Positivity Constraints on Chiral Perturbation Theory Pion-Pion Scattering Amplitudes

Petre Diţă[1]

National Institute of Physics & Nuclear Engineering
Bucharest, PO Box MG6, Romania

Abstract

We test the positivity property of the chiral perturbation theory (ChPT) pion-pion scattering amplitudes within the Mandelstam triangle. In the one-loop approximation, $\mathcal{O}(p^4)$, the positivity constrains only the coefficients b_3 and b_4, namely one obtains that b_4 and the linear combination $b_3 + 3b_4$ are positive quantities. The two-loops approximation gives inequalities involving all the six arbitrary parameters entering ChPT amplitude, but the corrections to the one-loop approximation results are small. ChPT amplitudes pass unexpectedly well all the positivity tests giving strong support to the idea that ChPT is the good theory of the low energy pion-pion scattering.

PACS: 13.75.Lb, 12.39.Fe, 11.55.Fv, 11.30.Rd

Key-words: Pion-Pion Scattering, Chiral Perturbation Theory, Positivity Constraints

1 Introduction

Chiral perturbation theory (ChPT) is considered a low-energy effective approximation of QCD, in particular it provides a representation of the elastic pion-pion scattering amplitudes that is crossing symmetric and has good analyticity properties. In a seminal paper[1] Gasser and Leutwyler developed ChPT which allows one to compute many Green functions involving

[1]email: dita@theor1.theory.ifa.ro

low-energy pions. It is well known that the physical pion-pion scattering amplitudes can be expressed in terms of a single function $A(s,t,u)$ whose form was obtained as a series expansion in powers of the external momenta and of the light quark masses. The first term of the series was given by Weinberg [2], the second by Gasser and Leutwyler [1] and only recently a two-loop calculation[3, 4] was obtained. In this approximation the function $A(s,t,u)$ has the following form

$$\begin{gathered} A(s,t,u) = a(s-1) + a^2\left[b_1 + b_2 s + b_3 s^2 + b_4 (t-u)^2\right] + \\ a^2\left[F^{(1)}(s) + G^{(1)}(s,t) + G^{(1)}(s,u)\right] + a^3\left[b_5 s^3 + b_6 s (t-u)^2\right] + \\ a^3\left[F^{(2)}(s) + G^{(2)}(s,t) + G^{(2)}(s,u)\right] + \mathcal{O}(a^4) \end{gathered} \tag{1.1}$$

where $a = (M_\pi/F_\pi)^2$, M_π is the mass of the physical pion, F_π the pion decay constant, s,t,u are the usual Mandelstam variables, expressed in units of the physical pion mass squared M_π^2

$$s = (p_1+p_2)^2/M_\pi^2, \;\; t = (p_1-p_2)^2/M_\pi^2, \;\; u = (p_1-p_3)^2/M_\pi^2,$$

$F^{(i)}(s)$ and $G^{(i)}(s,t)$ are known functions and $b_i,\ i = 1,\ldots,6$ are arbitrary parameters which cannot be determined by ChPT [1, 3, 4]. In any realistic comparison with experiment we have to provide some numerical values for all these parameters obtained from other sources. One hopes that by using unitarity this can be done although, until now, no program for implementing this property has been presented. The common belief is that imposing unitarity is not a simple matter since its implementation in one channel destroys crossing symmetry in other channels. However there is a weak form of unitarity, the positivity of the absorptive parts, which is a linear property and which can be imposed. This property was used thirty years ago to obtain constraints on the $\pi^0\pi^0$ s-wave partial amplitude, $f_0(s)$, in the unphysical region $0 < s < 4$ and on the d-wave scattering lengths. These constraints were useful because at that time almost nothing was known about the explicit form of the scattering amplitudes and they were used in testing models for pion-pion partial-wave amplitudes. The advantage of ChPT is that it furnishes an explicit form for the pion-pion scattering amplitudes whose unknown part is contained in a few numerical coefficients. Thus is of certain interest to see how these properties reflect into constraints on the b_i coefficients entering Eq.(1.1).

Beginning with the paper [5], Martin has used the positivity, analyticity and crossing symmetry to obtain constraints on the $\pi^0\pi^0$ s-wave partial amplitude, $f_0(s)$, in the unphysical region $0 < s < 4$; a few of them have the following form [6]

$$f_0(4) > f_0(0) > f_0(3.15) \qquad f_0(0) > \frac{1}{2}\int_2^4 f_0(s)\,ds$$

$$F(0, 2(1+\frac{1}{\sqrt{3}})) > f_0(2(1+\frac{1}{\sqrt{3}})) \tag{1.2}$$

where $F(s,t)$ denotes the $\pi^0\pi^0$ elastic scattering amplitude. A more complete set is found in ref. [7]. The most elaborate form of these constraints is the following result: the $\pi^0\pi^0$ s-wave, $f_0(s)$, has a minimum located between $1.218989 < s < 1.696587$ [6, 7, 8, 9, 10, 11, 12] and this result can be improved only by unitarity. These results can be translated into constraints on the parameters b_i entering ChPT pion-pion scattering amplitudes. As we will see later all the above inequalities are equivalent in the one-loop approximation, $\mathcal{O}(p^4)$, to a single constraint on the coefficients b_i whose typical form is

$$b_3 + 3b_4 \geq \frac{37}{1920\pi^2}$$

the only difference being the numerical value appearing on the right hand side. This is the explanation of the inefficacy of these apparently distinct constraints which was observed from the beginning by people constructing models for pion-pion partial-waves.

In this paper we work only in the unphysical region $|s,t,u| < 4$, i.e. there where the amplitude (1.1) is considered to be a very good approximation to the true amplitude. Other approaches use information from the physical region to obtain constraints on the same parameters b_i [13, 14, 15]. In the one-loop approximation, $\mathcal{O}(p^4)$, the positivity property constrains only the coefficients b_3 and b_4. By taking into account the $\mathcal{O}(p^6)$ contributions one gets constraints involving all the six parameters entering Eq.(1.1).

By using the unitarity bounds on $\pi^0\pi^0$ scattering amplitude in the unphysical region [16] one get upper and lower bounds on some linear combinations of the parameters b_i. These unitarity bounds are not very constraining; to see this we give that one obtained from the bound on $F(2,0)$, where as above $F(s,t)$ denotes the $\pi^0\pi^0$ amplitude. The bound is $-3.5 \leq F(2,0) \leq 2.9$

and it is equivalent to the following lower and upper bounds

$$-3.5 \times 32\pi \leq a + a^2 \left(3b_1 + 4b_2 + 8b_3 + 8b_4 + \frac{9}{32\pi^2}\right) + a^3 \left[16b_5 + 16b_6 + \right.$$

$$\left. \frac{(4-\pi)}{\pi^2}\left(\frac{5b_1}{16} + \frac{b_2}{2} + \frac{11b_3}{12} + \frac{5b_4}{12}\right) + \frac{965}{3456\pi^4} - \frac{251}{3456\pi^3} + \frac{41}{6144\pi^2}\right] \leq 2.9\times 32\pi$$

Because of the factor 32π appearing on left and right hand side the bounds are not very strong and we will not consider them here.

The physical isospin amplitudes F^I can be expressed in terms of the single function $A(s,t,u)$ as follows

$$\begin{aligned} F^0(s,t,u) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\ F^1(s,t,u) &= A(t,u,s) - A(u,s,t) \\ F^2(s,t,u) &= A(t,u,s) + A(u,s,t) \end{aligned}$$

where $A(s,t,u)$ is given by Eq.(1.1).

Having only three independent amplitudes one gets only three independent constraints since the crossing symmetry is an exact symmetry for the ChPT amplitudes. The construction of our positivity constraints is outlined in the next Section where we present an over-determined system of constraints. Their implications on the coefficients b_i are discussed in Section III. The paper ends with Conclusion.

2 Positivity constraints

Let $F^I(s,t)$ denote the $\pi\pi$ scattering amplitude with isotopic spin I in the s channel. In matrix notation $\mathbf{F}(s,t)$ satisfies the following crossing relation [17]

$$\mathbf{F}(s,t) = C_{st}\mathbf{F}(t,s) = C_{su}\mathbf{F}(u,t)$$

where the notations are

$$\mathbf{F}(s,t) = \begin{pmatrix} F^0(s,t) \\ F^1(s,t) \\ F^2(s,t) \end{pmatrix}$$

$$C_{st} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}, \; C_{su} = \begin{pmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix}$$

From the results of axiomatic field theory we know that the amplitudes $F^I(s,t)$ satisfy fixed-t dispersion relations with two subtractions [18] for $|t| < 4$. We may write them as

$$\mathbf{F}(s,t) = C_{st}[\mathbf{a}(t) + (s-u)\mathbf{b}(t)] + \frac{1}{\pi}\int_4^\infty \frac{dx}{x^2}\left(\frac{s^2}{x-s} + \frac{u^2}{x-u}C_{su}\right)\mathbf{A}(x,t) \tag{2.1}$$

where $\mathbf{A}(x,t)$ is the absorptive part of $\mathbf{F}(s,t)$ and the subtraction constants are of the form

$$\mathbf{a}(t) = \begin{pmatrix} a^0(t) \\ 0 \\ a^2(t) \end{pmatrix} \qquad \mathbf{b}(t) = \begin{pmatrix} 0 \\ b^1(t) \\ 0 \end{pmatrix}$$

due to crossing symmetry.

In the following we shall consider that s, t, u take values in the unphysical region $|s,t,u| < 4$. We calculate the difference

$$\mathbf{F}(s,t) - \mathbf{F}(s_1,t)$$

and we are looking for those combinations of isospin amplitudes for which this difference does not depend on the subtraction constants. From Eq.(2.1) we find

$$\frac{1}{s-s_1}(\mathbf{F}(s,t) - \mathbf{F}(s_1,t)) = 2C_{st}\,\mathbf{b}(t) + f(A) \tag{2.2}$$

where $f(A)$ denotes the complicated term containing the integration over the absorptive parts. The first term on the right hand side of Eq.(2.2) is

$$C_{st}\,\mathbf{b}(t) = \begin{pmatrix} b(t) \\ \frac{1}{2}b(t) \\ -\frac{1}{2}b(t) \end{pmatrix}$$

The last relation shows that there are three combinations of isospin amplitudes for which the difference (2.2) have no dependence on the subtraction constants. They are F^0+2F^2, F^1+F^2 and F^0-2F^1 and we shall denote them as F_i, $i=1,2,3$ in this order. The first one is the well known $\pi^0\pi^0$ elastic amplitude. One easily obtains from Eq.(2.2) the relation

$$F_i(s,t)-F_i(s_1,t)=\frac{(s-s_1)(s-u_1)}{\pi}\int_4^\infty \frac{(2x+t-4)A_i(x,t)\,dx}{(x-s)(x-s_1)(x-u)(x-u_1)}$$

$i=1,2,3$. From this relation we get

$$\frac{\partial F_i(s,t)}{\partial s}=\frac{s-u}{\pi}\int_4^\infty \frac{(2x+t-4)A_i(x,t)\,dx}{(x-s)^2(x-u)^2}$$

Because the absorptive parts A_1 and A_2 are positive we find that

$$\frac{1}{s-u}\frac{\partial F_i(s,t)}{\partial s}\geq 0 \quad i=1,2$$

The third combination involves the absorptive parts $A^0(x,t)-2A^1(x,t)$ whose sign is not defined and we cannot say anything about the sign of the derivatives of $F_3(s,t)$. The precedent relations show us that on the line $s=u$, $F_i(s,t)$, $i=1,2$ attain their minimum values. Indeed we obtain from them the second derivatives

$$\frac{\partial^2 F_i(s,t)}{\partial s^2}=\frac{2}{\pi}\int_4^\infty\left(\frac{1}{(x-s)^3}+\frac{1}{(x-u)^3}\right)A_i(x,t)\,dx\geq 0,\ \ i=1,2 \qquad (2.3)$$

which are positive definite, implying that the functions $F_i(s,t)$ have a minimum on the line $s=u$. From the last relation we obtain also

$$\frac{\partial^{2n-1}F_i(s,t)}{\partial s^{2n-1}}=\frac{(2n-1)!}{\pi}\int_4^\infty\left(\frac{1}{(x-s)^{2n}}-\frac{1}{(x-u)^{2n}}\right)A_i(x,t)\,dx=$$

$$\frac{(2n-1)!\,(s-u)}{\pi}\int_4^\infty[(x-s)^n+(x-u)^n]$$

$$\left[\frac{(x-s)^{n-1}+(x-s)^{n-2}(x-u)+\ldots(x-u)^{n-1}}{(x-s)^{2n}(x-u)^{2n}}\right]A_i(x,t)\,dx$$

In this way we obtain the set of positivity constraints

$$\frac{1}{s-u}\frac{\partial^{2n-1}F_i(s,t)}{\partial s^{2n-1}}\geq 0 \ \text{ and } \ \frac{\partial^{2n}F_i(s,t)}{\partial s^{2n}}\geq 0 \qquad (2.4)$$

$i = 1, 2,\ n = 1, 2, \ldots$

A first remark is the following, if the positivity constraints have to be fulfilled it is sufficient to test them only on the line $s = u$, i.e. $2s + t - 4 = 0$, where the functions $F_i(s,t)$ attain their minimum values. In this way we have only one free parameter, $0 < |s| < 4$, and on this line the odd and even derivatives give the same information. From the point of view of computation it is simpler to work with even derivatives.

Up to now we have obtained two constraints given by Eq.(2.3). Because we have three independent isospin amplitudes it follows that we can obtain another one at most.

The positivity constraints can be imposed even on the isospin amplitudes themselves. This can be easily seen from the relation (2.1) for $F^2(s,t)$ which after derivation gives

$$\frac{\partial^2 F^2(s,t)}{\partial s^2} = \frac{2}{\pi}\int_4^\infty dx \left[\left(\frac{1}{(x-s)^3} + \frac{1}{6}\frac{1}{(x-u)^3}\right) A^2(x,t) + \frac{1}{3}\frac{1}{(x-u)^3}A^0(x,t) + \frac{1}{2}\frac{1}{(x-u)^3}A^1(x,t)\right]$$

The right hand side of the previous relation is a positive quantity and by iteration we obtain that

$$\frac{\partial^{2n} F^2(s,t)}{\partial s^{2n}} \geq 0 \qquad n = 1, 2, \ldots \tag{2.5}$$

Unfortunately the numerical calculations show that this relation is not independent of the previous two ones. An other way to obtain them is to make use of the Gribov-Froissart representation for the partial wave amplitudes. One writes dispersion relations for the isospin amplitudes, the subtraction constants being given by the s- and p-wave partial amplitudes, and one finds

$$F^I(s,t) = f_0^I(s) + \frac{1}{\pi}\int_0^\infty A^I(x,s) g(x,s,t)\, dx \qquad I = 0, 2 \tag{2.6}$$

where

$$g(x,s,t) = \frac{1}{x-t} + \frac{1}{x-u} + \frac{2}{4-s} ln(1 + \frac{s-4}{x})$$

For $I = 1$ we can write similarly

$$F^1(s,t) = 3f_1^1(s)(1 + \frac{2t}{s-4}) + \frac{1}{\pi}\int_0^\infty A^1(x,s) h(x,s,t)\, dx \tag{2.7}$$

where

$$h(x,s,t)=\frac{1}{x-t}-\frac{1}{x-u}+\frac{6(2t+s-4)}{(4-s)^2}\left[\frac{2x+s-4}{4-s}ln(1+\frac{s-4}{x})+2\right]$$

The absorptive parts entering (2.6)-(2.7) are the t-channel ones and in the following we make use of the $t\longleftrightarrow u$ crossing symmetry. From the relation (2.6) we find analogous formulas to (2.4), namely

$$\frac{1}{t-u}\frac{\partial^{2n-1}F^I(s,t)}{\partial t^{2n-1}}\geq 0 \text{ and } \frac{\partial^{2n}F^I(s,t)}{\partial t^{2n}}\geq 0 \tag{2.8}$$

$n=1,2,\ldots,\ I=0,2$, which are not numerically independent of the previous ones. More interesting is the relation (2.7) which can be written as

$$\frac{F^1(s,t)}{t-u}=\widehat{F^1}(s,t)=\frac{3f_1^1(s)}{s-4}+\frac{1}{\pi}\int_0^\infty A^1(x,s)\left[\frac{1}{(x-t)(x-u)}+\right.$$

$$\left.\frac{6}{(4-s)^2}\left(\frac{2x+s-4}{4-s}ln(1+\frac{s-4}{x})+2\right)\right]dx$$

This relation provides us with another independent relation. Because $F^1(s,t)$ is antisymmetric in $t\longleftrightarrow u$, $\widehat{F^1}(s,t)$ is an analytic function within the Mandelstam triangle. Deriving it with respect of t we obtain

$$\frac{\partial\widehat{F^1}(s,t)}{\partial t}=\frac{t-u}{\pi}\int_0^\infty\frac{A^1(x,s)}{(x-t)^2(x-u)^2}dx$$

From this relation we obtain similar formulas to Eq.(2.4), namely

$$\frac{1}{t-u}\frac{\partial^{2n-1}F^I\widehat{(s,t)}}{\partial t^{2n-1}}\geq 0 \quad \text{and} \quad \frac{\partial^{2n}F^I\widehat{(s,t)}}{\partial t^{2n}}\geq 0 \tag{2.9}$$

Finally the positivity may also be expressed as the positivity of the partial wave amplitudes

$$f_l^I(s)=\frac{1}{4-s}\int_4^\infty A^I(x,s)Q_l(\frac{2x}{4-s}-1)\,dx \tag{2.10}$$

for $l\geq 2$ inside the unphysical region $0\leq s\leq 4$, but the numerical calculations show that these last constraints are weaker than those derived above.

3 Numerical Results

In the previous section we derived a complete set of positivity constraints. In an exact theory many of them are consequence of the others as we will see later. Because ChPT does not completely specify the amplitude and on the other hand the power of different constraints is not the same it is useful to derive as many constraints as possible. Since all of them have to be satisfied we will select the strongest one in every case.

To test the method we worked first in the one-loop approximation, $\mathcal{O}(p^4)$, i.e. we retained terms up to a^2 in Eq.(1.1). In this order the obtained constraints do not depend on the value taken by a; in the two-loop approximation the constraints will be linear in a. We consider first the constraints on the $\pi^0\pi^0$ scattering amplitude.

The first two constraints (1.2) on the $\pi^0\pi^0$, s-wave $f_0(4) > f_0(0) > f_0(3.15)$ are equivalent to

$$b_3 + 3b_4 \geq -\frac{9}{1024} - \frac{29}{384\pi^2} \approx -1.64 \times 10^{-2} \quad \text{and} \quad b_3 + 3b_4 \geq -1.47 \times 10^{-2}$$

respectively. The third relation $f_0(0) > \frac{1}{2}\int_2^4 f_0(s)\, ds$ is equivalent to

$$b_3 + 3b_4 \geq -4.64 \times 10^{-4}$$

the strongest result being the last one. Already the last relation (1.2) furnishes a better result, the above combination of coefficients gets positive

$$b_3 + 3b_4 \geq 7.36 \times 10^{-4}$$

The derivative of the s-wave has the form

$$f_0'(s) = \frac{2}{3}(b_3 + 3b_4)(5s - 8) + h(s)$$

where the function $h(s)$ has a long expression which we do not write it here. Since $5s - 8 < 0$, for $s < 1.6$, the upper and lower bounds on the derivative $f_0'(s)$ describing the position of the minimum are equivalent to a single lower bound on the combination $b_3 + 3b_4$. The best result is obtained at $s = 1.696587$ and is

$$b_3 + 3b_4 \geq 6.67 \times 10^{-3}$$

The last result is the strongest constraint upon the combination $b_3 + 3b_4$ obtained from inequalities satisfied by the $\pi^0\pi^0$ s-wave within the unphysical region $0 \leq s \leq 4$. We have given all the above results to understand why these constraints were easily satisfied by the phenomenological models for partial-wave amplitudes constructed in the past years; satisfying a few of them the others are automatically fulfilled.

A similar analysis with analogous results was done in ref.[19] including in the game the s- ans d-waves.

Stronger constraints are obtained from the positivity of the second derivative of the full amplitudes. From Eq.(2.3) we find for $i = 1$, i.e. the $\pi^0\pi^0$ amplitude again, a relation of the form

$$b_3 + 3b_4 + h_1(s) \geq 0$$

where $h_1(s)$ is a decreasing function for $s < 4$. Thus a problem arises, at what point has to be considered the above relation. We decided to limit the range of s within the interval $|s| < 4$ since we are aware that the amplitude (1.1) is only an approximation of the true amplitude, approximation truly not valid for values $s > 16$ in the physical region. $s = -4$ is equivalent to $t = 12$ in the physical region of the t-channel. One gets

$$b_3 + 3b_4 \geq \frac{37}{1920\pi^2} \approx 1.92 \times 10^{-3}, \quad \text{for} \quad s = 0 \quad \text{and}$$

$$b_3 + 3b_4 \geq 8.28 \times 10^{-3} \quad \text{for} \quad s = -4$$

which is stronger than the previous inequality. In the following we will list numerical values only at $s = 0$, the numerical values at $s = -4$ being not very different although they are a little better such as the previous relations show.

We will work now in the two-loops approximation, $\mathcal{O}(p^6)$, and consider only the constraints derived in Section 2 which give the strongest results. The constraints have the form

$$\sum_{i=1}^{6} c_i^k(s, a)\, b_i + f_k(s, a) \geq 0$$

where $k = 1, 2, 3$ labels the independent constraints. As we said before there are only three possible constraints from the positivity of the absorptive parts, because we have only three independent amplitudes and the crossing symmetry is an exact symmetry for the ChPT amplitudes. In the previous section we derived six constraints but only three are numerically independent. The above inequalities are obtained for every value of s within the unphysical region $0 \leq |s| \leq 4$. Each inequality defines a half space where can live the parameters b_i. Thus the true result would be that obtained by constructing the intersection of these half spaces. Unfortunately the envelope cannot be found in an analytic form the functions $f_k(s,a)$ being very complicated.

The constraint (2.3) for $i = 1$ and $s = 0$ is equivalent to

$$b_3 + 3b_4 - \frac{37}{1920\pi^2} + a\left[\frac{7b_1}{320\pi^2} - \frac{b_2}{60\pi^2} - \frac{2b_3}{45\pi^2} + \frac{b_4}{180\pi^2} + 16b_6 - \frac{367}{552960\pi^2} + \frac{6869}{1658880\pi^4}\right] \geq 0 \tag{3.1}$$

Making $a \to 0$ one gets the one-loop result.

For $i = 2$ we find the relation

$$b_4 - \frac{31}{5760\pi^2} - a\left[\frac{b_1}{240\pi^2} + \frac{43b_2}{2880\pi^2} + \frac{b_3}{24\pi^2} + \frac{23b_4}{180\pi^2} - 4b_6 - \frac{67}{276480\pi^2} - \frac{707}{331776\pi^4}\right] \geq 0 \tag{3.2}$$

The inequality (2.5) for $n = 2$ gives

$$b_3 + 5\,b_4 - \frac{173}{5760\pi^2} + a\left[\frac{13\,b_1}{960\pi^2} - \frac{67\,b_2}{1440\pi^2} - \frac{23\,b_3}{180\pi^2} - \frac{b_4}{4\pi^2} + 24\,b_6 - \frac{11}{61440\pi^2} + \frac{13939}{1658880\pi^4}\right] \geq 0$$

and it is easily seen that it is a consequence of the previous two ones being the sum of the first and of the second one multiplied by 2. The relations (2.6)-(2.8) give us, in principle, three new inequalities, but only one will be independent of the first two already obtained. For $I = 0$ we obtain

$$b_3 + 7\,b_4 - \frac{47}{1152\pi^2} + a\left[\frac{b_1}{192\pi^2} - \frac{11\,b_2}{144\pi^2} - \frac{19\,b_3}{90\pi^2} - \frac{91\,b_4}{180\pi^2} + 32\,b_6 +\right.$$

$$\frac{169}{552960\pi^2}+\frac{7003}{550960\pi^4}\Big]\geq 0$$

which again is a linear combination of the first two ones.

For $I=1$ the relation has the form

$$\frac{11}{2688\pi^2}+a\Big[\frac{b_1}{224\pi^2}+\frac{17b_2}{1344\pi^2}-\frac{151b_3}{2016\pi^2}-\frac{653b_4}{2016\pi^2}+b_5+b_6+$$

$$\frac{37}{215040\pi^2}+\frac{4111}{290304\pi^4}\Big]\geq 0 \tag{3.3}$$

This is the third independent relation as can easily be seen because the one-loop approximation gives a positive number independent of b_i. More important is the fact that the function $f_3(t,a)$ appearing in the relation (3.1) in the one loop approximation is positive over (presumably) all the negative real axis which proves that the positivity is very well satisfied even by the lowest approximation of the ChPT amplitudes!

For $I=2$ the inequality is

$$b_3+b_4-\frac{49}{5760\pi^2}+a\Big[\frac{29\,b_1}{960\pi^2}+\frac{19\,b_2}{1440\pi^2}+\frac{7\,b_3}{180\pi^2}+\frac{47\,b_4}{180\pi^2}+8\,b_6-$$

$$\frac{127}{110592\pi^2}-\frac{67}{552960\pi^4}\Big]\geq 0 \tag{3.4}$$

We have written all the inequalities since they were useful in checking the calculations.

A first remark is the following, the coefficient b_5 appears only in the inequality (3.3). Thus we can say that it is unimportant and make it vanish also in the amplitudes! It is true that the above inequalities have been obtained by an extremal property, they are calculated on the line where the corresponding amplitudes are taking their minimal values. This may be a suggestion that the physical partial waves satisfy also an extremal princ iple which has to be found. This is also supported by the findings of Wanders, who could not obtain a reliable value for this parameter[15]. What the above results suggest is that a good determination of b_5 can be obtained only from $I=1$ data.

We have tested how the parameters b_i found in literature compare with the inequalities. Unfortunately there are only two papers that gives values for all b_i [14, 20]. The values given by Bijnens *et al.* are in the domain allowed by

Eqs.(3.1)-(3.4), but the values obtained by Ananthanaryan strongly violate the inequality (3.4), the previous ones being satisfied. The values obtained by Knecht *et al.*[3] for the last four parameters can be used to obtain constraints on the b_1 an d b_2 but the allowed domain is rather large; the same is true for the values given by Wanders[15].

Using Roy equation analysis of the available $\pi\pi$ phase shift data Ananthanaryan and Büttiker[21] obtained values for the chiral coupling constants $\bar{l}_1$, $\bar{l}_2$ but their results are not easily translated into constrain ts on the b_i coefficients We have tested also the inequalities (2.4), (2.8), (2.9) and (2.10) for a few values $n \geq 3$ and $l \geq 2$ respectively and we found that they are very well fulfilled.

4 Conclusion

We have tested the positivity properties of the ChPT pion-pion amplitudes and we have obtained a number of inequalities which express this property. We conclude that the pion-pion amplitudes given by the relation (1.1) satisfy this property unexpectedly well. In the $\mathcal{O}(p^4)$ approximation the positivity implies two constraints: b_4 is a positive quantity and so is the combination $b_3 + 3b_4$. As concerns the positivity of the $I = 1$ amplitude this was tested numerically up to $t = -10^5$ where the derivative is still positive. Including $\mathcal{O}(p^6)$ contributions one get constraints involving all the six parameters b_i, but the corrections to the $\mathcal{O}(p^4)$ results are small which support the idea that the expansion (1.1) is the best candidate for the true amplitude. It might seem surprising but we consider that the main result of this study is the conclusion that the most important parameters to be determined are b_3 and b_4. This is a consequence of the good properties near threshold of the Weinberg approximation together with the very powerful property of positivity of the scattering amplitudes. Let us remind that this property was essential in deriving the analyticity domain of pion-pion amplitudes [22]. This means that the amplitude (1.1) in which all but b_3, b_4 are zero will give a fair description of the low energy phenomenology up to about 6-700 MeV and the contribution of the other coefficients will be seen at higher energies. In conclusion a good determination of the above two parameters will be a good starting point in the comparison of the theory with experiment. Work on this line is in progress.

Acknowledgements. This work started when the author was a visitor at the Institute of Theoretical Physics, University of Bern in the frame of the Swiss National Science Foundation Program. I take this opportunity to thank SNSF for support. It is a pleasure to thank Prof. H. Leutwyler and Prof. G. Wanders for discussions.

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