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Five-leg photon–neutrino interactions

A. Abada, J. Matias and R. Pittau
Theory Division, CERN, CH-1211 Geneva 23, Switzerland

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A. Abada, J. Matias, R. Pittau

Theory Division, CERN, CH-1211 Geneva 23, Switzerland
E-mail: abada@mail.cern.ch, matias@mail.cern.ch, pittau@mail.cern.ch

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1 Introduction

Processes involving photons and neutrinos are potentially of interest in astrophysics and cosmology. However, it was realized some time ago in [1] that 4-leg processes ($\gamma\nu \to \gamma\nu$, $\gamma\gamma \to \nu\bar{\nu}$ and $\nu\bar{\nu} \to \gamma\gamma$) are too strongly suppressed to be of relevance. The reason for this suppression is the prohibition of a two-photon coupling to a $J = 1$ state. This is because of Yang's theorem [2]. On the other hand, this theorem does not apply to 5-leg processes involving two neutrinos and three photons, such as

$$\begin{aligned} &\gamma\nu \to \gamma\gamma\nu \\ &\gamma\gamma \to \gamma\nu\bar{\nu} \\ &\nu\bar{\nu} \to \gamma\gamma\gamma\,, \end{aligned} \tag{1}$$

The extra α in the cross section is compensated by an interchange of the $1/M_W$ suppression by an $1/m_e$ enhancement [3,4]. The relative enhancement of the 5-leg process versus the 4-leg one happens to be of several orders of magnitude, depending on the energy.

In [3], Dicus and Repko derived an effective Lagrangian for the above five-leg photon–neutrino interactions by using the Euler–Heisenberg Lagrangian that describes the photon–photon scattering [5]. Moreover, in the literature there already existed a computation by Shabalin and Hieu [6] of the second process in (1) whose result disagrees with the one given in [3].

To settle this question, in a recent work [4] we computed the first and the second processes (1), in the framework of the effective theory, confirming the results reported in ref. [3]. In section 2, we justify the feasibility of this approach [4] and give another derivation of the 5-leg effective vertex starting from the Euler–Heisenberg Lagrangian [5].

The effective approach gives reliable results for energies below the threshold for e^+e^- pair production, while its extrapolation to energies above 1 MeV, interesting to study, for example, supernova dynamics, is suspect. Therefore, an exact calculation of the processes (1) is important to see their role in astrophysics and the range of validity of the effective theory. This calculation [7] is summarized in section 3, with the main results.

Very recently, parallel work in the same direction has been carried out by Dicus, Kao and Repko [8], so that we had a chance to compare our numerical results, finding complete agreement between the two independent calculations.

We end up with a discussion on some of the implications of the exact results of these 5-leg processes in astrophysics and cosmology, and we give our conclusions.

2 Effective theory

The starting point is the leading term of the Euler–Heisenberg Lagrangian [5], which describes the photon–photon scattering of Fig. 1a:

$$\mathcal{L}_{\rm E-H} = \frac{\alpha^2}{180 m_e^4}\left[5\left(F_{\mu\nu}F^{\mu\nu}\right)^2 - 14 F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}\right] \tag{2}$$

where $F_{\mu\nu}$ is the photon field-strength tensor and α the QED coupling constant.

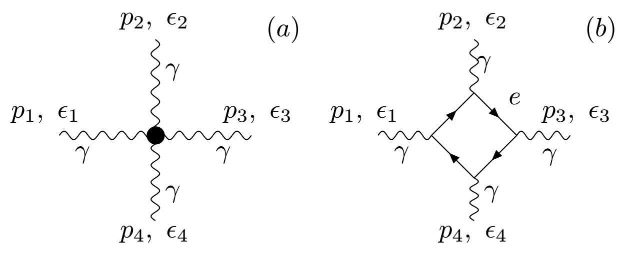

Figure. 1: Four-photon interaction (a) in the effective Lagrangian and (b) in the SM.

In what follows we will show the relation between the effective Lagrangian of eq. (2) and the one describing processes (1), which is given by

$$\mathcal{L}_{\rm eff} = \frac{C}{180}\Big[5\left(\tilde{F}_{\mu\nu}F^{\mu\nu}\right)\left(F_{\lambda\rho}F^{\lambda\rho}\right) - 14\tilde{F}_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}\Big] \tag{3}$$

where

$$C = \frac{g^5 s_W^3 (1+v_e)}{32\pi^2 m_e^4 M_W^2} = \frac{2 G_F \alpha^{3/2}(1+v_e)}{\sqrt{2\pi} m_e^4}\,, \qquad (4)$$

and $\tilde{F}_{\mu\nu}$ stands for the field strength of the new "gauge field" $\tilde{A}_\nu \equiv \bar{\psi}\gamma_\nu(1-\gamma_5)\psi = 2\Gamma_\nu$, with Γ_ν the neutrino current

$$\Gamma_\mu = \bar{v}_+(5)\gamma_\mu w_- u_-(4)\,.$$

We use the notation $w^\pm = (1\pm\gamma_5)/2$ and $v_e = -1/2 + 2s_W^2$, where s_W is the sine of the Weinberg angle.

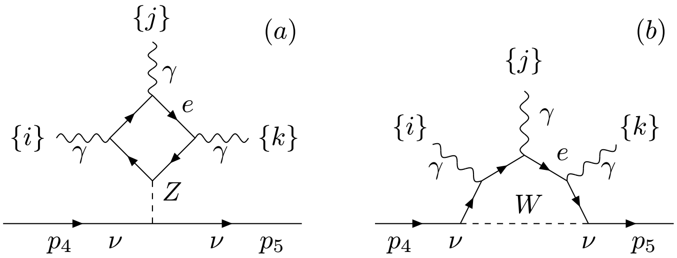

Figure 2. SM leading diagrams contributing to five-leg photon–neutrino processes.

The keypoint of the proof is to demonstrate that the SM amplitudes of diagrams (a) and (b) of Fig. 2, which we called A_{ijk} and B_{ijk}, respectively, can be rewritten in terms of diagram (b) of Fig. 1.

The total amplitude $M(\lambda,\rho,\sigma)$ of the 5-leg process reads

$$\begin{aligned} M(\lambda,\rho,\sigma) = [&(A_{123}+A_{321}) + (B_{123}+B_{321}) \\ &+ (A_{132}+A_{231}) + (B_{132}+B_{231}) + \\ &+ (A_{213}+A_{312}) + (B_{213}+B_{312})]\,, \end{aligned} \qquad (5)$$

where we denote the photon polarization by λ, ρ and σ (see [4] for the explicit form of A_{ijk} and B_{ijk}).

The reason why the terms in eq. (5) are collected in pairs is because, when adding the two terms in each pair, and using the reversing invariance of the γ-matrix traces, changing q to $-q$ and m_e to $-m_e$, the γ_5 piece of the Zee vertex cancels [4]. For instance, the couple

$$A_{123}+A_{321} = \epsilon^\alpha(\vec{P}_1,\lambda)\epsilon^\beta(\vec{P}_2,\rho)\epsilon^\gamma(\vec{P}_3,\sigma)(A_{123}^{\alpha\beta\gamma}+A_{321}^{\alpha\beta\gamma}),$$

where

$$A_{123}^{\alpha\beta\gamma}+A_{321}^{\alpha\beta\gamma} = -\frac{2}{(2\pi)^4}(g s_W)^3\left(\frac{g}{2c_W}\right)^2 v_e \Gamma_\mu \frac{1}{\Delta_Z} L_1^{\mu\alpha\beta\gamma}$$

and

$$L_1^{\mu\alpha\beta\gamma} = \int dq^n\, \mathrm{Tr}\left[\gamma^\mu \frac{1}{Q_{23}^-}\gamma^\gamma\frac{1}{Q_2^-}\gamma^\beta\frac{1}{Q_0^-}\gamma^\alpha\frac{1}{Q_1^-}\right],$$

cancels completely its axial part. We are using the notations of Refs. [4,7]:

$$\begin{aligned} &Q_{\pm i}^\mp = \not{Q}_{\pm i} \mp m_e\,, && Q_{\pm(ij)}^\mp = \not{Q}_{\pm(ij)} \mp m_e\,, \\ &Q_{\pm i} = Q_0 \pm p_i\,, && Q_{\pm(ij)} = Q_0 \pm p_i \pm p_j\,, \quad Q_0 = q\,, \\ &D_{\pm i} = Q_{\pm i}^+ \cdot Q_{\pm i}^-\,, && D_{\pm(ij)} = Q_{\pm(ij)}^+ \cdot Q_{\pm(ij)}^-\,. \end{aligned} \qquad (6)$$

and $1/\Delta_Z = 1/\left((p_4+p_5)^2 - M_Z^2\right) \sim -1/M_Z^2$.

The same trick can be applied to the couples of B's. Let us take, for instance,

$$B_{123}^{\alpha\beta\gamma}+B_{321}^{\alpha\beta\gamma} = -\frac{4}{(2\pi)^4}(g s_W)^3\left(\frac{g}{2\sqrt{2}}\right)^2 \Gamma_\mu L_2^{\mu\alpha\beta\gamma} \qquad (7)$$

$$L_2^{\mu\alpha\beta\gamma} = \int dq^n\, \mathrm{Tr}\left[\gamma^\mu \frac{1}{Q_{23}^-}\gamma^\gamma\frac{1}{Q_2^-}\gamma^\beta\frac{1}{Q_0^-}\gamma^\alpha\frac{1}{Q_1^-}\right]\frac{1}{\Delta_W(q)}$$

where $\Delta_W(q) = (q+p_2+p_3+p_5)^2 - M_W^2$.

The second step of the proof is to shrink diagram (b) of Fig. 2 to diagram (b) of Fig. 1 with the photon leg substituted by the neutrino current. This is done by expanding the W propagator inside the loop, using

$$\frac{1}{\Delta_W(q)} = \frac{1}{q^2-M_W^2} - \frac{k^2+2q\cdot k}{(q^2-M_W^2)((q+k)^2-M_W^2)} \qquad (8)$$

where $\Delta_W(q) \equiv (q+k)^2 - M_W^2$ and $k = p_2+p_3+p_5$. Once introduced in eq. (7), the first term in the r.h.s. of eq. (8) gives rise to an L_1 type of integral plus a contribution of next order in $1/M_W^2$. On the other hand, the second term on the r.h.s. of eq. (8) is of order $1/M_W^4$, and can therefore be neglected.

In conclusion, at leading order in $1/M_W^2$, the set of four diagrams (from a total of 12) is always proportional to L_1:

$$A_{123}^{\alpha\beta\gamma}+A_{321}^{\alpha\beta\gamma}+B_{123}^{\alpha\beta\gamma}+B_{321}^{\alpha\beta\gamma} = -\frac{g^5 s_W^3}{2}\frac{(1+v_e)}{\Delta_Z c_W^2}\Gamma_\mu L_1^{\mu\alpha\beta\gamma}$$

We have shown that in the SM the 5-leg processes (1) at leading order correspond up to a global factor to the 4-photon scattering process; both theories should thus be described at this order by the same effective theory, after substitution of one polarization vector by the neutrino current

$$\epsilon_\mu(\overrightarrow{P_4},\lambda_4) \to \frac{1}{2}\Gamma_\mu \qquad (9)$$

and fixing the overall constant C. This constant C (eq. (4)) is fixed unambiguously by considering the following ratios of amplitudes in the large-m_e limit

$$\lim_{\text{large } m_e} \frac{\mathcal{A}_{4\gamma}^{\rm SM}}{\mathcal{A}_P^{\rm SM}} = \frac{\mathcal{A}_{4\gamma}^{\rm eff}}{\mathcal{A}_P^{\rm eff}}\,, \qquad (10)$$

where P stands for our 5-leg processes.

2.1 Cross sections

Using the obtained effective Lagrangian eq. (3) that coincides with the one used in [3] we have evaluated the cross section for the first two processes:

$$\sigma^{\rm eff}(\gamma\nu\to\gamma\gamma\nu) = \frac{262}{127575}\frac{G_F^2 a^2 \alpha^3}{\pi^4}\left(\frac{\omega}{m_e}\right)^8 \omega^2\,.$$

$$\sigma^{\rm eff}(\gamma\gamma \to \gamma\nu\bar{\nu}) = \frac{2144}{637875}\frac{G_F^2 a^2 \alpha^3}{\pi^4}\left(\frac{\omega}{m_e}\right)^8 \omega^2\,. \quad (11)$$

In [4] we have also computed the unpolarized differential cross sections and the polarized cross sections, using as a check for each polarized cross section of the second process the well-known polarized cross sections of the 4-photon scattering process. The results in [4] coincide for the differential and total cross sections with the ones obtained in [3] and are, therefore, in disagreement with the ones obtained by Shabalin et al. in [6].

3 Direct computation of the 5-leg processes

The range of energy in which the processes (1) are relevant is well below the W mass, so that we treated the neutrino–electron coupling as a four-Fermi interaction. We also assumed massless neutrinos. The total amplitude reads

$$M(\lambda,\rho,\sigma) = -2(1+v_e)\,\Gamma'_\mu \sum_{i=1}^{3} I_i^\mu(p_1,p_2,p_3,\lambda,\rho,\sigma) \quad (12)$$

I_2^μ and I_3^μ are obtained from I_1^μ by the replacements $p_2 \leftrightarrow p_3$ and $\epsilon(p_2,\rho) \leftrightarrow \epsilon(p_3,\sigma)$ for I_2^μ, and $p_2 \leftrightarrow p_1$ and $\epsilon(p_2,\rho) \leftrightarrow \epsilon(p_1,\lambda)$ for I_3^μ. The factor Γ'_μ is

$$\Gamma'_\mu = (g s_W)^3 \left(\frac{g}{2c_W}\right)^2 \left(\frac{1}{\Delta_Z}\right) \frac{1}{(2\pi)^4}\, \bar{v}_+(5)\gamma_\mu u_-(4). \quad (13)$$

A detailed description of the method used for the computation of the total amplitude eq. (12) can be found in [7]. The reduction of $M(\lambda,\rho,\sigma)$ to scalar one-loop integrals is performed with the help of the technique described in [9]. The general philosophy of such a method is using the γ algebra in the traces to reconstruct the denominators appearing in the loop integrals, rather than making a more standard tensorial decomposition [10]. The algorithm can be iterated in such a way that only scalar and rank-one functions appear at the end of the reduction, at worst together with higher-rank two-point tensors. In that way, all the results are expressed only in terms of scalar functions with 3 and 4 denominators, rank-1 integrals with 3 and 4 denominators, rank-2 integrals with 3 denominators, and rank-3 functions with 3 denominators. This already provides an important simplification with respect to the standard decomposition, in that the computation of tensors such as

$$T^{\mu\nu;\,\mu\nu\rho;\,\mu\nu\rho\sigma} = \int d^n q\, \frac{q^\mu q^\nu;\; q^\mu q^\nu q^\rho;\; q^\mu q^\nu q^\rho q^\sigma}{D_0\, D_{-1}\, D_2\, D_{(23)}} \quad (14)$$

is completely avoided. A suitable choice of the polarization vectors [11] as explained in [7] is the key ingredient in the case at hand. To obtain compact expressions, we made a large use of the Kahane–Chisholm manipulations over γ matrices [12]. Such identities are strictly four-dimensional, while we are, at the same time, using dimensional regularization. Our solution is splitting, *before any trace manipulation*, the n-dimensional integration momentum appearing in the traces as [9] $q \to q + \tilde{q}$, where q and $\tilde{q}$ are the four-dimensional and ϵ-dimensional components ($\epsilon = n-4$), respectively, so that $q \cdot \tilde{q} = 0$. The γ algebra can then be safely performed in four dimensions, at the price of having additional terms. In fact, the splitting $q \to q + \tilde{q}$ is equivalent to redefining $m_e^2 \to m_e^2 - \tilde{q}^2$ from the beginning. The net effects are then the extra integrals containing powers of $\tilde{q}^2$ in the numerator, whenever m_e^2 is present in the formulae. The computation of such integrals in the limit $\epsilon \to 0$ is straightforward [9]. Finally, a standard Passarino–Veltman decomposition [10] of the simple remaining tensorial structures in terms of scalar loop functions, concludes our calculation. We implemented the outcoming formulae in a Fortran code, performing the phase-space integration by Monte Carlo. Numerical results are reported in the next section.

3.1 Results

Our formulae remain valid also when including all neutrino species. In this case, only the first diagram in Fig. 2a contributes, at leading order in ω/m_e, because the second one is suppressed by powers of $\omega/m_{\mu,\tau}$. Therefore, the inclusion of all neutrinos can be achieved by simply replacing $(1+v_e)$ with $(1+3\,v_e)$ in eq. (12). However, we only considered ν_e in our numerical results. Effective and exact computations are compared in Figs. 3, 4 and 5, for the three processes. Furthermore, Table 1 shows the ratio between exact cross section and $\sigma^{\rm eff}$ for several values of ω/m_e, then exhibiting the range of validity of the effective theory.

From the above figures and numbers it is clear that, for all three processes, the effective theory is valid only when, roughly, $\omega/m_e \leq 2$, as expected. At larger ω, the exact computation predicts a softer energy dependence with respect to the $(\omega/m_e)^{10}$ behaviour given by the effective Lagrangian.

All these results are in full agreement with those reported in [8]. In order that the results be useful, the only [a] option is to fit the curves in Figs. 3, 4 and 5. The results of the fits are

$$\frac{\sigma(\gamma\nu \to \gamma\gamma\nu)}{\sigma^{\rm eff}(\gamma\nu \to \gamma\gamma\nu)} = r^{-2.76046} \times \exp[2.13317 -$$
$$2.12629\log^2(r) + 0.406718\log^3(r) - 0.029852\log^4(r)],$$

[a] A large-m_e expansion has been performed. The first term does not give numerical predictions that are useful to extend the effective theory beyond $\omega = m_e$.

ω/m_e	$\gamma\nu \to \gamma\gamma\nu$	$\gamma\gamma \to \gamma\nu\bar{\nu}$	$\nu\bar{\nu} \to \gamma\gamma\gamma$
0.3	0.969(8)	1.09(1)	1.20(1)
0.4	0.923(6)	1.17(1)	1.37(1)
0.5	0.888(6)	1.28(1)	1.68(1)
0.6	0.852(4)	1.47(1)	2.20(1)
0.7	0.826(5)	1.80(1)	3.17(1)
0.8	0.811(5)	2.41(1)	5.31(2)
0.9	0.819(6)	3.95(2)	11.88(3)
1.0	0.880(7)	23.1(1)	176.3(3)
1.1	1.19(1)	18.2(1)	94.7(2)
1.3	1.71(2)	8.31(5)	31.6(1)
1.5	1.44(1)	3.37(2)	11.9(1)
1.7	0.996(8)	1.40(1)	4.96(3)
1.9	0.635(4)	0.622(3)	2.23(1)
2.0	0.503(3)	0.424(2)	1.54(1)

Table 1: Ratio between exact and effective results for the three cross sections. The error on the last digit comes from the phase-space Monte Carlo integration.

$$\frac{\sigma(\gamma\gamma \to \gamma\nu\bar{\nu})}{\sigma^{\rm eff}(\gamma\gamma \to \gamma\nu\bar{\nu})} = r^{-7.85491} \times \exp[4.42122 + 0.343516\log^2(r) - 0.114058\log^3(r) + 0.0103219\log^4(r)],$$

$$\frac{\sigma(\nu\bar{\nu} \to \gamma\gamma\gamma)}{\sigma^{\rm eff}(\nu\bar{\nu} \to \gamma\gamma\gamma)} = r^{-6.57374} \times \exp[5.27548 - 0.689808\log^2(r) + 0.15014\log^3(r) - 0.0123385\log^4(r)] \quad (15)$$

where the effective cross sections $\sigma^{\rm eff}$ are given in eq. (11) and ref. [3] and $r = \omega/m_e$. All the above fits are valid in the energy range $1.7 < r < 100$.

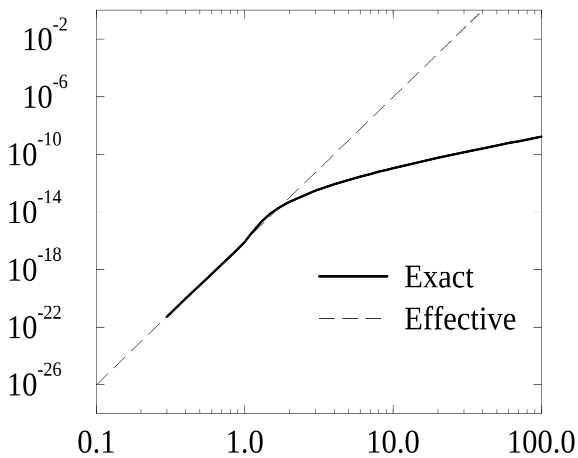

Figure 3: $\gamma\nu \to \gamma\gamma\nu$ cross section in fb as a function of ω/m_e.

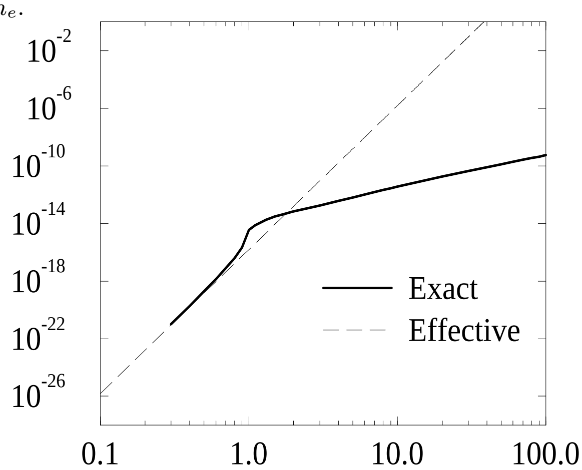

Figure 4: $\gamma\gamma \to \gamma\nu\bar{\nu}$ cross section in fb as a function of ω/m_e.

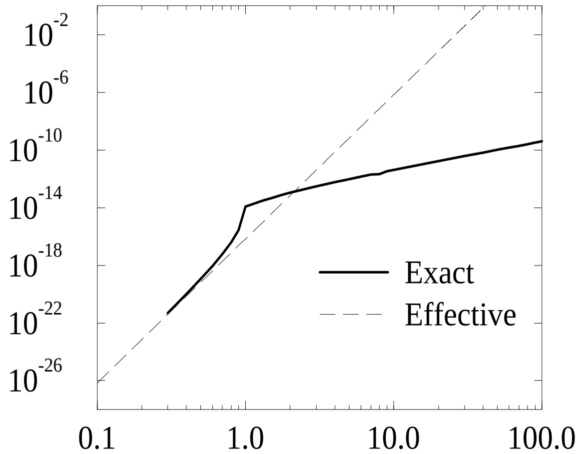

Figure 5: $\nu\bar{\nu} \to \gamma\gamma\gamma$ cross section in fb as a function of ω/m_e.

4 Astrophysical and cosmological possible implications

The 5-leg photon–neutrino processes are of interest in astrophysics. The processes $\nu\gamma \to \nu\gamma\gamma$ and $\nu\bar{\nu} \to \gamma\gamma\gamma$ can affect the mean free path of neutrinos inside the supernova core, while the process $\gamma\gamma \to \gamma\nu\bar{\nu}$ is a possible cooling mechanism for hot objects [13].

Basing their results on the assumption

$$\sigma(\gamma\nu \to \gamma\gamma\nu) = \sigma_0 \left(\frac{\omega}{1\ \text{MeV}}\right)^{\gamma}, \ \sigma_0 = 10^{-52}\,\text{cm}^2, \quad (16)$$

and on the data collected from supernova 1987 A, the authors of [13] fitted the exponent γ in eq. (16) to be less than 8.4, for ω of the order of a few MeV. The physical requirement behind this is that neutrinos of a few MeV should immediately leave the supernova, so that their mean free path is constrained to be larger than 10^{11} cm.

The effective theory predicts $\gamma = 10$, while, using the curves from the exact calculation, a softer energy dependence is observed in the region of interest (see Fig. 3). A fit to the exact curve gives $\gamma \sim 3$ for 1 MeV $< \omega <$ 10 MeV, thus confirming the expectations of [13].

A second interesting quantity is the range of parameters for which the neutrino mean free path for such reactions is inside the supernova core (of 10 km typical size), therefore affecting its dynamics. Always in [13] it was found, with the help of Monte Carlo simulations, that for several choices of temperature and chemical potential, and assuming the validity of the effective theory ($\gamma = 10$), this happens when $\omega \geq 5$ MeV. Since the exact results are now available, it would be of extreme interest to see how the above prediction is affected. More in general, we think that the reactions in (1) should be included in supernova codes.

On the basis of the effective theory results, the authors of [3], suggested that these processes could also have some relevance in cosmology. Consider, in fact, the mean number $\bar{N}$ of neutrino collisions, via the first of processes (1), in an expansion time t equal to the age of the Universe [14]:

$$\bar{N} = \sigma(\gamma\nu \to \gamma\gamma\nu) n_\nu \; c\, t \,, n_\nu = \text{neutrino number density.}$$

By writing n_ν and t in terms of the photon energy at thermal equilibrium ($\omega \sim kT$), expressed in units of 10^{10} K and denoted by T_{10}, we get

$$n_\nu = 1.6 \times 10^{31} T_{10}^3 \text{ cm}^{-3}\,, \quad t = 2T_{10}^{-2} \text{ s.} \tag{17}$$

When $\bar{N}$ is large, the neutrinos are in thermal contact with matter and radiation, while, for $\bar{N} \sim 1$ (namely $\sigma \sim 10^{-42} T_{10}^{-1}$ cm^2), the neutrinos decouple. Applying the effective formula in eq. (11), the resulting decoupling temperature is $T_{10} \sim 9.5$, namely $\omega \sim 8.2$ MeV, therefore outside the range of validity of the effective theory. By repeating the same analysis with the exact result, we found instead that $\bar{N}$ becomes of the order of 1 at $\omega \sim 1$ GeV, and at these energies, other processes enter the game. In conclusion, the five-leg reactions in eq. (1) are unlikely to be important for a study of the neutrino decoupling temperature, contrary to what the effective theory seemed to suggest.

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