quad S = \mathrm{Tr}H^2 - \mathrm{Tr}U^{-1}HUH + \tfrac{1}{2}\xi \mathrm{Tr}H.$$

Using the Harish-Chandra-Itzykson-Zuber formula [17][18] we perform the U-integration and find

$$Z(N,\xi) = \frac{1}{N!} \int_{\mathbb{R}_+^N} d^N y \; e^{-\sum_i \left(y_i^2 + \frac{1}{2}\xi y_i\right)} \mathrm{Det}_{ij}\left(e^{y_i y_j}\right). \tag{3.10}$$

The grand partition function will be expressed in terms of the quantities

$$W_l = \int_{\mathbb{R}_+^l} \prod_{i=1}^{l} dy_i \; e^{-\frac{1}{2}\left[\xi y_i + (y_- y_{i+1})^2\right]}, \qquad (y_{l+1} \equiv y_1), \tag{3.11}$$

as

$$\mathcal{Z}(\mu,\xi) = \exp \sum_{l=1}^{\infty} (-)^{l-1} e^{\mu l} \frac{W_l}{l}. \tag{3.12}$$

Clearly $W_l = \mathrm{Tr}\mathcal{K}^l$, where $\mathcal{K}$ now acts on the functions on the positive semi-axis as

$$\mathcal{K}f(y) = e^{-\frac{\xi}{2}y} \int_0^\infty e^{-\frac{1}{2}(y-y')^2} f(y') dy'. \tag{3.13}$$

Therefore we arrive at the same determinant representation (3.7) where the kernel K is replaced by its Fourier transform $\mathcal{K}$.

For small ξ,

$$(-)^{l-1} W_l = \frac{1}{l\xi} g_l(\xi) \tag{3.14}$$

where g_l is an analytic function.

4. The grand partition function as a tau-function

In this section we show that grand partition function

$$\mathcal{Z}(\mu,\epsilon,V)=\sum_{N=0}^{\infty}\int_{-\infty}^{\infty}\prod_{i=1}^{N}dx_i\ e^{-V(x_i)}\ \frac{\prod_{i<j}(x_i-x_j)^2}{\prod_{i,j}(x_i-x_j-i\epsilon)} \tag{4.1}$$

can be represented as a tau-function of the KP hierarchy.

4.1. Vertex operator construction - bosonic representation

Introduce the bosonic field $\varphi(z)$ with mode expansion

$$\varphi(z)=\hat{q}+\hat{p}\ln z+\sum_{n\neq 0}\frac{J_n}{n}z^{-n}, \tag{4.2}$$

$$[J_n,J_m]=n\delta_{m+n,0};\quad [\hat{p},\hat{q}]=1 \tag{4.3}$$

and the vacuum state $|l\rangle$ defined by

$$J_n|l\rangle=0,\quad (n>0);\quad \hat{p}|l\rangle=l|l\rangle. \tag{4.4}$$

The associated normal ordering is defined by putting J_n with $n>0$ to the right. Define the vertex operator

$$\mathcal{V}_\epsilon(z)=:e^{\varphi(z+i\epsilon/2)}::e^{-\varphi(z-i\epsilon/2)}: \tag{4.5}$$

satisfying the OPE

$$\mathcal{V}_\epsilon(z)\mathcal{V}_\epsilon(z')=\frac{(z-z')^2}{(z-z')^2+\epsilon^2}:\mathcal{V}_\epsilon(z)\mathcal{V}_\epsilon(z'):, \tag{4.6}$$

the Hamiltonian

$$H[t]=\sum_{n>0}t_nJ_n, \tag{4.7}$$

and the operator

$$\Omega_\mu=\exp\Big(e^\mu\int_{-\infty}^{\infty}\frac{dz}{2\pi i}\mathcal{V}_\epsilon(z)\Big). \tag{4.8}$$

Then the vacuum expectation value

$$\tau_0[t]=\langle 0|e^{H[t]}\Omega_\mu|0\rangle \tag{4.9}$$

is equal to the canonical partition function (4.1) with chemical potential μ and potential

$$V(z) = U(z + \frac{i}{2}\epsilon) - U(z - \frac{i}{2}\epsilon) = -\sum_{n=0}^{\infty} T_n z^n,$$
$$U(z) = -\sum_{n=1}^{\infty} t_n z^n, \tag{4.10}$$

where the chemical potential μ can be reabsorbed into the definition of V and where the coefficients T_n and t_n are related via:

$$T_{n-1} = i\sum_{k=0}^{\infty} \binom{n+2k}{n} (-)^k \frac{\epsilon^{2k+1}}{4^k} t_{n+2k}$$
$$t_{n+1} = \frac{-i}{(n+1)!}\sum_{k=0}^{\infty}(n+2k)!\frac{\epsilon^{2k-1}}{4^k}\mathbf{b}_k T_{n+2k}, \tag{4.11}$$

where the numbers $\mathbf{b}_n$ (they are related to Bernoulli numbers B_n) are given by the generating function:

$$\frac{x}{\sin(x)} = \sum_{n=0}^{\infty} \mathbf{b}_n x^{2n}. \tag{4.12}$$

4.2. Fermionic representation

The fermionic representation of the partition function is constructed using the bosonization formulas

$$\psi(z) =: e^{-\varphi(z)} : \quad \psi^*(z) =: e^{\varphi(z)} : \quad \partial\varphi(z) =: \psi^*(z)\psi(z) : \tag{4.13}$$

where the fermion operators

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}}, \qquad \psi^*(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi^*_{-r} z^{-r-\frac{1}{2}} \tag{4.14}$$

satisfy the anticommutation relations

$$[\psi_r, \psi^*_s]_+ = \delta_{rs}. \tag{4.15}$$

The operators (4.7) and (4.8) are represented by

$$H[t] = \sum_{n>0} t_n \sum_r : \psi^*_{r-n}\psi_r : \quad (n \in \mathbb{Z}) \tag{4.16}$$

$$\Omega_\mu = \exp\left[e^\mu \int_{-\infty}^{\infty} \frac{dx}{2\pi i} :\psi(x+i\frac{\epsilon}{2})\psi^*(x-i\frac{\epsilon}{2}):\right]. \tag{4.17}$$

and the vacuum states with given electric charge l satisfy

$$\begin{aligned} \langle l|\psi_{-r} = \langle l|\psi^*_r = 0 \qquad (r>l) \\ \psi_r|l\rangle = \psi^*_{-r}|l\rangle = 0 \qquad (r>l). \end{aligned} \tag{4.18}$$

The original expression (4.1) is obtained from the expectation value (4.9) by first commuting the operator e^H to the right until it hits the right vacuum by using the formulas

$$\begin{aligned} e^{H[t]}\psi(z)e^{-H[t]} &= e^{\sum_{n=1}^\infty t_n z^n}\psi(z) \\ e^{H[t]}\psi^*(z)e^{-H[t]} &= e^{-\sum_{n=1}^\infty t_n z^n}\psi^*(z) \end{aligned} \tag{4.19}$$

and then applying the Wick theorem to calculate the expectation value

$$\left\langle l \left| \prod_i \psi(z_i)\psi^*(w_i) \right| l \right\rangle = \prod_i \left(\frac{z_i}{w_i}\right)^l \prod_{i<j} \frac{(z_i-z_j)(w_i-w_j)}{(z_i-w_j)(w_i-z_j)}. \tag{4.20}$$

4.3. The KP hierarchy

The partition function (4.9) is a particular case of the “general solution” of the KP hierarchy obtained as the limit $N\to\infty$ of a general N-soliton solution [19]

$$\tau_l[t] = \langle l|e^{\sum_{n>0} t_n J_n}\Omega_a|l\rangle, \tag{4.21}$$

where the $GL(\infty)$ rotation

$$\Omega_a = \exp\Big(\int dxdy a(x,y)\psi(x)\psi^*(y)\Big) \tag{4.22}$$

is parametrized by an arbitrary integrable function $a(x,y)$[1].

The tau-functions $\tau_l,\ l\in\mathbb{Z}$ are Fredholm determinants

$$\tau_l = \det(1+e^\mu K_l) \tag{4.23}$$

of the kernels K_l

$$K_l(x,y) = \frac{E_l(x+i\epsilon)}{E_l(x-i\epsilon)}\ \frac{1}{x-y-2i\epsilon}, \qquad E_l(x) = x^l\ \exp\left(\sum_n t_n x^n\right). \tag{4.24}$$

[1] In the soliton solutions it is a sum of delta-functions, $a(x,y)=\sum_{k=1}^N a_i\delta(x-p_i)\delta(y-q_i)$ so that the operator Ω_a is a finite sum, $\Omega_a = \sum\sum_{k=1}^N a_i\psi(p_i)\psi^*(q_i)$.

For $l=0$ we get precisely the operator K (3.6).

The KP hierarchy of differential equations is generated by the *Hirota bilinear equations* [20]:

$$\oint \frac{dz}{2\pi i} z^{l-l'} \exp\Big(\sum_{n>0}(t_n - t'_n)z^n\Big)\tau_l(t_n - \frac{1}{n}z^{-n})\ \tau_{l'}(t'_n + \frac{1}{n}z^{-n}) = 0 \qquad (l' \le l). \quad (4.25)$$

Let us sketch the proof of (4.25). First we remark that each element $\Omega \in GL(\infty)$ is represented by an infinite c-number matrix $a = \{a_{rs}\}$

$$\Omega\psi_r\Omega^{-1} = \sum_s \psi_s a_{sr}, \qquad \Omega^{-1}\psi^*_r\Omega = \sum_s a_{rs}\psi^*_s. \quad (4.26)$$

As a consequence, there exists a tensor Casimir operator

$$S_{12} = \sum_r \psi_r \otimes \psi^*_r = \oint \frac{dz}{2\pi i}\psi(z)\otimes\psi^*(z) \quad (4.27)$$

which satisfies for any $\Omega \in GL(\infty)$

$$S_{12}\ \Omega\otimes\Omega = \Omega\otimes\Omega\ S_{12}. \quad (4.28)$$

On the other hand $S_{12}\ |l\rangle\otimes|l\rangle = 0$ because, according to (4.18), for each r either ψ_r or ψ^*_r is annihilated by the right vacuum $|l\rangle$. Therefore (4.28) implies that $S_{12}\ \Omega|l\rangle\otimes\Omega|l\rangle = 0$. Taking the scalar product with $\langle l+1|e^{H[t]}\otimes\langle l-1|e^{H[t']}$ we find

$$\oint \frac{dz}{2\pi i}\langle l+1|e^{H[t]}\psi(z)\Omega_\mu|l\rangle\langle l-1|e^{H[t']}\psi^*(z)\Omega_\mu|l\rangle = 0 \quad (4.29)$$

where the integration contour surrounds the origin. Eq. (4.29) simply reflects the fact that the tensor Casimir (4.27) is constant on the orbits of $GL(\infty)$. Finally we use the bosonization formulas (4.13) to represent the fermions as vertex operators, $\psi(z)\to\mathbf{V}_-(z)$, $\psi^*(z)\to\mathbf{V}_+(z)$, where $\mathbf{V}_\pm(z)$ act in the space of the coupling constants as

$$\mathbf{V}_\pm(z) = \exp\Big(\pm\sum_{n=0}^{\infty}t_n z^n\Big)\exp\Big(\mp\ln\frac{1}{z}\frac{\partial}{\partial\mu}\mp\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial}{\partial t_n}\Big). \quad (4.30)$$

The general case $l \neq l'$ is treated similarly and one obtains the following identity

$$\oint dz\ \Big(\mathbf{V}_+(z)\cdot\tau_l[t]\Big)\ \Big(\mathbf{V}_-(z)\cdot\tau_{l'}[t']\Big) = 0 \qquad (l'\le l) \quad (4.31)$$

which is identical to the Hirota equation (4.25).

The differential equations of the KP hierarchy are obtained by expanding (4.25) in the differences $y_n = \frac{1}{2}(t_n - t'_n)$. In the case $l' = l$ the first nontrivial equation (the KP equation) is obtained by requiring that y_1^3 term vanishes:

$$\Big(\frac{\partial^4}{\partial y_1^4} + 3\frac{\partial^2}{\partial y_2^2} - 4\frac{\partial}{\partial y_1}\frac{\partial}{\partial y_3}\Big)\tau_l[t+y]\tau_l[t-y]\Big|_{y=0} = 0. \tag{4.32}$$

In terms of the "specific heat"

$$u[t] = 2\frac{\partial^2}{\partial t_1^2}\log\tau_l \tag{4.33}$$

the KP equation reads

$$3\frac{\partial^2 u}{\partial t_2^2} + \frac{\partial}{\partial t_1}\Big[-4\frac{\partial u}{\partial t_3} + 6u\frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3}\Big] = 0. \tag{4.34}$$

5. The $d = 4$ integral with quadratic potential: KP equation, weak coupling and double scaling

In this section we study the case $t_n = 0,\ n > 3$.

5.1. Reduction to a single equation

The potential of the form $V(x) = \mu + \lambda x^2$, $\lambda = \xi^2$ is related to the three couplings t_1, t_2, t_3 by

$$\mu = i\epsilon(t_1 - \frac{1}{4}\epsilon^2 t_3 - \frac{t_2^2}{3t_3}),\quad \xi^2 = 3i\epsilon t_3. \tag{5.1}$$

Rescale $\epsilon \to 1$. Then $u = -2\partial_\mu^2\mathcal{F}$ and it is easy to show that (4.34) implies the following partial differential equation for the function $\psi = \xi u(\mu,\xi)$

$$\psi_\xi - \psi\psi_\mu - \frac{\xi}{6}\left(\psi - \psi_{\mu\mu}\right)_\mu = a(\xi). \tag{5.2}$$

By comparing to the expansion in (3.12)–(3.14) we show that $a(\xi) \equiv 0$. If we expand

$$\psi(\mu,\xi) = \sum_{l=1}^{\infty} e^{\mu l} e^{-\frac{\xi^2}{12}(l^3-l)}\eta_l(\xi), \tag{5.3}$$

then equation (5.2) is equivalent to the infinite system of recursive first-order differential equations:

$$\eta'_l = \frac{l}{2}\sum_{p+q=l}\eta_p\eta_q \exp\frac{\xi^2 lpq}{4}. \tag{5.4}$$

The form (5.3) is dictated by the semi-classical approximation to the integral (2.10). Indeed, the expression $\frac{\xi^2}{12}(l^3-l)$ is nothing but the classical action evaluated on the solution to the equations of motion:

$$
\begin{aligned}
[X_1, X_2] &= -2i\lambda\phi, \\
[\phi, X_2] &= -iX_1, \\
[\phi, X_1] &= +iX_2.
\end{aligned}
\tag{5.5}
$$

The solutions to (5.5) are classified by the decompositions of N-dimensional representation into irreducibles of $SU(2)$. The logarithm of the grand partition function takes into account only irreducible l-dimensional representations, the rest is generated by the exponentiation. The functions η_l therefore describe the quantum fluctuations around the saddle points.

We conclude this section by listing the two equivalent forms of the equation obeyed by u:

$$
\begin{aligned}
&\psi_\xi - \psi\psi_\mu - \frac{\xi}{6}(\psi - \psi_{\mu\mu})_\mu = 0 \\
&2u_\lambda + \frac{1}{\lambda}u - uu_\mu - \frac{1}{6}(u - u_{\mu\mu})_\mu = 0.
\end{aligned}
\tag{5.6}
$$

5.2. Weak coupling limit

For low values of l the equations (5.4) can be solved explicitly. It is interesting to look at the large $\lambda = \xi^2$ asymptotics of the solutions. We expect that as $\lambda \to \infty$ the partition function at fixed N has the following scaling behavior of $Z(N,\lambda) = e^{F(N,\lambda)}$

$$
Z(N,\lambda) = \frac{1}{\lambda^{N^2/2}}\left(1+\ldots\right) = \frac{1}{\xi^{N^2}}\left(1+\ldots\right). \tag{5.7}
$$

On the other hand one finds from (5.4)

$$
u = -\frac{e^\mu}{\xi} + \frac{e^{2\mu}}{\xi^2}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{\xi^{2n}} + \ldots \tag{5.8}
$$

and therefore indeed

$$
Z(1,\lambda) = \frac{1}{2\xi}, \quad Z(2,\lambda) = -\frac{1}{\xi^4}\sum_{n=1}^{\infty}\frac{(2n-1)!!}{\xi^{2n}}, \tag{5.9}
$$

in accord with the scaling (5.7).

It turns out that equation (5.2) has another interesting property. Suppose we are studying the 't Hooft's limit, where the free energy has an expansion of the form

$$
F(N,\lambda) = \sum_{g=0}^{\infty} N^{2-2g} F_g(N/\lambda). \tag{5.10}
$$

Given $\mathcal{Z}(\mu,\lambda) = e^{\mathcal{F}(\mu,\lambda)}$ we extract the fixed–N partition function via Fourier transform

$$Z(N,\lambda) = \oint \frac{d\mu}{2\pi i} e^{-i\mu N + \mathcal{F}(i\mu,\lambda)} \tag{5.11}$$

which in the large N limit can be taken using the saddle point approximation. In the planar limit ($N = \infty$, with λ/N finite) the functions $\mathcal{F}(\mu,\lambda)$ and $F(N,\lambda)$ are Legendre transforms of each other. If we keep all the $1/N$ corrections then the corresponding expansion of $\mathcal{F}(\mu,\lambda)$ is:

$$\mathcal{F}(\mu,\lambda) = \sum_{g=0}^{\infty} \lambda^{2-2g} \mathcal{F}_g\left(\mu/\lambda\right) \tag{5.12}$$

and

$$u(\mu,\lambda) = \sum_{g=0}^{\infty} u_g\left(\mu/\lambda\right) \lambda^{-2g}, \qquad u_g(x) = -2\mathcal{F}_g''(x). \tag{5.13}$$

Let us introduce the variables

$$x = \mu/\lambda, \quad y = \lambda^{-2}, \quad \chi(x,y) = u(\mu,\lambda) \tag{5.14}$$

relevant for the 't Hooft limit. Then equation (5.2) may be rewritten as:

$$\chi - 2\left(x + \frac{1}{12}\right)\chi_x \; - 4y\chi_y - \chi\chi_x + \frac{y}{6}\chi_{xxx} = 0 \tag{5.15}$$

which after the expansion

$$\chi(x,y) = \sum_{g=0}^{\infty} y^g \chi_g(x) \tag{5.16}$$

reduces to the infinite system of the recursive equations for χ_g's:

$$\begin{aligned} \left(-\frac{1}{6} - 2x - \chi_0\right)\chi_g' + (1 - 4g + \chi_0') &= -\frac{\chi_{g-1}'''}{6} + \sum_{a+b=g,a,b\neq 0} \chi_a\chi_b', \quad g > 0 \\ \chi_0 - 2\left(x + \frac{1}{12}\right)\chi_0' - \chi_0\chi_0' &= 0. \end{aligned} \tag{5.17}$$

The equation for χ_0 is the only non-linear one. Its solution is:

$$x = \frac{\alpha_0}{2}\chi_0^2 - \chi_0 - \frac{1}{12}. \tag{5.18}$$

Of course equation (5.2) has more general solutions, in particular those for which the expansion (5.16) is not bounded as $g \geq 0$. It turns out that the solution corresponding to the matrix integral in question does have the form (5.16). In Appendix **A** we show that $\alpha_0 = \frac{\pi^2}{2}$.

It follows that for large N and λ the free energy has 't Hooft-like behaviour[2]:

$$F(N,\lambda) = -\frac{1}{10}\left(\frac{243\pi^2}{4}\right)^{1/3} N^2 \left(\frac{\lambda}{N}\right)^{\frac{1}{3}} + \dots \tag{5.19}$$

5.3. Double scaling limit near the quadratic singularity in $\chi(x)$

So far we investigated (3.1) only in the large N ('t Hooft) limit in the canonical ensemble or in the equivalent large μ limit for the grand canonical ensemble.

As we know from [21] and the Appendix **B**, the universal scaling behaviour of higher $1/N$ corrections sometimes can be summed up to some functions obeying nonlinear differential equations, like Painleve II for the pure $2d$ gravity.

It is reasonable to ask whether we can do the same with the $1/\mu$ expansion for our model starting from the general KP equation (5.15) and what is the physical or geometrical meaning of this expansion (we recall that in the pure gravity described by the one matrix model of the Appendix **B** the corresponding $1/N$-expansion has the meaning of the expansion over the genera of the topologies of the two dimensional manifold).

Let us concentrate on the square root singularity of (5.18) at $\chi_c = \frac{1}{\alpha_0}$, $x_c = -\frac{2}{\alpha_0}$. We try the following ansatz:

$$\chi = \chi_c + y^a v(z), \quad z = y^b\,(x - x_c) \tag{5.20}$$

As in the case of one-matrix model the presence of quadratic singularity implies that $b = -2a$. In the full analogy with the Appendix **B** (with only difference that y takes the place of $1/N^2$) inserting this ansatz into (5.15) and neglecting the subleading corrections we obtain $b = -\frac{2}{5}$, $a = \frac{1}{5}$ and that function $v(z)$ satisfies the Painleve II equation:

$$v'' - 3vv' = -\frac{12}{\pi^2} z. \tag{5.21}$$

From this equation we find the following coefficients of the $1/\lambda$ expansion in (5.12) (which is the same as $1/\mu$ expansion in this approximation) for the singular part of the free energy near:

$$\mathcal{F}_0 = -\sqrt{\Delta},\ \mathcal{F}_1 = 1/4 \log \Delta,\ \cdots \tag{5.22}$$

[2] Recall that $F = \mathcal{F} - \mu N$ in our conventions

where $\Delta = const(x - x_c)$ (we choose *const* in such a way that the coefficient in front of $\sqrt{\Delta}$ was 1, then the next coefficients, $1/4, \cdots$ are universal constants).

So everything goes just like in the pure $2d$ quantum gravity . The $1/\mu$ expansion looks like the topological $1/N$ expansion with the coefficients giving the leading scaling behaviour of the partition functions of successive topologies (see the details in [21]). It is tempting to speculate that the quadratic singularity in $\chi(x)$ corresponds to the pure gravity. It may be related to the large planar graph expansion with respect to g in the model of dense selfavoiding random paths (we will argue at the end of the section 6 that our matrix integral describes such a model in the large N limit). It would be interesting to demonstrate it by passing from the grand canonical to the canonical ensemble for the free energy.

6. Saddle-point approach

6.1. $d = 4$ integral

So far we managed to calculate the grand canonical version of the integral (3.5) in the large μ limit by the use of the KP equations. It is not clear whether from this asymptotics we can derive the large N limit of canonical partition function. In fact, we shall show that it is possible by comparing to the results of more direct approach, originally proposed in this context by J. Hoppe [22]. Namely, in the case of the gaussian potential $V(x) = \frac{1}{2}\lambda x^2$ it is possible to solve the integral saddle point equation for the distribution of the eigenvalues of a matrix in (3.1). We work out the details of the solution (correcting some minor mistakes in [22] and actually deriving the result) and extract interesting critical behaviours of our system.

It is natural to scale the coupling λ as N, and rescale ϵ to 1 i.e. to set:

$$\lambda = \frac{N}{g^2}, \quad \epsilon \to 1 \tag{6.1}$$

Indeed, we can rewrite the integral (3.1) as:

$$Z(N,\lambda) = \frac{1}{\mathrm{vol}(G)\lambda^{\frac{N^2}{2}}} \int dX d\phi \exp \tfrac{1}{2} \left(\mathrm{Tr}\left(\phi^2 + X^2\right) + \frac{1}{\lambda}\mathrm{Tr}[X,\phi]^2 \right) \tag{6.2}$$

We see that $\frac{1}{\sqrt{\lambda}}$ plays the role of the coupling constant while $\frac{N}{\lambda}$ is 't Hooft coupling. One of the interesting quantities is the one-point function:

$$\gamma = \frac{g^2}{4N^3} \langle \mathrm{Tr}[X,\phi]^2 \rangle \tag{6.3}$$

which is related to $\langle \mathrm{Tr}\phi^2 \rangle$ due to scaling properties of (3.1). In the large N limit the integral (3.1) localizes onto the critical point of the effective potential:

$$V_{eff}(\phi) = \sum_i V(\phi_i) + \sum_{i<j} \log\left(1 + \frac{1}{(\phi_i - \phi_j)^2}\right) \tag{6.4}$$

Its critical point is found from the equation:

$$\frac{\phi_k}{g^2} = \frac{1}{N} \sum_{j \neq k} \frac{1}{(\phi_k - \phi_j)\left(1 + (\phi_k - \phi_j)^2\right)} \tag{6.5}$$

In the usual fashion one assumes that in the large N limit the eigenvalues form a continious medium with the density:

$$\rho(x) = \frac{1}{N} \sum_i \delta(x - \phi_i) \tag{6.6}$$

We expect that ρ vanishes outside of the interval $(-a, +a)$, and also that it is an even function $\rho(x) = \rho(-x)$. We introduce the companion function - generating function of momenta $\langle \mathrm{Tr}\phi^k \rangle$:

$$F(x) = \int_{-a}^{+a} dy \frac{\rho(y)}{x - y}, \tag{6.7}$$

and rewrite the equation (6.5) as:

$$\frac{x}{g^2} = \hat{F}(x) - \tfrac{1}{2}\left(F(x+i) + F(x-i)\right) \tag{6.8}$$

where $\hat{F}(x) = \frac{1}{2}\left(F(x+i0) + F(x-i0)\right)$. By definition $F(x)$ has a cut at $(-a, +a)$:

$$F(x+i0) - F(x-i0) = 2\pi i \rho(x) \tag{6.9}$$

Introduce the functions:

$$F^{(1)}(x) = F(x + \frac{i}{2}) - F(x - \frac{i}{2}), \quad G(x) = -\frac{x^2}{g^2} - iF^{(1)}(x) \tag{6.10}$$

The saddle point equation (6.8) can be rewritten now as:

$$G(x + \frac{i}{2}) = G(x - \frac{i}{2}), \quad x \in (-a, +a) \tag{6.11}$$

The definitions (6.7)(6.10) imply that:

$$\begin{aligned} F(z) &= \overline{F(\bar{z})} \quad F(z) = -F(-z) \\ G(z) &= \overline{G(\bar{z})} \quad G(z) = G(-z) \end{aligned} \tag{6.12}$$

and also that the function $G(z)$ has the cuts at $(\pm\frac{i}{2}-a, \pm\frac{i}{2}+a)$. It is also clear from (6.11)(6.12) that $G(z)$ is real when $z \in \mathbb{R}, i\mathbb{R}, (\pm\frac{i}{2}-a, \pm\frac{i}{2}+a)$. Hence $G(z)$ defines a holomorphic map of the region $\mathcal{U}$ bounded by $\mathbb{R}_+, i\mathbb{R}_+$ and by the sides of the interval $(\frac{i}{2}, \frac{i}{2}+a)$ onto the upper half-plane $\mathcal{H}$. The inverse map is given by the following integral formula: $G(z) = -\zeta$, where:

$$z = A\int_{x_1}^{\zeta} \frac{dt(t-x_3)}{\sqrt{(t-x_1)(t-x_2)(t-x_4)}} \tag{6.13}$$

The map sends the special points $x_1 > x_2 > x_3 > x_4$ and ∞ as follows:

ζ	z
$+\infty$	$+\infty$
x_1	0
x_2	$\frac{i}{2}$
x_3	$\frac{i}{2}+a$
x_4	$\frac{i}{2}$
$-\infty$	$+i\infty$

These conditions imply the following equations on x_i, A, a:

$$\begin{aligned}
\tfrac{1}{2} &= A\int_{x_2}^{x_1} dt \frac{t-x_3}{\sqrt{(t-x_2)(x_1-t)(t-x_4)}}\\
a &= A\int_{x_3}^{x_2} dt \frac{x_3-t}{\sqrt{(x_2-t)(x_1-t)(t-x_4)}}\\
a &= A\int_{x_4}^{x_3} dt \frac{x_3-t}{\sqrt{(x_2-t)(x_1-t)(t-x_4)}}
\end{aligned} \tag{6.14}$$

From (6.10) we know the large z asymptotics of ζ:

$$\zeta = \frac{1}{g^2}z^2 + z^{-2} + \delta z^{-4} + \ldots, \qquad z = g\zeta^{\frac{1}{2}} - \frac{1}{2g}\zeta^{-\frac{3}{2}} - \frac{\delta}{2g^3}\zeta^{-\frac{5}{2}} + \ldots \tag{6.15}$$

where

$$\delta = -\frac{1}{4} + 3\nu, \quad \nu = \int_{-a}^{+a} \rho(y)y^2 dy \tag{6.16}$$

At large t the function $z(\zeta)$ as given by (6.13) has the following expansion:

$$z = 2A\left(\zeta^{\frac{1}{2}} + a_0 + a_1\zeta^{-\frac{1}{2}} + a_2\zeta^{-\frac{3}{2}} + a_3\zeta^{-\frac{5}{2}} + \ldots\right) \tag{6.17}$$

where:

$$
\begin{aligned}
a_0 =& \int_{x_1}^{\infty} dt \left(\frac{(t-x_3)}{\sqrt{(t-x_1)(t-x_2)(t-x_4)}} - \frac{1}{\sqrt{t-x_1}} \right) \\
a_k =& \frac{(-)^{k-1}}{2k-1} \left(\gamma_k + x_3 \gamma_{k-1} \right), \quad k > 0 \\
\gamma_k =& \sum_{p+q+r=k} \binom{-\frac{1}{2}}{p} \binom{-\frac{1}{2}}{q} \binom{-\frac{1}{2}}{r} x_1^p x_2^q x_4^r
\end{aligned}
\tag{6.18}
$$

By comparing (6.15) and (6.18) we get the following equations on x_i, A:

$$
\begin{aligned}
A &= \frac{g}{2} \\
a_0(x_i) &= 0 \\
x_1 + x_2 + x_4 &= 2x_3 \\
x_1^2 + x_2^2 + x_4^2 - 2x_3^2 &= \frac{6}{g^2}
\end{aligned}
\tag{6.19}
$$

which in addition to the equations in (6.14) fix everything completely. As shown in [22] the equation $a_0 = 0$ follows from (6.14) by a contour deformation argument. Introduce more notations:

$$
\begin{aligned}
& y_i = g x_i, \qquad \lambda_i = \frac{y_i}{y_1 - y_4} \\
& m = \frac{y_2 - y_4}{y_1 - y_4}, \quad 1 > m > 0, \quad m' = 1 - \frac{1}{m},
\end{aligned}
\tag{6.20}
$$

The equations (6.14) assume the following form:

$$
\begin{aligned}
(y_2 + y_4 - y_1)\mathbf{K}(m) + 2(y_1 - y_4)\mathbf{E}(m) &= 0 \\
-(y_1 + y_2 - y_4)\mathbf{K}(m') + 2(y_2 - y_4)\mathbf{E}(m') &= \sqrt{\frac{y_2 - y_4}{g}}
\end{aligned}
\tag{6.21}
$$

where we use the standard elliptic functions:

$$
\mathbf{K}(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - m \sin^2\theta}}, \quad \mathbf{E}(m) = \int_0^{\frac{\pi}{2}} d\theta \sqrt{1 - m \sin^2\theta}, \tag{6.22}
$$

which have the following crucial properties:

$$
\begin{aligned}
& \mathbf{K}(m') = \sqrt{m}\mathbf{K}(1-m), \qquad \mathbf{E}(m') = \frac{1}{\sqrt{m}}\mathbf{E}(1-m) \\
& \mathbf{E}(m)\mathbf{K}(1-m) + \mathbf{E}(1-m)\mathbf{K}(m) - \mathbf{K}(m)\mathbf{K}(1-m) = \frac{\pi}{2}
\end{aligned}
\tag{6.23}
$$

In the sequel we use the short-hand notations: $\mathbf{E} = \mathbf{E}(m), \mathbf{K} = \mathbf{K}(m), \vartheta = \mathbf{E}/\mathbf{K}$. The first equation in (6.21) allows us to express λ_2 in terms of m, while the second together with (6.23) gives $g(y_1 - y_4)$:

$$\lambda_2 \;=\; 1 - 2\vartheta, \quad g(y_1 - y_4) = \frac{1}{\pi^2}\mathbf{K}^2 \tag{6.24}$$

From (6.19) and the equations $\lambda_4 = \lambda_2 - m, \quad \lambda_1 = \lambda_4 + 1$ we get an expression for $y_1 - y_4$ and consequently for g:

$$\begin{aligned} g(m)^2 &= \frac{\mathbf{K}^4}{12}\left(4m\lambda_2 + 1 - 3\lambda_2^2 - 2\lambda_2\right) \\ &= \frac{\mathbf{K}^4}{12}\left(-3\vartheta^2 + 2(2-m)\vartheta - (1-m)\right). \end{aligned} \tag{6.25}$$

We can now compute $\langle \mathrm{Tr}\phi^2 \rangle$ and $F(N, g)$:

$$\begin{aligned} \nu(m) = \; & 2\frac{g^4}{N^2}\frac{\partial F(N,g)}{\partial g^2} = \\ & \frac{1}{12} + \frac{\mathbf{K}^2}{5\pi^2}\left(\frac{4m\lambda_2(1-m) + (5\lambda_2^2 - 1)(2m - 1 - \lambda_2)}{4m\lambda_2 + 1 - 3\lambda_2^2 - 2\lambda_2}\right) \\ \nu(m) - \frac{1}{12} = \; & \frac{\mathbf{K}^2}{5\pi^2}\times \\ & \times\frac{2\vartheta\left(5\vartheta^2 + 5m(\vartheta - 1) + 6 - 10\vartheta + m(m-1)\right) + (1-m)(m-2)}{-3\vartheta^2 + 2(2-m)\vartheta - (1-m)} \end{aligned} \tag{6.26}$$

The formulae (6.25)(6.26) provide the exact analytic solution of the large N model in the parametric form.

Small g expansion. In this case we expect to get a regular planar graph expansion of the matrix integral (3.8) with respect to the quartic term in the action. The careful analysis shows that $g \to 0$ limit corresponds to $m \to 0$. In this limit we can expand:

$$\begin{aligned} g =& \frac{1}{8\sqrt{2}}m + \frac{1}{16\sqrt{2}}m^2 + \frac{87}{2048\sqrt{2}}m^3 + \ldots \\ \nu =& \frac{1}{256}m^2 + \frac{1}{256}m^3 + \ldots \end{aligned} \tag{6.27}$$

which implies

$$\begin{aligned} \nu =& \frac{1}{2}g^2 - \frac{1}{2}g^4 + \ldots \\ F(N,g) =& \frac{N^2}{2}\left(\log g - 2g^2 + \ldots\right) \end{aligned} \tag{6.28}$$

in the perfect agreement with the planar graph expansion and the formula (5.7).

Large g limit. It corresponds to the situation when we are far beyond the convergency radius of the g-series. The integral (3.8) is dominated by the commutator term in the action and hence the fluctuations of the matrices are very large due to the zero modes. It follows from very careful study of (6.24)(6.25) that $g \to \infty$ limit corresponds to $m \to 1$. In this limit ($m = 1 - \varepsilon$):

$$\begin{aligned} g =& \frac{1}{2\sqrt{3}\pi^2}\log^{\frac{3}{2}}\left(\frac{16}{\varepsilon}\right) - \frac{\sqrt{3}}{4\pi^2}\log^{\frac{1}{2}}\left(\frac{16}{\varepsilon}\right)\dots \\ \nu =& + \frac{1}{20\pi^2}\log^2\left(\frac{16}{\varepsilon}\right) - \frac{7}{20\pi^2}\log\left(\frac{16}{\varepsilon}\right) + \dots \end{aligned} \tag{6.29}$$

Hence:

$$\begin{aligned} \nu =& \frac{(12\pi)^{\frac{2}{3}}}{20} g^{\frac{4}{3}} - \frac{3}{(12\pi)^{\frac{2}{3}}} g^{\frac{2}{3}} \dots \\ F(N,g) =& -N^2\left(\frac{3(12\pi)^{\frac{2}{3}}}{40} g^{-\frac{2}{3}} - \frac{9}{5(12\pi)^{\frac{2}{3}}} g^{-\frac{5}{3}} + \dots\right) \end{aligned} \tag{6.30}$$

in perfect agreement with (5.19)![3]. In fact, the strong coupling expansion can be greatly simplified if we choose $L = \frac{1}{\log\left(\frac{16}{\varepsilon}\right)}$ as an expansion parameter and systematically neglect all non-perturbative in L corrections, i.e. we consider the leading logarithmic approximation. We then get very simple formulae:

$$\begin{aligned} g^2 =& \frac{1}{12\pi^4 L^3}\left(1 - 3L\right) \\ \nu =& \frac{1}{20\pi^2 L^2}\left(\frac{1 - 10L + 20L^2}{1 - 3L}\right) + \frac{1}{12} \end{aligned} \tag{6.31}$$

The surprise is that the formula (6.31) is *exactly* equivalent to (5.18).
Proof. Eq. (5.18) with $\alpha_0 = \frac{\pi^2}{2}$ gives:

$$\mathcal{F}_0' = \frac{\pi^2}{12}\chi_0^3 - \frac{1}{4}\chi_0^2, \quad \mathcal{F}_0 = \frac{\pi^4}{120}{\chi_0}^5 - \frac{5\pi^4}{96}{\chi_0}^4 + \frac{1}{12}{\chi_0}^3. \tag{6.32}$$

The Legendre transform leading from $\mathcal{F}_0$ to F_0 yields: $\nu = 2x - \frac{\mathcal{F}_0}{\mathcal{F}_0'}$, while $g^2 = -\mathcal{F}_0'$. Clearly, it leads to (6.31) if we substitute

$$\chi_0 = \frac{\log\left(\frac{16}{\varepsilon}\right)}{\pi^2} \tag{6.33}$$

[3] Indeed, $\frac{1}{10}\left(\frac{243\pi^2}{4}\right)^{\frac{1}{3}} = \frac{3(12\pi)^{\frac{2}{3}}}{40}$

This confirms once again that indeed $\alpha_0 = \frac{\pi^2}{2}$.

Remarks.

1. It is very tempting to speculate that the relation to supersymmetric gauge theories, which was one of the original motivations of this work, is somehow revealed by the appearence of the family of elliptic curves, parametrized by the value of coupling g just like in [23]. Notice that the solution which we studied here has the flavour with application of mirror symmetry. Indeed, the naive coordinates in the space of our Lagrangians (which is just the coupling g for quadratic potential) have been replaced by the period of a certain differential on elliptic curve. It is conceivable that the similar construction takes place for more general potentials.
2. One can show by considering the planar graph expansion that in the large N limit the partition function does not change if we substitute the commutator by the anticommutator in the action in (3.8). The latter model describes the statistical ensemble of ϕ^4 type random graphs covered by dense nonoriented selfavoiding random loops. This is the dense phase of the $O(n)$ loop-gas model [24] with $n = 1$. The critical behaviour (thermodynamic limit) is due to the dominance of graphs of infinite size which renders the g-expansion of the partition function divergent and is therefore determined by the closest to the origin singularity in g^2. The latter is defined by It is a solution of the equation $g'(m) = 0$. It should appear for the negative g^2 and corresponds to the situation when all three cuts are located on the real axis symmetrically with respect to the origin. When g increases, the end-points of the cuts get closer and singularity occurs when they touch each other. It is known [25] that the critical behavior of the dense $O(1)$ model is in the universality class of the pure $2d$ quantum gravity . For example, the one-point function behaves as

   $$\nu \sim (g_c - g)^{\frac{3}{2}}.$$

   For the sake of convenience we list here the explicit formula for $(g^2)'$:

   $$(g^2)' = -9\pi^4 \mathbf{K}^4 \frac{\vartheta(\vartheta - 1)(\vartheta - 1 + m)}{m(1-m)}. \tag{6.34}$$

3. The last assertion can be partially confirmed by the study of the specific heat at the fixed chemical potential. From the equation (5.18) we get the critical point $x_c = -\left(\frac{1}{\pi^2} + \frac{1}{12}\right)$, which can be substituted into(6.31)–(6.33) to yield $g_c^2 = -\frac{1}{3\pi^4}$ which is negative indeed. Note however that the corresponding value of $\varepsilon = 2.16$ is way larger then the leading log approximation allows us to look at.

7. On the correlation functions in the $d = 6, 10$ cases and directions for future

This section is devoted to the work in progress and will be inconclusive. We sketch the possible similar saddle-point approach to the $d = 6$ integral. We also attempt a fermionic representation for the $d = 10$ integral.

7.1. Saddle-point approach to $d = 6$ integral

We keep the same notations for the resolvent F and density ρ. We set $\epsilon_1 + \epsilon_2 = 1$, $\epsilon_1 = \beta$, $\epsilon_2 = \gamma$. The equation (6.8) is replaced by:

$$\begin{aligned}\frac{x}{g^2} =& \hat{F}(x) - \tfrac{1}{2}\left(F(x+i\beta) + F(x-i\beta)\right) \\ & - \tfrac{1}{2}\left(F(x+i\gamma) + F(x-i\gamma)\right) + \tfrac{1}{2}\left(F(x+i) + F(x-i)\right)\end{aligned} \tag{7.1}$$

Introduce the derived functions:

$$\begin{aligned} f(x) &= \frac{ix^2}{2g^2\gamma} + F(x + \frac{i\beta}{2}) - F(x - \frac{i\beta}{2}) \\ g(x) &= \frac{1}{2}\left(f(x+\frac{i\gamma}{2}) - f(x-\frac{i\gamma}{2})\right)\end{aligned} \tag{7.2}$$

The saddle point equation (7.1) is equivalent to

$$g(x+\frac{i}{2}) + g(x-\frac{i}{2}) = 0 \tag{7.3}$$

The function $g(x)$ has four cuts: at $x \in \pm\frac{i}{2}, \pm\frac{i}{2}(\beta-\gamma) + (-a, +a)$, it is real: $g(\bar{z}) = \overline{g(z)}$ and it is purely imaginary for $z \in i\mathbb{R}$. It would be nice to guess the correct function from the stated properties. We plan to return to this problem in the future.

7.2. Fermionic representation for the $d = 10$ integral

We now proceed with fermionic representation of the $d = 10$ integral (2.17). Unfortunately we were not able to find such a representation for all values of $\epsilon_1, \epsilon_2, \epsilon_3$. However, let us consider the limit $\epsilon_3 \to -\epsilon_1, \epsilon_4 \to -\epsilon_2$. At the same time we keep

$$e^{\mu}(\epsilon_1 + \epsilon_3) = e^{\bar{\mu}}$$

finite. We claim that in this limit

$$\mathcal{Z}^{d=10}_{V,\mu,\epsilon} = \langle 0| e^{H[t]} e^{\bar{\Omega}_{\mu}} |0\rangle \tag{7.4}$$

where

$$\bar{\Omega}_\mu = e^\mu \int_{-\infty}^{+\infty} \frac{dz}{2\pi i} : \psi(z-a)\psi(z+a)\psi^*(z-b)\psi^*(z+b) : \tag{7.5}$$

with $a = \frac{1}{2}(\epsilon_1 + \epsilon_2), b = \frac{1}{2}(\epsilon_1 - \epsilon_2)$. Now the relation between V and U is modified into:

$$V(z) = U(z+a) + U(z-a) - U(z+b) - U(z-b) \tag{7.6}$$

As in the index computation it is possible that by making the appropriate mass perturbation we reduce the $d = 10$ integral for the gauge group $U(N)$ to the products of $d = 4$ integrals for gauge groups $U(n_1) \times \ldots \times U(n_k)$ with $N = \sum_{l=1}^{\infty} l n_l$. Under this assumption:

$$\mathcal{Z}^{d=10}(\mu, \epsilon, V) = \prod_{l=1}^{\infty} \mathcal{Z}^{d=4}(l\mu, \epsilon, V) = \prod_{l=1}^{\infty} \mathrm{Det}(I + e^{l\mu} K) = \Theta(\mu, K). \tag{7.7}$$

The latter expression is very interesting, since it posesses certain modular properties and allows one to deduce the large N asymptotics using very little information about the operator K itself:

$$\Theta(\mu, K) \sim \exp 2\sqrt{N|\mathrm{Li}_2(-K)|} \tag{7.8}$$

where

$$\mathrm{Li}_2(-K) = \sum_{l=1}^{\infty} \frac{1}{l^2} \mathrm{Tr}(-K)^l \tag{7.9}$$

8. Conclusions

Here we summarize the results of our computations. The integral (3.8) which is a cousin of (3.1) is studied in two regimes: at fixed N and at fixed μ. In the first case we got the large N asymptotics in the 't Hooft limit, see (6.25)–(6.26). In the second case we showed that the grand partition function is a particular tau-function of KP hierarchy. In particular, we obtained the equation (5.2) for the specific heat $u = -2\partial_\mu^2 \mathcal{F}$ in the case of quadratic potential $V \sim \lambda z^2$:

$$2u_\lambda + \frac{1}{\lambda} u - u u_\mu - \frac{1}{6}(u - u_{\mu\mu})_\mu = 0.$$

In the large μ, λ limit we obtained the simple explicit formula for the specific heat as a function of $x = \mu/\lambda$:

$$\frac{\pi^2}{4} u^2 - u - \frac{1}{12} = x.$$

We observed various similarities to the properties of supersymmetric gauge theories in four dimensions and we hope that our results will find their place in the study of dynamics of D-particles in various dimensions.

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Appendix A. Determination of α_0

We now present a trick allowing to get the value of the unknown coefficient α_0 in (5.18). If $\alpha_0 \neq 0$ then for large x one has $u \sim \pm\sqrt{\frac{2x}{\alpha_0}}$ and

$$\partial_\mu \mathcal{F} \sim \pm \frac{\sqrt{2}\mu^{\frac{3}{2}}}{3\xi\sqrt{\alpha_0}}. \tag{A.1}$$

Below we rescale $\lambda \to \lambda/2$ to be in agreement with notations of (3.13).

Let e^{-E_k} be the eigenvalues of the integral operator $\mathcal{K}$. It follows from the determinantal representation (3.7) of the partition function that $e^{\mathcal{F}(\mu,\xi)} = \prod_k \left(1 + e^{\mu - E_k}\right)$. Correspondingly, the mean value of the number of particles is given by

$$\langle N \rangle \equiv \partial_\mu \mathcal{F} = \sum_k \frac{1}{1 + e^{E_k - \mu}}. \tag{A.2}$$

We are interested in the limit where both μ and ξ (and therefore E_k, see below) are very large, i.e. a kind of low temperature limit for Fermi-gas with energy levels given by the spectrum of the operator $\log \mathcal{K}$. In the low temperature limit we simply need to count the number of energy levels below the Fermi level μ.

The eigenvalue problem for the operator $\mathcal{K}$ is similar (although far from being equivalent in general) to the eigenvalue problem of the particle of unit mass which is confined to move at the positive semi-axis $y > 0$ and subject to the spike-like potential

$$U(z,t) = \frac{\xi}{2} y \sum_{n \in \mathbb{Z}} \delta(t - n), \quad \xi > 0.$$

The operator $\mathcal{K}$ is to be compared with the operator $\mathcal{U}_1$ of the evolution during the unit imaginary time. The latter can be easily diagonalized:

$$\mathcal{U}_1 f_E = e^{-E} f_E \tag{A.3}$$

with $f_E(y) = A(y - \frac{2\tilde{E}}{\xi})$, $\tilde{E} = E + \log\sqrt{2\pi} + \frac{\xi^2}{48}$, and $A(y)$ is the modified Airy function

$$A(y) = \int_\gamma \frac{dp}{2\pi} e^{ipy + \frac{p^2}{4} + i\frac{p^3}{3\xi}} \tag{A.4}$$

where the contour γ is such that $\Im p^3 > 0$ as $p \to \infty$ along γ. The spectrum is determined from the condition that $f_E(0) = 0$, i.e. $A(-\frac{2\tilde{E}}{\xi}) = 0$. For large values of E this equation can be solved using quasi-classics. It gives

$$E_k \sim \tfrac{1}{2}\left(\frac{3\pi\xi k}{2}\right)^{\frac{2}{3}}, \qquad k \to \infty. \tag{A.5}$$

Assuming that for our problem we may use the same asymptotics we conclude that

$$\langle N \rangle \sim \frac{2^{\frac{5}{2}}}{3\pi\xi} \mu^{\frac{3}{2}}, \tag{A.6}$$

which means that

$$\alpha_0 = \frac{\pi^2}{2}. \tag{A.7}$$

Remark. It is interesting to note that similar Schröedinger probelm arises in the Born-Oppenheimer approximation to the quantum mechanics of particle in two dimensions confined by the potential x^2y^2, which is a good model for the matrix potential $\mathrm{Tr}[X,Y]^2$, see [22].

Appendix B. Solution of one-matrix model from KP equation and double scaling limit.

As an example illustrating the application of the KP hierarchy to matrix integrals we will derive the critical singularity and the double scaling limit of the one-matrix integral

$$Z_N(t) = \int \mathcal{D}M \exp N\mathrm{Tr} \sum_{q=1}^{3} t_q M^q \tag{B.1}$$

in the case of cubic potential. The fact that the matrix integral (B.1) is a τ-function of the KP hierarchy has been established in [26]. A direct derivation of the Hirota equations (4.25) from the matrix integral can be found in [27]. The free energy

$$F_N = \frac{1}{N^2}\log Z_N \tag{B.2}$$

and the specific heat

$$u(t) = 2\partial_{t_1}^2 F_N \tag{B.3}$$

can be obtained from the KP equation (4.34) if we take into account that the specific heat is actually a function of a single parameter:

$$u = t_3^{-\frac{2}{3}}\psi\left(\frac{t_1}{t_3^{\frac{1}{3}}} - \frac{t_2^2}{3t_3^{\frac{4}{3}}}\right) \tag{B.4}$$

and is given in the Gaussian limit by

$$t_3 = 0: \quad u = -\frac{1}{t_2}. \tag{B.5}$$

By substituting (B.4) into (4.34) we derive that the function f obeys the following ordinary differential equation

$$\psi + 2x\psi' + 9\psi\psi' + \frac{3}{2N^2}\psi''' = \psi_0 \tag{B.6}$$

where ψ_0 is a constant,

$$x = \frac{t_1}{t_3^{\frac{1}{3}}} - \frac{t_2^2}{3t_3^{\frac{4}{3}}}, \quad \psi = t_3^{\frac{2}{3}}u. \tag{B.7}$$

In the large N limit we get an algebraic equation for $x(\psi)$

$$(\psi - \psi_0)\frac{dx}{d\psi} + 2x + 9\psi = 0 \tag{B.8}$$

whose solution depends on two constants (ψ_0, β):

$$x = \frac{\beta}{(\psi - \psi_0)^2} - 3\left(\psi - \frac{1}{2}\psi_0\right). \tag{B.9}$$

By comparing with (B.5) we get: $\psi_0 = 0$, $\beta = -\frac{1}{3}$. Hence the large N result is:

$$t_1 t_3 - \frac{1}{3}t_2^2 = -3t_3^2 u - \frac{1}{3u^2}. \tag{B.10}$$

The critical point is at:

$$x_c = -\left(\frac{9}{2}\right)^{\frac{2}{3}}, \quad \psi_c = \left(\frac{2}{9}\right)^{\frac{1}{3}} \tag{B.11}$$

To compare with the results of Brezin et al. [28] we set:

$$t_1 = 0, \quad t_2 = \frac{1}{2N}, \quad t_3 = \frac{g}{N^{\frac{3}{2}}}$$

and get $x = -\frac{1}{12g^{\frac{4}{3}}}$. Hence

$$g_c^2 = \frac{1}{108\sqrt{3}}$$

which is exactly the result of the tedious computations [28] who used the distribution of the eigenvalues in the large N limit. Eq. (4.34) also allows a trivial derivation of the double scaling limit for the pure $2d$ quantum gravity [21]. This limit consists in sending $x - x_c$ to zero and N to infinity in such a way that the double scaling variable $z = N^b(x - x_c)$ remains finite. We try the ansatz: $\psi = \psi_c + N^a v(z)$. Since $\psi \simeq \psi_c + \text{const} \cdot \sqrt{x - x_c}$ we have: $b = -2a$. We then plug this ansatz into (4.34) and get

$$v''' + 6N^{2+5a} vv' + \frac{2}{3}\psi_c N^{2+5a} + (\frac{4}{3}zv' + \frac{2}{3}v)N^{2+6a} = 0. \tag{B.12}$$

To keep here the nonlinear term (the source of all higher genus corrections $\sim 1/N^{2g}$) we impose the condition: $2 + 5a = 0$ which gives

$$a = -\frac{2}{5}, \;\; b = \frac{4}{5}.$$

One immediately sees that the last term in (B.12) vanishes in the double scaling limit ($N^{2+6a} = N^{\frac{-2}{5}} \to 0$.) Integrating (B.12) once with respect to z we finally obtain the Painleve II equation

$$v'' + 3v^2 = cz \tag{B.13}$$

where $c = -\frac{2}{3}\psi_c = -\frac{2^{\frac{4}{3}}}{3^{\frac{5}{3}}}$.

Remark. We obtained the equation (B.13) without any use of the method of orthogonal polynomials which is so far the only technique known to work for that purpose. It raises the hope that the method of Hirota equations proposed here will allow to obtain the non-perturbative description (beyond the loop expansion) of some interesting non-critical string theories, such as $O(2)$ model [24] whose grand partition is known to satisfy the KdV hierarchy [29].

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