# Cosmological Phase Transitions and Radius Stabilization in Higher Dimensions

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**Abstract**

Recently there has been considerable interest in field theories and string theories with large extra spacetime dimensions. In this paper, we explore the role of such extra dimensions for cosmology, focusing on cosmological phase transitions in field theory and the Hagedorn transition and radius stabilization in string theory. In each case, we find that significant distinctions emerge from the usual case in which such large extra dimensions are absent. For example, for temperatures larger than the scale of the compactification radii, we show that the critical temperature above which symmetry restoration occurs is reduced relative to the usual four-dimensional case, and consequently cosmological phase transitions in extra dimensions are delayed. Furthermore, we argue that if phase transitions do occur at temperatures larger than the compactification scale, then they cannot be of first-order type. Extending our analysis to string theories with large internal dimensions, we focus on the Hagedorn transition and the new features that arise due to the presence of large internal dimensions. We also consider the role of thermal effects in establishing a potential for the radius of the compactified dimension, and we use this to propose a thermal mechanism for generating and stabilizing a large radius of compactification.

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# 1   Introduction

The possibility of large extra spacetime dimensions has recently received considerable attention. There are two fundamentally different ways in which such extra dimensions might arise. First, they may appear as extra dimensions felt by all of the particles and forces of nature, both gauge and gravitational. Such extra dimensions are therefore universal, and apply to all observable physics. These sorts of large extra dimensions can have important consequences. For example, in Ref. [1], it was shown that large extra spacetime dimensions of this type could be used to lower the grand unification (GUT) scale. This demonstrates that extra universal dimensions have the power to alter one of the fundamental high energy scales of physics. Large extra dimensions of this type also provide a natural way of explaining the fermion mass hierarchy by permitting the fermion masses to evolve with a power-law dependence on the energy scale [1]. Moreover, as first investigated in Ref. [2], large extra universal dimensions can also be used to induce supersymmetry-breaking via the Scherk-Schwarz mechanism [3]. Such supersymmetry-breaking scenarios have a number of interesting signatures [2, 4, 5, 6]. Other phenomenological properties of string theories with large extra universal dimensions have been discussed in Refs. [7, 8].

However, these are not the only types of extra dimensions that might arise. For example, there may also be extra dimensions that are felt only by the gravitational force, with the observable world of the Standard Model restricted to a "brane". Such extra dimensions can also play a key role. For example, they emerge naturally in describing the strong-coupling behavior of certain string theories [9]. More recently, it has even been proposed that large extra dimensions of this type may be used to lower the fundamental Planck scale to the TeV range and thereby avoid the gauge hierarchy problem [10]. Such extra dimensions may also be used in a field-theoretic and string-theoretic context to transmit supersymmetry-breaking between four-dimensional boundaries [11, 12, 13], and indeed this leads to a new "world-as-brane" perspective which has been investigated in Refs. [14, 10, 15].

Finally, large extra dimensions of both types also play an important role in lowering the fundamental string scale, as first pointed out in Ref. [16]. This idea of lowering the string scale was later dramatically extended to the TeV range in Ref. [17], and subsequently pursued in the context of string theories with extra large dimensions in Refs. [10, 18, 1, 19].

Extra dimensions of both types can be expected to have a profound effect on the dynamics of the early universe. There are many possible effects which come into play in the context of a higher-dimensional cosmology [20]. In addition to the issue of inflation occurring in $D > 4$ dimensions, one might think of the effects of extra dimensions on cosmological phase transitions, cosmological density perturbations, and topological defects.

In this paper, we shall consider several aspects of large extra dimensions as they relate to the dynamics of the early universe. In doing so, we shall follow two comple-

mentary approaches.

First, we shall consider the effects of large extra dimensions through a field-theoretic analysis. Most of the applications of field theories are based upon the theory of phase transitions [21]. In particular, the concepts of spontaneous symmetry breaking in gauge theories [22] and symmetry restoration at high temperatures [23] play a fundamental role. At temperatures above a certain critical temperature, gauge and/or global symmetries are restored and the order parameter — usually the vacuum expectation value of a scalar field — vanishes. Of particular interest for cosmology is the nature of the phase transition, whether it is first-order or not. In most models, this depends upon the mass of the scalar field. If the phase transition is strongly first-order, the universe may be dominated by the vacuum energy and undergo a period of inflation [24]. A first-order phase transition proceeds by nucleation of bubbles of the true vacuum, and this dynamics might provide the local out-of-equilibrium conditions that are a necessary ingredient for the formation of the baryon asymmetry in the early universe [25]. On the other hand, if the phase transition is higher-order or very weakly first-order, thermal fluctuations may drive the transition. Spontaneous symmetry-breaking phase transitions may also lead to the formation of topological defects, which may take the form of domain walls, cosmic strings, and magnetic monopoles in Grand Unified Theories (GUTs). These cosmological objects may be either very insalubrious or have great potential for cosmological relevance. The latter case is particularly true for cosmic strings.

In Sect. 2, we will take a field-theoretic approach in order to explore the role of extra dimensions in cosmological phase transitions. Specifically, we will study the issue of restoration of spontaneously broken symmetries above a critical temperature in the case in which the temperature of the system is larger than the inverse of the compactification radii. As we shall see, an interesting distinction emerges between the critical temperature above which symmetry restoration takes place in four dimensions, and the corresponding critical temperature in $D > 4$: the former may be much larger, *i.e.*, cosmological phase transitions in extra dimensions can be delayed. We will also argue that the cosmological phase transitions, if they happen to take place at temperatures larger than the inverse radii, cannot be of the first order — *i.e.*, they do not proceed by nucleation of critical bubbles. We will also provide general formulae for the effective potential of the order parameter at finite temperature, discuss the applicability of our approximations, and argue about some possible cosmological implications of our findings.

Ultimately, however, a field-theoretic analysis of the role of extra large spacetime dimensions is limited by the fact that higher-dimensional gauge theories are non-renormalizable, and require the introduction of ultraviolet cutoffs which in turn signal the appearance of new physics. Since string theory is the only known consistent higher-dimensional theory which lacks the divergences ordinarily associated with non-renormalizable field theories, it is natural to consider the corresponding effects of large extra dimensions in string theory.

There also exists another reason why it is important to consider the extension to string theory. As we indicated above, one of the primary motivations for considering large extra dimensions is that they can lower the fundamental GUT and Planck scales [1, 10]. However, as discussed in Refs. [17, 10, 18, 1, 19], such scenarios must ultimately be embedded into reduced-scale string theories in order to be consistent. For example, if the GUT and Planck scales are reduced to the TeV range, then this ultimately requires a TeV-scale string theory as well. Therefore, the "stringy" behavior ordinarily associated with string thermodynamics will now become important at far lower energies than previously thought relevant in the discussion of the early universe, and hence will have heightened significance.

In Sect. 3, therefore, we shall consider some aspects of string cosmology in the presence of large extra spacetime dimensions. One crucial issue that arises in string theory and string cosmology is the role of the Hagedorn transition. As we will review in Sect. 3.1, the Hagedorn transition is a phenomenon that arises in any theory containing an exponentially growing number of states as a function of mass, and string theory is no exception. Normally, the Hagedorn transition does not play a crucial role in string cosmology because it occurs only at temperatures which are roughly equal to the string scale, and this is usually taken to be near the Planck scale. However, if the string scale is now significantly lowered (perhaps even to the TeV range), then the nature and properties of the Hagedorn transition become of paramount importance. Moreover, as we shall see, the presence of large extra dimensions within a Hagedorn-type framework has a number of interesting cosmological effects.

Another crucial issue that must be addressed in any discussion of large extra dimensions is the *radius* of these dimensions. Normally, in string theory, spacetime supersymmetry ensures that the radius is a modulus — *i.e.*, that it has a flat potential. However, as is well-known, thermal effects necessarily break supersymmetry, and therefore it is possible that thermal effects can themselves create a potential for the radius which might explain how compactified radii can become large. In Sect. 3.2, therefore, we shall calculate such thermal effects within the framework of Type I (open) string theory, and show that finite-temperature effects might indeed be able to generate and stabilize the desired large radii.

## 2    Cosmological Phase Transitions in Field Theory

We begin by discussing some of the effects of large extra dimensions using a field-theoretic approach.

### 2.1    The general setup

In any discussion of extra spacetime dimensions, we know that these extra dimensions must be compactified in order to be consistent with the observed low-energy world consisting of only four flat dimensions. For the sake of simplicity, we shall

assume that there is only one extra dimension, which is compactified on a circle with fixed radius $R$ where $R^{-1}$ exceeds presently observable energy scales. The generalization to more than one extra dimension is straightforward.

The appearance of an extra dimension of radius $R$ implies that a given complex quantum field $\Phi$ now depends not only on the usual four-dimensional spacetime coordinates $x$, but also the additional coordinate $y$. Demanding the periodicity of $\Phi$ under

$$y \ \rightarrow \ y + 2\pi R \qquad (2.1)$$

implies that $\Phi(x, y)$ takes the form

$$\Phi(x, y) \ = \ \sum_{n=-\infty}^{\infty} \Phi^{(n)}(x) \exp\left(iny/R\right) \ , \qquad (2.2)$$

where $n \in \mathbb{Z}$. The "four-dimensional" fields $\Phi^{(n)}(x)$ are the so-called Kaluza-Klein modes, and $n$ is the corresponding Kaluza-Klein excitation number. In general, the mass of each Kaluza-Klein mode is given by

$$m_n^2 \ \equiv \ m_0^2 + \frac{n^2}{R^2} \ , \qquad (2.3)$$

where $m_0$ is the mass of the zero mode. At energies far below $R^{-1}$, one expects the extra dimension to be unobservable. However, at energies or temperatures much larger than $R^{-1}$, excitations of many Kaluza-Klein modes become possible and the contributions of these Kaluza-Klein modes must be included in all physical computations. It is clear that only the lowest-lying Kaluza-Klein modes play an important role, because the contributions of the very heavy modes are suppressed by their large masses. In particular, at temperatures $T \gg R^{-1}$, one expects the relevant number of Kaluza-Klein modes to be $\sim RT$. This expectation is confirmed by the explicit computation of physical quantities such as the critical temperature.

It is important to note that not every state can have Kaluza-Klein excitations. This complication arises because it is necessary for the Kaluza-Klein excitations to fall into representations that permit suitable Kaluza-Klein mass terms to be formed. This issue is particularly important for chiral fermionic states which cannot be given a Kaluza-Klein mass. One therefore has two choices at this stage: either the chiral fermionic states do not have Kaluza-Klein excitations, or chiral-conjugate mirror fermions need to be introduced to form a massive Kaluza-Klein tower. We also note that if the extra dimension is compactified on $S^1/\mathbb{Z}_2$ (a circle subjected to the further identification $y \rightarrow -y$), the Kaluza-Klein excitations can be decomposed into even fields $\Phi_+(x, y) = \sum_{n=0}^{\infty} \Phi^{(n)}(x) \cos(ny/R)$ and odd fields $\Phi_-(x) = \sum_{n=1}^{\infty} \Phi^{(n)}(x) \sin(ny/R)$. Since the appropriate transformation of the fields under the discrete parity $\mathbb{Z}_2$ is determined by the interactions, half of the original Kaluza-Klein theory may be projected out according to the $\mathbb{Z}_2$ parity of the fields. If only the odd tower is left, the zero mode is missing.

## 2.2 Computing the one-loop effective potential

Given this setup, we are now in a position to compute the one-loop effective potential $V^{1-\text{loop}}(\varphi_c)$ for a generic order parameter $\varphi_c$. Let us suppose that our theory contains a set of scalar fields $\chi_i$ $(i = 1, ..., n)$ which, because of their interactions with the quantum field $\hat{\varphi}$, accrue a mass squared in the background of the classical field $\varphi_c = \langle \hat{\varphi} \rangle$ given by

$$M_i^2(\varphi_c) \;=\; m_i^2 + m_i^2(\varphi_c) \,, \tag{2.4}$$

where $m_i^2$ is a bare mass which does not depend upon the background field. In four dimensions, the one-loop effective potential at finite temperature assumes the familiar form [23]

$$
\begin{aligned}
V_{\text{bos}}^{1-\text{loop}}(\varphi_c) \;&=\; \frac{T}{2} \sum_i n_i \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln\left[\omega_n^2 + k^2 + M_i^2(\varphi_c)\right] \\
&=\; T \sum_i n_i \int \frac{d^3k}{(2\pi)^3} \ln\left[1 - e^{-\beta\sqrt{k^2+M_i^2(\varphi_c)}}\right] \,.
\end{aligned} \tag{2.5}
$$

Here $\beta \equiv 1/T$, $n_i$ is the number of degrees of freedom of the field $\chi_i$, and the sum is over the Matsubara frequencies $\omega_n = 2\pi n T$. In passing to the second line, we have explicitly performed the Matsubara sum and dropped the zero-temperature one-loop term $\int d^3k/(2\pi)^3 \sqrt{k^2 + M_i^2(\varphi_c)}/2$.

Let us now suppose that there is an extra dimension which contributes a Kaluza-Klein tower with an extra mass term $\ell^2/R^2$, where $\ell \in \mathbb{Z}$. The one-loop effective potential, as seen from the four-dimensional world, then reads

$$V_{\text{bos}}^{1-\text{loop}}(\varphi_c) \;=\; \frac{T}{2} \sum_i n_i \sum_{\ell=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln\left[\omega_n^2 + k^2 + M_i^2(\varphi_c) + \frac{\ell^2}{R^2}\right] \,. \tag{2.6}$$

Our next step is to rewrite this expression in terms of a Schwinger proper-time parameter $s$ using the identity

$$\frac{d \ln A}{dA} \;=\; \int_0^{\infty} ds \, e^{-sA} \,. \tag{2.7}$$

After integration over the three-momentum, this yields

$$V_{\text{bos}}^{1-\text{loop}}(\varphi_c) \;=\; -\frac{T}{16\pi^{3/2}} \sum_i n_i \int_0^{\infty} \frac{ds}{s^{5/2}} \, e^{-sM_i^2(\varphi_c)} \, \vartheta_3\left(4\pi i T^2 s\right) \, \vartheta_3\left(\frac{is}{\pi R^2}\right) \tag{2.8}$$

where the $\Theta$-functions are defined as

$$\Theta_\beta^\alpha(\tau) \;=\; \sum_{n=-\infty}^{\infty} e^{2\pi i n \beta} \, e^{\pi i \tau (n+\alpha)^2} \tag{2.9}$$

with

$$\vartheta_1 \equiv \Theta_{1/2}^{1/2} \,, \quad \vartheta_2 \equiv \Theta_0^{1/2} \,, \quad \vartheta_3 \equiv \Theta_0^0 \,, \quad \vartheta_4 \equiv \Theta_{1/2}^0 \,. \tag{2.10}$$

These $\Theta$-functions have the remarkable property that

$$\Theta_\beta^\alpha(-1/\tau) \;=\; \sqrt{-i\tau}\; e^{-2\pi i \alpha \beta}\, \Theta_\alpha^{-\beta}(\tau) \tag{2.11}$$

where one chooses the branch of the square root with non-negative real part.

Let us focus on the limit $RT \gg 1$. By making the change of variable $s' = 4\pi T^2 s$ and subtracting the $T=0$ part of the one-loop potential, the expression (2.8) becomes

$$\Delta V_{\text{bos}} \;=\; -\frac{T^4}{2} \sum_i n_i \int_0^\infty \frac{ds}{s^{5/2}}\, e^{-sM_i^2(\varphi_c)/4\pi T^2} \left( \vartheta_3\,(is) - \frac{1}{\sqrt{s}} \right) \vartheta_3 \left( \frac{is}{4\pi^2 R^2 T^2} \right) \tag{2.12}$$

where we have introduced the notation

$$\Delta V \;\equiv\; V^{1-\text{loop}}(\varphi_c) \;-\; V^{1-\text{loop}}(\varphi_c)\Big|_{T=0} \,. \tag{2.13}$$

Here

$$\begin{aligned}
V_{\text{bos}}^{1-\text{loop}}(\varphi_c)\Big|_{T=0} &= -\frac{1}{32\pi^2} \sum_i n_i \int_{\Lambda^{-2}}^\infty \frac{ds}{s^3}\, e^{-sM_i^2(\varphi_c)}\, \vartheta_3\left( \frac{is}{\pi R^2} \right) \\
&= \sum_i n_i (R\Lambda) \left\{ \frac{-\Lambda^4}{80\pi^{3/2}} + \frac{M_i^2 \Lambda^2}{48\pi^{3/2}} - \frac{M_i^4}{32\pi^{3/2}} + \frac{M_i^5}{60\pi\Lambda} + \mathcal{O}(M_i^6) \right\} \\
&\quad + V_{\text{bos}}^{D=4}
\end{aligned} \tag{2.14}$$

where $\Lambda$ is an ultraviolet cutoff and where $V_{\text{bos}}^{D=4}$ represents the usual bosonic contribution to the four-dimensional one-loop Coleman-Weinberg effective potential [26]. Note that the zero-temperature potential scales as $R\Lambda$, which is the effective number of Kaluza-Klein states below the cutoff $\Lambda$. If we now use the fact that

$$\vartheta_3(4\pi i\tau) \;\approx\; \frac{1}{\sqrt{4\pi\tau}} \left[ 1 + \mathcal{O}\left( e^{-1/4\tau} \right) \right] \qquad \text{as} \;\; \tau \to 0 \tag{2.15}$$

and perform a high-temperature expansion, $M_i(\varphi_c)/T \ll 1$, we find that the expression (2.12) reduces to

$$\begin{aligned}
\Delta V_{\text{bos}} \;=\;\; & \sum_i n_i \,(RT) \left\{ -\frac{3}{2\pi}\zeta(5)T^4 + \frac{\zeta(3)}{4\pi}T^2 M_i^2(\varphi_c) \right. \\
& + \; \frac{1}{64\pi}M_i^4(\varphi_c)\left[ -3 + 4\ln(M_i(\varphi_c)/T) \right] - \frac{M_i^5(\varphi_c)}{60\pi T} \left. \right\} + \; \mathcal{O}(M_i^6) \\
& + \; V_{\text{bos}}^{D=4}(T)
\end{aligned} \tag{2.16}$$

where $\zeta(p) \equiv \sum_{n=1}^\infty 1/n^p$ is the Riemann zeta-function and where $V_{\text{bos}}^{D=4}(T)$ is the usual four-dimensional finite-temperature effective potential [23]. The first term in

this expansion accounts for the total pressure of the relativistic bosonic particles in the gas. The fact that the terms of the expansion are multiplied by the factor $RT$ does not come as a surprise. At temperatures $T \gg R^{-1}$, there are approximately $RT$ Kaluza-Klein states which may be treated as massless and which are, therefore, excited in the thermal bath. This set of states contributes to the effective potential. By contrast, the Kaluza-Klein states whose masses exceed the temperature are essentially decoupled from the thermal bath.

It is also important to note that in the $RT \gg 1$ expansion of the effective potential, we do not recover any odd powers of the mass $M_i(\varphi_c)$. This is very different from what happens in $D = 4$ field theory at finite temperature, where the infrared limit $|k| \to 0$ becomes problematic around $M_i(\varphi_c) = 0$. At any order of perturbation theory, the infrared divergence comes from the Feynman diagrams where the momenta of the particles in the loop correspond to the $n = 0$ Matsubara mode, and give rise to odd powers of $M_i(\varphi_c)$. These odd powers of $M_i(\varphi_c)$ play a fundamental role in the dynamics of cosmological phase transitions in $D = 4$ because their presence induces an energy barrier which separates the extremum of the scalar potential associated with the symmetric phase from a local minimum of the broken phase. At the critical temperature $T_c$, both phases are equally favored energetically, and at later times the broken-phase minimum becomes the global minimum. The phase transition proceeds by nucleation of bubbles of the true vacuum, signalling a first-order phase transition.

Our findings indicate that for one extra compactified dimension, the $n = 0$ Matsubara frequency mode induces a term $\mathcal{O}(TM_i^4(\varphi_c))$, which has even powers of $M$. At finite $R$, one also obtains the term $\mathcal{O}(RM_i^5(\varphi_c))$, but this is cancelled by the contribution from the $T = 0$ one-loop effective potential. This can be seen explicitly in (2.14) and (2.16). This observation can also be seen and generalized through a simple scaling argument. In the flat-space limit $R \to \infty$, it is sufficient to consider the effective potential in $D$ spacetime dimensions

$$V = \frac{T}{2} \sum_n \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \ln(4\pi^2 T^2 n^2 + k^2 + M^2) . \qquad (2.17)$$

Formally taking derivatives of the effective potential leads to the convergent expressions

$$\frac{\partial^{N+1} V}{\partial (M^2)^{N+1}} = \begin{cases} \dfrac{(-1)^N}{2^{2N+1}\pi^N} T \left[ \dfrac{1}{M^2} + 2\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(M^2 + 4\pi^2 T^2 n^2)} \right] & \text{for } D = 2N+1 \\ \dfrac{(-1)^N}{2^{2N+1}} \sqrt{\pi} T \left[ \dfrac{1}{M^3} + 2\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{(M^2 + 4\pi^2 T^2 n^2)^{3/2}} \right] & \text{for } D = 2N . \end{cases}$$
$$(2.18)$$

Integrating the above expressions for the derivatives and expanding in powers of

$(M/T)$, we find that

$$V(M^2) \sim \begin{cases} TM^{2N} + TM^{2N}\left(\dfrac{M^2}{T^2} + \dfrac{M^4}{T^4} + \dfrac{M^6}{T^6} + ...\right) & \text{for } D = 2N+1 \\[2em] TM^{2N-1} + M^{2N}\left(\dfrac{M^2}{T^2} + \dfrac{M^4}{T^4} + \dfrac{M^6}{T^6} + ...\right) & \text{for } D = 2N \end{cases} \tag{2.19}$$

where we have neglected all numerical coefficients in the expansion. Note that for $D = 4$, we obtain the usual cubic term in the four-dimensional finite-temperature effective potential. However, when the number of extra dimensions is odd ($D = 5, 7, 9, ...$), we see that no odd powers of $M$ appear in the high-temperature expansion of the potential. This agrees with our explicit calculation in $D = 5$ (note that (2.17) implicitly includes the $T = 0$ one-loop effective potential). On the other hand, for even dimensions ($D = 6, 8, 10, ...$), an odd power of $M$ exists in the potential and specifically arises from the Matsubara zero-mode. When these even dimensions are compactified with a finite radius $R$, this term becomes $(RT)^{D-4}T^4(M/T)^{D-1}$ where we have inserted a factor of $R^{D-4}$ resulting from the compactification. Note that the factors of $(RT)^{D-4}$ take into account the effective numbers of Kaluza-Klein states at temperature $T$. This term is suppressed relative to the term $(RT)^{D-4}M^4$ by a factor $(M/T)^{D-5}$ for even dimensions $D > 4$. At finite $R$, there is also the usual $TM^3$ term which cannot be seen in the $R \to \infty$ limit. In the limit $MR \gg 1$, the term $TM^3$ is suppressed by a factor $1/[(RT)^{D-5}(MR)]$ compared to the term $(RT)^{D-4}M^4$, and for even dimensions $D > 4$ this term is suppressed by $1/(MR)^{D-4}$ relative to the term $(RT)^{D-4}T^4(M/T)^{D-1}$. Thus, once again there is no barrier, and we conclude that for $M \gg R^{-1}$ no barrier is present in the effective potential in $D > 4$ dimensions at high temperature. This fact might have dramatic consequences for the dynamics of phase transitions, such as the electroweak phase transition, in the early universe. We leave this subject for future investigation.

In an analogous way, we can also compute the contribution to the one-loop effective potential from a set of fermion fields $\psi_i$ ($i = 1, ..., n$) which, because of their interactions with the quantum field $\hat{\varphi}$, accrue a mass squared $M_i^2(\varphi_c)$ in the background. For fermions, (2.5) is replaced by

$$\begin{aligned} V_{\text{fer}}^{1-\text{loop}}(\varphi_c) &= -\frac{T}{2} \sum_i n_i \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \ln\left[\omega_n^2 + k^2 + M_i^2(\varphi_c)\right] \\ &= -T \sum_i n_i \int \frac{d^3k}{(2\pi)^3} \ln\left[1 + e^{-\beta\sqrt{k^2+M_i^2(\varphi_c)}}\right] \end{aligned} \tag{2.20}$$

where the sum is now over the shifted (fermionic) Matsubara frequencies $\omega_n = (2n+1)\pi T$. Incorporating the effects of an extra dimension is handled as before. In the general case of chiral fermions, chiral-conjugate mirror fermions need to be introduced if the fermions are to have a Kaluza-Klein tower. However, for simplicity, we shall

consider Kaluza-Klein states with Dirac masses $\ell^2/R^2$. Let us again consider the limit $RT \gg 1$. Proceeding just as we did below (2.5), we obtain

$$\Delta V_{\text{fer}} = +\frac{T^4}{2} \sum_i n_i \int_0^\infty \frac{ds}{s^{5/2}} e^{-sM_i^2(\varphi_c)/4\pi T^2} \left(\vartheta_2(is) - \frac{1}{\sqrt{s}}\right) \vartheta_3\left(\frac{is}{4\pi^2 R^2 T^2}\right) . \tag{2.21}$$

This is the fermionic analogue of (2.12). In the limit $RT \gg 1$, we can use (2.11) to evaluate this integral, obtaining

$$\begin{aligned}
\Delta V_{\text{fer}} = \sum_i n_i (RT) &\left\{ -\frac{45}{32\pi}\zeta(5)T^4 + \frac{3\zeta(3)}{16\pi}T^2 M_i^2(\varphi_c) \right. \\
&\left. - \frac{\ln 2}{16\pi} M_i^4(\varphi_c) + \frac{M_i^5(\varphi_c)}{60\pi T} \right\} + \mathcal{O}(M_i^6) + V_{\text{fer}}^{D=4}(T) \tag{2.22}
\end{aligned}$$

where $V_{\text{fer}}^{1-\text{loop}}(\varphi_c)|_{T=0} = -V_{\text{bos}}^{1-\text{loop}}(\varphi_c)|_{T=0}$ and where $V_{\text{fer}}^{D=4}(T)$ is the fermionic contribution to the four-dimensional finite-temperature effective potential [23]. Note, in particular, that the fermionic contribution to the squared mass term carries the same sign as the bosonic piece in (2.16). This is a typical feature of high-temperature field theories where, even starting with a supersymmetric theory, supersymmetry is broken by finite-temperature effects. Also note that although the sign of the quartic term $\varphi_c^4$ in (2.22) is negative, the potential is not destabilized because the tree-level quartic term continues to dominate in the limit $\lambda(RT) \lesssim 1$ where $\lambda$ is the coefficient of the quartic term in $V(\varphi_c)$. This happens to be the limit in which the one-loop computation is reliable, as we shall discuss in the next subsection.

Finally, for completeness, let us discuss some issues in the context of Scherk-Schwarz supersymmetry breaking by compactification. This has some features that are similar to those of the finite-temperature calculation. At $T = 0$, combining the contributions from bosons and fermions leads to the the one-loop effective potential

$$\begin{aligned}
V_{\text{SS}}^{1-\text{loop}}(\varphi_c) = \frac{1}{2} \sum_i n_i \sum_{n=-\infty}^\infty \int \frac{d^4k}{(2\pi)^4} &\left\{ \ln\left[k^2 + M_i^2(\varphi_c) + \frac{n^2}{R^2}\right] \right. \\
&\left. - \ln\left[k^2 + M_i^2(\varphi_c) + \frac{(n+1/2)^2}{R^2}\right] \right\} \\
= -\frac{1}{32\pi^2} \sum_i n_i \int_0^\infty \frac{ds}{s^3} e^{-sM_i^2} &\left[\vartheta_3\left(\frac{is}{\pi R^2}\right) - \vartheta_2\left(\frac{is}{\pi R^2}\right)\right] . \tag{2.23}
\end{aligned}$$

This result is free of ultraviolet divergences, as expected. In the limit $RM \ll 1$, we can perform the change of variable $s' = (s/\pi R^2)$ to find that the contribution from the Kaluza-Klein states to the one-loop effective potential reads

$$\begin{aligned}
V_{\text{SS}}^{1-\text{loop}}(\varphi_c) = \sum_i n_i &\left\{ -\frac{93}{1024\pi^6}\frac{\zeta(5)}{R^4} + \frac{7\zeta(3)}{128\pi^4}\frac{M_i^2(\varphi_c)}{R^2} \right. \\
&\left. + \frac{M_i^4(\varphi_c)}{128\pi^2}[-3 + 4\ln(\pi M_i(\varphi_c)R)] \right\} + \mathcal{O}(M_i^6) . \tag{2.24}
\end{aligned}$$

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The first term in (2.24) is a Casimir force term, which is expected to arise in such a non-supersymmetric model. More interestingly, however, we observe from (2.24) that the squared mass term receives a finite contribution which scales as $1/R^2$. If the order parameter $\varphi_c$ is associated with the low-energy Standard Model Higgs field, this implies (by naturalness arguments) that $1/R$ has to be smaller than about 10-100 TeV. This raises the interesting possibility of breaking the Standard-Model gauge group via the contributions of the Kaluza-Klein states if the $1/R^2$ term happens to carry a negative sign. This is possible in more general Scherk-Schwarz compactification scenarios.[*]

At finite temperature, we can perform the explicit sum over the Matsubara frequency modes that appear in (2.23) after compactifying the time coordinate. Performing a high-temperature expansion in the limit $RT \gg 1$ then leads to the result

$$
\begin{aligned}
V_{\text{SS}}^{1-\text{loop}}(T) \;=\; & \sum_i n_i \, (RT) \left\{ -\frac{93\zeta(5)}{32\pi} T^4 + \frac{7\zeta(3)}{16\pi} T^2 M_i^2(\varphi_c) \right. \\
& \left. + \; \frac{M_i^4(\varphi_c)}{64\pi} \left[ -3 + 4\ln(M_i(\varphi_c)/2T) \right] \right\} + \mathcal{O}(M_i^6) \; . \quad (2.25)
\end{aligned}
$$

Again we notice the overall factor $RT$ which is the effective number of Kaluza-Klein states. In addition, we find that to leading order, the result is simply the sum of the terms (2.16) and (2.22). This is expected, for in the high-temperature expansion, the supersymmetry-breaking mass difference between the bosons and fermions is negligible.

## 2.3 Multi-loop corrections

Let us now consider what happens when multi-loop corrections are included in the computation of the effective potential. For this purpose, we shall assume that our theory is a simple $(\lambda/4)\hat{\varphi}^4$-theory. In such a case, the corrections to the tree-level potential are provided by the quantum and/or thermal excitations of the $\varphi$-field itself and, in particular, by its Kaluza-Klein tower. As opposed to large-angle scattering processes, forward-scattering processes do not alter the distribution function of particles traversing a gas of quanta; they instead simply modify the dispersion relation. Forward scattering is manifest, for example, as ensemble and scalar-background corrections to the masses of the particles in the plasma. In other words, when they propagate, the particles in equilibrium in the thermal gas acquire a plasma mass $\delta m(T)$ through forward scatterings. Now, if the thermal environment is at some

---

[*]In this connection, note that in compactifying from five dimensions to four dimensions, there are two symmetries which can be exploited for the Scherk-Schwarz breaking. The first is the fermion number operator $(-1)^F$, which gives $\sum_i n_i = n_V + n_H$ where $n_V$ ($n_H$) is the number of five-dimensional $N = 2$ vector (hyper-) multiplets. The second symmetry, by contrast, is the $\mathbb{Z}_2$ $R$-parity, which gives $\sum_i n_i = n_V - n_H$. It is the second symmetry which can lead to a negative contribution for the $1/R^2$ term. Similar considerations can also be found in Ref. [27].

temperature $T$, only those excitations which are lighter than approximately $T$ may be in thermal equilibrium and present in the plasma. Other thermal excitations with masses much larger than $T$ are decoupled from the thermal bath, and do not alter the potential.

Given the expression (2.16), the one-loop plasma mass of the $\varphi$-quanta is easily found to be $\delta m^2(T) \sim \lambda(RT)T^2$. This means that for $\lambda(RT) \gg 1$, the quanta responsible for the one-loop correction to the potential of the order parameter $\varphi_c$ are in fact much heavier than $T$. This means that they should decouple, and give no contribution to the effective potential! In other words, the one-loop high-temperature expansion that we used to derive (2.16) breaks down in a certain range of the parameters, i.e., $\lambda(RT) \gg 1$, which is where the one-loop effective potential should receive large contributions from two- and higher-loop orders of perturbation theory. These contributions will be even larger than the one-loop contribution. Perturbation theory is therefore invalid unless a proper resummation is done.

Given this observation, we see that in order to obtain more accurate information about the issue of cosmological phase transitions in $D > 4$ dimensions, we have to study an infinite series of diagrams in perturbation theory. This is exactly analogous to what happens in a simple $\lambda \hat{\varphi}^4$ theory in equilibrium at finite temperature in $D = 4$, where the leading contributions to the effective potential in the infrared region come from the daisy and superdaisy multi-loop graphs [23].

In order to deal with this problem, we need a self-consistent loop expansion of the effective potential in terms of the *full* propagator. Such a technique was developed some time ago by Cornwall, Jackiw, and Tomboulis (CJT) in their effective-action formalism for composite operators [28]. In the rest of this section, we will consider a scalar field theory where $\hat{\varphi}$ transforms as a vector under the action of $O(N)$, i.e., $\hat{\varphi}^2 = \hat{\varphi}_a \hat{\varphi}^a$ with $a = 1, ..., N$, with a potential given by $V(\varphi) = (\lambda/4)(\hat{\varphi}^2)^2$. For the sake of simplicity, we will assume that there are no fermions in the theory.

We consider a generalization $\Gamma[\varphi_c, G]$ of the usual effective action which depends not only on $\varphi_c(x)$, but also on $G(x, y)$, a possible expectation value of the time-ordered product $\langle T\varphi(x)\varphi(y)\rangle$. The physical solutions must satisfy the stationary conditions

$$\frac{\delta\Gamma[\varphi_c, G]}{\delta\varphi_c(x)} = 0 , \qquad \frac{\delta\Gamma[\varphi_c, G]}{\delta G(x, y)} = 0 . \tag{2.26}$$

The conventional effective action $\Gamma[\varphi_c]$ is given by $\Gamma[\varphi_c, G]$ at the solution $G_0(\varphi_c)$ of (2.26). In this formalism, it is possible to sum a large class of ordinary perturbation-series diagrams that contribute to the effective action $\Gamma[\varphi_c]$, and the gap equation which determines the form of the full propagator is obtained by a variational technique.

We now apply the CJT formalism in the limit of large $N$ when the next-to-leading terms can be exactly summed. At each order, we keep only the term dominant in $N$ for large values of $N$. This allows us to resum the series of the leading multi-loop diagrams exactly and to solve the gap equation for the full propagator without any

approximation.

In order to obtain a series expansion of the effective action, we introduce the functional operator

$$D_{ab}^{-1}(\varphi_c, x, y) = \frac{\delta^2 I}{\delta\varphi_{c,a}(x)\,\delta\varphi_{c,b}(y)} \tag{2.27}$$

where $I$ is the classical action. The required series obtained by CJT is then [28]

$$\Gamma[\varphi_c, G] = I(\varphi_c) + \frac{1}{2}\operatorname{Tr}\ln D_0\,G^{-1} + \frac{1}{2}\operatorname{Tr}\left[D^{-1}\,G - 1\right] + \Gamma_2[\varphi_c, G]\,, \tag{2.28}$$

where $D_0^{-1} = -(\partial_\mu\partial^\mu)\,\delta_{ab}\delta^4(x, y)$. Here $\Gamma_2[\varphi_c, G]$ is the sum of all two-particle-irreducible vacuum graphs in the theory, with vertices defined by the classical action with shifted fields $I[\varphi_c + \varphi]$ and propagators set equal to $G(x, y)$.

Previous calculations show that among the multi-loop graphs contributing to the effective potential in the $O(N)$-theory, only the daisy and superdaisy diagrams survive in the large-$N$ limit [29]. This enables us to consider in $\Gamma_2[\varphi_c, G]$ only the graph of $\mathcal{O}(\lambda)$. This is essentially the Hartree-Fock approximation, which is known to be exact in the many-body version of our large-$N$ limit.
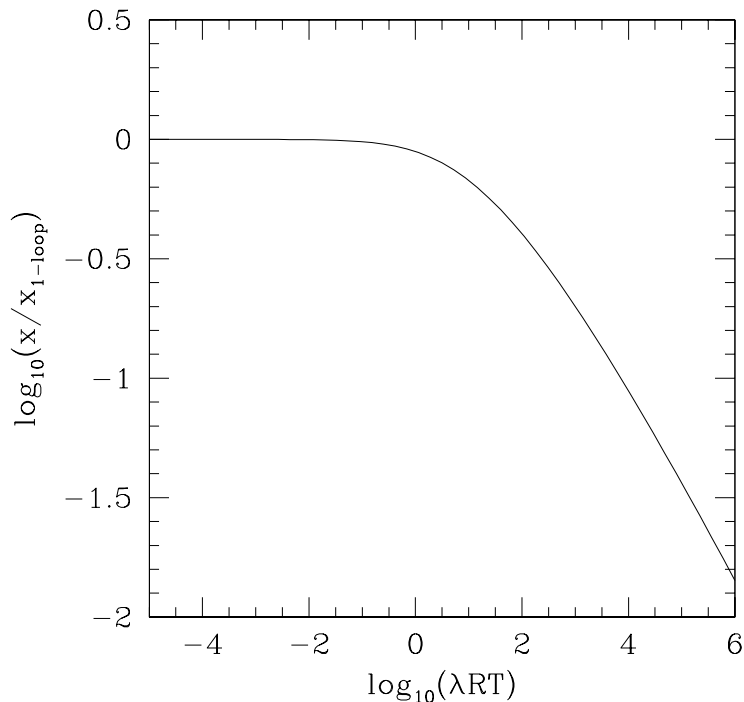


Figure 1: The ratio $x/x_{1-\mathrm{loop}}$ as a function of $\lambda RT$.

It turns out to be more convenient to concentrate on the effective masses rather than on the effective potential. By stationarizing the effective action $\Gamma[\varphi_c, G]$ with

respect to $G_{ab}$, we obtain the gap equation

$$G_{ab}^{-1}(x,y) = D_{ab}^{-1}(x,y) + 3\lambda \left[\delta_{ab}\, G_{cc}(x,x) + 2\, G_{ab}(x,x)\right]\delta^4(x,y) \;. \qquad (2.29)$$

This equation is exact in the limit of large $N$, and contains all the information about the dominant-$N$ contributions to the full propagator. Indeed, the exact Schwinger-Dyson equation reduces to (2.29) for large $N$. Next, we Fourier-transform (2.29) and take $\varphi_c = 0$. The gap equation then reads

$$M^2 = \frac{3\lambda}{2\pi}(RT)T^2\left[\frac{M}{T}\,\mathrm{Li}_2\left(e^{-M/T}\right) + \mathrm{Li}_3\left(e^{-M/T}\right)\right] \qquad (2.30)$$

where $\mathrm{Li}_n(x) \equiv \sum_{k=1}^{\infty} x^k/k^n$ are the polylogarithm functions. As we discussed above, in the limit $\lambda(RT) \ll 1$ we expect that the value of $M^2$ solving the gap equation is well-approximated by the one-loop mass $M_{1-\mathrm{loop}}^2 = (3\zeta(3)/2\pi)\lambda(RT)T^2$. On the other hand, for $\lambda(RT) \gg 1$, we expect the solution $M^2$ to be quite different from $M_{1-\mathrm{loop}}^2$. Our expectations are indeed confirmed in Fig. 1, where we have defined $x \equiv M/T$ to be the solution of the gap equation (2.30) and plotted the value of $x$ normalized to $x_{1-\mathrm{loop}} = \sqrt{3\zeta(3)\lambda(RT)/2\pi}$. Note, in particular, that at large values of $\lambda(RT)$, the value of $x$ is much smaller than the value indicated by the one-loop computation, indicating that the plasma mass is screened by higher-order corrections. This means that the usual perturbative expansion fails for $\lambda(RT) \gtrsim 1$, and has to be replaced by an improved perturbative expansion where an infinite number of diagrams are resummed at each order in the new expansion.

## 2.4 Implications of large extra dimensions

We have seen that in $D > 4$ dimensions, the effective potential of a given order parameter at high temperature has some peculiar features which are not present in the case in which only four dimensions are experienced. This means that the dynamics of cosmological phase transitions taking place in the early universe when the Compton wavelength $\sim T^{-1}$ of the thermal excitations was still smaller than the length scale of the extra dimension(s) is different from the usual dynamics in four dimensions.

To simplify our discussion, let us consider again the simplest scalar field theory with the potential

$$V(\varphi_c) = -\frac{\mu^2}{2}\varphi_c^2 + \frac{\lambda}{4}\varphi_c^4 \qquad (2.31)$$

where $\mu^2$ is a positive bare mass term. The vacuum expectation value of the scalar field in the present vacuum is therefore $\langle\varphi_c\rangle = \mu/\sqrt{\lambda}$. We know, however, that in $D = 4$ and at very high temperature, the bare mass $\mu$ receives a temperature-dependent correction $\delta m^2(T) = \lambda T^2/4$. As a result, for temperatures higher than the critical temperature

$$(T_c)_{D=4} = 2\frac{\mu}{\sqrt{\lambda}} \;, \qquad (2.32)$$

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the vacuum expectation value of the scalar field vanishes. This is the signal of a phase transition.

Let us now suppose that there is an extra dimension which opens up at a certain length scale $R$, and let us follow the dynamics of the system. Since at energies smaller than $R^{-1}$ one should recover the low-energy effective theory in four dimensions, it is reasonable to assume that $\mu$ is smaller than $R^{-1}$.

When the thermal gas was extremely hot, such that $RT \gg 1$, the universe was effectively five-dimensional. Now, if there appear $T = 0$ corrections to the effective potential of the form (2.24) — and if the correction $\mathcal{O}(1/R^2)$ to the bare squared mass $-\mu^2$ is negative — then it is clear that the phase transition will occur at a temperature $T_c = \mathcal{O}(R^{-1})$, independently of the value of $\mu$. This is already an interesting result if we believe, for instance, that there is an extra dimension at the TeV-scale and the order parameter is the Standard-Model Higgs field. Under these circumstances, the electroweak phase transition will be very different from what is usually expected.

If, on the other hand, the corrections of the form (2.24) are not present (*e.g.*, as would arise without Scherk-Schwarz supersymmetry-breaking), we can easily convince ourselves that for $\lambda RT \gtrsim 1$, the plasma mass $\delta m(T)$ is much smaller than the value suggested by the one-loop analysis, but nevertheless too large for the phase transition to occur.

When the universe cools down to values of the temperature $T \lesssim (\lambda R)^{-1}$, but still larger than $R^{-1}$, the phase transition may occur when the plasma squared mass becomes larger than the negative bare squared mass $-\mu^2$. This takes place at the critical temperature

$$(T_c)_{D=5} \;=\; \left( \frac{2\pi}{3\zeta(3)} \frac{\mu^2}{\lambda R} \right)^{1/3} . \tag{2.33}$$

This estimate is valid as long as $T_c R \gtrsim 1$ and the high-temperature expansion is valid. This translates into the bound $\sqrt{\lambda} \lesssim \mu R \lesssim \lambda^{-1}$, which is not very stringent if $\lambda \lesssim 1$. In addition, one should also be aware of the power-law running of the four-dimensional couplings [1], since this will affect the determination of the critical temperature. Indeed the critical temperature (2.33) satisfies $RT_c \lesssim 1/\lambda$, and for $\lambda \lesssim 1$ the power-law running caused by the large number of Kaluza-Klein states can drastically affect the couplings in the tree-level potential. Of course, this issue should be addressed in more realistic theories than we are considering here.

Note that the ratio between the critical temperatures (2.33) and (2.32) is

$$r \;\equiv\; \frac{(T_c)_{D=5}}{(T_c)_{D=4}} \;\approx\; 0.6 \left( \frac{\lambda^{1/2}}{\mu R} \right)^{1/3} , \tag{2.34}$$

and lies in the range $\sqrt{\lambda} \lesssim r \lesssim 1$. We thus see that the effect of the extra dimension — besides preventing a phase transition from being first-order — is to delay the instant at which the phase transition occurs. This is not surprising, because a large fraction

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of the Kaluza-Klein tower now contributes to the plasma mass squared, increasing it by a factor $\sim RT$.

## 2.5  Other features

Even though these results have been obtained for a toy model, we expect these features to be present in more realistic theories. They should therefore have a profound impact on our understanding of the early universe, and many aspects of early-universe cosmology should be now reconsidered under the supposition that the universe might experience extra dimensions at early epochs. Here, we shall briefly outline some of these features.

We know that the monopole problem is one of the central issues in modern astroparticle physics. The problem of monopoles is especially serious since it is generic to the idea of GUTs where the GUT gauge group is broken via the Higgs mechanism.[†] The production of magnetic monopoles in cosmological GUT phase transitions by the Kibble mechanism seems almost unavoidable, and is very much akin to the mechanism for the production of various defects in ordinary laboratory phase transitions. In the more familiar $D = 4$ cosmology, approximately one monopole per horizon should arise at the GUT phase transition, so that the resulting monopole-to-entropy ratio is expected to be of the order of $n_M/s \sim (T_c/M_{\mathrm{Pl}})^3$, where $M_{\mathrm{Pl}} \approx 1.2 \times 10^{19}$ GeV is the Planck mass. Barring significant monopole-antimonopole annihilation, entropy production, or the presence of large global charges at early epochs which may prevent the phase transition [31, 32], the relic monopole density today is unacceptable. Let us suppose, however, that the energy scale of the extra dimension is close to the GUT scale, so that the GUT phase transition is actually occurring in five dimensions. A naïve estimate then leads to a monopole-to-entropy ratio

$$\frac{n_M}{s} \;\sim\; \left(\frac{R^{1/2}T_c^{7/2}}{M_{\mathrm{Pl}}^3}\right)_{D=5} \,, \tag{2.35}$$

where we have used the fact that the entropy density scales like $RT^4$ and that one expects the formation of one monopole per horizon volume. This ratio (2.35) may be much smaller than the corresponding one in $D = 4$. Even though the formation of magnetic monopoles in extra-dimensional cosmology needs to be addressed more rigorously before any firm conclusion can be drawn, our estimates seem to suggest that the monopole problem may be ameliorated in the scenario depicted in this paper.

Other issues include the formation of cosmic strings in $D > 4$ dimensions as the result of the spontaneous breaking of abelian symmetries; the possibility of extreme supercooling and a subsequent period of inflation if the energy density of the plasma becomes smaller than the vacuum energy density associated with a scalar (inflaton)

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[†]Note, however, that in higher dimensions [1], the GUT symmetry may be broken via an alternative mechanism involving orbifolds (*e.g.*, Wilson lines). In such cases, GUT monopoles may have different properties [30].

potential; and the dynamics of the electroweak phase transition if the extra dimensions open up at the TeV-scale.

# 3    Extension to string theory

In the previous section, we studied the behavior of higher-dimensional phase transitions using a field-theoretic approach. However, such an approach ultimately faces an important limitation. As we mentioned in the Introduction, the fact that such higher-dimensional gauge theories are non-renormalizable implies that their properties depend on ultraviolet cutoffs which in turn signal the appearance of new ultraviolet physics. String theory is the only known consistent higher-dimensional theory which lacks the divergences ordinarily associated with non-renormalizable field theories. In this section, therefore, we shall consider an extension of our analysis to string theory.

There also exists another reason why it is important to consider the extension to string theory. As mentioned in the Introduction, one of the primary motivations for considering large extra dimensions is that they can lower the fundamental GUT scale [1] and Planck scale [10]. However, as discussed in Refs. [17, 10, 18, 1, 19], such scenarios must ultimately be embedded into reduced-scale string theories in order to be consistent. For example, if the GUT and Planck scales are reduced to the TeV range, then this ultimately requires a TeV-scale string theory as well. Therefore, the "stringy" behavior ordinarily associated with string thermodynamics will now become important at far lower energies than previously thought relevant in the discussion of the early universe, and hence will have heightened significance.

In this section, we shall focus on two such "stringy" effects. The first of these concerns the Hagedorn transition, while the second concerns the possible generation and stabilization of a large radius of compactification due to thermal effects.

## 3.1    The Hagedorn phenomenon:
## limiting temperature vs. phase transition

One of the most profound differences between string theory and field theory is the presence of an exponentially growing number of string states as a function of mass. These states arise as string oscillator modes which are not present in a theory in which the fundamental degrees of freedom are point particles. As first pointed out in the 1960's by Hagedorn [33], theories with exponentially growing numbers of states exhibit a remarkable phenomenon, namely a critical temperature beyond which the thermodynamic partition function (and indeed all subsequent thermodynamic quantities) cannot be defined. This critical temperature is called the Hagedorn temperature, and can be interpreted either as a limiting temperature or as the location of a phase transition. It turns out that this ultimately depends on the details of the physical system in question, and will be discussed in detail below. Thus, the

nature of the Hagedorn phenomenon is ultimately one of the focal points of any discussion of string theory at finite temperature.

Let us begin by briefly reviewing some of the aspects of the Hagedorn transition in arbitrary numbers of uncompactified dimensions $D$. While the situation in the critical dimension $D = 10$ is well-understood, there is apparently some confusion in the literature regarding the effects caused by the compactification to $D < 10$ and the proper treatment of the corresponding Kaluza-Klein excitations. This will be particularly important for theories with large radii of compactification. Therefore, one of our aims in this section will be to resolve these discrepancies.

Let us begin, as in the previous section, by recalling the general expression for the free energy at finite temperature in $D$ spacetime dimensions

$$\ln Z \ \sim \ \int_0^\infty dM \ \rho(M) \ \int d^{D-1}k \ \ln \left( \frac{1 + e^{-\beta\sqrt{k^2+M^2}}}{1 - e^{-\beta\sqrt{k^2+M^2}}} \right) \ . \tag{3.1}$$

This expression is the $D$-dimensional analogue of (2.5) and (2.20), where we have assumed a supersymmetric configuration of bosonic and fermionic states whose density is given by $\rho(M)$ at mass $M$. As usual, the total energy of any state with mass $M$ is given by $E^2 = k^2 + M^2$, and we have neglected (and will continue to neglect) overall numerical coefficients. Note that (3.1) is simply the expression for the logarithm of the macrocanonical partition function $Z$, as indicated; this is related to the potentials $V$ discussed in Sect. 2 via $V = -T \ln Z$. Also note that unlike the situation in field theory, where we considered the dependence of $M$ on a background field $\varphi_c$, in string theory are forced to set $\varphi_c = 0$ and treat $M$ as a free parameter.

In our analysis in Sect. 2, we considered only a discrete set of bosonic or fermionic states, so that $\rho(M)$ was essentially a delta-function $\delta(M - M(\varphi_c))$. However, in string theory, we have a much more complicated set of states which consist of not only Kaluza-Klein modes, but also string oscillator and winding modes. As a result, $\rho(M)$ takes the more complicated form

$$\rho(M) \ \sim \ aM^{-b}e^{cM} \quad \text{as } M \to \infty \ . \tag{3.2}$$

Here $(a, b, c)$ are presumed to be constant, positive coefficients whose values depend on the particular system under study. It is this change in $\rho(M)$ which leads to the important difference between field theory and string theory. As we shall see, the parameter $c$ ultimately determines the Hagedorn temperature of the system, while the parameter $b$ determines whether this temperature is to be interpreted as a limiting temperature or as the site of a phase transition.

Note that for the purposes of this analysis, we are not dealing with a full string theory. Rather, we are implicitly dealing with a gas of particles whose properties (such as the density of states) match the individual modes of the string. This approximation will be sufficient for our purposes.

Type I, Type II, and heterotic strings all have critical dimensions $D = 10$, and the result (3.1) applies directly in this case. The density given in (3.2) then includes the contributions from only string oscillator states, since there are no Kaluza-Klein or winding-mode states resulting from compactification. However, once we compactify to spacetime dimensions $D < 10$, we must properly incorporate the contributions of Kaluza-Klein and winding-mode states. There are two equivalent ways in which this can be done. The first way, as in the previous section, is to replace the momentum integrations in (3.1) that correspond to compactified directions with discrete summations over Kaluza-Klein and winding modes. This then leads, as before, to products of $\vartheta$-functions in the integrand, and we should continue to demand that $\rho(M)$ include the contributions of string oscillator states only. However, in string theory it turns out to be simpler to choose a second method: we can neglect the contributions from Kaluza-Klein states to the momentum integration altogether, and simply incorporate their effects in a string calculation of $\rho(M)$. It turns out that this changes the value of $b$ without affecting the value of $c$. These methods are ultimately equivalent because the Kaluza-Klein $\vartheta$-functions in the momentum integrand effectively shift the value of $b$ in $\rho(M)$ by a $D$-dependent amount, and this amount can be most easily calculated using the underlying conformal symmetry of the string directly. Therefore, in this section, we shall *neglect the contributions of the Kaluza-Klein states in the momentum integral, and compensate for this by including their effects in the value of $b$*. This is an important point which has been missed in several prior analyses of the string Hagedorn transition in $D < 10$ dimensions. We will also follow the same procedure for the string winding states, which have no analogue in a field theory based on point particles.

With this understanding, let us now proceed to evaluate (3.1), taking $D$ to represent the number of *uncompactified* dimensions only. From this result, we will then be able to calculate the internal energy $U(T)$ as well as other thermodynamic quantities. We shall follow parts of the approaches outlined in Refs. [34, 35, 36, 37]. Because we are interested in the high-temperature behavior, the contributions from the extremely massive string states dominate. Therefore, we can Taylor-expand the logarithm, obtaining

$$\ln Z \;\sim\; \int_0^\infty dM \; \rho(M) \; \int d^{D-1}k \; e^{-\beta\sqrt{k^2 + M^2}} \; . \tag{3.3}$$

Next, we perform the momentum integrations, obtaining

$$\ln Z \;\sim\; \int_0^\infty dM \; \rho(M) \; \beta^{1-D/2} \, M^{D/2} \, K_{D/2}(\beta M) \; . \tag{3.4}$$

Here $K_{D/2}(z)$ is the modified Bessel function of third kind, with asymptotic behavior $K_\nu(z) \sim z^{-1/2} \, e^{-z}$ as $|z| \to \infty$. We thus obtain

$$\begin{aligned}
\ln Z \;&\sim\; \int_0^\infty dM \; \rho(M) \; (M/\beta)^{(D-1)/2} \, e^{-\beta M} \\
&\sim\; \int_0^\infty dM \; M^{-b+(D-1)/2} \, \beta^{(1-D)/2} \, e^{-(\beta-c)M}
\end{aligned} \tag{3.5}$$

where we have substituted the density (3.2) into the last line.

The divergence of this integral at the $M \to 0$ endpoint is unphysical, reflecting the failure of the asymptotic form (3.2) to properly reflect the true number of physical states in the $M \to 0$ limit. For a proper treatment, a more precise functional form should be used in this limit; methods for deriving such forms can be found in Ref. [38]. What concerns us here, however, is the opposite extreme as $M \to \infty$. Here the asymptotic form (3.2) is presumed to be accurate, whereupon we see that the partition function $Z(\beta)$ necessarily diverges unless $\beta > c$. This then defines the critical (Hagedorn) temperature, given by

$$T_H \; \equiv \; c^{-1} \; . \tag{3.6}$$

For $T < T_H$, the partition function is finite and the corresponding thermodynamic properties can be defined without difficulty. At $T = T_H$, however, this description based on the canonical ensemble fails, and one must resort to a more fundamental description of the physics (*e.g.*, one based on the microcanonical ensemble) in order to determine the nature of this transition.

However, as indicated above, one clue can already be determined directly from the canonical ensemble. It is possible to study the behavior of the free energy $F(T)$ and the internal energy $U(T)$ as functions of the temperature as we approach the Hagedorn transition from below. If these thermodynamic quantities also diverge, then an infinite amount of energy would be required to propel the system past the Hagedorn temperature. In such cases, the Hagedorn temperature is a true limiting temperature of the system. On the other hand, if these thermodynamic quantities remain finite as $T \to T_H$, then infinite amounts of energy are not required, and $T_H$ is more appropriately interpreted as the site of a phase transition.

In order to derive the conditions that distinguish between these two cases, let us first consider the free energy itself and take $T \to T_H$ (or $\beta \to c$) in (3.5). The exponential term then cancels, and we are left with

$$\ln Z \; \sim \; \int_0^\infty dM \; M^{-b+(D-1)/2} \; . \tag{3.7}$$

This has an ultraviolet divergence for $b \leq (D+1)/2$, with the divergence becoming logarithmic when this inequality is saturated. Similarly, the internal energy $U(T) \equiv -\partial \ln Z/\partial \beta$ and the entropy $S(T) \equiv \beta^2 \partial(\beta^{-1} \ln Z)/\partial \beta$ have an ultraviolet divergence for $b \leq (D+3)/2$, and the specific heat $c_V \equiv \beta^2 \partial^2 \ln Z/\partial \beta^2$ has an ultraviolet divergence for $b \leq (D+5)/2$. These results agree with those found in Refs. [36, 37]. In each case, we then find that the relevant thermodynamical quantity $X(T)$ diverges as

$$X(T) \; \sim \; \begin{cases} |\ln(T_H - T)| & \text{for } b = b_{\text{crit}} \\ (T_H - T)^{b - b_{\text{crit}}} & \text{for } b < b_{\text{crit}} \end{cases} \tag{3.8}$$

where $b_{\text{crit}} = (D+1)/2 + n$, with $n$ denoting the number of $\beta$-derivatives of $\ln Z$ necessary to produce $X(T)$. We conclude that if $b \leq (D+3)/2$, it takes an infinite

amount of energy to raise the temperature of the system past $T_H$ — *i.e.*, in such cases, $T_H$ is to be interpreted literally as a physical limiting temperature. By contrast, for $b > (D+3)/2$, only a finite amount of energy is needed, whereupon $T_H$ is more appropriately interpreted as a site of a phase transition.

Let us now consider the values of $b$ and $c$ that arise in string theory. It is here that we shall have to be careful to properly incorporate the effects of the Kaluza-Klein states (and winding-mode states) resulting from compactification.

First, strictly speaking, string theory provides us not with a density of states $\rho(M)$, but rather a discrete set of energy levels characterized an oscillation number $n$ and a corresponding number of states $g_n$. In general, these degeneracies $g_n$ take the asymptotic form

$$g_n \sim A n^{-B} e^{C\sqrt{n}} \quad \text{as } n \to \infty \tag{3.9}$$

where once again $(A, B, C)$ are positive constants which depend on the particular string theory in question. In order to relate (3.9) to (3.2), we need to know how the pure number $n$ relates to the spacetime mass $M$. In general, this relation depends on the type of string theory under consideration, and is given by

$$n = \frac{f}{4}\alpha' M^2 \quad \text{where } f = \begin{cases} 1 & \text{for closed strings} \\ 4 & \text{for open strings.} \end{cases} \tag{3.10}$$

Here $\alpha' \equiv M_{\text{string}}^{-2}$ is the Regge slope, and the factor $f$ reflects the different conventional normalizations for the lengths of closed versus open strings. The second step is to extract a density $\rho(M)$ from the level degeneracies $g_n$. By equating the discrete partition function $Z \equiv \sum_n g_n e^{-\beta M}$ with the continuous partition function $Z \sim \int dM \rho(M) e^{-\beta M}$ in the limit $T \to T_H$, we obtain

$$\rho(M) = (\tfrac{1}{2} f \alpha' M) g_n . \tag{3.11}$$

Note, in particular, that we do not divide by $M$ to obtain the density; rather we multiply by $M$ and adjust the units via $\alpha'$. Thus, putting the pieces together, we are able to relate the coefficients $(B, C)$ in (3.9) to the coefficients $(b, c)$ in (3.2), yielding

$$b = 2B - 1 , \qquad c = \tfrac{1}{2}\sqrt{f\alpha'}\, C . \tag{3.12}$$

The string Hagedorn temperature is therefore given by

$$\sqrt{\alpha'}\, T_H = \frac{2}{\sqrt{f}} C^{-1} \tag{3.13}$$

and we find

$$\begin{aligned} B &\leq (D+5)/4 & \Longrightarrow & \quad \text{limiting temperature} \\ B &> (D+5)/4 & \Longrightarrow & \quad \text{phase transition.} \end{aligned} \tag{3.14}$$

The final step is to calculate the coefficients $B$ and $C$ for the different string theories, taking proper account of the Kaluza-Klein and winding modes as well as the usual string oscillator modes. In the case of closed strings, both $B$ and $C$ receive separate contributions from the left- and right-moving components of the worldsheet theory; these contributions are then added together. In the case of open strings, by contrast, the left- and right-moving oscillations are required to conspire to form standing waves, and hence only one such component (left or right) is sufficient to describe the states of the string. In either case, it turns out [38] that these left- and right-moving contributions depend on only the light-cone central charge $\gamma$ of the appropriate worldsheet conformal field theory and the modular weight $k$ of its characters:

$$C_{L,R} = \sqrt{\frac{2\gamma}{3}}\,\pi\,, \qquad B_{L,R} = \frac{3}{4} - \frac{k}{2}\,. \qquad (3.15)$$

Note that the role of the Kaluza-Klein states is to leave the light-cone central charge of the conformal field theory unaffected (thereby preserving the value of $C$), but to modify the modular weight of the characters (thereby affecting $B$). This occurs because the summation over Kaluza-Klein modes introduces additional $\vartheta$-functions, each with modular weight $+1/2$, into the full string one-loop partition function. Thus, while the value of $C$ is unaffected by the compactification of a given string theory from $D = 10$ to $D < 10$, the value of $B$ is changed in a dimension-dependent manner. It is this effect which was not incorporated into several prior analyses [36, 37].

Given this understanding, we shall now simply quote the results. For a Type II string compactified to $D$ spacetime dimensions, we find

$$C = 4\sqrt{2}\pi\,, \qquad B = \frac{11}{2} - \frac{10-D}{2} = \frac{1}{2}(D+1)\,. \qquad (3.16)$$

Here the first contribution to $B$ comes from the string oscillator modes, while the second comes from the Kaluza-Klein modes. This combined value of $B$ implies a phase transition for all spacetime dimensions $D \geq 4$ at the temperature $\sqrt{\alpha'}T_H = (2\sqrt{2}\pi)^{-1}$, or $T_H \approx M_{\text{string}}/9$. Likewise, for a heterotic string compactified to $D$ spacetime dimensions, we find

$$C = 2(2+\sqrt{2})\pi\,, \qquad B = \frac{1}{2}(D+1)\,, \qquad (3.17)$$

again implying a phase transition at all spacetime dimensions $D \geq 4$ at the slightly lower temperature $\sqrt{\alpha'}T_H = [(2+\sqrt{2})\pi]^{-1}$, or $T_H \approx M_{\text{string}}/11$. In general, the properties of such a phase transition and the physics beyond the Hagedorn temperature are not well-understood. Various discussions can be found in Refs. [35, 36, 37, 39, 40, 41, 42, 43, 44, 45, 46].

However, the situation is completely different for Type I strings. If we first consider the contributions from the perturbative open-string sectors (corresponding to

open strings stretched between the compactified nine-branes), we find

$$ C = 2\sqrt{2}\pi \ , \qquad B = \frac{1}{4}\left(D+1\right) \ . \qquad (3.18) $$

This leads to the same Hagedorn temperature as in the Type II case; of course, this is to be expected since the Type II theory is a subset of the Type I theory and corresponds to its closed-string sector. However, because of the different value of $B$, *we see that the open string theory has a true limiting temperature for all values of $D$*. Note, in particular, that this result disagrees with that found in Table 2 of Ref. [37], where the contributions of the Kaluza-Klein states were not taken into account. Thus, we see that within the context of an open string theory, we face the prospect of a true limiting Hagedorn temperature *for all values of $D$*. In other words, all energy pumped into the system goes into exciting high-mass open-string states rather than into increasing the thermal kinetic energy of the low-mass string states.

One natural question that arises in the case of open strings is whether this conclusion is affected by the presence of non-perturbative Dirichlet $p$-branes and their associated excitations. After all, it might seem that since the $p$-branes have different effective dimensionalities which depend on $p$, they might give rise to non-perturbative states whose thermodynamical properties depend on $p$ rather than on the full spacetime dimension $D$. Ultimately, however, it can be shown that this is not the case. Mathematically, this can be seen by analyzing the partition functions of the corresponding non-perturbative string sectors. Physically, however, we can easily see that although a given open string might have its endpoints restricted to a $p$-brane, the density of states to which it gives rise is determined by its *excitations*, *i.e.*, its varied embeddings into the external $D$-dimensional spacetime. Thus, the states that arise from the potentially non-perturbative $(p_1, p_2)$-sectors of open-string theory will obey the same properties as those from the perturbative nine-brane/nine-brane sectors discussed above, irrespective of the values of $(p_1, p_2)$.

Thus, to summarize, we see that the behavior of various thermodynamic quantities depends on the effective spacetime dimension in different ways, depending on whether we are dealing with closed or open strings. These results are summarized in Table 1.

These results have important implications for Type I string theories. Recall that Type I string theories contain both closed- and open-string sectors. The closed strings correspond to the gravitational sector (as well as those gauge symmetries resulting from compactification of the higher-dimensional gravity theory). By contrast, the open strings give rise to the gauge symmetries resulting from the nine-branes (Chan-Paton factors). Thus, we see that within the context of open-string theories, *it is possible for the gravitational and gauge sectors to experience different thermodynamic behaviors* as the Hagedorn temperature is approached. Specifically, we see that it is possible for the gravitational sector to undergo a Hagedorn phase transition and enter an (unknown) post-Hagedorn phase, while the Chan-Paton gauge sector instead feels a limiting temperature with divergent thermodynamic quantities.

| | closed strings | | | open strings |
|---|---|---|---|---|
| | $D=4$ | $D=5$ | $D=6$ | all $D$ |
| $F$ | $+$ | $+$ | $+$ | $-1$ |
| $U, S$ | $+$ | $+$ | $+$ | $-2$ |
| $c_V$ | $-1/2$ | $0$ | $+$ | $-3$ |

Table 1: Divergence behavior of the free energy $F$, the internal energy $U$, the entropy $S$, and the specific heat $c_V$ as $T \to T_H$, for both closed and open strings, as a function of the number $D$ of uncompactified spacetime dimensions. For each thermodynamic quantity $X$, we have listed the corresponding divergence exponent $x$, defined as $X(T) \sim (T_H - T)^x$ as $T \to T_H$. Here $x = 0$ indicates the logarithmic behavior $X(T) \sim |\log(T - T_H)|$, and '+' indicates a non-divergent quantity.

This situation might have various cosmological consequences. For instance, in the pre-big-bang cosmology [47], a period of dilaton-driven inflation is ended when the curvature becomes of order of the string scale, thus preventing the scale factor of the three-dimensional universe from reaching the singularity. A smooth transition to the standard hot big-bang cosmology is supposed to follow. It is possible, though, that in the phase of high curvature the Kaluza-Klein modes and the oscillator and winding modes of the string are efficiently excited. If these modes thermalize, one might expect that the resulting temperature is of the order of the string scale, leading to a Hagedorn phase transition in the gravitational sector. This stage might change the estimate of the total energy stored in the quantum fluctuations amplified by the pre-big-bang backgrounds, which might in turn change the way the universe enters the radiation-dominated phase.

There are also several novel features in the case of Type I strings with large-radius compactifications. Ordinarily, in a string theory whose compactification radii are close to the string scale, it is not possible to change the spacetime dimensionality as a function of the energy scale when the energy is below the Hagedorn temperature. This is because, as we have seen, the Hagedorn temperature is typically an order of magnitude below the string scale. Thus, the effective value of $D$ is fixed in such theories. However, if the string theory in question has an intermediate-scale radius whose energy scale $R^{-1}$ is substantially below the corresponding string scale, it is possible, upon increasing the energy and temperature of the system, to cross the radius threshold and thereby effectively increase the value of $D$.

This raises some intriguing possibilities. The electroweak phase transition might have taken place when the temperature of the universe was not far below the Hagedorn temperature. Furthermore, we see from Table 1 that our findings can have an important effect on the behavior of the specific heat $c_V$ as the temperature of the system is increased. Specifically, we can easily imagine a situation in which the specific heat is driven towards large values as energy is pumped into the system, until the energy exceeds the radius threshold and a new dimension opens up. This in turn

could change the thermodynamics of the system in such a way that large amounts of entropy are suddenly "released" when the universe cools and the number of extra dimensions decreases. This in turn could dilute the densities of unwanted relics such as domain walls and magnetic monopoles which were created at earlier epochs. Indeed, this is the stringy analogue of the idea of large entropy generation via dimensional compactification in field theory [20]. Moreover, in the present case, the decay of an exponentially large number of massive string states may be of further help.

Finally, note that in the case of open strings, the Hagedorn phenomenon provides us with a natural way of generating extremely large values of thermodynamic quantities such as energy and entropy — indeed, for temperatures approaching the Hagedorn temperature, these values will be much larger than would have been expected without string theory. This simple fact may also have important cosmological implications. For example, in order to solve the smoothness and flatness problems of the standard big-bang cosmology [48], one requires that the patch containing our present observed universe contain an entropy greater than about $10^{88}$. Therefore, creating a large amount of entropy close to the Hagedorn temperature may help in explaining why the universe looks so smooth and flat to us.

However, as we shall now discuss, perhaps the most useful implication of this fact is that it may be used to generate and stabilize a large radius of compactification.

## 3.2   Thermal generation of a large compactification radius

Let us now turn our attention to perhaps the most important problem that affects any discussion of theories with large extra spacetime dimensions: the generation (and indeed the stabilization) of such a large radius of compactification. While there are many mechanisms that might be imagined (see, for example, Refs. [49, 15]), in this section we shall explore a relatively simple idea. In string theory, spacetime supersymmetry forces all compactification radii to act as moduli — *i.e.*, they have an exactly flat potential to all orders in perturbation theory. However, because bosons and fermions feel different statistics at finite temperatures, thermal effects necessarily break supersymmetry. It is therefore natural to expect that thermal effects can produce a potential for the radii of compactification.

In this section, we shall explore this idea within the context of a simple toy string model, namely the ten-dimensional supersymmetric Type I $SO(32)$ string evaluated at finite temperature and toroidally compactified down to nine dimensions with a single radius of compactification. As a function of the temperature, we shall calculate the one-loop potential for the radius. It turns out that the behavior we shall find is generic, even for compactifications down to four dimensions.

Unlike the case in field theory, three new features arise in the case of Type I string theory. The first, of course, is the Hagedorn phenomenon, discussed above. The second is the presence of not only momentum modes whose energies are inversely proportional to the compactification radius $R$, but also *winding* modes whose

energies grow linearly with $R$. Such winding modes arise in only the closed-string (gravitational) sector of the Type I theory. Finally, in Type I string theory, the usual one-loop field-theory diagram now generalizes to receive four separate contributions from the four possible unoriented one-loop topologies: the torus, the Klein bottle, the cylinder, and the Möbius strip. Each gives rise to a different radius-dependence.

In this paper, we shall not review the construction of the $SO(32)$ Type I theory or the derivation of these one-loop amplitudes. Instead, we shall merely write down the results. In order to do this, we first define the circle-compactified partition function

$$Z_R(\tau) \;\equiv\; \sum_{m,n} \frac{1}{\overline{\eta}\eta} \, \overline{q}^{(ma-n/a)^2/4} \, q^{(ma+n/a)^2/4} \tag{3.19}$$

where $q \equiv e^{2\pi i \tau}$, where

$$\eta(\tau) \;\equiv\; q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \;, \tag{3.20}$$

and where we have defined the inverse dimensionless radius $a \equiv \sqrt{\alpha'}/R$. Here the sums over $(m, n)$ respectively represent the contributions from Kaluza-Klein momentum and winding modes of the string. Next, we define four related circle-compactified functions which are equivalent to $Z_R(\tau)$ except for various restrictions on their summation variables:

$$\begin{aligned}
\mathcal{E}'_0 &\equiv& \{m \in \mathbb{Z}, \; n \text{ even}\} \\
\mathcal{E}'_{1/2} &\equiv& \{m \in \mathbb{Z} + \tfrac{1}{2}, \; n \text{ even}\} \\
\mathcal{O}'_0 &\equiv& \{m \in \mathbb{Z}, \; n \text{ odd}\} \\
\mathcal{O}'_{1/2} &\equiv& \{m \in \mathbb{Z} + \tfrac{1}{2}, \; n \text{ odd}\} \;.
\end{aligned} \tag{3.21}$$

Third, we define the so-called characters of the transverse Lorentz group $SO(8)$ in terms of the Jacobi $\vartheta$-functions given in (2.10):

$$\begin{aligned}
\chi_I &=& \tfrac{1}{2}(\vartheta_3{}^4 + \vartheta_4{}^4)/\eta^4 \\
\chi_V &=& \tfrac{1}{2}(\vartheta_3{}^4 - \vartheta_4{}^4)/\eta^4 \\
\chi_S &=& \tfrac{1}{2}(\vartheta_2{}^4 + \vartheta_1{}^4)/\eta^4 \\
\chi_C &=& \tfrac{1}{2}(\vartheta_2{}^4 - \vartheta_1{}^4)/\eta^4 \;.
\end{aligned} \tag{3.22}$$

These characters correspond to spacetime scalars, vectors, spinors, and conjugate spinors of $SO(8)$ respectively. Note that $\chi_V = \chi_S = \chi_C$ as functions of $\tau$; this is a manifestation of $SO(8)$ triality. Finally, we note that the incorporation of finite-temperature effects can be achieved in the usual way, via the introduction of Matsubara modes which can be viewed as the Kaluza-Klein modes corresponding to the compactification of an additional (time) direction on a circle of radius $R_T = \beta/2\pi = (2\pi T)^{-1}$. We will therefore also define the corresponding inverse-radius variable

$$a_T \;\equiv\; \frac{\sqrt{\alpha'}}{R_T} \;=\; 2\pi \, \frac{T}{M_{\text{string}}} \tag{3.23}$$

and use the same circle-compactified functions as before. Note from (3.23) that $2\pi T$ serves as a "thermal mass". Also note, in this regard, that the Hagedorn temperature for the Type I system corresponds to $a_T^* = 1/\sqrt{2}$. We shall therefore be forced to restrict our attention to values $a_T < 1/\sqrt{2}$ in what follows.

Given these definitions, it is then straightforward to write down the string contributions to the one-loop effective potential $V$ (or equivalently, to the so-called cosmological constant $\Lambda \equiv V/T$) arising from the four sectors corresponding to the torus, the Klein-bottle, the cylinder, and the Möbius-strip one-loop amplitudes. We find

$$\Lambda = \Lambda_T + \Lambda_K + \Lambda_C + \Lambda_M \tag{3.24}$$

where these individual contributions are given as [6, 50]:

$$\Lambda_T = -\tfrac{1}{2} \int_{\mathcal{F}} \frac{d^2\tau}{(\operatorname{Im}\tau)^5} (\overline{\eta}\eta)^{-6} \left\{ [\overline{\chi}_V\chi_V + \overline{\chi}_S\chi_S] \, \mathcal{E}_0'(a_T) \;+\; [\overline{\chi}_I\chi_I + \overline{\chi}_C\chi_C] \, \mathcal{O}_0'(a_T) \right.$$
$$\left. -\; [\overline{\chi}_V\chi_S + \overline{\chi}_S\chi_V] \, \mathcal{E}_{1/2}'(a_T) \;-\; [\overline{\chi}_I\chi_C + \overline{\chi}_C\chi_I] \, \mathcal{O}_{1/2}'(a_T) \right\} Z_R(\tau)$$

$$\Lambda_K = -\tfrac{1}{2} \int_0^\infty \frac{dt}{t^5} \, \eta(q^4)^{-8} \, (\chi_V - \chi_S)(q^4) \left( \sum_{m=-\infty}^{\infty} q^{2m^2 a_T^2} \right) \left( \sum_{m=-\infty}^{\infty} q^{2m^2 a^2} \right)$$

$$\Lambda_C = -\tfrac{1}{2} N^2 \int_0^\infty \frac{dt}{t^5} \, \eta(q)^{-8} \left\{ \chi_V(q) \sum_{m=-\infty}^{\infty} q^{m^2 a_T^2} \;-\; \chi_S(q) \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2 a_T^2} \right\}$$
$$\times \sum_{m=-\infty}^{\infty} q^{m^2 a^2}$$

$$\Lambda_M = \tfrac{1}{2} N \int_0^\infty \frac{dt}{t^5} \, \eta(-q)^{-8} \left\{ \chi_V(-q) \sum_{m=-\infty}^{\infty} q^{m^2 a_T^2} \;-\; \chi_S(-q) \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2 a_T^2} \right\}$$
$$\times \sum_{m=-\infty}^{\infty} q^{m^2 a^2} \; . \tag{3.25}$$

In each case, these (dimensionless) cosmological constants are given in units of $\tfrac{1}{2}\mathcal{M}^8$, where $\mathcal{M}$ is the reduced string scale $M_{\text{string}}/2\pi$; thus they have the dimensions of inverse eight-volumes in our nine-dimensional theory. In (3.25), $N = 32$ is the number of nine-branes in the theory, with $q \equiv e^{2\pi i\tau}$ for the torus amplitude and $q \equiv e^{-\pi t}$ for the Klein-bottle, cylinder, and Möbius-strip amplitudes. Note that while the torus amplitude receives contributions from both Kaluza-Klein momentum and winding modes, the string orientifold projection removes all winding modes, and consequently the remaining three amplitudes receive contributions from only the compactification momentum modes. Also note that $\Lambda_K = 0$ as a result of the identity $\chi_V = \chi_S$, signalling that spacetime supersymmetry is not broken by thermal effects in the Klein-bottle sector of the theory. However, in each of the remaining sectors, the supersymmetry is broken through the relative half-shift in the Matsubara frequencies between bosonic states (corresponding to $\chi_{I,V}$) and fermionic states (corresponding to $\chi_{S,C}$).

Note that (3.25) is simply the stringy generalization of field-theoretic expressions such as (2.8). This can be shown (for example, for the cylinder amplitude) as follows. For simplicity, we shall take $R \to \infty$ (or $a \to 0$). Since $\chi_V = \chi_S$, we can write the resulting cylinder amplitude as

$$\Lambda_C \sim T^9 \int_0^\infty \frac{ds}{s^{11/2}} \left( \frac{\vartheta_2^4}{\eta^{12}} \right) (2\pi^2 is/a_T^2) \left[ \vartheta_3(4\pi^2 is) - \vartheta_2(4\pi^2 is) \right] \tag{3.26}$$

where $s \equiv a_T^2 t/(4\pi)$. In the field-theory limit $T \ll M_{\text{string}}$ (or $a_T \ll 2\pi$), the contribution $\vartheta_2^4/\eta^{12}$ from the string oscillators goes to a constant. By using the identity

$$\int_0^\infty \frac{d^{D-1}k}{(2\pi)^{D-1}} \ln \left( \frac{1 - e^{-\beta E}}{1 + e^{-\beta E}} \right) =$$
$$- \frac{1}{2(4\pi)^{(D-1)/2}} \int_0^\infty \frac{ds}{s^{(D+1)/2}} e^{-M^2 s} \left[ \vartheta_3(4\pi is T^2) - \vartheta_2(4\pi is T^2) \right] , \tag{3.27}$$

we see that (3.26) yields the effective potential of a supersymmetric ten-dimensional field theory at finite temperature. Thus, we see that the field-theory limit of our string calculation corresponds to $T \ll M_{\text{string}}$.

At a formal level, it is interesting to note that the $R \to \infty$ limit of this model is equivalent to the nine-dimensional non-supersymmetric Type I $SO(32)$ "interpolating model" which was constructed in Ref. [50] and used in order to construct strong/weak coupling Type I duals for non-supersymmetric heterotic strings. Indeed, as shown in Ref. [6], the temperature $T$ here plays the role of the compactification radius $R$ of Ref. [50] via the relation $T = 1/\pi R$. However, unlike Ref. [50], we shall hold $T$ fixed and calculate the dependence of $\Lambda$ on $R$ for fixed $T$. This will ultimately yield the desired finite-temperature potential for the radius $R$.

Evaluating these integrals is relatively straightforward, and proceeds using methods analogous to those discussed in Ref. [50]. The results are shown in Figs. 2 and 3. In Fig. 2, we show the separate torus, cylinder, and Möbius-strip contributions to the effective potential, as well as their total. For this calculation we have taken $a_T = 1/2$. It is clear that the cylinder contribution dominates the sum, due to the large multiplicity of nine-branes in the theory. It is also clear that while the torus contribution exhibits the expected $T$ duality under which $a \to 1/a$ or $R \to \alpha'/R$ (due to the presence of both momentum and winding modes in this sector), the open-string contributions do not. Instead, as $R$ becomes small, the compactified momentum modes become extremely heavy and their contributions to the potential vanish. Thus, as $R \to 0$, all radius dependence essentially "freezes out" of these open-string contributions, and their contributions to the total effective potential become flat in this regime.

It is important to note that although we are plotting values of $R$ for $R/\sqrt{\alpha'} < 1$ as well as $R/\sqrt{\alpha'} > 1$, their physical interpretation is completely different. For

$R/\sqrt{\alpha'} > 1$, the radius is larger than the fundamental string length, so the proper interpretation is indeed that of a compactified Type I theory. For $R/\sqrt{\alpha'} < 1$, by contrast, the radius is smaller than the fundamental string length. In this case the proper interpretation is the $T$-dual one, namely that of a compactified Type I$'$ theory where the radius $R$ now signifies the distance perpendicular to the nine-branes. Thus, in this way our analysis is applicable to both "universal" extra dimensions as well as those felt only by gravity.

All of the plots in Fig. 2 are calculated for $a_T = 1/2$. In Fig. 3, by contrast, we show how the total effective potential depends on the temperature $a_T = 2\pi T/M_{\text{string}}$. For small temperatures, we see that the effective potential becomes flat, reflecting the restoration of spacetime supersymmetry in this limit. For larger temperatures, however, we see that the effective potential becomes quite strong and steep. Of course, as expected, this potential ultimately diverges towards negative infinity for all radii $R$ as $a_T \to a_T^* = 1/\sqrt{2} \approx 0.707$. This is of course simply the Hagedorn temperature of the theory, and reflects the contribution of a tachyonic Matsubara winding-mode state which appears in the torus amplitude at this temperature [40, 41].

It is important to interpret the results in Figs. 2 and 3 correctly. It is clear, of course, that in all cases, *the finite-temperature effects set up an instability which pushes the radius away from the fundamental string scale at $R = \sqrt{\alpha'}$ and towards either extremely large or extremely small values*. (In this connection, recall that all of these potentials go to $-\infty$ as $R \to 0$.) Thus, we see that finite-temperature effects can provide a natural mechanism for generating hierarchically large or small radii within the context of Type I string theory. Moreover, we also see that finite-temperature effects tend to render the self-dual point $R = \sqrt{\alpha'}$ unstable. Our findings might therefore have important implications for the pre-big-bang scenario where the sizes of the extra dimensions are usually thought to converge towards their respective self-dual values.

Of course, there are several important caveats that must be mentioned if one tries to implement this mechanism within a cosmological model. First of all, these potentials represent only the contributions that come from thermal effects. While these might be supposed to dominate at high temperatures (*e.g.*, near the Hagedorn temperature), there will be other effects at lower temperatures that will come into play, such as the zero-temperature potentials that arise due to ordinary supersymmetry-breaking effects. Furthermore, as we shall discuss below, entropy conservation can be expected to play an important role if the universe undergoes an adiabatic evolution. However, it is interesting that these finite-temperature effects by themselves are capable of generating either large or small radii at an early epoch, while the universe is presumably still dominated by thermal behavior. This mechanism could therefore be useful in setting up the initial large- or small-radius *pre-conditions* before other potentials come into play.

Although these potentials clearly generate large radii of compactification, this still leaves one further question unanswered: how are such radii ultimately *stabilized*?
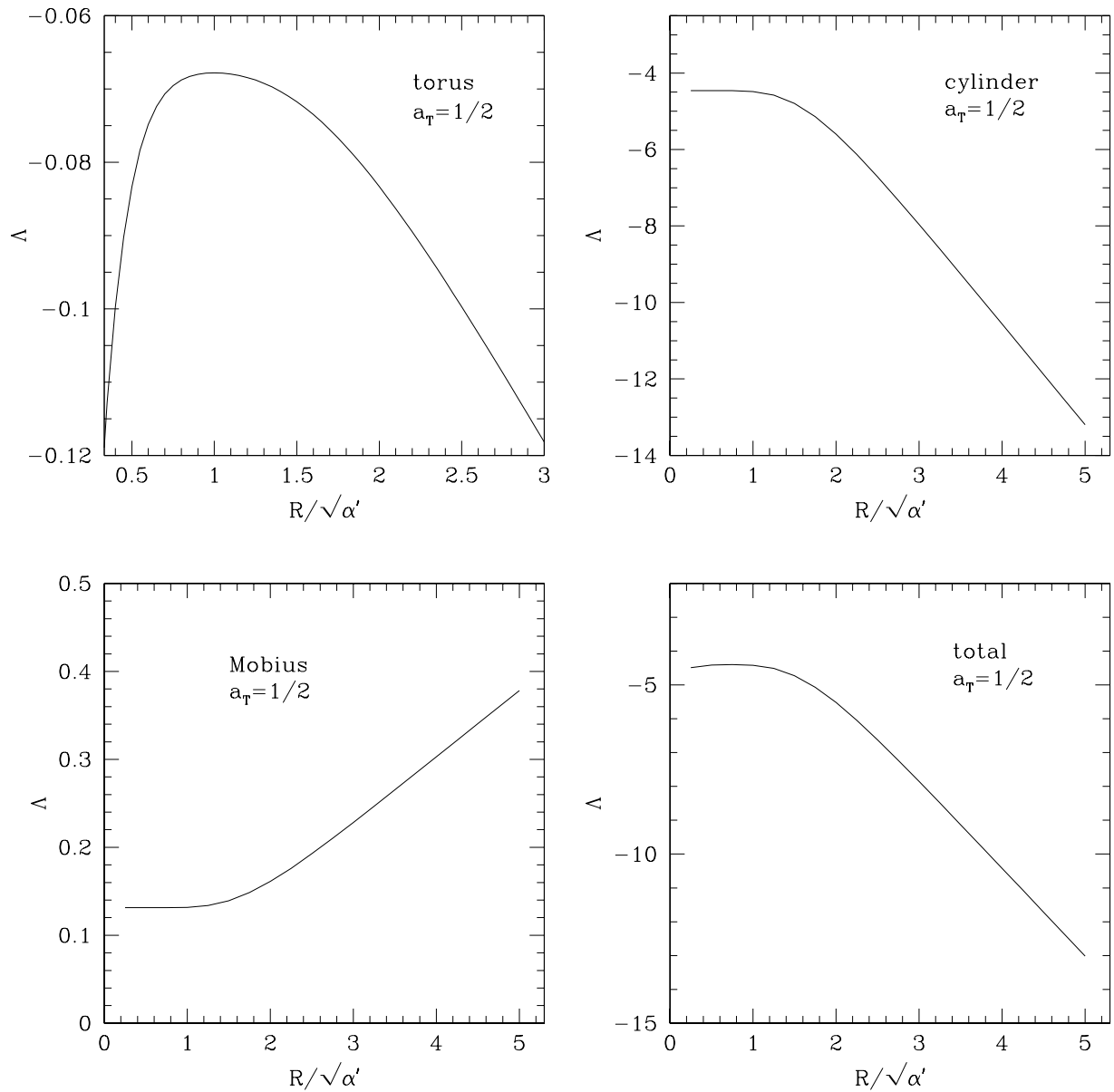
Figure 2: Individual contributions to the effective potential as a function of the radius of compactification, evaluated at the temperature $a_T = 1/2$ or $T/M_{\text{string}} = 1/4\pi$. The total effective potential (lower right) shows a clear tendency to push the radius out to large values.
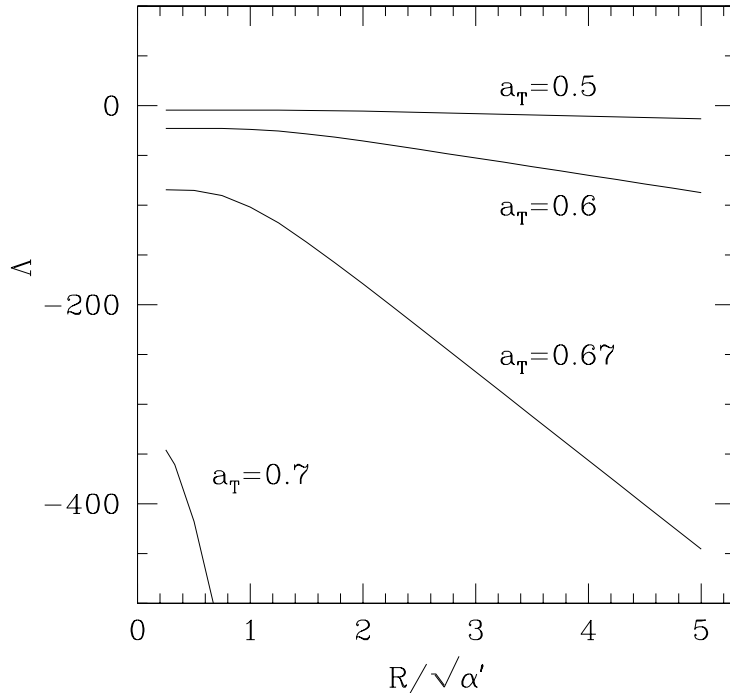
Figure 3: The effective potential as a function of the radius of compactification, for temperatures ranging from $a_T = 0.5$ (or $T/M_{\text{string}} = 1/4\pi$) to $a_T = 0.7$ (or $T/M_{\text{string}} = 7/20\pi$). In all cases, the radius is pushed out to large values. Note that the potential diverges to negative values at the Hagedorn temperature $a_T = 1/\sqrt{2} \approx 0.707$.

While one can imagine many possible effects that could intercede and stabilize the compactification radii as the universe cools, one natural way to stabilize the radii at all temperatures is through the constraint of entropy conservation. Indeed, such a constraint is appropriate when the universe undergoes an adiabatic evolution. Given the cosmological constants $\Lambda$ that we have calculated, it is a straightforward procedure to calculate the total entropy $S$ via the relation

$$ S \; = \; -\frac{\partial V}{\partial T} \; = \; -\left(\Lambda + T\frac{\partial}{\partial T}\Lambda\right) \; . \tag{3.28} $$

We then obtain the results shown in Fig. 4(a) through Fig. 4(e). Note that $S$ has the same units as $\Lambda$, namely those of an inverse eight-volume, and should be interpreted as an entropy density with respect to the eight-dimensional flat space. However, this quantity represents the *total* entropy with respect to the compactified dimension with radius $R$.

It is clear from these plots that the entropy $S$ rises linearly as a function of the compactification radius in the limit $R/\sqrt{\alpha'} \gg 1$. It also clear that the entropy
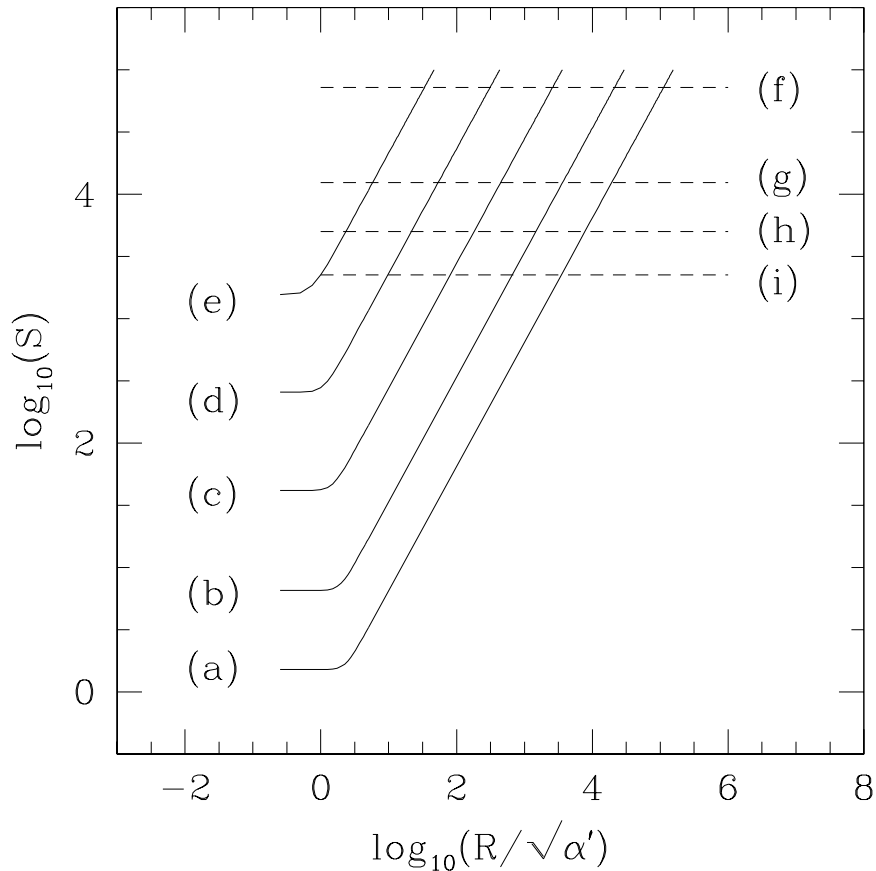
30

Figure 4: *Solid lines*: Entropy as a function of the string compactification radius, for temperatures (a) $a_T = 0.333$; (b) $a_T = 0.4$; (c) $a_T = 0.5$; (d) $a_T = 0.6$; and (e) $a_T = 0.667$. *Dashed lines*: Values of the entropy for temperatures (f) $a_T = 0.7$; (g) $a_T = 0.69$; (h) $a_T = 0.68$; and (i) $a_T = 0.667$, all calculated at $R/\sqrt{\alpha'} = 1$ and held constant as a function of radius. If we assume that a cooling phase of the universe becomes adiabatic (entropy-conserving) at a given initial temperature [(f) through (i)] when the compactification radius is at the string scale, then hierarchically large compactification radii are generated at lower temperatures [(a) through (e)]. Note that it is the Hagedorn phenomenon at $a_T^* = 1/\sqrt{2} \approx 0.707$ that leads to the dramatic rise in the initial entropy as a function of temperature which in turn generates such hierarchically large radii of compactification.

rises dramatically as a function of the temperature. Of course, this behavior is not surprising, for we expect on the basis of field theory alone that in a ten-dimensional spacetime where one dimension has been compactified (such as in our toy string model), the total entropy should scale as

$$S \sim R T^9 \tag{3.29}$$

for $RT \gg 1$. However, the important point is that in string theory, this dependence is even stronger than in field theory. Indeed, this enhanced dependence becomes increasingly evident as the temperature approaches the Hagedorn temperature $T_H$, for in this limit the "stringy" behavior begins to dominate and we expect from the results in Table 1 that

$$S \approx \frac{S_0}{(T_H - T)^2} \qquad \text{as} \quad T \to T_H \tag{3.30}$$

for some (radius-dependent) $S_0$. This behavior is shown from Fig. 5, where we plot the entropy as a function of the temperature, with fixed radius $R/\sqrt{\alpha'} = 1$. For small temperatures $a_T \lesssim 0.5$, the entropy indeed grows as a power of the temperature, in accordance with (3.29). For larger temperatures, however, the entropy begins to exhibit the Hagedorn behavior (3.30), with $S_0 \approx 3.63$ and $T_H = 1/\sqrt{2}$.

Combining these two observations, we see that this leads to a natural cosmological mechanism for generating and stabilizing a hierarchically large radius of compactification. Of course, we cannot speculate with any certainty about the nature of the physics at the Hagedorn temperature. However, let us imagine the universe cooling below this temperature. We shall begin by assuming a radius of compactification near the string scale, so that $R/\sqrt{\alpha'} = 1$. As the universe cools, we expect that at some temperature $T_{\mathrm{ad}}$ the evolution becomes adiabatic, so that the total entropy is conserved. Presumably this might happen quite early, while the temperature is still relatively close to the Hagedorn temperature and the entropy is therefore extremely high. In Figs. 4(f) through Fig. 4(i), for example, we have illustrated various values of the (fixed) entropy that would result. We then find that large compactification radii are generated for temperatures below $T_{\mathrm{ad}}$. For example, consulting Fig. 4, we see that if $T_{\mathrm{ad}}$ corresponds to $a_T = 0.7$ (as shown in Fig. 4(f)), then the compactification radius will have grown to $\gtrsim 10^5$ in string units by the time the temperature has dropped to $a_T = 1/3$. Further cooling will produce compactification radii that are even hierarchically larger.

Once again, there are several important comments and caveats that must be mentioned in this connection. First, it may seem that much of this radius-enhancement effect is purely field-theoretic. After all, according to (3.29), entropy conservation alone implies the relation $R_1/R_2 = (T_2/T_1)^9$. However, after compactification to four dimensions (and assuming that all six compactified dimensions have equal compactification radii $R$), the total entropy scales as $S \sim R^6 T^9$, whereupon the field-theoretic entropy/temperature scaling relation weakens to $R_1/R_2 = (T_2/T_1)^{3/2}$. By contrast,
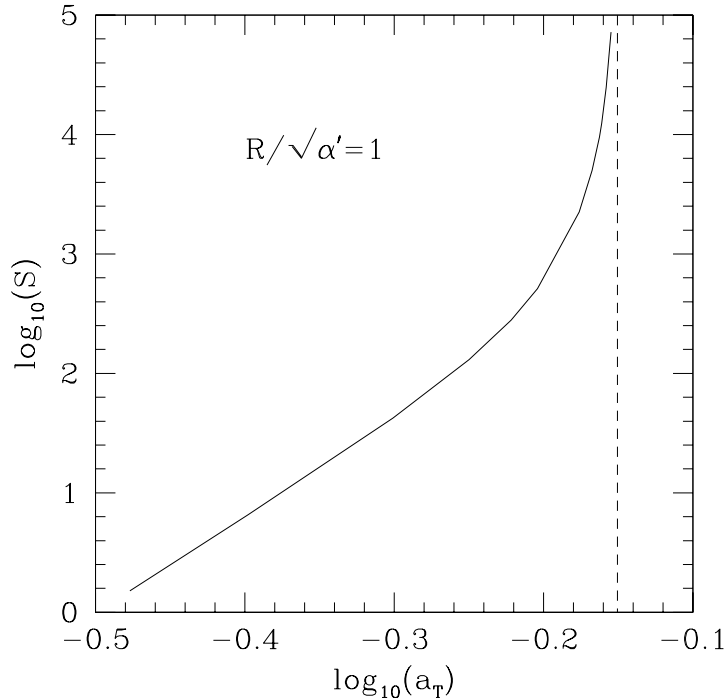
Figure 5: *Solid line*: Entropy as a function of the temperature, evaluated for fixed radius $R/\sqrt{\alpha'} = 1$. For small temperatures, the entropy behaves as expected in field theory, while the stringy Hagedorn behavior becomes dominant at higher temperatures. *Dashed line*: the Hagedorn limiting temperature.

for sufficiently high temperatures, the string-theoretic scaling behavior (3.30) is, as we have seen in Sect. 3.1, valid for all spacetime dimensions *regardless* of their radii of compactification. Thus, within the context of string theories near the Hagedorn temperature, we have a natural mechanism for boosting the total entropy to values that exceed those possible in field theory, and this in turn can generate large compactification radii as the universe cools.

Finally, we must also mention another important caveat. In the above analysis, we assumed that our underlying nine- or four-dimensional spacetime is flat, with fixed (infinite) radius. Of course, in a realistic cosmological setting, this will not be the case, and we can expect to deal with at least *two* radii, $R_>$ and $R_<$, corresponding to the three large spatial dimensions and six small dimensions respectively. In this case, the field-theoretic entropy relation becomes $S \sim R_>^3 R_<^6 T^9$. In string theory, of course, the entropy continues to diverge at the Hagedorn temperature. Thus, we once again expect to generate hierarchically large values of entropy at large temperatures. However, the implications of this fact for the generation of large compactification radii will depend on how this extra entropy is ultimately distributed between the large and

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small dimensions. This in turn will depend on some additional outside input, such as the dynamics of the large radii as a function of time or temperature. For example, it might be that at primordial epochs, some of the extra spatial dimensions are somehow frozen or contracting very slowly. Under these circumstances, if the universe is cooling, the change of the size of the radius of the remaining extra dimension(s) may be dramatic near the Hagedorn transition.

However, as pointed out in Ref. [49], this issue may be further complicated due to various stringy effects, such as the radius-stabilizing effects of string winding modes. Such string modes can wind around the compactified dimensions, in the process essentially halting their expansion. Thus, it is far from clear what generic predictions can be made in such cases, and we leave this issue for further study.

It is also interesting to note that different effective potentials arise for different sectors of the theory. In particular, the gravitational sector feels only the torus contribution, while the gauge sector feels the full sum of the contributions. This means that different radii of compactification might be generated for the gravitational and the gauge sectors at finite temperature. For instance, the gauge sector of the theory might feel extra dimensions compactified at the scale of $(\text{TeV})^{-1}$, as discussed in Ref. [1], while gravity might live in a bulk of dimension of millimeter-length [10]. Therefore, the dynamics of dimensional compactification might be different for the gravitational and gauge sectors. Indeed, we have already seen in Sect. 3.1 that the gravitational (closed) string sector experiences only a phase transition at the Hagedorn temperature, while the gauge (open) string sector actually feels a limiting temperature. This might have important implications for some crucial cosmological issues such as the production of gravitons in the bulk and the transition to the standard hot big-bang.

Indeed, many other related issues also arise in this context. For example, many strong/weak coupling duality relations in string theory provide non-perturbative connections between open strings and closed strings. The most famous example of this is the strong/weak coupling duality between the $SO(32)$ Type I (open) string and the $SO(32)$ heterotic (closed) string. It would be interesting to understand the implications of such duality relations for the distinction between a Hagedorn limiting temperature and phase transition. Likewise, it would be interesting to understand how temperature duality (the symmetry under which $T \to M_{\text{string}}^2/T$) emerges on the heterotic side as the Type I coupling is increased. Clearly, as the coupling increases on either side of the duality relation, interactions (and the effects of non-perturbative $D$-brane states) should change the thermodynamic behavior. Of course, it remains an important issue as to whether duality even holds for $T \neq 0$, since supersymmetry is broken. Despite certain pieces of evidence (see, *e.g.*, Ref. [50]), it is not yet clear whether duality holds without supersymmetry or at finite temperature.

Thus, we conclude that within the context of finite-temperature string theory, thermal effects provide a natural way of not only generating large (in some cases, hierarchically large) radii of compactification, but also stabilizing these radii at these

large values. The main new feature relative to the field-theory case is the generation of large amounts of entropy near the string Hagedorn temperature. Of course, our analysis was only in the context of a nine-dimensional toy string model in which we took the underlying nine-dimensional spacetime to be fixed in volume. However, we expect that the mechanism that we have illustrated (whereby hierarchically large values of entropy are converted to hierarchically large compactification radii) is interesting, and may also find application a more realistic setting.

## Acknowledgments

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