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Matrix String Partition Functions

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We evaluate quasi-classically the Ramond partition function of Euclidean $D = 10$ $U(N)$ super-Yang–Mills theory reduced to a two-dimensional torus. The result can be interpreted in terms of free strings wrapping the space-time torus, as expected from the point of view of Matrix string theory. We demonstrate that, when extrapolated to the ultraviolet limit (small area of the torus), the quasi-classical expressions reproduce exactly the recently obtained expression for the partition function of the completely reduced SYM theory, including the overall numerical factor. This is an evidence that our quasi-classical calculation might be exact.

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1. Introduction

The interpretation [1] of the dimensional reductions of ten-dimensional supersymmetric Yang–Mills (SYM) theory as effective theories for the dynamics of p-dimensional extended objects (Dp-branes) initiated a new wave of interest in these theories. It culminated in the BFSS conjecture [2] that a system of interacting D0-branes, described by the $D = 10$ SYM theory reduced to one dimension, provides, in the large N limit, a constructive definition of M-theory, the hypothetical theory encompassing all known string theories and eleven-dimensional supergravity. It also led to Matrix string theory [3,4,5], which describes non-perturbatively type IIA string theory by $D = 10$ SYM theory reduced to two dimensions. Finally, an interpretation of the completely reduced SYM theory as type IIB string theory has been advanced in [6]. All three reduced SYM theories are closely related and, in the large N limits each one contains, in a certain sense, the other two. We will refer to the SYM theory reduced to $2, 1$ and 0 dimensions as the DVV, BFSS and IKKT model, correspondingly.

The most basic information about these theories, namely concerning their vacuum excitations, can be obtained by studying their partition functions.[1] The only partition function computed at present is that of the completely reduced theory, which we will denote by $\mathscr{Z}_{\mathrm{IKKT}}$. It was studied by several groups [7,8,9,10] in order to prove the existence of bound states in the BFSS model [7,8,11]. It was conjectured by Green and Gutperle [11] that

$$\mathscr{Z}_{\mathrm{IKKT}}(g) = g^{-\frac{7}{2}(N^2-1)} \mathcal{F}_N \sum_{m|N} \frac{1}{m^2}, \tag{1}$$

where the numerical factor $\mathcal{F}_N$ was computed later by Krauth, Nicolai and Staudacher [9]. Recently, this conjecture was rigorously proved by Moore, Nekrasov and Shatashvili [10].

In this paper we present the quasi-classical calculation of the partition function of the Euclidean matrix string theory (the DVV model) compactified on a rectangular torus

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[1] By partition function of a supersymmetric matrix model we understand the volume form in the sector with a minimal number of fermionic zero-modes.

$\mathcal{T}^2$ with periods R and T, with Ramond-Ramond boundary conditions.[2] Replacing the fermionic and bosonic potentials by constraints, we obtain

$$\mathcal{Z}_{\mathrm{DVV}} = \sum_{m|N} \frac{1}{m} \sum_{p\in\mathbb{Z}} e^{-\frac{RT}{2}Np^2}. \tag{2}$$

Further we argue that this expression which, by its definition, is valid only in the limit of large area RT, is actually exact. Our argument is based on the quasi-classical calculation of the DVV partition function in which the constant mode $A_\sigma^{(0)}$ of the $U(1)$-component of the gauge field subtracted by inserting a delta-function, and which we denote by $\langle \delta(A_\sigma^{(0)}/\sqrt{2\pi})\rangle_{\mathrm{DVV}}$. In the limit $RT \to 0$ the modified partition function coincides with the partition function (1) of the completely reduced theory

$$\left\langle \delta\left(A^{(0)}/\sqrt{2\pi g}\right)\right\rangle_{\mathrm{DVV}} \xrightarrow[R,T\to 0]{} \frac{(RT)^{-\frac{7}{2}(N^2-1)}}{T\ \mathcal{F}_N} \mathcal{Z}_{\mathrm{IKKT}}\left(1/RT\right). \tag{3}$$

Comparing our quasi-classical result

$$\left\langle \delta\left(A_\sigma^{(0)}/\sqrt{2\pi}\right)\right\rangle_{\mathrm{DVV}} = \frac{1}{T}\sum_{m|N}\frac{1}{m^2}\sum_{E\in\mathbb{Z}} e^{-\frac{1}{2}\frac{E^2}{RTN}}. \tag{4}$$

extrapolated to the $RT \to 0$ limit with the exact expression (1) we find perfect agreement, including the numerical factor computed in [9,12]. It is therefore very plausible that our quasi-classical results are valid everywhere, *i.e.*, that these models possess the property of having *exact quasi-classics*.

The excitations that contribute to the DVV partition function can be interpreted as free type IIA Green-Schwarz strings wrapping the torus, with additional abelian gauge degrees of freedom on the world-sheet. The sum over all possible wrappings reproduces, in the large N limit, the integration over the string moduli.

The paper is organized as follows. In Section 2, we define the DVV partition function and make a quantitative description of the dimensional reductions that give the BFSS and IKKT partition functions. In Sections 3 and 4, we present the derivation of (2) and (4). Section 5 contains discussions of our results.

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[2] The gauge coupling constant can be absorbed in the area of the torus and therefore is considered to be equal to 1.

2. The partition functions of the DVV, BFSS and IKKT matrix models

2.1. The functional integral of Matrix string theory (the DVV model) compactified on a torus

The Euclidean DVV model is ten-dimensional $U(N)$ SYM theory dimensionally reduced to a two-dimensional cylinder [5]. The field content of the theory includes the two-dimensional $U(N)$ gauge field $\{A_a\}_{a=1}^2$, the bosonic Hermitian Higgs fields $\{X^I\}_{I=1}^8$, and the Majorana–Weyl fermions $\{\Psi_\alpha\}_{\alpha=1}^{16}$. In order to study the partition function, we compactify the theory on a rectangular[3] two-torus $\mathcal{T}^2$ with periods R and T,

$$\mathcal{T}^2 = \{\vec{\sigma} = (\sigma, \tau) \mid \sigma \in [0, R], \tau \in [0, T]\}.$$

The DVV partition function is defined by the functional integral

$$\mathcal{Z}_{\text{DVV}} = \int \frac{\mathcal{D}A_a}{\text{Vol}(\mathcal{G})} \ \mathcal{D}X\mathcal{D}\Psi \ \prod_{I=1}^{8} \delta\left(\frac{X_I^{(0)}}{\sqrt{2\pi}}\right) \prod_{\alpha=1}^{16} \Psi_\alpha^{(0)} \ \exp\left(-\mathcal{S}_{\text{DVV}}[A, X, \Psi]\right) \quad (5)$$

where the action is

$$\begin{aligned}
\mathcal{S}_{\text{DVV}} = \int_{\mathcal{T}^2} d^2\sigma \text{Tr}\Big[&\frac{F_{ab}^2}{4} + \frac{1}{2}[D_a, X_I]^2 + \frac{i}{2}\Psi^T[\slashed{D}, \Psi] \\
&+ \frac{1}{2}\Psi^T\Gamma_I[X_I, \Psi] - \frac{1}{4}\sum_{I,J}[X_I, X_J]^2\Big],
\end{aligned} \quad (6)$$

$D_a = \partial_a - iA_a$ is the covariant derivative, and a minimal set of pairs of fermionic and bosonic zero-modes:

$$\Psi_\alpha^{(0)} = \frac{\text{Tr}\int_{\mathcal{T}^2} d^2\sigma\,\Psi_\alpha}{\sqrt{NRT}}, \quad X_I^{(0)} = \frac{\text{Tr}\int_{\mathcal{T}^2} d^2\sigma\,X_I}{\sqrt{NRT}}. \quad (7)$$

is subtracted from the integration measure of the matter fields. The integration measure over the gauge fields is divided as usual by the volume $\text{Vol}(\mathcal{G})$ of the gauge group.

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[3] The matrix string action assumes that the metric on the two-dimensional space-time is $g_{ab} = \delta_{ab}$.

2.2. Relation to the partition function of the BFSS matrix model compactified on a circle

After shrinking one of the periods of the torus, the theory degenerates to a SYM theory reduced to one dimension. In the limit $R \to 0$, the component A_σ of the gauge field enters in the action in the same way as the eight Higgs fields, and the $O(8)$ symmetry of the action is enhanced to $O(9)$. The functional integral (5) describes, in this limit, the partition function of a one-dimensional reduction of the SYM theory, with A_σ playing the role of X_9. However, this is not yet the partition function of the BFSS model because the *integration measure* of the field A_σ is not identical to that of the eight remaining Higgs fields. In order to obtain the same integration measure, we have to introduce in the original functional integral (5) a delta-function of the constant mode

$$A_\sigma^{(0)} = \frac{\text{Tr} \int_{\mathcal{T}^2} d^2\sigma A_\sigma}{\sqrt{NRT}} \tag{8}$$

σ-component of the gauge field. Let us denote by $\langle \delta(A_\sigma^{(0)}/\sqrt{2\pi}) \rangle_{\text{DVV}}$ the DVV functional integral (5) with the inserted delta-function. In the limit $R \to 0$ it indeed reduces to the partition function $\mathcal{Z}_{\text{BFSS}}$ of the BFSS model at finite temperature T, defined by the action

$$\mathcal{S}_{\text{BFSS}} = R \int_0^T d\tau \, \text{Tr}\Big(\frac{1}{2} [D_\tau, X_I]^2 + \frac{i}{2} \, \Psi^T [D_\tau, \Psi] - \frac{1}{4} \sum_{I,J} [X_I, X_J]^2 + \frac{1}{2} \, \Psi^T \Gamma^I [X_I, \Psi] \Big), \tag{9}$$

$$\Big\langle \delta \Big(A_\sigma^{(0)}/\sqrt{2\pi} \Big) \Big\rangle_{\text{DVV}} \quad \underset{R \to 0}{\longrightarrow} \quad \mathcal{Z}_{\text{BFSS}}. \tag{10}$$

2.3. The IKKT model as the high-temperature limit of the BFSS model

The IKKT matrix integral

$$\mathcal{Z}_{\text{IKKT}}(g) = \prod_{\mu=1}^{10} [dX^\mu] \prod_{\alpha=1}^{16} [d\Psi_\alpha] \, \exp\Big(-\frac{1}{g} \mathcal{S}_{\text{IKKT}}[X, \Psi] \Big) \tag{11}$$

with the action

$$\mathcal{S}_{\text{IKKT}} = -\frac{1}{4} \sum_{\mu,\nu} \text{Tr}\, [X_\mu, X_\nu]^2 + \frac{1}{2} \sum_\mu \text{Tr}\, \Psi^T [\Gamma^\mu X_\mu, \Psi]. \tag{12}$$

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depends on a single parameter, the coupling constant $g = g_{\rm YM}^2$. In refs. [9,10] the integral was understood as an integral over traceless matrices. Here we use the same normalizations as in [9] but, in order to facilitate the comparison with the previous integrals, we define the integration measures by

$$[dX^\mu] \;=\; dX^\mu \, \delta\Big(\frac{{\rm Tr}X^\mu}{\sqrt{2\pi N}}\Big), \quad [d\Psi_\alpha] \;=\; d\Psi_\alpha \, \frac{{\rm Tr}\Psi_\alpha}{\sqrt{N}}, \tag{13}$$

where dX and $d\Psi$ are the flat measures with normalization

$$\int dX e^{-\frac12 {\rm Tr}\, X^2} = 1, \quad \int d\Psi_\alpha d\Psi_\beta e^{-{\rm Tr}\, \Psi_\alpha^T \Psi_\beta} = 1. \tag{14}$$

The partition function of the IKKT model, with the above normalization of the measure, is equal to [9,10][4]

$$\mathcal{Z}_{\rm IKKT}(g) \;=\; g^{-\frac{7}{2}(N^2-1)} \, \mathcal{F}_N \, \sum_{m|N} \frac{1}{m^2} \tag{15}$$

where $\mathcal{F}_N$ is the "group factor" [9]

$$\mathcal{F}_N = \frac{\sqrt{N}\, N!}{(\sqrt{\pi})^{N^2-1}} \, \frac{1}{2\pi N} \prod_{k=1}^{N} \frac{(2\pi)^k}{k!}. \tag{16}$$

The $T \to 0$ limit of the BFSS model has been analysed by Sethi and Stern [8,11] in connection with the computation of the Witten index. The BFSS action (9) reduces, in the limit $T \to 0$ and after identifying the constant mode of the gauge field with the Higgs field X_{10}, to the $O(10)$-symmetric IKKT action (12) multiplied by the area RT of the torus. The limit for the measure is less trivial. Let us first consider the measure of the gauge field.

The one-dimensional gauge field A corresponds, by the exponential map, to a generic element of the local gauge group $U(\tau) = \hat{T} e^{i \int_0^\tau A(t)dt}$. It is therefore convenient to

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[4] The overall power of g can be determined by dimensional arguments. The integration over the fermions gives a pfaffian, which is a homogeneous polynomial in X/g of degree $8(N^2 - 1)$. The rescaling $X \to g^{1/4}X$ makes the action g-independent and produces a factor $g^{(N^2-1)(8(1-1/4)-10/4)} = g^{7/2(N^2-1)}$.

parametrize the constant mode A of the gauge field in terms of the group element $U = e^{iTA}$ and integrate over the Haar measure on $SU(N)$ (normalized as $\int_{SU(N)} dU = 1$). In the vicinity of any of the N central elements of $SU(N)$ the group element can be parametrized by $U = e^{2\pi ik/N} e^{iTA}$ and the Haar measure becomes

$$\frac{dU}{\text{Vol}[SU(N)]} \;\to\; \frac{T^{N^2-1}}{N\mathcal{F}_N}[dA] \tag{17}$$

where $\mathcal{F}_N$ is given by (16) and the measure $[dA]$ is identical to the measure $[dX]$ defined in (13). (The details of the calculation can be found in [12].) As was explained by Sethi and Stern [8], in the limit $T \to 0$ the integral over the gauge field is saturated by the vicinity of the N central elements of $SU(N)$ and therefore by eq. (17)

$$\mathcal{D}A \;\xrightarrow[T\to0]{}\; \frac{T^{N^2-1}}{\mathcal{F}_N}[dX_{10}]. \tag{18}$$

For the measures of the matter fields we find, from the kinetic part of the action (9) and the normalization (14),

$$\begin{aligned}
\mathcal{D}X &\to \left(\frac{T}{R}\right)^{-\frac{9}{2}N^2} \prod_{I=1}^{9} dX_I \\
\mathcal{D}\Psi &\to R^{-\frac{16}{2}N^2} \prod_{\alpha} d\Psi_\alpha.
\end{aligned} \tag{19}$$

Taking into account the rescaling of the zero-modes

$$\begin{aligned}
\prod_I \delta\left(\frac{X_I^{(0)}}{\sqrt{2\pi}}\right) &\to (RT)^{-\frac{9}{2}} \prod_I \delta\left(\frac{\text{Tr}X_I}{\sqrt{2\pi N}}\right) \\
\prod_\alpha \Psi_\alpha^{(0)} &\to (RT)^{\frac{16}{2}} \prod_\alpha \delta\left(\frac{\text{Tr}\Psi_\alpha}{\sqrt{N}}\right)
\end{aligned} \tag{20}$$

and combining all factors of T and R, we finally obtain

$$\begin{aligned}
\left\langle \delta\left(A^{(0)}/\sqrt{2\pi g}\right) \right\rangle_{\text{DVV}} \;\xrightarrow[R,T\to0]{}\;& \frac{(RT)^{-\frac{7}{2}(N^2-1)}}{T\,\mathcal{F}_N} \, \mathcal{Z}_{\text{IKKT}}\,(1/RT) \\
=\;& \frac{1}{T}\sum_{m|N} \frac{1}{m^2}.
\end{aligned} \tag{21}$$

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3. Quasi-classical calculation of the partition function of the DVV model

As argued by Dijkgraaf, Verlinde and Verlinde [5], in the infrared limit $RT \to \infty$ the non-diagonal components of the gauge and matter fields become infinitely massive, and the bosonic and fermionic potentials turn into constraints. Under the constraint that all matrices $\Phi = \{X_I, \Psi_\alpha, iD_a = i\partial_a + A_a\}$ are simultaneously diagonalizable, for each field configuration there exists a unitary matrix $V(\sigma, \tau)$ such that

$$\Phi(\sigma, \tau) = V^{-1}(\sigma, \tau)\Phi^D(\sigma, \tau)V(\sigma, \tau),$$

where $\Phi^D = \mathrm{diag}\{\Phi_1, ..., \Phi_N\}$. We have therefore

$$\Phi^D(R, \tau) = \hat{S}^{-1}\Phi^D(0, \tau)\hat{S},$$
$$\Phi^D(\sigma, T) = \hat{T}^{-1}\Phi^D(\sigma, 0)\hat{T} \tag{22}$$

where $\hat{S} = V(0, \tau)V^{-1}(R, \tau)$ and $\hat{T} = V(\sigma, 0)V^{-1}(\sigma, T)$. By construction, $\hat{S}\hat{T} = \hat{T}\hat{S}$. Assuming that all the eigenvalues are distinct, the only unitary transformations relating two diagonal matrices represent permutations of their diagonal elements. Therefore the matrices $\hat{S}$ and $\hat{T}$ act as two commuting permutations $\hat{s} : i \to s_i$ and $\hat{t} : i \to t_i$ of the symmetric group S_N [5]

$$\left[\hat{T}^{-1}\Phi\hat{T}\right]_i = \Phi_{t_i}, \quad \left[\hat{S}^{-1}\Phi\hat{S}\right]_i = \Phi_{s_i} \tag{23}$$

and, in particular, do not depend on the coordinates σ and τ.

Each pair of permutations describes an N-covering of the target-space torus consisting, in general, of several connected components. Each connected component can be interpreted as the world-sheet of a string wrapping several times the torus in both directions. A q-component covering corresponds to a decomposition

$$\hat{s} = \prod_{k=1}^q \hat{s}^{(k)}, \quad \hat{t} = \prod_{k=1}^q \hat{t}^{(k)},$$

[5] After our manuscript was finished, we learned about the paper [13], where a similar interpretation of the partition function of *pure $U(N)$* YM theory on the torus is presented.

where all factors commute with each other and satisfy

$$(\hat{s}^{(k)})^{j_k}(\hat{t}^{(k)})^{m_k} = (\hat{s}^{(k)})^{n_k}(\hat{t}^{(k)})^{l_k} = 1, \quad \sum_{k=1}^{q}(n_k m_k - l_k j_k) = N. \tag{24}$$

The k-th world-sheet torus is the complex plane $\omega = \sigma + i\tau$ factored by the periods $\omega_1 = n_k R + i\,(l_k T)$ and $\omega_2 = j_k R + i\,(m_k T)$; it covers the target torus $N_k = n_k m_k - l_k j_k$ times. In this way the orbifold structure of the target space generates the sum over all twisted boundary conditions, which becomes, in the limit $N \to \infty$ and for the "long" strings only, the integral over all complex structure of the world-sheet tori. Owing to the periodicity condition (22) on the fermions, each connected component has 16 fermionic zero modes. In our problem we are only interested in contributions with exactly 16 fermionic zero-modes; therefore only one-sheet ($q = 1$) coverings will be relevant. A covering with periods $\omega_1 = nR + i(lT)$ and $\omega_2 = jR + i(mT)$ wrapping the target torus $N = nm - jl$ times is defined by two permutations, $\hat{s}$ and $\hat{t}$, satisfying

$$\hat{s}\hat{t} = \hat{t}\hat{s}, \quad \hat{s}^n \hat{t}^l = \hat{s}^j \hat{t}^m = 1.$$

By a mapping class transformation we can reduce this to

$$\hat{s}\hat{t} = \hat{t}\hat{s}, \quad \hat{s}^n = \hat{s}^j \hat{t}^m = 1 \quad (mn = N, j = 0, 1, ..., n-1). \tag{25}$$

The explicit solution of (25) is, up to an internal homomorphism,

$$\hat{s} = \{i \to i + m (\text{mod } N)\}, \quad \hat{t} = \begin{cases} \{i \to i + 1 (\text{mod } m)\} & \text{if } j = 0 \\ \{i \to i - j (\text{mod } N)\} & \text{if } j = 1, ..., n-1. \end{cases} \tag{26}$$

After having identified the distinct topological sectors, we can write the partition function as a sum over all pairs m and n such that $mn = N$ and $j = 0, 1, ..., n-1$. The diagonal matrix variables $\Phi^D(\sigma, \tau)$, with boundary conditions belonging to the equivalence classes $[m, n; j]$, can be considered as scalar variables defined on the torus with periods $\omega_1 = nR$ and $\omega_2 = jR + i(mT)$ in the $\omega = \sigma + i\tau$ plane.

Now let us proceed to the evaluation of the partition function in the infrared limit. In the limit $RT \to \infty$ the measure over the gauge field reduces to the integral over the diagonal components of the gauge field, normalized by the volume of the diagonal gauge group $\mathcal{G}^D$ which is the localized $U(1)^N$. Taking into account the overall factor $1/N!$ because the eigenvalues are determined up to a permutation, we get

$$\int \frac{\mathcal{D}A}{\mathrm{vol}(\mathcal{G})} \mathcal{D}X \mathcal{D}\Psi \to \frac{1}{N!} \sum_{\hat{s}\hat{t}=\hat{t}\hat{s}} \int \frac{\mathcal{D}A^D}{\mathrm{vol}(\mathcal{G}^D)} \mathcal{D}X^D \mathcal{D}\Psi^D \tag{27}$$

where the boundary conditions in each term are twisted according to eq. (23). The number of permutations in each class $[m,n;j]$ is equal to the number of combinations of m and n elements times the number of cyclic permutations of order m and n,

$$\frac{N!}{n!m!}(m-1)!(n-1)! = (N-1)!.$$

Summing the contributions of all equivalence classes we get

$$\mathcal{Z}_{\mathrm{DVV}} = \frac{(N-1)!}{N!} \sum_{mn=N} \sum_{j=0,\ldots,n-1} \mathcal{Z}_{[m,n;j]} \tag{28}$$

where $\mathcal{Z}_{[m,n;j]}$ is the partition function of the Abelian $(N=1)$ DVV model defined on the torus of area NRT with periods $\omega_1 = nR$ and $\omega_2 = jR + i(mT)$. The latter is a product of the partition function of the Abelian gauge field and that associated with the diagonal components of the matter field

$$\mathcal{Z}_{[m,n;j]} = \mathcal{Z}_{[m,n;j]}^{\mathrm{gauge}} \mathcal{Z}_{[m,n;j]}^{\mathrm{matter}}. \tag{29}$$

The contribution of the $U(1)$ gauge field to the partition function is given, in the gauge $A_\tau^D = 0$, to the functional integral with respect to the angular variable

$$\begin{aligned}
\theta(\tau) &= \int_0^R \mathrm{Tr}A_\sigma^D(\tau,\sigma)d\sigma \\
&= \int_0^{nR} A_\sigma(\tau,\sigma)d\sigma.
\end{aligned} \tag{30}$$

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Here A_σ denotes the Abelian gauge field on the "world-sheet" torus that corresponds to the N-component field of A_σ^D on the space-time torus.

The gauge-field partition function is

$$\mathcal{Z}_{[m,n;j]}^{\text{gauge}} = \int_{\theta(mT)=\theta(0)} \mathcal{D}\theta \; e^{-\frac{1}{2nR}\int_0^{mT} d\tau(\partial_\tau\theta)^2} = \sum_{p\in\mathbb{Z}} e^{-\frac{RT}{2}Np^2}. \tag{31}$$

The result of the integration depends only on the area RT and not on the modular parameter of the torus, which reflects the symmetry of the two-dimensional gauge theory with respect to area-preserving diffeomorphisms. Further, with our conventions for the zero modes, the integral of the matter fields is exactly 1 because of supersymmetry [14]

$$\mathcal{Z}_{[m,n;j]}^{\text{matter}} = 1. \tag{32}$$

This gives the result (2) which can be written, after a Poisson resummation, as

$$\mathcal{Z}_{\text{DVV}} = \sum_{m|N} \frac{1}{m} \sqrt{\frac{2\pi}{RTN}} \sum_{E\in\mathbb{Z}} e^{-\frac{1}{2}\frac{(2\pi E)^2}{RTN}}. \tag{33}$$

4. The DVV partition function with subtracted constant mode of the gauge field

The evaluation of the modified DVV (8) can be done in the same way as in the previous section. In the topological sector $[m,n;j]$ the constant mode $A_\sigma^{(0)}$ is expressed through the angular variable (30) as

$$A_\sigma^{(0)} = \frac{\int_0^{mT} d\tau\,\theta(\tau)}{\sqrt{NRT}}. \tag{34}$$

The combinatorics is the same as in the previous section and the only difference is in the expression of the gauge-field partition function in each topological sector. The latter is given by the one-dimensional functional integral with respect to the field $\theta(\tau)$ taking its values in the unit circle

$$\tilde{\mathcal{Z}}_{[m,n;j]} = \int_{\theta(mT)=\theta(0)} \mathcal{D}\theta \; e^{-\frac{1}{2nR}\int_0^{mT} d\tau(\partial_\tau\theta)^2} \, \delta\left(\frac{\int_0^{mT} d\tau\,\theta(\tau)}{\sqrt{2\pi RTN}}\right), \tag{35}$$

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and is evaluated as

$$\tilde{\mathcal{Z}}_{[m,n;j]} = \sqrt{\frac{NRT}{2\pi}} \frac{1}{mT} \sum_{p \in \mathbb{Z}} e^{-\frac{1}{2} RTNp^2} = \frac{1}{mT} \sum_{E \in \mathbb{Z}} e^{-\frac{1}{2} \frac{(2\pi E)^2}{RTN}}. \tag{36}$$

Summing over all topological sectors we find, instead of (33),

$$\left\langle \delta \left(A_\sigma^{(0)} / \sqrt{2\pi} \right) \right\rangle_{\text{DVV}} = \frac{1}{T} \sum_{m|N} \frac{1}{m^2} \sum_{E \in \mathbb{Z}} e^{-\frac{1}{2} \frac{E^2}{RTN}}. \tag{37}$$

Comparing this expression, whose derivation implies the infrared limit $RT \to \infty$, with Eq. (21), we see that it is equally true in the ultraviolet limit $RT \to 0$.

5. Discussion

We have computed the quasi-classical partition function of the Euclidean Matrix string theory compactified on a two-dimensional torus $\mathcal{T}^2$, with doubly periodic boundary conditions and a minimal set of zero-modes removed. We have shown that the relevant degrees of freedom are described, in the limit of large area of the torus, by an Abelian supersymmetric sigma-model accompanied by a $U(1)$ gauge theory defined on the orbifold space $S^N \mathcal{T}^2 = (\mathcal{T}^2)^{\otimes N}/S_N$, where S_N is the symmetric group of N elements. The sigma-model and the gauge field are coupled through the boundary conditions, which are described by a pair of commuting permutations of S_N.

Each such configuration can be interpreted as a set of non-interacting strings in a light-cone gauge, with additional gauge degrees of freedom on the world-sheet. The world-sheet of each of such string defines a multiple covering of the space-time torus $\mathcal{T}^2$, characterized by a modular parameter ω_2/ω_1, with $\omega_{1,2}$ sweeping the lattice $R\mathbb{Z}+iT\mathbb{Z}$. In the large-N limit, the sum over coverings converges to an integral with a correct modular-invariant measure, under the condition that we are far from the boundary of the moduli space. This means that the partition function of a "long" string in the DVV model is indeed that of the Green-Schwarz string in light-cone gauge, with additional Abelian gauge degree of freedom on the world-sheet. To complete the proof of the (perturbative) matrix-string

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correspondence, one has to consider the matrix-field configurations that define branched coverings of the space-time torus which, according to the original suggestion in [5], should describe strings in interaction. Some steps in this direction was made by the authors [15,16,17].

On the other hand, near the boundary of the moduli space, that is when one of the periods is kept finite while $N \rightarrow \infty$, the sum keeps its discrete character and the corresponding excitations ("short" strings) describe particles. Thus the distinction between particle and string excitations in the DVV model can be made only after performing the large-N limit. The latter is not yet completely understood, in spite of the recent progress made in [18]. With the convention that a minimal set of zero-modes is deleted from the functional measure, which is the case considered in the present paper, the allowed field configurations of the orbifold theory are described by a single string whose world-sheet wraps N times the space-time torus $\mathcal{T}^2$. Our computation gives an intuitive understanding of the sum over the divisors of N in the expression for the partition function of the IKKT model (1). It is quite analogous to the argument presented by Green and Gutperle in which this last partition function was compared with the partition function of the matrix quantum mechanics at infinite temperature. Their calculation was based on the assumption of the existence of bound states of D0-branes, which lead to the sum over the divisors of N. In our case the sum over the divisors comes from the sum over the conjugacy classes of permutations defining the different topological sectors of the S_N-orbifold theory. This is a trivial example of the Hecke operator, defined by its action on a modular-invariant form $\mathcal{A}$ (actually a constant in our case)

$$\mathcal{H}_N[\mathcal{A}](\tau) = \frac{1}{N} \sum_{\substack{N=mn \\ 0 \le j < n}} \mathcal{A} \left(\frac{m\tau + j}{n} \right) \, , \tag{38}$$

which takes care of the inequivalent N-fold wrappings. (For more general discussion see [19,20].) Comparing our calculation with the argument of Green and Gutperle, we see that

the bound states of D0-branes can indeed be interpreted as strings winding several times around the spatial (σ-)dimension of the two-torus.

The partition functions of the DVV and the IKKT models are related to the half-BPS saturated amplitudes, such as the $t_8 t_8 \mathcal{R}^4$ term in type IIB theory [21,22]. Our computation confirms the rules for counting wrapped D-branes used in [21,22,23,24]. Moreover, we have shown that the high-temperature limit of the DVV model reproduces the D-instantons contributions predicted in [21], which confirms the dual interpretation of the DVV model as describing wrapped D1-branes.

We consider as the principal result of this paper not the computation of the IR partition function itself, which is rather trivial, but its comparison with the known result for the partition function of the completely reduced gauge theory (the IKKT model). For this purpose we have established the exact relations between the partition functions of the SYM theories reduced to 2, 1 and 0 dimensions, which we presented in Section 2. Then we were able to check that when extrapolated to the small-area limit, the quasi-classical result reproduces exactly the known expression of the IKKT partition function. Therefore we conjecture that expressions (33) and (37) are actually *exact*, which means, that the theory has the property of having exact quasi-classics. We expect that this can be proved by extending to the two-dimensional case the calculation of Moore, Nekrasov and Shatashvili [10] based on Witten's description of the two-dimensional gauge theory as cohomological field theory [25].

Finally, let us remark that it is possible to give a very simple explanation of the pre-exponential factors in expression (33) if the partition function of the DVV model is considered as a certain limit of the partition function of a supersymmetric Schild string defined on the orbifold $S^N \mathcal{T}^2$. We intend to report on this subject in the near future.

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References

[1] E. Witten, Nucl. Phys. **B460** (1996) 335, hep-th/9510135.

[2] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. **D55** (1997) 5112, hep-th/9610043.

[3] L. Motl, hep-th/9701025.

[4] T. Banks and N. Seiberg, Nucl. Phys. **B497** (1997) 41, hep-th/9702187.

[5] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B500** (1997) 43, hep-th/9703030.

[6] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. **B498** (1997) 467, hep-th/9612115.

[7] P. Yi, Nucl. Phys. **B505** (1997) 307, hep-th/9704098

[8] S. Sethi and M. Stern, Commun. Math. Phys. **194** (1998) 675, hep-th/9705046.

[9] V. Krauth, H. Nicolai and M. Staudacher, Phys. Lett. **431B** (1998) 31, hep-th/9803117.

[10] G. Moore, N. Nekrasov and S. Shatashvili, hep-th/9803265.

[11] M.B. Green and M. Gutperle, JHEP **01** (1998) 005, hep-th/9711107.

[12] V. Krauth and M. Staudacher, hep-th/9804199.

[13] M. Billó, M. Caselle, A. D'Adda and P. Provero, hep-th/9809095

[14] E. D'Hoker and D.H. Phong, Rev. Mod. Phys. **60** (1988) 917.

[15] S. Giddings, F. Hacqueboard and H. Verlinde, hep-th/9804121.

[16] T. Wynter, Phys. Lett. **415B** (1997) 349 , hep-th/9709029.

[17] G. Bonelli, L. Bonora and F. Nesti, hep-th/9807232.

[18] T. Wynter, hep-th/9806173.

[19] R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, Commun. Math. Phys. **185** (1997) 197, hep-th/9608096.

[20] C. Bachas, Nucl.Phys.Proc.Suppl. **68** (1998) 348, hep-th/9710102.

[21] M.B. Green and M. Gutperle, Nucl. Phys. **B498** (1997) 195, hep-th/9701093.

[22] M.B. Green and P. Vanhove, Phys. Lett. **408B** (1997) 122, hep-th/9704145; C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers and P. Vanhove, Nucl. Phys. **B509** (1998) 33, hep-th/9707126; B. Pioline and E. Kiritsis, Nucl. Phys. **B508** (1997) 509, hep-th/9707018; Phys. Lett. **418B** (1998) 61, hep-th/9710078; E. Kiritsis and N.A. Obers, JHEP **10** (1997) 004, hep-th/9709058; I. Antoniadis, B. Pioline and T.R. Taylor, Nucl. Phys. **B512** (1998) 61, hep-th/9707222.

[23] K. Becker, M. Becker and A. Strominger, Nucl. Phys. **B456** (1995) 130, hep-th/9507158.

[24] H. Ooguri and C. Vafa, Phys. Rev. Lett. **77** (1996) 3296, hep-th/9608079.

[25] E. Witten, Commun. Math. Phys. **141** (1991) 153.