A QUANTUM THEORY OF 2D SPACE-TIME

J. AMBJØRN
The Niels Bohr Institute,
Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

Abstract
I review our understanding of the fractal structure of quantum space-time. The fractal structure of space-time arises in the same way as the "path" of a scalar particle is "fractal", i.e. has Hausdorff dimension 2. Presently we only have a mathematical well defined quantum theory for two-dimensional geometries and the review concentrates on presenting the most elementary aspects of this theory in a concise way.

1 INTRODUCTION

The free relativistic particle provides us with the simplest example of "quantum geometry". The action of a free relativistic particle is just the length of its world line[1] in R^d. The classical path between two space-time points x and y is just the straight line. The system is quantized by summing over all paths P_{xy} from x to y with the Boltzmann weight determined by the classical action, which is simply the length $L(P_{xy})$ of the path. We write for the relativistic two-point function:

$$G(x,y) = \int \mathcal{D}P_{xy}\; \mathrm{e}^{-mL(P_{xy})}, \tag{1}$$

where m is the mass of the particle. The measure on the set of *geometric paths* P_{xy} can be defined and are related in a simple way (see [1]) to the ordinary Wiener measure on the set of parameterized paths[2]. One of the main features of this measure is that a "typical" path has a length

$$L_{xy} \sim \frac{1}{\varepsilon}\,|x-y|^2, \tag{2}$$

where ε is some cut-off. We say that the fractal dimension of a typical *random path* is two.

The generalizations of (2) go in various directions: one can consider higher dimensional objects like strings. The action of a string will be the area A of the world sheet F swept out by the string moving in R^d. If we consider closed strings the quantum propagator between two boundary loops L_1 and L_2 will be

$$G(L_1,L_2) = \int \mathcal{D}F_{L_1L_2}\; \mathrm{e}^{-A(F_{L_1L_2})}, \tag{3}$$

where the integration is over all surfaces in R^d with boundaries L_1 and L_2. Alternatively, we can for manifolds of dimensions higher than one consider actions which depend only on the intrinsic geometry of the manifold. The simplest such action is the Einstein-Hilbert action, here written for a n-dimensional manifold $\mathcal{M}$:

$$S(g) = \Lambda \int_{\mathcal{M}} \mathrm{d}^n\xi \sqrt{g(\xi)} - \frac{1}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^n\xi \sqrt{g(\xi)}\, R(\xi), \tag{4}$$

where g is the metric on $\mathcal{M}$ and R the scalar curvature defined from g. Quantization of geometry means that we should sum over all geometries g with the weight $\mathrm{e}^{-S(g)}$. The partition function will be

$$Z(\Lambda,G) = \int \mathcal{D}[g]\; \mathrm{e}^{-S(g)}, \tag{5}$$

[1] In the following we will always be working in Euclidean space-time.
[2] The geometric paths are just parameterized paths up to diffeomorphisms.

where the integration is over all equivalence classes of metrics, i.e. metrics defined up to diffeomorphisms. One can add matter coupled to gravity to the above formulation. Let $S_m(\phi, g)$ be the diffeomorphism invariant Lagrangian which describes the classical dynamics of the matter fields in a fixed background geometry defined by g and let λ denote the coupling constants of the scalar fields. The quantum theory will be defined by

$$Z(\Lambda, G, \lambda) = \int \mathcal{D}[g]\mathcal{D}\phi \; \mathrm{e}^{-S(g)-S_m(\phi,g)}. \tag{6}$$

Two-dimensional quantum gravity is particularly simple. As long as we do not address the question of topology changes of the underlying manifold $\mathcal{M}$, the Einstein-Hilbert action (4) simplifies since the curvature term is just a topological constant, and we can write

$$S(g) = \Lambda \int_{\mathcal{M}} \mathrm{d}^2\xi \sqrt{g(\xi)} \quad \text{(two dimensions)}. \tag{7}$$

Classical string theory, as defined by the area action $A(F)$, has an equivalent formulation where an independent intrinsic metric $g(\xi)$ is introduced on the two-dimensional manifold corresponding to the world sheet and where the coordinates of the surface, $x(\xi) \in R^d$, are viewed as d scalar fields on the manifold with metric $g(\xi)$. The quantum string theory will then be a special case of two-dimensional quantum gravity coupled to matter, as defined by (6), with $S(g)$ given by (7). In the following we will study this theory, with special emphasis on pure two-dimensional quantum gravity, i.e. two-dimensional quantum gravity without any matter fields.

2 A TOY MODEL: THE FREE PARTICLE

It is instructive first to perform the same exercise for the free relativistic particle given by (1). In this case one can approximate the integration over random paths by the summation and integration over the class of piecewise linear paths where the length of each segment of the path is fixed to a, i.e. we make the replacement

$$\int \mathcal{D}P_{xy} \to \sum_{P_{xy}} \equiv \sum_n \int \prod \mathrm{d}\hat{e}_i \; \delta\Big(a \sum_i \hat{e}_i - (x-y)\Big), \tag{8}$$

where $\hat{e}_i$ denote unit vectors in R^d and $\sum_{P_{xy}}$ is a symbolic notation of the summation and integration over the chosen class of paths. The action is simply $m_0 \cdot na$ for a path with n "building blocks". A "discretized" two-point function is then defined by

$$G_a(x, y; m_0) = \sum_{P_{xy}} \mathrm{e}^{-m_0 L(P_{xy})} \equiv \sum_n \mathrm{e}^{-m_0 an} \int \prod \mathrm{d}\hat{e}_i \; \delta\Big(a \sum_i \hat{e}_i - (x-y)\Big). \tag{9}$$

The integration over the unit vectors is most easily performed by a Fourier transformation with removes the δ-function:

$$G_a(p; m_0) = \int dx \, \mathrm{e}^{-ip\cdot(x-y)} G_a(x, y; m_0) = \sum_n \mathrm{e}^{-m_0 an} \int \prod \mathrm{d}\hat{e}_i \; \mathrm{e}^{-ia\,p\cdot\hat{e}_i}. \tag{10}$$

Since

$$\int \mathrm{d}\hat{e}_i \; \mathrm{e}^{-iap\cdot\hat{e}_i} = 2\pi^{d/2} \left[\frac{J_{(d-1)/2}(ap)}{(ap)^{(d-1)/2}}\right] \equiv f(ap), \tag{11}$$

the final expression for $G_a(p; m_0)$ becomes

$$G_a(p; m_0) = \sum_n \Big(\mathrm{e}^{m_0 a} f(ap)\Big)^n = \frac{1}{1 - \mathrm{e}^{-m_0 a} f(ap)}. \tag{12}$$

In the following we only need the following properties of $f(ap)$:

$$f(ap) = f(0)(1 - c^2(ap)^2 + \cdots), \qquad f(0) > 0.$$

In order to obtain the continuum two-point function we have to take $a \to 0$ and this involves a *renormalization of the bare mass* m_0 *as well as a wave-function renormalization.* Let us define the *physical mass* m_{ph} by

$$\mathrm{e}^{-m_0 a} f(ap) \to 1 - c^2 m_{ph}^2 a^2, \quad \text{i.e.} \quad m_0 = \frac{\log f(0)}{a} + c^2 m_{ph}^2 a. \tag{13}$$

With this fine tuning of the *bare mass* m_0 we obtain for $a \to 0$

$$G_a(p; m_0) \sim a^{-2}\, G_{cont}(p; m_{ph}), \tag{14}$$

where the continuum two-point function of the free relativistic particle is

$$G_{cont}(p; m_{ph}) \equiv \frac{1}{p^2 + m_{ph}^2}.$$

The prefactor $1/a^2$ in eq. (14) is a so-called wave-function renormalization. It is related to the short distance behavior of the propagator as will be discussed below.

2.1 Scaling relations and geometry

It is worth rephrasing the results obtained so far in terms of dimensionless quantities and in this way make the statistical mechanics aspects more visible. Introduce $\mu = m_0 a$ and $q = ap$ and view the coordinates in R^d as dimensionless. The steps in the discretized random walk will then be of length 1 and (12) reads

$$G_\mu(q) = \sum_n \mathrm{e}^{-\mu n} f(q) = \frac{1}{1 - \mathrm{e}^{-\mu} f(q)}. \tag{15}$$

It is seen that μ acts like a chemical potential for inserting additional sections in the piecewise linear random walk and that we have a *critical value* $\mu_c = \log f(0)$ such that the average number of steps of the random walk diverge for $\mu \to \mu_c$ from above. This is why we can take a continuum limit when $\mu \to \mu_c$. In fact, the relation (13) becomes

$$\mu - \mu_c = m_{ph}^2 a^2, \tag{16}$$

which defines a as a function of μ:

$$a(\mu) = m_{ph}^{-1} (\mu - \mu_c)^{1/2}. \tag{17}$$

Further, we see that the so-called *susceptibility* diverges as $\mu \to \mu_c$:

$$\chi(\mu) \equiv \int \mathrm{d}^d x\, G_\mu(x) = G_\mu(q = 0) = \frac{1}{1 - \mathrm{e}^{-(\mu - \mu_c)}} \sim \frac{1}{\mu - \mu_c}. \tag{18}$$

These considerations can be understood in a more general framework. It is not difficult to show that $G_\mu(x)$ has to fall off exponentially for large x under very general assumptions concerning the probabilistic nature of the (discretized) random walk. It follows from standard subadditivitive arguments. In essence, they say that the random walks from x to y which pass through a given point z constitute a subset of the total number of random walks from x to y. This implies that

$$G_\mu(x, y) \sim \mathrm{e}^{-m(\mu)|x-y|} \quad \text{for} \quad |x - y| \gg \frac{1}{m(\mu)}. \tag{19}$$

Let us now *assume* that

$$m(\mu) \to (\mu - \mu_c)^\nu \quad \text{for} \quad \mu \to \mu_c. \tag{20}$$

In order that $G_\mu(x,y)$ has a non-trivial limit for $\mu \to \mu_c$ we have to introduce the following generalization of (16)

$$m(\mu) = m_{ph} a(\mu), \quad x_{ph} = x\, a(\mu), \quad \text{i.e. } a(\mu) \sim (\mu - \mu_c)^\nu. \tag{21}$$

It is clear that $m(\mu)$ has the interpretation as inverse correlation length (or a mass). If the mass $m(\mu)$ goes to zero as $\mu \to \mu_c$ the two-point function $G_\mu(x,y)$ will in general satisfy a power law for $|x-y|$ much less that the correlation length:

$$G_\mu(x,y) \sim \frac{1}{|x-y|^{d-2+\eta}} \quad \text{for} \quad |x-y| \ll \frac{1}{m(\mu)}. \tag{22}$$

Finally the susceptibility is defined as in (18):

$$\chi(\mu) = \int d^d x\; G_\mu(x,y) \sim \frac{1}{(\mu-\mu_c)^\gamma}, \tag{23}$$

where the *critical exponents* ν, η and γ (almost) by definition satisfy

$$\gamma = \nu(2-\eta) \quad (\text{Fisher's scaling relation}). \tag{24}$$

For the random walk representation of the free particle considered above we have:

$$\nu = \frac{1}{2}, \quad \eta = 0, \quad \gamma = 1. \tag{25}$$

Let us now show that $1/\nu$ is the extrinsic Hausdorff dimension of the random walk between x and y. The average length of a path between x and y is equal

$$\langle L_{xy} \rangle = \frac{\sum_{P_{xy}} L(P_{xy})\, \mathrm{e}^{-\mu L(P_{xy})}}{\sum_{P_{xy}} \mathrm{e}^{-\mu L(P_{xy})}} = -\frac{\mathrm{d} \log G_\mu(x,y)}{\mathrm{d}\mu}. \tag{26}$$

For $|x-y|$ sufficiently large, such that (19) can be used, we have

$$\langle L_{xy} \rangle \sim m'(\mu)|x-y|. \tag{27}$$

However, the continuum limit has to be taken in such a way that

$$m(\mu)|x-y| = m_{ph}|x_{ph} - y_{ph}|, \tag{28}$$

i.e. independent of μ for $\mu \to \mu_c$. From (20) and (28) we obtain

$$\langle L_{xy} \rangle \sim \frac{m'(\mu)}{m(\mu)} \sim \frac{1}{\mu - \mu_c} \sim |x-y|^{1/\nu}. \tag{29}$$

We define the extrinsic Hausdorff dimension by

$$\langle L_{xy} \rangle \sim |x-y|^{d_H^{(e)}}, \tag{30}$$

and we conclude that the critical exponent ν is related to the extrinsic Hausdorff dimension $d_H^{(e)}$ by

$$d_H^{(e)} = \frac{1}{\nu} \tag{31}$$

2.2 Summary

Above it has been shown how it is possible by a simple, appropriate choice of regularization of the set of geometric paths from x to y to define the measure $\mathcal{D}P_{xy}$. One of the basic properties of this measure, namely that a generic path has $d_H^{(e)} = 2$ was easily understood. It is important that the regularization is performed directly in the set of geometric paths. In this way it becomes a reparameterization invariant regularization of $\mathcal{D}P_{xy}$. The regularization can be viewed as a grid in the set of geometric paths, which becomes uniformly dense in the limit $\mu \to \mu_c$ or alternatively $a(\mu) \to 0$. The Wiener measure itself is defined on the set of *parameterized paths* and will not lead to the relativistic propagator..

3 THE FUNCTIONAL INTEGRAL OVER 2D-GEOMETRIES

As described above the partition function for two-dimensional geometries is

$$Z(\Lambda) = \int \mathcal{D}[g] \; \mathrm{e}^{-\Lambda V_g}, \qquad V_g \equiv \int_{\mathcal{M}} d^2\xi \sqrt{g(\xi)}. \tag{32}$$

It is sometimes convenient to consider the partition function where the volume V of space-time is kept fixed. We define it by

$$Z(V) = \int \mathcal{D}[g] \; \delta(V - V_g), \tag{33}$$

such that

$$Z(\Lambda) = \int_0^\infty dV \; \mathrm{e}^{-V\Lambda} Z(V). \tag{34}$$

It is often said that two-dimensional quantum gravity has little to do with four-dimensional quantum gravity since there are no dynamical gravitons in the two-dimensional theory (the Lagrangian is trivial since it contains no derivatives of the metric). However, all the problems associated with the definition of reparameterization invariant observables are still present in the two-dimensional theory, and the theory is in a certain sense *maximal quantum*: from (33) it is seen that *each equivalence class of metrics is included in the path integral with equal weight*, i.e. we are as far from a classical limit as possible. Thus the problem of defining genuine reparameterization invariant observables in quantum gravity is present in two dimensional quantum gravity as well. Here we will discuss the so-called Hartle-Hawkings wave-functionals and the two-point functions. The Hartle-Hawking wave-functional is defined by

$$W(L;\Lambda)) = \int_L \mathcal{D}[g] \; \mathrm{e}^{-S(g;\Lambda)} \tag{35}$$

where L symbolizes the *boundary* of the manifold $\mathcal{M}$. In dimensions higher than two one should specify (the equivalence class of) the metric on the boundary and the functional integration is over all equivalence classes of metrics having this boundary metric. In two dimensions the equivalence class of the boundary metric is uniquely fixed by its length and we take L to be the length of the boundary. It is often convenient to consider boundaries with variable length L by introducing a *boundary cosmological term* in the action:

$$S(g;\Lambda,\Lambda_b) = \Lambda \int_{\mathcal{M}} \mathrm{d}^2\xi \sqrt{g(\xi)} + \Lambda_B \int_{\partial\mathcal{M}} \mathrm{d}s, \tag{36}$$

where $\mathrm{d}s$ is the invariant line element corresponding to the boundary metric induced by g and Λ_B is called the boundary cosmological constant. We can then define

$$W(\Lambda_B,\Lambda) = \int \mathcal{D}[g] \; \mathrm{e}^{-S(g;\Lambda,\Lambda_B)}. \tag{37}$$

The wave-functions $W(L;\Lambda)$ and $W(\Lambda_B,\Lambda)$ are related by a Laplace transformation in the boundary length:

$$W(\Lambda_B,\Lambda) = \int_0^\infty \mathrm{d}L \; \mathrm{e}^{-\Lambda_B L} \, W(L;\Lambda). \tag{38}$$

The two-point function is defined by

$$G(R;\Lambda) = \int \mathcal{D}[g]\, \mathrm{e}^{-S(g,\Lambda)} \iint \mathrm{d}^2\xi \sqrt{g(\xi)}\, \mathrm{d}^2\eta \sqrt{g(\eta)}\ \delta(D_g(\xi,\eta) - R), \tag{39}$$

where $D_g(\xi,\eta)$ denotes the geodesic distance between ξ and η in the given metric g. Again, it is sometimes convenient to consider a situation where the space-time volume V is fixed. This function, $G(R;V)$ will be related to (39) by a Laplace transformation, as above for the partition function Z:

$$G(R;\Lambda) = \int_0^\infty \mathrm{d}V\ \mathrm{e}^{-V\Lambda}\, G(R;V). \tag{40}$$

It is seen that $G(R;\Lambda)$ and $G(R;V)$ has the interpretation of partition functions for universes with two marked points separated a given geodesic distance R. If we denote the average volume of a spherical shell of geodesic radius R in the class of metrics with space-time volume V by $S_V(R)$, we have by definition

$$S_V(R) = \frac{G(R;V)}{V Z(V)}. \tag{41}$$

One can define an intrinsic fractal dimension, d_H, of the ensemble of metrics by

$$\lim_{R\to 0} S_V(R) \sim R^{d_H - 1}(1 + O(R)). \tag{42}$$

Alternatively, one could take over the random walk definition of d_H. According to this definition

$$\langle V \rangle_R \sim -\frac{\partial \log G(R;\Lambda)}{\partial \Lambda} \sim R^{d_H} \tag{43}$$

for a suitable range of R related to the value of Λ. I will show that the two definitions agree in the case of pure gravity. Eq. (42) can be viewed as a "local" definition of d_H, while eq. (43) is "global" definition. Since the two defintions result in the same d_H two-dimensional gravity has a genuine fractal dimension over all scales.

Eq. (33) shows that the calculation of $Z(V)$ is basically a counting problem: each geometry, characterized by the equivalence class of metrics $[g]$, appears with the same weight. The same is true for the other observables defined above. One way of performing the summation is to introduce a suitable regularization of the set of geometries by means of a cut-off, to perform the summation with this cut-off and then remove the cut-off, like in the case of geometric paths considered above.

3.1 The regularization

The integral over geometric paths were regularized by introducing a set of basic building blocks, "rods of length a", which were afterwards integrated over all allowed positions in R^d. Let us imitate the same construction for two-dimensional space-time [2, 3, 4]. The natural building blocks will be equilateral triangles with side lengths ε, but in this case there will be no integration over positions in some target space[3]. We can glue the triangles together to form a triangulation of a two-dimensional manifold $\mathcal{M}$ with a given topology. If we view the triangles as flat in the interior, we have in addition a unique piecewise linear metric assigned to the manifold, such that the volume of each triangle is $dA_\varepsilon = \sqrt{3}\varepsilon^2/4$ and the total volume of a triangulation T consisting of N_T triangles will be $N_T dA_\varepsilon$, i.e. we can view the triangulation as associated with a Riemannian manifold $(\mathcal{M}, g)$. In the case of a one-dimensional manifold the total volume is the only reparameterization invariant quantity. For a two-dimensional manifold $\mathcal{M}$ the scalar curvature R is a *local* invariant. This local invariance in present in a natural way when we consider various triangulations. Each vertex v in a triangulation has a certain order n_v. In the context of

[3]We could introduce such embedding in R^d, but in that case we would not consider two-dimensional gravity but rather bosonic string theory, where the embedded surface was the world sheet of the string, as already mentioned above [3, 5].

two-dimensional piecewise linear geometry, curvature is located at the vertices and is characterized by a *deficit angle*

$$\Delta_v = \frac{\pi}{3}(6 - n_v), \tag{44}$$

such that the total curvature of the manifold is

$$\int \sqrt{g} R = \sum_v \Delta_v. \tag{45}$$

From this point of view a summation over triangulations of the kind mentioned above will form a grid in the class of Riemannian geometries associated with a given manifold $\mathcal{M}$. The hope is that the grid is sufficient dense and uniform to be able the describe correctly the functional integral over all Riemannian geometries when $\varepsilon \to 0$.

We will show that it is the case by explicit calculations, where some of the results can be compared with the corresponding continuum expressions. They will agree. But the surprising situation in two-dimensional quantum gravity is that the analytical power of the regularized theory seems to exceed that of the formal continuum manipulations. Usually the situation is the opposite: regularized theories are either used in a perturbative context to remove infinities order by order, or introduced in a non-perturbative setting in order make possible numerical simulations. Here we will derive analytic (continuum) expressions with an ease which can presently not be matched by formal continuum manipulations.

3.2 The Hartle-Hawking wave-functional

Let us calculate the discretized version, $w(\lambda, \mu)$ of the Hartle-Hawking wave-functional $W(\Lambda_B, \Lambda)$, defined by (37). We assume the underlying manifold $\mathcal{M}$ has the topology of the disk. First note that the discretized action corresponding to (36) can be written as

$$S_T(\mu, \lambda) = \mu N_T + \lambda l_T, \tag{46}$$

where the given triangulation T also defines the metric, N_T and l_T denote the number of triangles and the number of links at the boundary of T, respectively, while μ and λ are the dimensionless "bare" cosmological and boundary cosmological coupling constants corresponding to Λ and Λ_B. We can now write

$$w(\lambda, \mu) = \sum_T \mathrm{e}^{-S_T(\mu,\lambda)}, \tag{47}$$

where the summation is over all triangulations of the disk. Until now I have not specified the class of triangulations. The precise class should not be important, by universality, since any structure not allowed at the smallest scale by one class of triangulations can be imitated at a somewhat larger scale. Thus, it is convenient to choose a class of "triangulations" which results in the simplest equation. They are defined as the class of complexes homeomorphic to the disk that can be obtained by successive gluing together of triangles and a collection of double-links which we consider as (infinitesimally narrow) strips, where links, as well as triangles, can be glued onto the boundary of a complex both at vertices and along links. Gluing a double-link along a link makes no change in the complex. An example of such a complex is shown in fig. 1.

By introducing

$$g = \mathrm{e}^{-\mu}, \quad z = \mathrm{e}^{\lambda}, \tag{48}$$

we can write (47) as

$$w(z, g) = \sum_{l,k} w_{l,k} g^k z^{-l-1} = \sum_l \frac{w_l(g)}{z^{l+1}}, \tag{49}$$

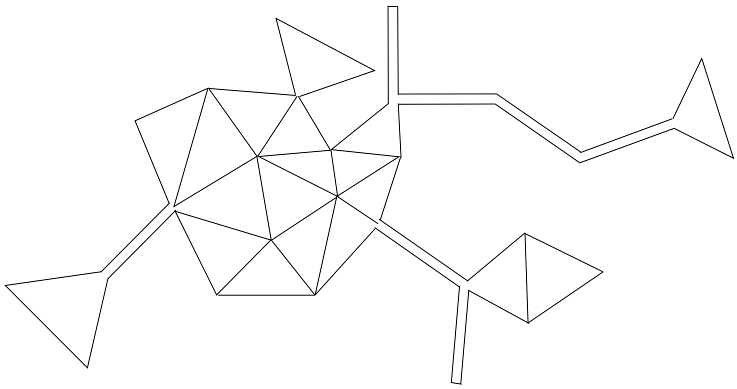

Fig. 1: A typical unrestricted "triangulation".

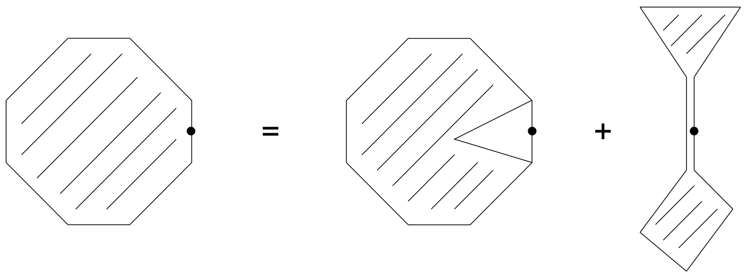

Fig. 2: Graphical representation of eq. 51.

where $w_{k,l}$ is the number of triangulations of the disk with k triangles and a boundary of l links. We see that $w(z,g)$ is the *generating function* [4] for $\{w_{l,k}\}$. The generating function $w(z,g)$ satisfies the following equation, depicted graphically in fig. 2,

$$w(z,g) = zg\, w(z,g) + \frac{1}{z} w^2(z,g). \tag{50}$$

This equation is not correct from the smallest values of of the boundary-length l, as is clear from fig. (2), since all boundaries on the right-hand of the equation have a boundary length $l > 1$. Denote by $w_1(g)$ the generating function for triangulations of the disk with a boundary with only one link (see eq. (49)). The correct equation which replaces (50) is

$$w(z,g) = \frac{1}{z} + zg\left(w(z,g) - \frac{1}{z} - \frac{w_1(g)}{z^2}\right) + \frac{1}{z} w^2(z,g), \tag{51}$$

if we use the normalization that a single vertex is represented by $1/z$. This equation is similar in spirit to the equation studied by Tutte in his seminal paper[6] from 1962, and it can by shown that it has a unique solution where all coefficients $w_{l,k}$ are positive. The solution is given by

$$w(z,g) = \frac{1}{2}\left(z - gz^2 + (gz - c_2(g))\sqrt{(z - c_+(g))(z - c_-(g))}\right), \tag{52}$$

where $c_-(g)$, $c_+(g)$ and $c_2(g)$ are analytic functions of g in a neighborhood of $g = 0$, with the initial conditions

$$c_2(0) = 1, \quad c_+(0) = 2, \quad , c_-(0) = -2. \tag{53}$$

[4] In (49) I have used $1/z$ rather than z as indeterminate for $\{w_{l,k}\}$ for later convenience, and for the same reason multiplied (49) by an additional factor $1/z$ relatively to (47).

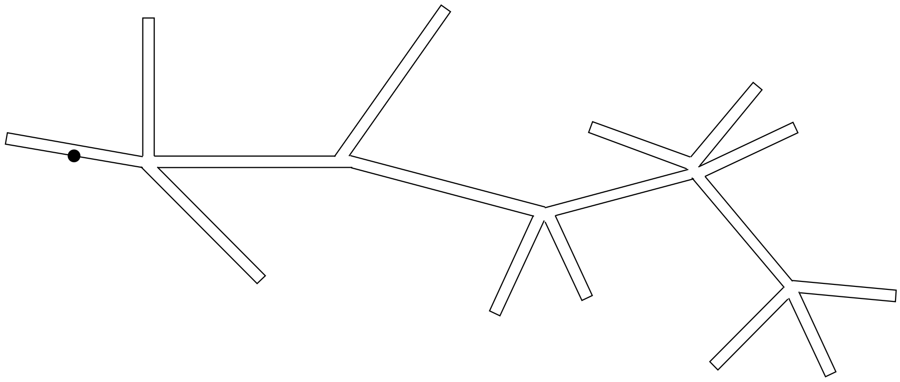

Fig. 3: A boundary graph with no internal triangles.

Thus, for $g = 0$ we have

$$w(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right) = \sum_{l=0}^{\infty} \frac{w_{2l}}{z^{2l+1}}, \tag{54}$$

where the coefficients w_{2l} have the interpretation as the number of boundaries with no internal triangles, see fig. 3. We have

$$w_{2l} = \frac{2l!}{(l+1)!\,l!} = \frac{1}{\pi} l^{-3/2}\, 2^{2l}(1 + O(l)), \tag{55}$$

i.e. the number of such boundaries grows exponentially with the length l. We can view $1/z$ as the so-called fugacity[5] for the number of boundary links, and the radius of convergence (here 1/2) can be viewed as the maximal allowed value of the fugacity. When z approaches $z_c(0) = 2$ the average length of a typical boundary will diverge. In the same way g acts as the fugacity for triangles. As g increases the average number of triangles will increase, and at a certain critical value g_c some suitable defined average value of triangles will diverge. In terms of the coefficients $w_{l,k}$ in (49) it reflects an exponential growth of $w_{l,k}$ for $k \to \infty$, independent of l, i.e. the functions $w_l(g)$ all have the same radius of convergence g_c. For a given value $g < g_c$ we have a critical value $z_c(g)$ at which the average boundary length will diverge. As g increases towards g_c, $z_c(g)$ will increase towards $z_c \equiv z_c(g_c)$.

From the explicit solutions for $c_\pm(g)$ and $c_0(g)$ it is found that

$$c_+(g_c) = z_c = c_2(g_c)/g_c, \tag{56}$$

and near g_c we have, with $\Delta g \equiv g_c - g$:

$$c_+(g) = z_c\Big(1 + \frac{1}{2}\sqrt{\Delta g}\Big), \quad c_2(g) = z_c g_c\Big(1 - \sqrt{\Delta g}\Big). \tag{57}$$

In particular, g_c is the radius of convergence for $c_+(g)$ and $c_2(g)$.

It is now possible to define a continuum limit of the above discretized theory by approaching the critical point in a suitable way:

$$g(\Lambda) = g_c(1 - \Lambda\varepsilon^2), \qquad z = z_c(1 + \Lambda_B \varepsilon). \tag{58}$$

If we return to the relations (48) between g and μ and z and λ, respectively, we can write (58) as follows:

$$\mu - \mu_c = \Lambda\varepsilon^2, \qquad \lambda - \lambda_c = \Lambda_B \varepsilon, \tag{59}$$

[5] The fugacity f is related to the chemical potential μ by $f = \mathrm{e}^{-\mu}$.

where μ_c and λ_c correspond to g_c and z_c, respectively. We can now, as is standard procedure in quantum field theory, relate coupling constants μ and λ to Λ and Λ_B by an *additive renormalization.* The dimensionless coupling constants μ and λ are associated with so-called *bare coupling constants* Λ_0 and Λ_{B0} as follows:

$$\mu N_T + \lambda l_T = \frac{\mu}{\varepsilon^2} N_T \varepsilon^2 + \frac{\lambda}{\varepsilon} l\varepsilon \equiv \Lambda_0 \int_{\mathcal{M}} \mathrm{d}^2\xi \sqrt{g} + \Lambda_{B0} \int_{\partial\mathcal{M}} ds. \tag{60}$$

We can now interpret (59) as an additive renormalization of the bare coupling constants:

$$\Lambda_0 = \frac{\mu_c}{\varepsilon^2} + \Lambda, \quad \Lambda_{B0} = \frac{\lambda_c}{\varepsilon} + \Lambda_B. \tag{61}$$

This additive renormalization is to be expected from a quantum field theoretical point of view since both coupling constants have a mass-dimension.

Using the known behavior (57) of $c_\pm(g)$ and $c_2(g)$ in the neighborhood g_c, we get from (52) (except for the first two terms with are analytic in g and therefore "non-universal" terms [6] which can be shown to play no role for continuum physics):

$$w(z, g) \sim \varepsilon^{3/2} W(\Lambda_B, \Lambda) \tag{62}$$

where [7, 8]

$$W(\Lambda_B, \Lambda) \sim (\Lambda_B - \frac{1}{2}\sqrt{\Lambda})\sqrt{\Lambda_B + \sqrt{\Lambda}}. \tag{63}$$

Again, the factor $\varepsilon^{3/2}$ has a standard interpretation in the context of quantum field theory: it is a wave-function renormalization.

By an inverse discrete Laplace transformation one obtains $w(l, g)$ from $w(z, g)$, and by an ordinary inverse Laplace transformation one obtains

$$W(L, \Lambda) = L^{-5/2}(1 + L\sqrt{\Lambda})\, \mathrm{e}^{-L\sqrt{\Lambda}}. \tag{64}$$

3.3 The two-point function

Let us return to the calculation of $G(R; \Lambda)$. Using the regularization we define a *geodesic two-loop function* by

$$G_\mu(l_1, l_2; r) = \sum_T \mathrm{e}^{-\mu N_T}, \tag{65}$$

and the class of triangulations which enters in the sum have the topology of a cylinder with an "entrance loop" of length l_1 and with one marked linked, and an "exit loop" of length l_2 and without a marked link, the loops separated by a geodesic distance r, see fig. 4. We say the geodesic distance between the exit loop and the entrance loop is r if each point on the exit loops has a minimal geodesic distance r to the set of points on the entrance loop. Note the asymmetry between exit and entrance loops in the definition. On the piecewise linear manifolds geodesic distances are uniquely defined. However, it is often convenient to use a graph-theoretical definition, since this makes combinatorial arguments easier. Here I define the geodesic distance between links (or vertices) as the shortest path along neighboring triangles.

$G_\mu(l_1, l_2; \mu)$ satisfies an equation [9], which is essentially equivalent to the equation satisfied by the Hartle-Hawking wave function $w(l, \mu)$ for a disk with boundary length l. It is obtained by a deformation of the entrance loop:

$$G_\mu(l_1, l_2; r) = gG_\mu(l+1, l; r) + 2\sum_{l=0}^{l_1-2} w(l, \mu)\, G_\mu(l_1 - l - 2, l_2; r). \tag{66}$$

In fig. 5 the possible elementary deformations of the entrance loops is shown. It is analogous to fig. 2.

[6] Analytic terms are usually non-universal since trivial analytic redefinitions of the coupling constants can change these terms completely.

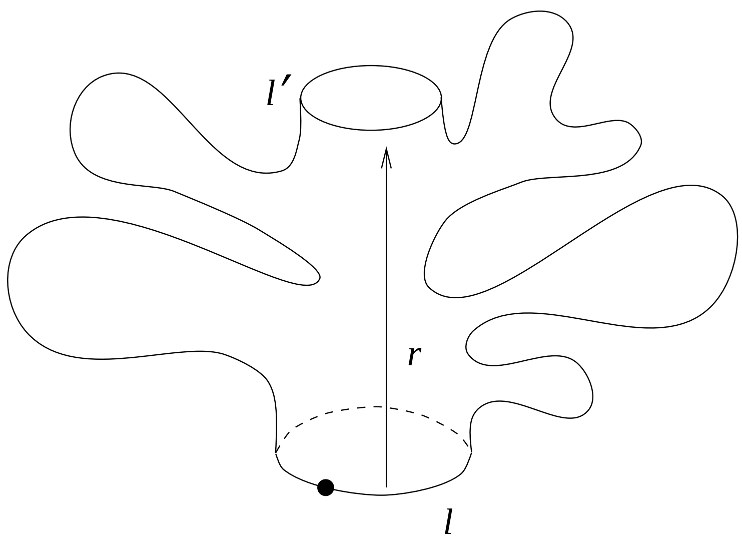

Fig. 4: A typical surface contributing to $G_\mu(l, l' : r)$. The "dot" on the entrance loop signifies that the entrance loop has one marked link.

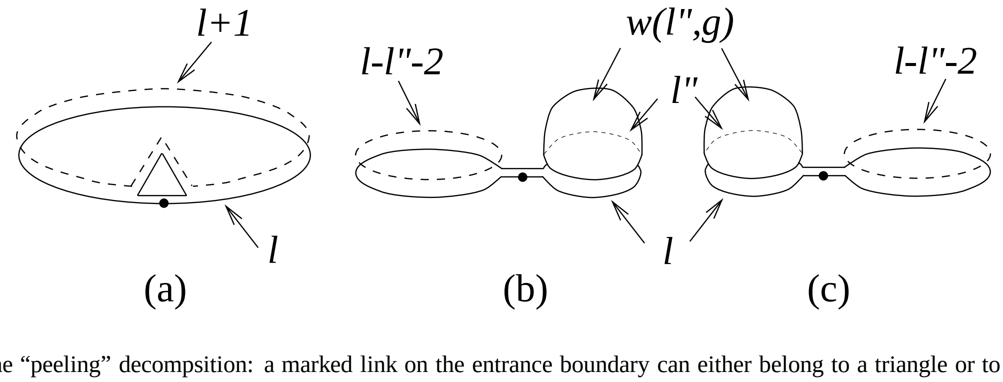

Fig. 5: The "peeling" decompsition: a marked link on the entrance boundary can either belong to a triangle or to a "double" link. The dashed curved indicates the new entrance loop.

The second term in eq. (66) corresponds to the case where the surface splits in two after the deformation. We can view the process as a "peeling" of the surface, which occasionally chops off outgrows with disk topology as shown in fig. 6. The application of the one-step peeling l_1 times should on average correspond to cutting a slice (see fig. 6), of thickness one (or ε, which we have chosen equal 1 for convenience in the present considerations) from the surface. Thus we identify the change caused by one elementary deformation with

$$\left[\frac{\partial}{\partial r}\, G_\mu(l_1,l_2;r)\right]\; \frac{1}{l_1}, \tag{67}$$

forgetting for the moment that r is an integer. It follows that we can write

$$\begin{aligned} \frac{\partial}{\partial r}\, G_\mu(l_1,l_2;r) \;&=\; -l_1 G_\mu(l_1,l_2;r) + g l_1 G_\mu(l_1+1,l_2;r) \\ &\quad +2\sum_l l w(l,\mu)\, G_\mu(l_1-l-2,l_2;r). \end{aligned} \tag{68}$$

To solve the combinatorial problem associated with (68) it is convenient (as for $w(l,\mu)$) to introduce the generating function $G_\mu(z_1,z_2;r)$ associated with (65):

$$G_\mu(z_1,z_2;r) = \sum_{l_1,l_2} \frac{G_\mu(l_1,l_2;r)}{z_1^{l+1} z_2^{l_2+1}}. \tag{69}$$

With this notation eq. (68) becomes

$$\frac{\partial}{\partial r}\, G_\mu(z_1,z_2;r) = \frac{\partial}{\partial z_1}\Big[\Big(z_1 - g z_1^2 - 2w(z,g)\Big)\, G_\mu(z_1,z_2;r)\Big]. \tag{70}$$

This differential equation can be solved since we know $w(z,g)$ (for details see [10, 9]). However, we are interested in the two-point function. It is obtained from the two-loop function be closing the exit loop with a "cap" (i.e. the full disk amplitude $w(l,\mu)$) and shrinking the entrance loop to a point. The corresponding equation is

$$\begin{aligned} G_\mu(r) \;&=\; \sum_{l_2} G_\mu(l_1=1,l_2;r)\; l_2 w(l_2,g) \\ &=\; \oint \frac{\mathrm{d}z'}{2\pi i z'}\Big[z^2 G_\mu\Big(z,\frac{1}{z'};r\Big)\Big]\;\;\Big[\frac{\partial}{\partial z'}[z' w(z',g)]\Big]\Big|_{z=\infty}. \end{aligned} \tag{71}$$

Since $w(z,g)$ and $G(z_1,z_2;r)$ are known we can find $G_\mu(r)$, see[11] for details. For $\mu \to \mu_c$, i.e. in the continuum limit, we obtain:

$$G_\mu(r) \sim (\mu-\mu_c)^{3/4}\, \frac{\cosh\Big[r\sqrt[4]{\mu-\mu_c}\Big]}{\sinh^3\Big[r\sqrt[4]{\mu-\mu_c}\Big]}. \tag{72}$$

If we introduce the following *continuum geodesic distance* $R = r\sqrt{\varepsilon}$, it follows that we can write:

$$G_\mu(r) \sim \varepsilon^{3/2} G(R;\Lambda), \qquad G(R;\Lambda) = \Lambda^{3/4}\frac{\cosh\Big[R\sqrt[4]{\Lambda}\Big]}{\sinh^3\Big[R\sqrt[4]{\Lambda}\Big]}. \tag{73}$$

The factor $\varepsilon^{3/2}$ is again a wave-function renormalization which connects the dimensionless, regularized $G_\mu(r)$ and the continuum two-point function $G(R;\Lambda)$.

We can compare the behavior of $G_\mu(r)$ (or $G(R;\Lambda)$) with that of the random walk two-point function. All conclusions and interpretations remain valid here, except that we only work with intrinsic

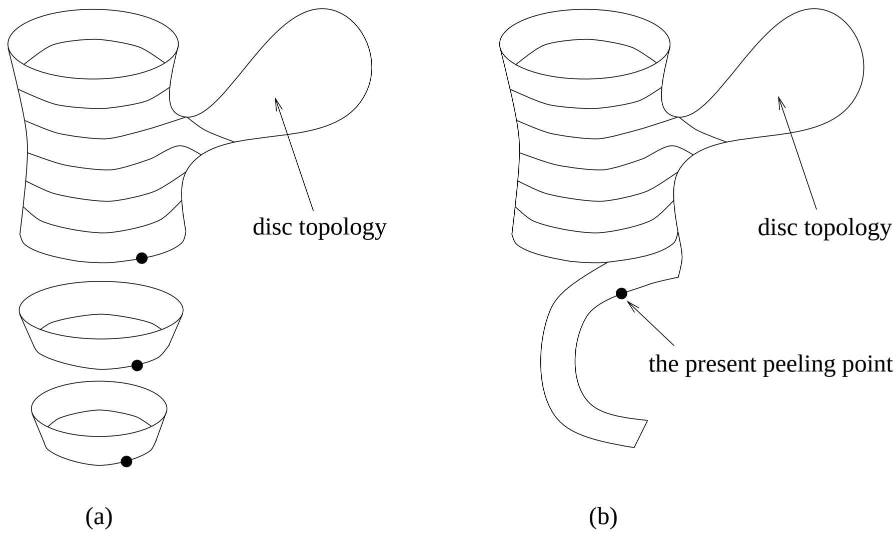

Fig. 6: Decomposition of a surface by (a) slicing and (b) peeling.

geometric objects. First note that $G_\mu(r)$ falls off exponentially for large r (see (19) for the random walk). As for the random walk it follows from general subadditive properties of $G_\mu(r)$. In addition the associated mass satisfies (20) since $m(\mu) \to 0$ for $\mu \to \mu_c$ as $(\mu - \mu_c)^\nu$ with $\nu = 1/4$. The behavior of $G_\mu(r)$ for $r \ll 1/m(\mu)$ is purely power-like corresponding to $\eta = 4$ in (22), and finally

$$\chi(\mu) = \int \mathrm{d}r \; G_\mu(r) \sim (\mu - \mu_c)^{1/2} + \text{less singular terms}, \tag{74}$$

i.e. $\gamma = -1/2$ according to definition (23). Needless to say, Fisher's scaling relation (24) is satisfied and the exponents for two-dimensional quantum gravity:

$$\nu = \frac{1}{4}, \quad \eta = 4, \quad \gamma = -\frac{1}{2}, \tag{75}$$

should be compared the the values for the random walk (see (25)). In particular it follows that *the intrinsic fractal dimension, d_H, of two-dimensional quantum space-time is*

$$d_H = \frac{1}{\nu} = 4. \tag{76}$$

This d_H is a "globally defined" Hausdorff dimension in the sense discussed below (43) as is clear from (72) or (73). We can determine the "local" d_H, defined by eq. (42), by performing the inverse Laplace transformation of $G(R;\Lambda)$ to obtain $G(R;V)$. The average volume $S_V(R)$ of a spherical shell of geodesic radius R in the ensemble of universes with space-time volume V can then calculated from (41). One obtains

$$S_V(R) = R^3 F(R/V^{\frac{1}{4}}), \quad F(0) > 0, \tag{77}$$

where $F(x)$ can be expressed in terms of certain generalized hyper-geometric functions [12]. Eq. (77) shows that also the "local" $d_H = 4$.

4 DISCUSSION

It has been shown how it is possible to calculate the functional integral over two-dimensional geometries, in close analogy to the functional integral over random paths. One of the most fundamental results from the latter theory is that the generic random path between two points in R^d, separated a geodesic distance

R, is *not* proportional to R but to R^2. This famous result has a direct translation to the theory of random two-dimensional geometries: the generic volume of a closed universe of radius R is *not* proportional to R^2 but to R^4.

It is presently an open question how to generalize these results to higher dimensional geometries. In particular, our space-time world seems to be four-dimensional. What is the genuine fractal dimension in the class of all four-dimensional geometries of fixed topology ? Numerical simulations seem to indicate that *the typical four-dimensional spherical geometry has infinite intrinsic Hausdorff dimension*.

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