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U-duality and M-Theory

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This work is intended as a pedagogical introduction to M-theory and to its maximally supersymmetric toroidal compactifications, in the frameworks of 11D supergravity, type II string theory and M(atrix) theory. U-duality is used as the main tool and guideline in uncovering the spectrum of BPS states. We review the 11D supergravity algebra and elementary 1/2-BPS solutions, discuss T-duality in the perturbative and non-perturbative sectors from an algebraic point of view, and apply the same tools to the analysis of U-duality at the level of the effective action and of the BPS spectrum, with a particular emphasis on Weyl and Borel generators. We derive the U-duality multiplets of BPS particles and strings, U-duality invariant mass formulae for 1/2- and 1/4-BPS states for general toroidal compactifications on skew tori with gauge backgrounds, and U-duality multiplets of constraints for states to preserve a given fraction of supersymmetry. A number of mysterious states are encountered in $D \leq 3$, whose existence is implied by T-duality and 11D Lorentz invariance. We then move to the M(atrix) theory point of view, give an introduction to Discrete Light-Cone Quantization (DLCQ) in general and DLCQ of M-theory in particular. We discuss the realization of U-duality as electric–magnetic dualities of the Matrix gauge theory, display the Matrix gauge theory BPS spectrum in detail, and discuss the conjectured extended U-duality group in this scheme.

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1 Introduction

1.1 Setting the scene

Since its invention in the late sixties, string theory has grown up in a tumultuous history of unexpected paradigm shifts and deceptive lulls. Not the least of these storms was the discovery that the five anomaly-free perturbative superstring theories were as many glances on a single eleven-dimensional *theoria incognita*, soon baptized M-theory, awaiting a better name [307, 319]. The genus expansion of each string theory corresponds to a different perturbative series in a particular limit $g_s \to 0$ in the M-theory parameter space, much in the same way as the genus expansion arises in 't Hooft large-N, fixed-$g^2_{\rm YM}N$ regime of Yang–Mills theory [302]. M-theory can be defined by the superstring expansions on each patch, and the superstring (perturbative or non-perturbative) dualities allow a translation from one patch to another, in a way analogous to the definition of a differential manifold by charts and transition functions. This analogy overlooks the fact that string theories are only defined as asymptotic series in $g_s \to 0$, and some analyticity is therefore required to move into the bulk of parameter space.

This definition has been effective in uncovering a number of features of M-theory, or rather its BPS sector, which behaves in a controlled way under analytic continuation at finite-g_s. In particular, M-theory is required to contain Cremmer, Julia and Scherk's eleven-dimensional supergravity [76] in order to account for the Kaluza–Klein-like tower of type IIA D0-branes as excitations carrying momentum along the eleventh dimension of radius $R_s \sim g_s^{2/3}$, as shown by Townsend and Witten [307, 319]; it should also contain membrane and fivebrane states, in order to reproduce the D2- and D4-brane, as well as the NS5-brane and the type IIA "fundamental" string. Which of these states is elementary is not decided yet, although M2-branes and D0-branes are favourite candidates [86, 24]. It may even turn out that none of them may be required, and that 11D SUGRA may emerge as the low-energy limit of a non-gravitational theory [163].

While the *dualities* between string theories relate different languages for the same physics, the *symmetries* of string theory provide a powerful guide into M-theory, which is believed to hold beyond the BPS sector. The best established of them is certainly T-duality, which identifies seemingly distinct string backgrounds with isometries (see for instance Refs. [135, 3] and references therein). Throughout this review, we shall restrict ourselves to maximally supersymmetric type II or M theories, and accordingly T-duality will reduce to the inversion of a radius on a d-dimensional torus. To be more precise, a *T-duality* maps to each other type IIA and type IIB string theories compactified on circles with inverse radii, while a *T-symmetry* consists of an even number of such inversions (together with Kalb–Ramond spectral flows to which we shall return), and therefore corresponds to a symmetry of type II string theories and of their M-theory extension[‡1]. As we shall recall, such T-symmetries on a torus T^d generate a $SO(d,d,\mathbb{Z})$ discrete symmetry group, the continuous version of which $SO(d,d,\mathbb{R})$ appears as a symmetry of the low-energy

[‡1]Having emphasized this point, we shall henceforth omit the distinction between dualities and symmetries.

effective action.

On the other hand, the action of 11D or type IIA supergravity compactified on a torus T^d as well as of the equations of motion of uncompactified type IIB supergravity have for long been known to exhibit continuous non-compact global symmetries, namely the exceptional symmetry $E_{d(d)}(\mathbb{R})$ of Cremmer and Julia and the $Sl(2,\mathbb{R})$ symmetry of Schwarz and West respectively [72, 182, 185, 277]. These symmetries transform the scalar fields and in general do not preserve the weak coupling regime, which puts them out of reach of perturbation theory, in contrast to the well established target-space T-duality.

In analogy with the electric–magnetic $Sl(2,\mathbb{Z})$ Montonen-Olive-Sen duality of four-dimensional $N=4$ super Yang–Mills theory [231, 281], Hull and Townsend have proposed [175] that a discrete subgroup $E_{d(d)}(\mathbb{Z})$ (resp. $Sl(2,\mathbb{Z})_B$) remains as an exact quantum symmetry of M-theory compactified on a torus T^d (resp. of ten-dimensional type IIB string theory and compactifications thereof)[‡2]. The two statements are actually equivalent, since after compactification on a circle the type IIB string theory becomes equivalent under T-duality on the (say) tenth direction to type IIA, and the symmetry $E_{d(d)}(\mathbb{Z})$ can be obtained by intertwining the $Sl(2,\mathbb{Z})_B$ non-perturbative symmetry with the T-duality $SO(d-1,d-1,\mathbb{Z})$. Conversely, the $Sl(2,\mathbb{Z})_B$ symmetry of type IIB theory can be obtained from the M-theory description as the modular group of the 2-torus in the tenth and eleventh directions [273, 16], and is a particular subgroup of the modular group $Sl(d,\mathbb{Z})$ of the d-torus. This, being a remnant of eleven-dimensional diffeomorphism invariance after compactification on the torus T^d, has to be an exact symmetry as soon as M-theory contains the graviton. The T-duality symmetry $SO(d-1,d-1,\mathbb{Z})$ is however not manifest in the M-theory picture. All in all, the U-duality group reads

$$E_{d(d)}(\mathbb{Z}) = Sl(d,\mathbb{Z}) \bowtie SO(d-1,d-1,\mathbb{Z}) \ , \tag{1.1}$$

where the symbol $\bowtie$ denotes the group generated by the two non-commuting subgroups.

The structure of the group (1.1) will be discussed at length in this review, and a set of Weyl and Borel generators will be identified. The former preserve the rectangularity of the torus and the vanishing of the gauge background, while the latter allow a move to arbitrary tori. States are classified into representations of the U-duality group $E_{d(d)}(\mathbb{Z})$, whether BPS or not, and we will derive U-duality invariant mass and tension formulae for 1/2- and 1/4-BPS states, as well as conditions for a state to preserve a given fraction of the supersymmetries. Besides the entertaining encounter with discrete exceptional groups, this will actually teach us about the spectrum of M-theory, since the more M-theory is compactified, the more degrees of freedom come into play. In particular, we will show the need to include states with masses that behave as $1/g_s^n,\ n\geq 3$, which are unconventional in perturbative string theory. An important application of these results is the exact determination of certain physical amplitudes in M-theory, such as the four-graviton R^4 coupling, which can be interpreted as traces over M-theory BPS states [21, 142, 253]. The weak coupling analysis of these exact couplings provides a very useful insight into the rules of semi-classical calculus in string theory [34, 246, 140, 146, 20, 199, 255].

[‡2] The first example of string duality actually appeared in the context of heterotic string theory [119].

A proposal has recently been put forward by Banks, Fischler, Shenker and Susskind to define M-theory *ab initio* on the (discrete) light front, as the large-N limit of a supersymmetric matrix model given by the dimensional reduction of 10D $U(N)$ super Yang–Mills (SYM) theory to $0+1$ dimension [24, 300]. This model also describes low-energy interactions of D0-branes induced by open string fluctuations [82, 186, 100], and, as shown by Seiberg, arises from considering M-theory on the light front as a particular limit of M-theory compactified on a circle, i.e. type IIA theory [278]. D0-branes are therefore identified as the partons of M-theory in this framework. This proposal has passed numerous tests, and has been shown to incorporate membrane and (transverse) fivebrane solutions, and to reproduce 11D SUGRA computations. The invariance under eleven-dimensional Lorentz invariance remains, however, to be demonstrated (see [217] for a step in that direction).

Upon compactification of d dimensions, the D0-branes interact by open strings wrapped many times around the compactification manifold, and the infinite-dimensional quantum mechanics can be rephrased as a gauge theory in $d+1$ dimensions [305, 127]. This dramatic increase of degrees of freedom certainly removes part of the appeal of the proposal, but becomes even more serious for $d \geq 4$, where the gauge theory loses its asymptotic freedom and becomes ill-defined at small distances. We will briefly discuss the proposals for extending this definition to $d = 4, 5$. We will also discuss the interpretation of M-theory BPS states in the gauge theory, and show the occurrence of unconventional states with energy $1/g_{\rm YM}^{2n}$, $n \geq 2$. Despite these difficulties, the Matrix gauge theory gives a nice understanding of U-duality as the electric–magnetic duality of the gauge theory, together with the modular group of the torus on which it lives [299, 127, 110]. The interpretation of finite-N matrix theory as the compactification of M-theory on a light-like circle implies that the U-duality group $E_{d(d)}(\mathbb{Z})$ be enlarged to $E_{d+1(d+1)}(\mathbb{Z})$ [172, 54, 244, 173]; we will show that this extra symmetry mixes the rank N of the gauge group with charges in a way reminiscent of Nahm duality. All these features are guidelines for a hypothetical fundamental definition of Matrix gauge theory.

1.2 Sources and omissions

This review is intended as a pedagogical introduction to M-theory, from the point of view of its 11D SUGRA low-energy limit, its strongly coupled type II string description, and its purported M(atrix) theory definition. It is restricted to maximally supersymmetric toroidally compactified M-theory, and uses U-duality as the main tool to uncover the part of the spectrum that is annihilated by half or a quarter of the 32 supersymmetries. The exposition mainly follows [110, 255, 244], but relies heavily on [318, 319, 310, 278, 91]. It is usefully supplemented by other presentations on supergravity solutions [296, 309, 310], M(atrix) theory [22, 51, 94], D-branes [257, 267, 306, 19], string dualities [98, 275, 316, 87, 238, 284] and perturbative string theory [247, 198] and general introductions [254, 14, 276]. The following topics are beyond the scope of this work:

- Black hole entropy: the modelisation of extremal black-holes by D-brane bound states has allowed a description of their microscopic degrees of freedom and a derivation of their Bekenstein–Hawking entropy [297] (see [224, 252] for reviews). The latter can be related to a U-duality invariant of the black hole charges [167, 79, 189, 5, 9], and U-duality can even allow the control of non-extremal states [294].

- Gauge dynamics: the study of D-brane configurations has also led to a qualitative understanding of gauge theories dynamics as world-volume dynamics of these objects; see [133] for a thorough review. We will mainly consider configurations of parallel branes, as describing the Matrix gauge theory description of M-theory on the light cone.

- BPS-saturated amplitudes: A special class of terms in the effective actions of M-theory and string theory receives contributions from BPS states only. We will briefly discuss an application of the M-theory mass formulae that we derived to the computation of exact R^4 couplings in Subsection 5.8, and refer to the existing literature for more details on the exact non-perturbative computation of these couplings, and their interpretation at weak coupling as a sum of instanton effects. Relevant references include [34, 246, 280] for two-derivative terms in $N = 2$ type II strings, [156, 157, 147, 12] for four-derivative terms in type II/heterotic theories, [21, 20, 199], for F^4 (and related) terms in type I/heterotic theories, and [140, 146, 142, 200, 12, 48, 192, 255, 253, 141, 145] for R^4 (and related) terms in type IIB/M-theory. Infinite series of higher-derivative BPS-saturated R^2F^{2g-2} or R^4H^{4g-4}, and R^n terms have also been computed or conjectured in Refs. [11, 226] and [49, 268, 180, 191].

- Scattering amplitudes: in order to validate the M(atrix) theory conjecture of BFSS, a number of scattering computations have been carried out both in the Matrix model and in 11D supergravity; they have shown agreement up to two loops, see for instance [32, 33, 258, 66, 96, 193, 317, 245, 229, 112]. This agreement is better than naively expected, and indicates the existence of non-renormalization theorems [248] for these interactions.

- D-instanton matrix model: an alternative formulation of M-theory as a statistical matrix model has been proposed by Ishibashi, Kawai, Kitazawa and Tsuchiya [179]. It has so far not been developed to the same extent as the BFSS proposal, and in particular the origin of U-duality has not been explicited. See Refs. [64, 114, 160, 113, 270, 203, 63, 65, 123, 201, 53, 225, 301, 13, 4, 124] for further discussion.

- Twelve dimensions and beyond: the structure of U-duality symmetry has led to speculate on the existence of a 12D [313, 27, 30, 242] or higher [28, 29, 26, 241, 293, 266, 240] dimensional parent of M-theory, with extra time directions. The $N = 2$ heterotic strings suggest an appealing construction of this theory (see [227] for a review). However, the full higher-dimensional Lorentz symmetry is partially reduced to its U-duality subgroup, and its usefulness remains unclear at present. We shall, however, encounter in Subsection 4.6 a tantalizing hint for an extra time-like direction with "length" l_p^3.

- String networks: a construction of 1/4-BPS states based on three-string junctions [275, 60, 83] has been suggested [288], that reproduces the U-duality invariant mass formula in 8 dimensions [50, 50, 208]. These solutions have been constructed from the M2-brane [206, 228] and their dynamics discussed in [264, 61], but their supergravity description is still unclear.

- Non-commutative geometry: it has been argued that non-commutative geometry [69] is the appropriate framework to discuss D-brane dynamics, and is even required in the presence of Kalb–Ramond two-form background [70, 99, 67]. This description incorporates T-duality [261, 271] and even U-duality in its Born–Infeld generalization [162]. It should in particular (see Subsection 7.8) extend Nahm's duality of ordinary two-dimensional Maxwell theory to higher-dimensional cases [237]. Related discussions can be found in Refs. [161, 213, 190, 44, 233, 211, 71].

- Gauged supergravity: 11D SUGRA possesses maximally supersymmetric backgrounds other than tori, namely compactifications on products of spheres and anti-deSitter spaces [120]. These correspond to the near-horizon geometry of M2- and M5-branes, and have been argued to provide a dual description to the gauge theory on these extended objects [223]. They will be ignored in this review.

- String theories with non maximal supersymmetry: the $E_8 \times E_8$ heterotic string and type I string can be obtained from M-theory by orbifold compactification [165, 166], while the $SO(32)$ heterotic string is related to $E_8 \times E_8$ by a T-duality, or to type I string theory by a non-perturbative duality [259]. M(atrix) theory descriptions have been proposed both in the heterotic [25, 235, 215, 194, 262, 164, 137, 187, 216, 205, 204, 90] and the type II [117, 197, 46, 138, 195, 89, 90] cases, as well as on non-orientable surfaces [196, 322], and will not be treated here.

- Non-BPS states: The study of stable non-BPS states has been iniated in [287] and further examined [286, 289, 285, 36]. It would be interesting to investigate the implications of U-duality symmetry on the spectrum of non-BPS states.

1.3 Outline

Section 2 introduces the superalgebra and fundamental BPS states of M-theory in the context of 11D SUGRA and type IIA/B superstring theories. T-duality is recalled and revisited in Section 3 from an algebraic point of view, at the level of the effective action and of the perturbative and non-perturbative BPS spectrum. The same techniques are used in Section 4 to introduce U-duality and its action on the spectrum of particles and strings, restricting to Weyl generators. Borel generators are incorporated in Section 5, where U-duality invariant mass and tension formulae for general toroidal compactification with arbitrary gauge backgrounds are derived, as well as U-duality multiplets of BPS constraints. Section 6 introduces Matrix gauge theory as the Discrete Light-Cone Quantization of M-theory following an argument by Seiberg, and discusses the dictionary between M-theory and Matrix gauge theory. The U-duality symmetry is finally discussed in Section 7 from the perspective of the Matrix gauge theory, as well as the extended U-duality symmetry arising from the extra light-like direction.

2 M-theory and BPS states

2.1 M-theory and type IIA string theory

M-theory was originally introduced as the strong coupling limit of type IIA superstring theory. The latter has been argued [318, 307] to dynamically generate an extra compact dimension at finite coupling of radius $R_s \sim g_s^{2/3}$ in units of an eleven-dimensional Planck length l_p:

$$R_s/l_p = g_s^{2/3} \ , \quad l_p^3 = g_s l_s^3 \tag{2.1}$$

where $1/l_s^2 = \alpha'$ denotes the string tension and g_s its coupling constant. The strong coupling limit $g_s \to \infty$ should therefore exhibit eleven-dimensional $N=1$ super-Poincaré invariance.

While a consistent eleven-dimensional theory of quantum gravity is still missing, it has been known for a long time that type IIA supergravity can be obtained from the eleven-dimensional $N=1$ supergravity of Cremmer, Julia and Scherk, by dimensional reduction on a circle. M-theory is therefore required to reduce at energies much smaller than $1/l_p$ to 11D SUGRA, in the same way as type IIA (or type IIB) superstring theory reduces to type IIA [62, 178, 131] (or type IIB [277, 168, 272]) supergravity at energies much smaller than $1/l_s$ (which is also smaller than both the ten-dimensional Planck mass $g_s^{-1/4}/l_s$ and the eleven-dimensional Planck mass $g_s^{-1/3}/l_s$ at weak coupling). This is summarized in the following diagram:

$$\begin{array}{ccc} \text{M-theory} & \underset{S^1}{\longrightarrow} & \text{type IIA string theory} \\ \downarrow & & \downarrow \\ \text{11D supergravity} & \underset{S^1}{\longrightarrow} & \text{10D type IIA supergravity} \end{array}$$

where compactification on a circle occurs from left to right and the energy decreases from top to bottom.

The matching relations (2.1) can be easily obtained by studying the Kaluza–Klein reduction of 11D SUGRA, described by the action

$$S_{11} = \frac{1}{l_p^9} \int \mathrm{d}^{11}x \sqrt{-g} \left(R - \frac{l_p^6}{48} (d\mathcal{C})^2 \right) + \frac{\sqrt{2}}{2^7 \cdot 3^2} \int \mathcal{C} \wedge d\mathcal{C} \wedge d\mathcal{C} \tag{2.2}$$

up to fermionic terms that we will ignore in the following. In addition to the usual Einstein–Hilbert term involving the scalar curvature R of the metric g_{MN}, the action contains a kinetic term for the 3–form gauge potential $\mathcal{C}_{MNR}$ (which we shall often denote by $\mathcal{C}_3$) as well as a topological Wess–Zumino term required by supersymmetry. The action (2.2) does not contain any dimensionless parameter, and the normalization of the Wess–Zumino term with respect to the Einstein term is fixed by supersymmetry. The dependence on the Planck length l_p has been reinstated by dimensional analysis, with the following conventions:

$$[dx] = 0, \ [g_{MN}] = 2, \ [\sqrt{-g}] = 11, \ [R] = -2, \ [\mathcal{C}_{MNR}] = 0, \ [d] = 0 \ , \tag{2.3}$$

relegating the dimension to the metric only. In particular, the relation between the eleven-dimensional Planck length and Newton's constant is given by $\kappa_{11}^2 = l_p^9/(2(2\pi)^8)$. We will generally ignore all numerical factors.

Dimensional reduction is carried out by substituting an ansatz

$$ds_{11}^2 = R_s^2(dx^s + \mathcal{A}_\mu dx^\mu)^2 + ds_{10}^2 \tag{2.4}$$

for the metric, where R_s stands for the fluctuating radius of compactification (as measured in the eleven-dimensional metric) and $\mathcal{A}$ describes the Kaluza–Klein $U(1)$ gauge field arising from the isometry along x^s [‡3], and splitting the three-form $\mathcal{C}_{MNR}$ in a two-form $B_{\mu\nu} = \mathcal{C}_{\mu\nu s}$ and a 3-form $\mathcal{C}_{\mu\nu\rho}$. Only the zero Fourier component (i.e. the zero Kaluza–Klein momentum part) of these fields along x^s is kept. On dimensional grounds the scalar curvature becomes

$$R(g_{MN}) = R(g_{\mu\nu}) + \left(\frac{\partial R_s}{R_s}\right)^2 + R_s^2(d\mathcal{A})^2 \ , \tag{2.5}$$

so that the reduced action reads

$$S_{10} = \frac{1}{l_p^9}\int \mathrm{d}^{10}x R_s\sqrt{-g}\left[R + \left(\frac{\partial R_s}{R_s}\right)^2 + R_s^2(d\mathcal{A})^2 + l_p^6(d\mathcal{C})^2 + \frac{l_p^6}{R_s^2}(dB)^2\right] + \int B\wedge d\mathcal{C}\wedge d\mathcal{C} \ . \tag{2.6}$$

On the other hand, the low-energy limit of type IIA string theory is given (in the string frame) by the action

$$S_{\mathrm{IIA}} = \frac{1}{l_s^8}\int d^{10}x\sqrt{-g}\left[e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{l_s^4}{12}(dB)^2\right) - \frac{l_s^2}{4}(d\mathcal{A})^2 - \frac{l_s^6}{48}(d\mathcal{C})^2\right] + \int B\wedge d\mathcal{C}\wedge d\mathcal{C} \ , \tag{2.7}$$

which describes the dynamics of the (bosonic) massless sector

$$\text{NS-NS} \quad : \quad g_{\mu\nu} \ , \quad B_{\mu\nu} \ , \quad \phi \tag{2.8a}$$

$$\text{R-R} \quad : \quad \mathcal{A}_\mu \ , \quad \mathcal{C}_{\mu\nu\rho} \tag{2.8b}$$

denoting the metric, antisymmetric tensor and dilaton from the Neveu–Schwarz square sector, and the one- and three-form gauge potentials from the Ramond square sector (indices μ run over $1\ldots 10$). Ramond p-form gauge fields will be generically denoted by $\mathcal{R}_p$. The dependence on the string length l_s is again instated on dimensional grounds, while the dependence on the coupling

$$g_s^2 = e^{2\phi} \tag{2.9}$$

[‡3] The subscript s is used to indicate that $s$tring theory is obtained in this way.

stems from the fact that the two-derivative action originates from string tree level (hence the $e^{-2\phi}$ factor), with each Ramond field coming with an additional power of e^{ϕ}, ensuring the correct Maxwell and Bianchi identities (see [260] for a recent discussion). In particular, the ten-dimensional Newton's constant is given by $\kappa_{10}^2 = g_s^2 l_s^8$. Identifying the dilaton ϕ with the scalar modulus $\ln R_s$ up to a numerical factor, and matching the two actions (2.6) and (2.7) leads to the relations

$$\frac{R_s}{l_p^9} = \frac{1}{g_s^2 l_s^8}\ , \qquad \frac{1}{R_s l_p^3} = \frac{1}{g_s^2 l_s^4}\ , \qquad \frac{R_s^3}{l_p^9} = \frac{1}{l_s^6}\ , \qquad \frac{R_s}{l_p^3} = \frac{1}{l_s^2}\ , \tag{2.10}$$

obtained by comparing the terms R, dB, $d\mathcal{A}$ and $d\mathcal{C}$ respectively in Eqs. (2.6) and (2.7). Two of these four relations turn out to be redundant as a consequence of supersymmetry, and they can be reduced to the matching relations already stated in Eq. (2.1), or equivalently

$$R_s = l_s g_s\ , \quad \frac{R_s}{l_p^3} = \frac{1}{l_s^2}\ , \tag{2.11}$$

which summarize the relation between the M-theory parameters $\{l_p, R_s\}$ and the string theory parameters $\{l_s, g_s\}$.

Using (2.11) and (2.9) in the metric (2.4) we find the alternative form

$$\frac{ds_{11}^2}{l_p^2} = e^{4\phi/3}(dx^s + \mathcal{A}_\mu \mathrm{d}x^\mu)^2 + e^{-2\phi/3}\frac{\mathrm{d}s_{10}^2}{l_s^2}\ , \tag{2.12}$$

which will be used to relate low-energy solutions of M-theory and type IIA string theory.

2.2 M-theory superalgebra and BPS states

While M-theory has to reduce to 11D SUGRA in the low-energy limit, little is known about its microscopic degrees of freedom. It is however postulated that the $N=1$ supersymmetry of 11D SUGRA should be valid at any energy, and the spectrum is therefore organized into representations of the super-Poincaré algebra [308]:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= (C\Gamma^M)_{\alpha\beta} Z_M + \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta} Z^{MN} \\ &\quad + \frac{1}{5!}(C\Gamma_{MNPQR})_{\alpha\beta} Z^{MNPQR} \end{aligned} \tag{2.13a}$$

$$\left[Q_\alpha, Z^{M\dots}\right] = 0\ . \tag{2.13b}$$

Here Q_α denotes the 32-component Majorana spinor generating the supersymmetry (see [207] for a general account on spinorial representations), and $\Gamma_{MN\dots}$ the antisymmetric product of Γ matrices, i.e. $\Gamma_M\Gamma_N\dots$ for distinct indices and zero otherwise. See Appendix A.1 for our gamma matrix conventions.

In addition to the usual translation operator P_M, which we denoted by Z_M for uniformity, the right-hand side of Eq. (2.13a) contains "central charges" Z^{MN} and Z^{MNRST} in non-trivial representations of the Lorentz group. These charges appear as irreducible representations (irreps) in the decomposition $528 = \mathbf{11} + \mathbf{55} + \mathbf{462}$ of the symmetric tensor product $\{Q_\alpha, Q_\beta\}$, and the simplest assumption is that they should commute with the SUSY charges Q_α [‡4] (their commutation properties with the Lorentz generators are encoded in their index structure). They can be identified as the electric and magnetic charges of extended objects [169, 84] with respect to the gauge potential $\mathcal{C}_{MNP}$ and the metric g_{MN} and their Kaluza–Klein descendants.

The various components of the central charges, their corresponding potentials, as well as the nature of the solution, are summarized in Table 2.1. Here, $\mathcal{E}_6$ denotes the six-form dual to $\mathcal{C}_3$ and $\mathcal{K}_{I;IMNPQRST}$ the 7-form[‡5] dual to the Kaluza–Klein gauge potential g_{IM} after compactifying the direction I. This hints toward the existence of extended states charged under these gauge fields, namely 2-branes, 5-branes, 6-branes and 9-branes. The 9-branes, which are not charged under a gauge potential, are not dynamical and correspond to the "end-of-the-world" branes in compactifications of M-theory with lower supersymmetry [165]. They will not be further considered in this review, but we will shortly return to the M2, M5 and KK6-brane.

Z_0	Z_I	Z^{IJ}	Z^{IJKLM}	Z^{0I}	Z^{0IJKL}
g_{00}	g_{0I}	$\mathcal{C}_{0IJ}$	$\mathcal{E}_{0IJKLM}$	none	$\mathcal{K}_{M;MNPQRST0}$
mass	momentum	M2-brane	M5-brane	9-brane	KK6-brane

Table 2.1: M-theory central charges, gauge fields and extended objects.

The generic representation of the superalgebra (2.13) is generated by the action of 16 fermionic creation operators on a vacuum $|0\rangle$ in a given representation of the Lorentz group; it is therefore 2^{16}-dimensional, i.e. contains 32768 bosonic states and 32768 fermionic states. The positivity of the matrix $\langle 0|\{Q_\alpha, Q_\beta\}|0\rangle$ implies a bound on the rest mass Z_0 known as the Bogomolny bound. When this bound is saturated, part of the supersymmetries annihilate the vacuum $|0\rangle$:

$$\sum_Z Z^{MN\dots}(C\Gamma_{MN\dots})^{\alpha\beta} Q_\beta |0\rangle = 0 \ , \tag{2.14}$$

resulting in a reduced degeneracy. Equation (2.14) requires that the 32×32 matrix $Z^{MN\dots}(\Gamma_{MN\dots})^{\alpha\beta}$ has at least one zero eigenvalue, and implies in particular the BPS condi-

[‡4] It is possible to introduce non-Abelian relations while still preserving the Jacobi identity [292], but the status of this possibility is still unclear.

[‡5] No antisymmetry is assumed for indices separated by a semi-colon. The peculiar index structure $\mathcal{K}_{1;8} = \mathcal{K}_{I;IMNPQRST}$ ensures that the seven-form indices $M, \dots, T$ are distinct from the compact direction I, and the double occurrence of I has the same origin as the square radius R^2 in Eq. (2.28).

tion

$$\det_{\alpha\beta}\left(\sum_Z (Z\cdot\Gamma)^{\alpha\beta}\right) = 0\ , \tag{2.15}$$

which determines the rest mass Z_0 in terms of the other charges.

The dimension can be further reduced if the zero eigenvalue is degenerate, and this requires more relations between the various charges. Since only Z_0 contributes to the trace on the right-hand side of Eq. (2.13a), the maximum number of zero eigenvalues is 16, corresponding to a state annihilated by half the supersymmetries, or in short a 1/2-BPS state. Because of its reduced dimension, a BPS state with smallest charge cannot decay, except if it can pair up with another state of opposite charge to make a representation twice as long [321]. These states can therefore be followed at strong coupling (in the M-theory language, this means for arbitrary geometries of the compactification manifold) and serve as the basis for many duality checks.

As an illustration, we wish to investigate the case where, besides the mass $\mathcal{M} = Z_0$, only the two-form central charges Z^{IJ} do not vanish. This will be later interpreted as an arbitrary superposition of M2-branes. We therefore have to solve the eigenvalue equation:

$$\Gamma\epsilon = \mathcal{M}\epsilon\ , \qquad \Gamma \equiv Z^{IJ}\Gamma_{0IJ} \tag{2.16}$$

Squaring this equation yields

$$\Gamma^2 = Z^{IJ}Z^{IJ} + Z^{IJ}Z^{KL}\Gamma_{IJKL} \mathrel{\hat{=}} \mathcal{M}^2\ , \tag{2.17}$$

where the symbol $\mathrel{\hat{=}}$ denotes the equality when acting on ϵ. The space of solutions now depends on the value of $k^{IJKL} \equiv Z^{[IJ}Z^{KL]} = Z\wedge Z$. If $k=0$, Eq. (2.17) implies $(Z^{IJ})^2 = \mathcal{M}^2$ and $\Gamma^2 = \mathcal{M}^2$. Since $\mathrm{Tr}\ \Gamma = 0$, the 32×32 matrix Γ has 16 eigenvalues $\mathcal{M}$ and 16 eigenvalues $-\mathcal{M}$, and therefore Eq. (2.16) is satisfied for a dimension-16 space of vectors ϵ. The state with charges Z^{IJ} is therefore annihilated by half the supersymmetry generators Q_α, and has a mass

$$\mathcal{M}_0^2 = Z^{IJ}Z^{IJ}\ . \tag{2.18}$$

The condition $Z\wedge Z = 0$ means that the antisymmetric charge matrix Z^{IJ} has rank 2, i.e. that only parallel M2-branes are superposed.

If on the other hand $Z\wedge Z \neq 0$, we may rewrite Eq. (2.17) as

$$\Gamma'\epsilon = \left(\mathcal{M}^2 - \mathcal{M}_0^2\right)\epsilon\ , \quad \Gamma' = k^{IJKL}\Gamma_{IJKL}\ , \tag{2.19}$$

and we are lead back to an equation similar to Eq. (2.16). Squaring again yields

$$\Gamma'^2 = (k^{IJKL})^2 + (k\cdot k)^{IJKL}\Gamma_{IJKL} + (k\wedge k)^{IJKLMNPQ}\Gamma_{IJKLMNPQ} \mathrel{\hat{=}} \left(\mathcal{M}^2 - \mathcal{M}_0^2\right)^2\ , \tag{2.20}$$

where $(k \cdot k)^{IJKL} = k^{IJMN} k^{KLMN}$. As before, if $k \cdot k = k \wedge k = 0$, this equation implies $(k^{IJKL})^2 = (\mathcal{M}^2 - \mathcal{M}_0^2)^2 = \Gamma'^2$. Since Tr $\Gamma' = 0$, Eq. (2.19) is satisfied by half the supersymmetries, but Eq. (2.16) by a quarter only. We therefore get a 1/4-BPS state with mass squared:

$$\mathcal{M}^2 = Z^{IJ} Z^{IJ} + \sqrt{k^{IJKL} k^{IJKL}} \tag{2.21a}$$

$$k^{IJKL} = Z^{[IJ} Z^{KL]} \ . \tag{2.21b}$$

This expression reduces to Eq. (2.18) for a 1/2-BPS state, i.e. when $k^{IJKL} = 0$. On the other hand, if $k \cdot k$ or $k \wedge k \neq 0$ do not vanish, the state is at most 1/8-BPS and we have to carry the same analysis one step further. Note that the conditions $k \cdot k \neq 0$ (resp. $k \wedge k \neq 0$) can only be satisfied when $d \geq 6$ (resp. $d \geq 8$), in agreement with the absence of 1/8-BPS states in more than five space-time dimensions.

2.3 BPS solutions of 11D SUGRA

In want of a microscopic formulation of M-theory (or of non-perturbative type IIA string theory), it is certainly difficult to determine what representations of the eleven-dimensional Poincaré superalgebra actually occur in the spectrum. However, this is achievable for BPS states, since supersymmetry protects these from quantum effects and in particular determine their exact mass formula. They can be studied at arbitrarily low energy, and in particular in the 11D SUGRA limit of M-theory. Instead of describing the equations implied by the BPS condition on the supergravity configuration, we refer the reader to existing reviews in the literature [106, 104, 309, 296, 310], and content ourselves with recalling the four 1/2-BPS standard solutions: the *pp*-wave and three extended solutions, the membrane (or M2-brane), fivebrane (M5-brane) and the Kaluza–Klein monopole, also known as the KK6-brane.

The eleven-dimensional metric describing the extended solutions splits into two parts: the world-volume, denoted by $E^{1,p}$, including the time and p world-volume directions, and the transverse Euclidean part E^{10-p}. These solutions are given in terms of a harmonic function H on the transverse space, which we choose as a single pole

$$H(r) = 1 + \frac{k}{r^{8-p}} \ , \tag{2.22}$$

although any superposition of such poles would do (this is stating the no-force condition between static BPS states; the constant shift in Eq. (2.22) ensures the asymptotic flatness of space-time, required for a soliton interpretation). The constant k depends on Newton's constant κ_{11} and on the p-brane tension, and is quantized by the requirement that the space-time be smooth (we will henceforth choose the smaller quantum).

The *pp*-wave[‡6] and KK6-brane solutions only involve the metric, and read [170, 307]

$$\text{pp-wave}: \quad \mathrm{d}s_{11}^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + (H-1)(\mathrm{d}t + \mathrm{d}\rho)^2 + \mathrm{d}s^2(E^9) \tag{2.23a}$$

[‡6]The name *pp-wave* stands for plane fronted wave with parallel rays [55]. The solution (2.23) was generalized in [41] to include excitations of the three-form potential.

$$H = 1 + \frac{k}{r^7} \tag{2.23b}$$

$$\text{KK6-brane}: \quad \mathrm{d}s_{11}^2 = \mathrm{d}s^2(E^{1,6}) + \mathrm{d}s_{\mathrm{TN}}^2(y) \tag{2.24a}$$

$$\mathrm{d}s_{\mathrm{TN}}^2 = H \mathrm{d}y^i \mathrm{d}y^i + H^{-1}\left(\mathrm{d}\psi_{\mathrm{TN}} + V_i(y)\mathrm{d}y^i\right)^2 \ , \quad i = 1,2,3 \tag{2.24b}$$

$$\nabla \times V = \nabla \cdot H \ , \quad H = 1 + \frac{k}{|y|} \ . \tag{2.24c}$$

The KK6-brane solution is analogous to the five-dimensional Kaluza–Klein monopole [295], and is built out from the four-dimensional Taub–NUT gravitational instanton (see Ref. [109] for a review of this topic), which is asymptotically of the form $\mathbb{R}^3 \times S^1$, where ψ_{TN} is the compact coordinate of S^1 with period $2\pi R$. Consequently, this solution only arises when at least one direction is compact. It is localized in the four Taub–NUT directions, as should be the case for a 6-brane, and magnetically charged under the graviphoton $g_{\mu \mathrm{TN}}$. It can be considered as the electromagnetic dual of a *pp*-wave, electrically charged under the graviphoton arising after compactification on a circle of radius R. *pp*-waves in compact directions will be called indifferently Kaluza–Klein excitations or momentum states.

The corresponding solutions for the M2- and M5-brane read [108, 149]:

$$\text{M2-brane}: \quad \mathrm{d}s_{11}^2 = H^{-2/3}\mathrm{d}s^2(E^{1,2}) + H^{1/3}\mathrm{d}s^2(E^8) \tag{2.25a}$$

$$d\mathcal{C}_3 = \mathrm{Vol}(E^{1,2}) \wedge dH^{-1} \tag{2.25b}$$

$$H = 1 + \frac{k}{r^6} \ , \quad k = \frac{\kappa_{11}^2 \mathcal{T}_2}{3\Omega_7} \tag{2.25c}$$

$$\text{M5-brane}: \quad \mathrm{d}s_{11}^2 = H^{-1/3}\mathrm{d}s^2(E^{1,5}) + H^{2/3}\mathrm{d}s^2(E^5) \tag{2.26a}$$

$$d\mathcal{C}_3 = \star_5 dH \tag{2.26b}$$

$$H = 1 + \frac{k}{r^3} \ , \quad k = \frac{\kappa_{11}^2 \mathcal{T}_5}{3\Omega_4} \ , \tag{2.26c}$$

which also show that the M2-brane (resp. M5-) is electrically (resp. magnetically) charged under the 3-form gauge potential. The symbol $\star_q$ denotes Hodge duality in q dimensions, and Ω_n the volume of the sphere S^n with unit radius:

$$\Omega_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \ . \tag{2.27}$$

The tensions (or mass per unit world-volume) of these four basic BPS configurations can be easily evaluated from ADM boundary integrals and Dirac quantization, or more easily yet by dimensional analysis:

$$\begin{aligned} &\text{KK-state}: \quad \mathcal{T}_0 = \frac{1}{R}\ , \quad \text{KK6-brane}: \quad \mathcal{T}_6 = \frac{R^2}{l_p^9}\ , \\ &\text{M2-brane}: \quad \mathcal{T}_2 = \frac{1}{l_p^3}\ , \quad \text{M5-brane}: \quad \mathcal{T}_5 = \frac{1}{l_p^6}\ . \end{aligned} \tag{2.28}$$

The tension (i.e. mass) of the *pp*-wave with momentum along a compact direction of radius R (occasionally denoted as R_{TN}) is the one expected for a massless particle in eleven dimensions; the tension of the KK6-brane is easily obtained from the latter by electric–magnetic duality, after reading off from Eq. (2.6) the Kaluza–Klein gauge coupling $1/g_{\mathrm{KK}}^2 = R^3/l_p^9$:

$$\mathcal{T}_6 = \frac{\mathcal{T}_0}{g_{\mathrm{KK}}^2} = \frac{R^2}{l_p^9}\ . \tag{2.29}$$

All these BPS states have been inferred from a classical analysis of 11D supergravity. They should in principle arise from a microscopic definition of M-theory, which would allow a full account of their interactions. Nevertheless, it is still possible to formulate their dynamics in terms of their collective coordinates which result from the breaking of global symmetries in the presence of the soliton [130]. Supersymmetry gives an important guideline, since (the unbroken) half of the 32 supercharges has to be realized linearly on the world-volume, while the other half is realized non-linearly. This fixes the dynamics of the M5-brane to be described in terms of the chiral $(2,0)$ six-dimensional tensor theory [59], while the membrane is described by the 2+1 supermembrane action [42, 86]. Unfortunately, the quantization of these two theories remains a challenge. As for the KK6-brane, the description of its dynamics is still an unsettled problem [155].

2.4 Reduction to type IIA BPS solutions

Upon compactification on a circle (with periodic boundary conditions on the fermion fields), the supersymmetry algebra is unaffected and the generators merely decompose under the reduced Lorentz group. The 32-component Majorana spinor Q_α decomposes into two 16-component Majorana–Weyl spinors of $SO(1,9)$ with opposite chiralities, and the $N=1$ supersymmetry in 11D gives rise to non-chiral $N=2$ supersymmetry in 10D. However, it is convenient not to separate the two chiralities explicitly, and rewrite the supersymmetry algebra as

$$\begin{aligned} \{Q_\alpha, Q_\beta\} \;=\; & (C\Gamma^\mu)_{\alpha\beta} P_\mu + (C\Gamma_s)_{\alpha\beta} Z \\ & + \frac{1}{2}(C\Gamma_{\mu\nu})_{\alpha\beta} Z^{\mu\nu} + (C\Gamma_\mu \Gamma_s)_{\alpha\beta} Z^\mu \\ & + \frac{1}{5!}(C\Gamma_{\mu\nu\rho\sigma\tau})_{\alpha\beta} Z^{\mu\nu\rho\sigma\tau} + \frac{1}{4!}(C\Gamma_{\mu\nu\rho\sigma}\Gamma_s)_{\alpha\beta} Z^{\mu\nu\rho\sigma}\ , \end{aligned} \tag{2.30}$$

where the eleventh Gamma matrix Γ_s is identified with the 10D chirality operator $\Gamma_0\Gamma_1\dots\Gamma_9$. The eleven-dimensional central charges give rise to the charges Z, Z^μ, $Z^{\mu\nu}$, $Z^{\mu\nu\rho\sigma}$, $Z^{\mu\nu\rho\sigma\tau}$ whose interpretation is summarized in Table 2.2, where we omitted the momentum charge P_μ. In this table, $\mathcal{K}_{m;mnpqrst}$ denotes the 6-form dual to $g_{\mu m}$ after compactification of the direction m.

Z	Z^{ij}	Z^i	Z^{ijklm}	Z^{ijkl}	Z^{0i}	Z^0	Z^{0ijkl}	Z^{0ijk}
$\mathcal{A}_0$	$\mathcal{C}_{0ij}$	B_{0i}	$\mathcal{E}_{0ijklm}$	$\mathcal{R}_{0ijkl}$	none	none	$\mathcal{K}_{m;mpqrs0}$	$\mathcal{R}_{0lmnpqr}$
D0	D2	F1	NS5	D4	D8	9-brane	KK5	D6

Table 2.2: Type IIA central charges, gauge fields and extended objects

Under Kaluza–Klein reduction, the BPS solutions of 11D SUGRA yield BPS solutions of type IIA supergravity. This reduction can, however, be carried out only if the eleventh dimension is a Killing vector of the configuration. This is automatically obeyed if the eleventh direction is chosen along the world-volume $E^{1,p}$, and reduces the eleven-dimensional p-brane to a ten-dimensional $(p-1)$-brane with tension $\mathcal{T}_{p-1} = R\mathcal{T}_p$; this procedure is called *diagonal* or *double* reduction [105], and we shall call the resulting solutions *wrapped* or *longitudinal* branes. One may also want to choose the eleventh direction transverse to the brane, but this is not an isometry, since the dependence of the harmonic function H on the transverse coordinates is non-trivial. However, this can be easily evaded by using the superposition property of BPS states, and constructing a continuous stack of parallel p-branes along the eleventh direction. The harmonic function on E^{10-p} turns into an harmonic function on E^{9-p}:

$$\int_{-\infty}^{\infty} \frac{dx^s}{\left[(x^s)^2+\rho^2\right]^{\frac{8-p}{2}}} \sim \frac{1}{\rho^{7-p}} \ . \tag{2.31}$$

We therefore obtain an *unwrapped* or *transverse* p-brane in ten dimensions with the same tension $\mathcal{T}_p$ as the one we started with. This procedure is usually called *vertical* or *direct* reduction. It has also been proposed to reduce along the isometry that arises when the sphere S^{9-p} in the transverse space E^{10-p} is odd-dimensional, hence given as a $U(1)$ Hopf fibration [107], but the status of the solutions obtained by this *angular* reduction is still unclear.

Applying this procedure to the four M-theory BPS configurations, with tensions given in Eq. (2.28), we find, after using the relations (2.11), the set of BPS states of type IIA string theory listed in Table 2.3:

As the table shows, we recover the set of all 1/2 BPS solutions of type IIA string theory, which include the KK excitations, the fundamental string and the set of solitonic states comprised by the NS5-brane, KK5-brane and the Dp-branes[‡7] with $p = 0, 2, 4, 6$ [‡8]. The

[‡7]The letter D stands for the Dirichlet boundary conditions in the $9-p$ directions orthogonal to the world-volume of the Dp-brane, which force the open strings to move on this $(p+1)$-dimensional hyperplane.

[‡8]There is also an 8-brane coupling to a nine-form, whose expectation value is related to the cosmological constant [256, 40, 38].

M-theory	mass/tension	type IIA
longitudinal M2-brane	$\mathcal{T}_1 = \frac{R_s}{l_p^3} = \frac{1}{l_s^2}$	F-string
transverse M2-brane	$\mathcal{T}_2 = \frac{1}{l_p^3} = \frac{1}{g_s l_s^3}$	D2-brane
longitudinal M5-brane	$\mathcal{T}_4 = \frac{R_s}{l_p^6} = \frac{1}{g_s l_s^5}$	D4-brane
transverse M5-brane	$\mathcal{T}_5 = \frac{1}{l_p^6} = \frac{1}{g_s^2 l_s^6}$	NS5-brane
longitudinal KK mode	$\mathcal{T}_0 = \frac{1}{R_s} = \frac{1}{g_s l_s}$	D0-brane
transverse KK mode	$\mathcal{T}_0 = \frac{1}{R_i} = \frac{1}{R_i}$	KK mode
longitudinal KK6-brane	$\mathcal{T}_5 = \frac{R_s R_{\mathrm{TN}}^2}{l_p^9} = \frac{R_{\mathrm{TN}}^2}{g_s^2 l_s^8}$	KK5-brane
KK6-brane with $R_{\mathrm{TN}} = R_s$	$\mathcal{T}_6 = \frac{R_s^2}{l_p^9} = \frac{1}{g_s l_s^7}$	D6-brane
transverse KK6-brane	$\mathcal{T}_6 = \frac{R_{\mathrm{TN}}^2}{l_p^9} = \frac{R_{\mathrm{TN}}^2}{g_s^3 l_s^9}$	6_3^1-brane

Table 2.3: Relation between M-theory and type IIA BPS states.

NS5-brane is a solitonic solution that is magnetically charged under the Neveu–Schwarz B-field [59]. The Dp-branes are solitonic solutions, electrically charged under the RR gauge potentials $\mathcal{R}_{p+1}$ (or magnetically under $\mathcal{R}_{7-p}$) [256]. The tension of these BPS states does not receive any quantum corrections perturbative or non-perturbative, which is why these objects are useful when considering non-perturbative dualities. States electrically (resp. magnetically) charged under the Neveu–Schwarz gauge fields have tensions that scale with the string coupling constant as g_s^0 (resp. $1/g_s^2$), whereas states charged under the Ramond fields have tensions that scale as $1/g_s$.

The last line in Table 2.3 is an unconventional solution, which we call a 6_3^1-brane, obtained by vertical reduction of the KK6-brane in a direction in the $\mathbb{R}^3$ part of the Taub–NUT space [54]. The integration involved in building up the stack is, however, logarithmically divergent, and, if regularized, yields a non-asymptotically flat space. However, as we will see in more detail in Subsection 4.9, at the algebraic level this solution is required by U-duality symmetry. At that point we will also explain our nomenclature for this (and other) non-conventional solutions. It is also interesting to note that all the tensions obtained above are not independent, since they follow from the basic relations (2.11). This already hints at the presence of a larger structure that relates all these states, a fact that we will establish using the conjectured U-duality symmetry of compactified M-theory.

The dimensional reduction can also be carried out at the level of the supergravity configuration itself. For example, using the relation (2.12) between the 11D metric and 10D string metric, one finds that a solution with 11D metric of the form

$$\mathrm{d}s_{11}^2 = H^{\kappa}\mathrm{d}s^2(E^{1,p}) + H^{\lambda}\mathrm{d}s^2(E^{10-p}) \tag{2.32}$$

yields two 10D solutions with metric and dilaton

$$\mathrm{d}s_{10}^2 = H^{\alpha}\mathrm{d}s^2(E^{1,p'}) + H^{\beta}\mathrm{d}s^2(E^{9-p'})\ , \quad e^{-2\phi} = H^{\gamma} \tag{2.33a}$$

where

$$\text{diagonal}: \quad p' = p-1\ , \quad \alpha = \frac{3\kappa}{2}\ , \quad \beta = \lambda + \frac{\kappa}{2}\ , \quad \gamma = -\frac{3\kappa}{2}\ , \tag{2.33b}$$

$$\text{vertical}: \quad p' = p\ , \quad \alpha = \kappa + \frac{\lambda}{2}\ , \quad \beta = \frac{3\lambda}{2}\ , \quad \gamma = -\frac{3\lambda}{2}\ , \tag{2.33c}$$

for diagonal and vertical reduction respectively. As explained in the beginning of this subsection, in the first case the harmonic function is the same as the original one, and in the second case it is an harmonic function on a transverse space with one dimension less. The reduction of the gauge potentials can be worked out similarly.

The resulting 10D type IIA configurations are then described by the following solutions:

$$\text{F-string}: \quad \mathrm{d}s_{10}^2 = H^{-1}\mathrm{d}s^2(E^{1,1}) + \mathrm{d}s^2(E^8) \tag{2.34a}$$

$$B_{01} = H^{-1}\ , \quad e^{-2\phi} = H\ , \quad H = 1 + \frac{k}{r^6} \tag{2.34b}$$

$$\text{NS5-brane}: \quad \mathrm{d}s_{10}^2 = \mathrm{d}s^2(E^{1,5}) + H\mathrm{d}s^2(E^4) \tag{2.35a}$$

$$dB = \star_4 dH\ , \quad e^{-2\phi} = H^{-1}\ , \quad H = 1 + \frac{k}{r^2} \tag{2.35b}$$

$$\text{D}p\text{-brane}: \quad \mathrm{d}s_{10}^2 = H^{-1/2}\mathrm{d}s^2(E^{1,p}) + H^{1/2}\mathrm{d}s^2(E^{9-p}) \tag{2.36a}$$

$$e^{-2\phi} = H^{(p-3)/2}\ , \quad H = 1 + \frac{k}{r^{7-p}} \tag{2.36b}$$

$$F_e^{(p+2)} = \mathrm{Vol}(E^{1,p}) \wedge dH^{-1}\ , \quad p = 0,1,2 \tag{2.36c}$$

$$F_m^{(8-p)} = \star_{9-p} dH\ , \quad p = 4,5,6 \tag{2.36d}$$

$$F^{(5)} = F_e^{(5)} + F_m^{(5)}\ , \quad p = 3 \tag{2.36e}$$

where, for completeness, we have included the Dp-brane solutions for all $p = 0 \dots 6$, although we note that only even p occurs in type IIA. The subscripts e and m indicate whether the p-branes are electrically or magnetically charged under the indicated fields. One also finds the ten-dimensional gravitational solutions, consisting of the pp-waves and KK5-brane, which have a metric analogous to the eleven-dimensional case (see (2.23) and (2.24)), with harmonic functions on a transverse space with one dimension less. Of course, one may explicitly verify that all of these solutions are indeed solutions of the tree-level action (2.7).

In contrast to the M2-brane and M5-brane, the dynamics of Dp-branes has a nice and tractable description as $(p+1)$-dimensional hyperplanes on which open strings can end and exchange momentum with [256]. The integration of open string fluctuations around a single D-brane at tree level yields the Born–Infeld action [58, 210, 18],

$$S_{\mathrm{BI}} = \frac{1}{l_s^{p+1}} \int \mathrm{d}^{p+1}\xi e^{-\phi} \sqrt{\hat{g} + \hat{B} + l_s^2 F} \ . \tag{2.37}$$

Here, the hatted fields $\hat{g}, \hat{B}$ stand for the pullbacks of the bulk metric and antisymmetric tensor to the world-volume of the brane, and F is the field strength of the $U(1)$ gauge field living on the brane. The coupling to the RR gauge potentials is given by the topological term [97, 143][‡9]

$$S_{\mathrm{RR}} = i \int e^{\hat{B} + l_s^2 F} \wedge \mathcal{R} \ , \tag{2.38}$$

where $\mathcal{R} = \sum_p \mathcal{R}_p$ denotes the total RR potential.

In the zero-slope limit, the Born–Infeld action becomes the action of a supersymmetric Maxwell theory with 16 supercharges. In the presence of N coinciding D-branes the world-volume gauge symmetry gets enhanced from $U(1)^N$ to $U(N)$, as a consequence of zero mass strings stretching between different D-branes [320]. The non-Abelian analogue of the Born–Infeld action is not known, although some partial Abelianization is available [312], but its zero-slope limit is still given by $U(N)$ super-Yang–Mills theory.

2.5 T-duality and type IIA/B string theory

So far, we have discussed M-theory and its relation to type IIA string theory. In this subsection, we turn to type IIB string theory and its relation, via T-duality, to type IIA [95, 80]. We first recall that the massless sector of type IIB consists of the same Neveu–Schwarz fields (2.8a) as the type IIA string, but the Ramond gauge potentials of type IIB now include a 0-form (scalar), a 2-form and a 4-form with self-dual field strength,

$$a \ , \ \ \mathcal{B}_{\mu\nu} \ , \ \ \mathcal{D}_{\mu\nu\rho\sigma} \ , \tag{2.39}$$

[‡9] There is also a gravitational term required for the cancellation of anomalies [139], but it does not contribute on flat backgrounds.

with $*\mathcal{D}_4 = \mathcal{D}_4$. The low-energy effective action has a form similar to that in (2.7), with the appropriate field strengths of the even-form RR potentials in (2.39), as long as the 4-form is not included[‡10]. The standard 1/2-BPS solutions of type IIB are the fundamental string, NS5-brane, Dp-branes with odd p, pp-waves and KK5-brane.

In order to describe the precise T-duality mapping, we again write the ten-dimensional metric as a $U(1)$ fibration

$$ds_{10}^2 = R^2(dx^9 + A_\mu dx^\mu)^2 + g_{\mu\nu}dx^\mu dx^\nu \quad , \quad \mu,\nu = 0\dots 8 \ . \tag{2.40}$$

T-duality on the direction 9 relates the fields in the type IIA and type IIB theories in the Neveu–Schwarz sector as

$$T_9 : \qquad R \leftrightarrow \frac{l_s^2}{R} \ , \quad g_s \leftrightarrow g_s\frac{l_s}{R} \ , \quad A_\mu \leftrightarrow B_{9\mu} \ , \quad B_{\mu\nu} \leftrightarrow B_{\mu\nu} - A_\mu B_{9\nu} + A_\nu B_{9\mu} \ , \tag{2.41}$$

leaving $g_{\mu\nu}$ and the string length l_s invariant. The Ramond gauge potentials are furthermore identified on both sides according to

$$T_9 : \qquad \mathcal{R} \leftrightarrow \mathrm{d}x^9 \cdot \mathcal{R} + \mathrm{d}x^9 \wedge \mathcal{R} \ , \ \ \mathcal{R} = \sum_p \mathcal{R}_p \ , \tag{2.42}$$

where $\cdot$ and $\wedge$ denote the interior and exterior products respectively. In other words, the 9 index is added to the antisymmetric indices of $\mathcal{R}$ when absent, or deleted if it was already present. These identifications actually receive corrections when $B \neq 0$, and the precise mapping is [41, 143, 111]

$$e^B\mathcal{R} \to \mathrm{d}x^9 \cdot (e^B\mathcal{R}) + \mathrm{d}x^9 \wedge (e^B\mathcal{R}) \tag{2.43}$$

in accord with the T-duality covariance of the RR coupling in (2.38). Whereas one T-duality maps the type IIA string theory to IIB and should be thought of as a change of variable, an even number of dualities can be performed and correspond to actual global symmetries of either type IIA or type IIB theories. This symmetry will be discussed in Section 3, and its non-perturbative extension in Section 4.

The action on the BPS spectrum can again be easily worked out, at the level of tension formulae or of the supergravity solutions themselves. As implied by the exchange of the Kaluza–Klein and Kalb–Ramond gauge fields A_μ and $B_{9\mu}$, states with momentum along the 9th direction are interchanged with fundamental string winding around the same direction. On the other hand, T-duality exchanges Neumann and Dirichlet boundary conditions on the open string world-sheet along the 9th direction, mapping Dp-branes to D$(p+1)$- or D$(p-1)$-branes, depending on the orientation of the world-volume with respect to x^9 [80, 37]. This of course agrees with the mapping of Ramond gauge potentials in Eq. (2.42). Similarly, NS5-branes are invariant or exchanged with KK5-branes, according to whether

[‡10]A local covariant action for the self-dual four-form can be written with the help of auxiliary fields [81], but for most purposes the equations of motion are sufficient.

they are wrapped or unwrapped, respectively [111, 249] [‡11]. This can also be easily seen by applying the transformation (2.41) to the tension formulae, as summarized in Table 2.4 for a T-duality T_i on an arbitrary compact dimension with radius R_i.

type IIA (B)	tension	T_i-dual tension	type IIB (A)
KK mode	$\mathcal{M} = \frac{1}{R_i}$	$\mathcal{M} = \frac{R_i}{l_s^2}$	winding mode
wrapped Dp-brane	$\mathcal{T}_{p-1} = \frac{R_i}{g_s l_s^{p+1}}$	$\mathcal{T}_{p-1} = \frac{1}{g_s l_s^p}$	unwrapped D$(p-1)$-brane
wrapped NS5-brane	$\mathcal{T}_4 = \frac{R_i}{g_s^2 l_s^6}$	$\mathcal{T}_4 = \frac{R_i}{g_s^2 l_s^6}$	wrapped NS5-brane
unwrapped NS5-brane	$\mathcal{T}_5 = \frac{1}{g_s^2 l_s^6}$	$\mathcal{T}_5 = \frac{R_i^2}{g_s^2 l_s^6}$	unwrapped KK5-brane

Table 2.4: T-duality of type II BPS states.

T-duality can then be used to translate the relation between strongly coupled type IIA theory and M-theory in type IIB terms. In this way, it is found that the type IIB string theory is obtained by compactifying M-theory on a two-torus T^2, with vanishing area, and a complex structure τ equated to the type IIB complex coupling parameter [273]:

$$\tau = a + \frac{i}{g_s} \ . \tag{2.44}$$

Here, a is the expectation value of the Ramond scalar and g_s the type IIB string coupling.

We focus for simplicity on the case where the torus is rectangular, so that τ is purely imaginary and hence the RR scalar a vanishes. In this case, the relation between the M-theory parameters and type IIB parameters reads

$$g_s = \frac{R_s}{R_9} \ , \qquad l_s^2 = \frac{l_p^3}{R_s} \ , \qquad R_B = \frac{l_p^3}{R_s R_9} \ , \tag{2.45}$$

where R_s, R_9 are the radii of the M-theory torus and R_B the radius of the type IIB 9th direction. The uncompactified type IIB theory is obtained in the limit $(R_s, R_9) \to \infty$, keeping R_s/R_9 fixed. From Eq. (2.45), we can then identify the type IIB BPS states to those of M-theory compactified on T^2. The results are displayed in Table 2.5 for states still existing in uncompactified type IIB theory, and in Table 2.6 for states existing only for finite values of R_B.

[‡11] Whereas the worldvolume dynamics of type IIB NS5- and D5-branes is described by a non-chiral $(1,1)$ vector multiplet, the type IIB KK5-brane is chiral and supports a $(2,0)$ tensor multiplet. Indeed, it is T-dual to the chiral type IIA NS5-brane [10]. On the other hand, the type IIA KK5-brane, dual to the type IIB NS5-brane, is nonchiral.

M-theory	mass/tension	type IIB
M2-brane wrapped around x^s	$\frac{R_s}{l_p^3} = \frac{1}{l_s^2}$	fundamental string
M2-brane wrapped around x^9	$\frac{R_9}{l_p^3} = \frac{1}{g_s l_s^2}$	D1-brane (D-string)
M5-brane wrapped on x^s, x^9	$\frac{R_s R_9}{l_p^6} = \frac{1}{g_s l_s^4}$	D3-brane
KK6-brane wrapped on x^9, charged under $g_{\mu s}$	$\frac{R_s^2 R_9}{l_p^9} = \frac{1}{g_s l_s^6}$	D5-brane
KK6-brane wrapped on x^s, charged under $g_{\mu 9}$	$\frac{R_9^2 R_s}{l_p^9} = \frac{1}{g_s^2 l_s^6}$	NS5-brane

Table 2.5: Relations between M-theory and type IIB BPS states.

M-theory	mass/tension	type IIB
M2-brane wrapped on x^s, x^9	$\frac{R_9 R_s}{l_p^3} = \frac{1}{R_B}$	KK mode
unwrapped M5-brane	$\frac{1}{l_p^6} = \frac{R_B^2}{g_s^2 l_s^8}$	KK5-brane with $R_{\rm TN} = R_s$
unwrapped M2-brane	$\frac{1}{l_p^3} = \frac{R_B}{g_s l_s^4}$	wrapped D3-brane
M5-brane wrapped on x^s	$\frac{R_s}{l_p^6} = \frac{R_B}{g_s l_s^6}$	wrapped D5-brane
M5-brane wrapped on x^9	$\frac{R_9}{l_p^6} = \frac{R_B}{g_s^2 l_s^6}$	wrapped NS5-brane
unwrapped KK6-brane, charged under $g_{\mu s}$	$\frac{R_s^2}{l_p^9} = \frac{R_B}{g_s l_s^8}$	wrapped D7-brane
unwrapped KK6-brane, charged under $g_{\mu 9}$	$\frac{R_s^2}{l_p^9} = \frac{R_B}{g_s^3 l_s^8}$	wrapped 7_3-brane

Table 2.6: More relations between M-theory and type IIB BPS states.

As in Table 2.3, we see in the last entry of Table 2.6 a non-standard BPS state with tension scaling as g_s^{-3}, which we have called a 7_3-brane. As this brane will turn out to be related to the D7-brane by S-duality (see Subsection 4.5) it may also be referred to as a (1,0) 7-brane. This and other non-standard solutions will be discussed in more detail in Subsection 4.9.

3 T-duality and toroidal compactification

Having discussed how *dualities* of string theory lead to the idea of a more fundamental eleven-dimensional M-theory, we now turn to the *symmetries* that this theory should exhibit, with the hope of getting more insight into its underlying structure. For this purpose, it is convenient to consider compactifications on tori, which have the advantage of preserving a maximal amount of the original super-Poincaré symmetries, while bringing in degrees of freedom from extended states in eleven dimensions in a still manageable way.

The approach here is similar to the one that was taken for the perturbative string itself, where the study of T-duality in toroidal compactifications revealed the existence of spontaneously broken "stringy" gauge symmetries (see [135] for a review). Given the analogy between the two problems, we shall first review in this section how T-duality in string theory appears at the level of the low-energy effective action and of the spectrum, with a particular emphasis on the brane spectrum. We shall then apply the same techniques in Sections 4 and 5 in order to discuss U-duality in M-theory.

3.1 Continuous symmetry of the effective action

Compactification of string theory on a torus T^d can be easily worked out at the level of the low-energy effective action, by substituting an ansatz similar to (2.4)

$$\mathrm{d}s_{10}^2 = g_{ij}\left(dx^i + A^i_\mu dx^\mu\right)\left(dx^j + A^j_\nu dx^\nu\right) + g_{\mu\nu}dx^\mu dx^\nu \tag{3.1a}$$

$$i,j = 1\dots d \quad , \quad \mu,\nu = 0\dots(9-d) \tag{3.1b}$$

in the ten-dimensional action

$$S_{10} = \frac{1}{l_s^8}\int \mathrm{d}^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{l_s^4}{12}(dB)^2\right) \ , \tag{3.2}$$

where we omitted Ramond and fermion terms. We have also split the ten-dimensional two-form B into $d(d-1)/2$ scalars B_{ij}, d vectors $B_{i\mu}$ and a two-form $B_{\mu\nu}$.

Concentrating on the scalar sector, and redefining the dilaton as $Ve^{-2\phi} = l_s^d e^{-2\phi_d}$ where $V = \sqrt{\det g}$ [‡12] is the volume of the internal metric, we obtain

$$S_{\mathrm{scal}} = \frac{1}{l_s^{8-d}}\int \mathrm{d}^{10-d}x\sqrt{-g}e^{-2\phi_d}\left(4(\partial\phi_d)^2 + \frac{1}{4}\mathrm{Tr}\ \partial g\partial g^{-1} + \frac{1}{4}\mathrm{Tr}\ g^{-1}\partial B\ g^{-1}\partial B\right) \ . \tag{3.3}$$

This can be rewritten as

$$S_{\mathrm{scal}} = \frac{1}{l_s^{8-d}}\int \mathrm{d}^{10-d}x\sqrt{-g}e^{-2\phi_d}\left(4(\partial\phi_d)^2 + \frac{1}{8}\mathrm{Tr}\ \partial M\partial M^{-1}\right) \ , \tag{3.4}$$

[‡12] g denotes the internal metric g_{ij}, except in the space-time volume element $\sqrt{-g}$ multiplying the action density.

where M is the $2d \times 2d$ symmetric matrix

$$M = \begin{pmatrix} g^{-1} & g^{-1}B \\ -Bg^{-1} & g - Bg^{-1}B \end{pmatrix} \ , \quad M^t \eta M = \eta \ , \quad \eta = \begin{pmatrix} & \mathbb{I}_d \\ \mathbb{I}_d & \end{pmatrix} \ , \tag{3.5}$$

orthogonal for the signature (d, d) metric η. The scalars g_{ij} and B_{ij} therefore parametrize a symmetric manifold

$$\mathcal{H} = \frac{SO(d, d, \mathbb{R})}{SO(d) \times SO(d)} \ni M \ , \tag{3.6}$$

where $SO(d) \times SO(d)$ is the maximal compact subgroup of $SO(d, d, \mathbb{R})$. The matrix M is more properly thought of as the $SO(d) \times SO(d)$ invariant $M = \mathcal{V}^t \mathcal{V}$ built out from the vielbein in $SO(d, d, \mathbb{R})$

$$\mathcal{V} = \left(\begin{array}{ccc|ccc} 1/R_1 & & & & & \\ & 1/R_2 & & & & \\ & & \ddots & & & \\ \hline & & & R_1 & & \\ & & & & R_2 & \\ & & & & & \ddots \end{array} \right) \cdot \left(\begin{array}{ccc|ccc} 1 & -A_2^1 & \dots & B_{11} & B_{12} & \dots \\ & 1 & \ddots & B_{21} & B_{22} & \dots \\ & & \ddots & \vdots & \vdots & \\ \hline & & & 1 & A_2^1 & \dots \\ & & & & 1 & \ddots \\ & & & & & \ddots \end{array} \right) \tag{3.7}$$

corresponding to the Iwasawa decomposition of $SO(d, d, \mathbb{R})$, as will be discussed in more detail in Section 4.2. The two-derivative action for the scalars g_{ij}, B_{ij}, ϕ_d is therefore invariant [222] under the action $M \to \Omega^t M \Omega$ of $\Omega \in O(d, d, \mathbb{R})$, and so is the entire two-derivative action in the Neveu–Schwarz sector, if the $2d$ gauge fields A_μ^i and $B_{i\mu}$ transform altogether as a vector under $O(d, d, \mathbb{R})$, the dilaton ϕ_d, metric $g_{\mu\nu}$ and two-form $B_{\mu\nu}$ being invariant.

The action on the Ramond sector is more complicated, since the Ramond scalars and one-forms transform as a spinor (resp. conjugate spinor) of $SO(d, d, \mathbb{R})$, with the chirality depending on whether we consider type IIA or IIB. Elements of $O(d, d, \mathbb{R})$ with (-1) determinant flip the chirality of spinors; they therefore are *not* symmetries of the action in the Ramond sector, but *dualities*, exchanging type IIA and type IIB theories. Indeed it is easy to see that the $R \to 1/R$ dualities that we discussed in Subsection 2.5 belong to this class of transformations. The tree-level effective action is therefore invariant under the continuous symmetry $SO(d, d, \mathbb{R})$, which extends the symmetry $Sl(d, \mathbb{R})$ that would be present in the dimensional reduction of any Lorentz-invariant field theory.

3.2 Charge quantization and T-duality symmetry

Owing to the occurrence of particles charged under the gauge fields A_μ^i and $B_{i\mu}$, the continuous symmetry $SO(d, d, \mathbb{R})$ can, however, not exist at the quantum level. For instance, perturbative string states have integer momenta m_i and winding numbers m^i under these gauge fields, lying in an even self-dual Lorentzian lattice Γ_p. The 1/2-BPS states are obtained when the world-sheet oscillators $\alpha^\dagger_{\mu n}$ and $\bar{\alpha}^\dagger_{\mu n}$ are not excited, and satisfy the mass formula and matching condition

$$\begin{aligned}\mathcal{M}^2 &= m^t M m \\ &= (m_i + B_{ij} m^j) g^{ik} (m_k + B_{kl} m^l) + m^i g_{ij} m^j \qquad (3.8a)\\ \|m\|^2 &= 0 \ , \qquad (3.8b)\end{aligned}$$

where $m = (m_i, m^i)$ is the vector of charges, $\|m\|^2 = 2 m_i m^i$ its Lorentzian square-norm and M is the moduli matrix given in (3.5).

On the other hand, 1/4-BPS states are obtained when the world-sheet oscillators are excited on the holomorphic (or antiholomorphic) side only, and have mass

$$\mathcal{M}^2 = m^t M m + \left| \|m\|^2 \right| \ , \qquad (3.9)$$

where the norm $|\|m\|^2|$ is equated to the left or right oscillator number by the matching conditions. Only the discrete subgroup preserving Γ_p can be a quantum symmetry, and this group is $O(d,d,\mathbb{Z})$, the set of integer-valued $O(d,d,\mathbb{R})$ matrices. In particular, the subgroup $Sl(d,\mathbb{R})$ of $SO(d,d,\mathbb{R})$ is reduced to the modular group of the torus $Sl(d,\mathbb{Z})$, an obvious consequence of momentum quantization in compact spaces.

In addition to this perturbative spectrum, type II string theory also admits a variety of D-branes, which are charged under the Ramond gauge potentials. Their charges take value in another lattice, Γ_D, and transform as a spinor under $SO(d,d,\mathbb{R})$. Again, the determinant (-1) elements of $O(d,d,\mathbb{Z})$ flip the chirality of spinors, and therefore do not preserve Γ_D. As we shall see shortly, $SO(d,d,\mathbb{Z})$ however does preserve the lattice of D-brane charges. This is in agreement with the fact that this group can be seen as the Weyl group of the extended gauge symmetries that appear at particular points in the torus moduli space, and are spontaneously broken elsewhere [134].

3.3 Weyl and Borel generators

In order to better understand the structure of the T-duality symmetry, it is useful to isolate a set of generating elements of $SO(d,d,\mathbb{Z})$. We define *Weyl* elements as the ones that preserve the conditions

$$g_{ij} = R_i^2 \delta_{ij} \ , \quad B_{ij} = 0 \ , \qquad (3.10)$$

that is square tori with vanishing two-form background, and *Borel* elements as the ones that do not. Weyl generators include the exchanges of radii $S_{ij} : R_i \leftrightarrow R_j$, which belong to the $Sl(d,\mathbb{Z})$ modular group, as well as the simultaneous inversions of two radii $T_{ij} : (R_i, R_j) \to (1/R_j, 1/R_i)$.

We choose the following minimal set of Weyl generators:

$$S_i : R_i \leftrightarrow R_{i+1} \ , \quad i = 1 \dots d-1 \ , \tag{3.11a}$$

$$T : (g_s, R_1, R_2) \leftrightarrow \left(\frac{g_s}{R_1 R_2}, \frac{1}{R_2}, \frac{1}{R_1} \right) \ . \tag{3.11b}$$

For convenience, we followed the double T-duality on directions 1 and 2 by an exchange of the two radii, included the action on the coupling constant and set the string length l_s to 1. Altogether, the Weyl group of $SO(d, d, \mathbb{Z})$ is the finite group

$$\mathcal{W}\left(SO(d, d)\right) = \mathbb{Z}_2 \bowtie \mathcal{S}_d \tag{3.12}$$

generated by the T-duality transformation T and the permutation group $\mathcal{S}_d$ of the d directions of the torus[‡13].

On the other hand, Borel generators include the Borel elements of the modular subgroup, acting as $\gamma_i \to \gamma_i + \gamma_j$ on the homology lattice of the lattice, as well as the integer shifts of the expectation value of the two-form in the internal directions $B_{ij} \to B_{ij} + 1$. Any element in $SO(d, d, \mathbb{Z})$ can be reached by a sequence of these transformations.

Weyl and Borel generators can be given a more precise definition as operators on the weight space of the Lie group or algebra under consideration (see for instance Ref. [176] for an introduction to the relevant group theory) [‡14]. Weyl generators correspond to orthogonal reflections with respect to planes normal to any root and generate a finite discrete group, while Borel generators act on the weight lattice by translation by a positive root. Any finite-dimensional irreducible representation (of the complex Lie algebra) can then be obtained by action of the Borel group on a, so called, highest-weight vector, and splits into orbits of the Weyl group with definite lengths.

3.4 Weyl generators and Weyl reflections

Weyl generators encode the simplest and most interesting part of T-duality. It is very easy to study the structure of the finite group they generate, by viewing them as orthogonal reflections in a vector space (the weight space) generated by the logarithms of the radii. More precisely, let us represent the scalar moduli $(\ln g_s, \ln R_1, \dots, \ln R_d)$ as a form φ on a vector space V_{d+1} with basis $e_0, e_1, \dots, e_d$, and associate to any weight vector $\lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$, its tension[‡15]

$$\mathcal{T} = e^{\langle \varphi, \lambda \rangle} = g_s^{x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \ . \tag{3.13}$$

[‡13]The Weyl group of $SO(d, d)$ can actually be written as the semi-direct product $\mathcal{S}_d \ltimes (\mathbb{Z}_2)^{d-1}$, where the commuting $\mathbb{Z}_2$'s are the double inversions of R_i and, say, R_1.

[‡14]From this point of view, Weyl generators are not properly speaking elements of the group, but can be lifted to generators thereof, at the cost of introducing $\mathbb{Z}_2$ phases in their action on the step operators E_α. See for instance Appendix B in Ref. [212], for a discussion of this issue in the physics literature.

[‡15]One could omit the x^0 coordinate since g_s can be absorbed by a power of the invariant Planck length $\prod R_i / g_s^2$, but we include it for later convenience.

The vector λ should be seen as labelling a state in the BPS spectrum, with tension $\mathcal{T}$. The generators (3.11) are then implemented as linear operators on V_{d+1} with matrices

$$S_i = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & \mathbb{I}_{d-3} \end{pmatrix} \; , \quad T = \begin{pmatrix} 1 & & & \\ -1 & & -1 & \\ -1 & -1 & & \\ & & & \mathbb{I}_{d-3} \end{pmatrix} \tag{3.14}$$

These operators S_i and T in (3.14) are easily seen to be orthogonal with respect to the signature $(-+\cdots+)$ metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^i)^2 + \mathrm{d}x^0(dx^1 + \cdots + \mathrm{d}x^d) \; , \tag{3.15}$$

and correspond to Weyl reflections

$$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \ \lambda}{\alpha \cdot \alpha}\alpha \tag{3.16}$$

with respect to planes normal to the vectors

$$\alpha_i = e_{i+1} - e_i \;\; , \;\; i = 1 \ldots d-1 \tag{3.17a}$$

$$\alpha_0 = e_1 + e_2 \; . \tag{3.17b}$$

The group generated by S_i and T is therefore a Coxeter group, familiar from the theory of Lie algebras (see [176] for an introduction, and [177, 122] for a full account). Its structure can be characterized by the matrix of scalar products of these roots:

$$(\alpha_i)^2 = (\alpha_0)^2 = 2 \tag{3.18a}$$

$$\alpha_i \cdot \alpha_{i+1} = \alpha_2 \cdot \alpha_0 = -1 \; . \tag{3.18b}$$

This precisely reproduces the Cartan matrix D_d of the T-duality group $SO(d,d,\mathbb{R})$, summarized in the Dynkin diagram:

$$\begin{array}{ccccccc} \bigcirc_0 & & & & & & \\ & \diagdown & & & & & \\ & & \oplus_2 & - & \oplus_3 & - \cdots - & \oplus_{d-1} \\ & \diagup & & & & & \\ +_1 & & & & & & \end{array} \tag{3.19}$$

The only delicate point is that the signature of the metric (3.15) on V_{d+1} is not positive-definite. This can be easily evaded by noting that the invariance of Newton's constant $\prod R_i/g_s^2$ implies that all roots are orthogonal to the vector

$$\delta = e_1 + \cdots + e_d - 2e_0 \tag{3.20}$$

with negative proper length $\delta^2 = -(d+4)$, so that the reflections actually restrict to the hyperplane V_d normal to δ:

$$\delta \cdot x = x^0 = 0 \ . \tag{3.21}$$

The Lorentz metric on V_{d+1} then restricts to a positive-definite metric $g_{ij} = \delta_{ij}$ on V_d. The dualities S_i and T therefore generate the Coxeter group D_d, which is the same as the Weyl group of the Lie algebra of $SO(d,d,\mathbb{R})$. In order to distinguish the various real and discrete forms of D_d, one needs to take into account the Borel generators, which we defer to Subsection 3.7.

The Dynkin diagram (3.19) allows a number of simple observations. We may recognize the Dynkin diagram A_{d-1} of the Lorentz group $Sl(d,\mathbb{R})$ (denoted with $+$), extended with the root $\bigcirc$ into the Dynkin diagram of the T-duality symmetry $SO(d,d,\mathbb{R})$. T-duality between type IIA and type IIB corresponds to the outer automorphism acting as a reflection along the horizontal axis of the Dynkin diagram. The chain denoted with $\bigcirc$'s represents a dual $Sl(d,\mathbb{R})$ subgroup, which is nothing but the Lorentz group on the type IIB T-dual torus. The full T-duality group is generated by these two non-commuting Lorentz groups of the torus and the dual torus.

Decompactification of the torus T^d into T^{d-1} is achieved by dropping the rightmost root, which reduces D_d to D_{d-1}. When the root α_2 is reached, the diagram disconnects into two pieces, corresponding to the identity $SO(2,2,\mathbb{R}) = Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R})$, or to the decomposition of the torus moduli space into the T and U upper half-planes[‡16]. Finally, for $d=1$ the T-duality group $SO(1,1,\mathbb{Z})$ becomes trivial, while the generator of $O(1,1,\mathbb{Z})$ corresponds to the inversion $R \leftrightarrow 1/R$, *not* a symmetry of either type IIA or type IIB theories.

3.5 BPS spectrum and highest weights

Having proved that the transformations S_i and T indeed generate the Weyl group of $SO(d,d,\mathbb{Z})$, we can use the same formalism to investigate the orbit of the various BPS states of string theory. According to (3.13) the mass or tension can be represented as a *weight vector* in V_{d+1}, and one should let Weyl and Borel generators act on it to obtain the full orbit. Each orbit admits a *highest weight* from which all other elements can be reached by a sequence of Weyl and Borel generators (Weyl generators alone are not sufficient, because they preserve the length of the weight).

All highest weights can be written as linear combinations with positive integer coeffi-

[‡16] The extra $\mathbb{Z}_2$ exchanging the two $Sl(2,\mathbb{R})$ factors belongs to $O(2,2,\mathbb{R})$ but not to $SO(2,2,\mathbb{R})$.

cients of the fundamental weights

$$\lambda^{(1)} = e_1 - e_0 \quad \to \quad \mathcal{M}_{\mathrm{wD}} = \frac{R_1}{g_s} \tag{3.22a}$$

$$\lambda^{(2)} = e_1 + e_2 - 2e_0 \quad \to \quad \mathcal{M}_{\mathrm{NS}} = \frac{R_1 R_2}{g_s^2} \tag{3.22b}$$

$$\lambda^{(d-2)} = e_1 + \cdots + e_{d-2} - 2e_0 \quad \to \quad \mathcal{M}_{\mathrm{w...wNS}} = \frac{R_1 \ldots R_{d-2}}{g_s^2} \tag{3.22c}$$

$$\lambda^{(d-1)} = e_1 + \cdots + e_{d-1} - 2e_0 \doteq -e_d \quad \to \quad \mathcal{M}_{\mathrm{wF}} = \frac{1}{R_d} \tag{3.22d}$$

$$\lambda^{(0)} = -e_0 \quad \to \quad \mathcal{M}_{\mathrm{D}} = \frac{1}{g_s} \tag{3.22e}$$

dual to the simple roots, that is $\lambda^{(i)} \cdot \alpha_j = -\delta_{ij}$ [‡17]. We used the symbol $\doteq$ for equality modulo the invariant vector δ in Eq. (3.20), and the notation F,D and NS for fundamental, Dirichlet and Neveu–Schwarz states, respectively, depending on the power of the coupling constant involved, and w for each wrapped direction (the notation wF is justified by the fact that the Kaluza–Klein states are in the same multiplet as the string winding states). This is summarized in the Dynkin diagram

$$\begin{array}{ccccccc}
\frac{1}{g_s} & & & & & & \\
 & \diagdown & & & & & \\
 & & \frac{R_1 R_2}{g_s^2} & - & \frac{R_1 R_2 R_3}{g_s^2} & - \cdots - & \frac{1}{R_d} \\
 & \diagup & & & & & \\
\frac{R_1}{g_s} & & & & & &
\end{array} \tag{3.23}$$

which shows the highest weights associated to each node of the Dynkin diagram.

In particular, we see from (3.23) that the type IIA D-particle mass ($\mathcal{M} = 1/g_s l_s$) lies in the spinor representation dual to α_1, just as do the type IIB D-string tension ($\mathcal{T}_1 = 1/g_s l_s^2$) and D-instanton action ($\mathcal{T}_{-1} = 1/g_s$), whereas the type IIB D-particle mass ($\mathcal{M} = R_i/g_s l_s^2$) and type IIA D-string tension ($\mathcal{T}_1 = R_i/g_s l_s^3$) and D-instanton action ($\mathcal{T} = R_i/g_s l_s$) transform in the spinor representation dual to α_0, of opposite chirality. On the other hand, the Kaluza–Klein states lie in a vector representation. All highest-weight representations can be obtained from the tensor product of these "extreme" (from the point of view of the Dynkin diagram) representations. T-duality on a single radius exchanges the two spinor representations, as it should.

[‡17] The minus sign shows that we are really considering *lowest*-weight vectors, but we shall keep this abuse of language.

3.6 Weyl-invariant effective action

In the previous subsections, we have discussed how the Weyl group of $SO(d,d)$ arises as the finite group generated by the permutations and double T-duality (3.11), whereas the low-energy action itself is invariant under the continuous group $SO(d,d,\mathbb{R})$. This has been checked in the scalar sector in Eq. (3.4), by direct reduction of the 10D effective action on T^d. It is however possible to rewrite the full action in a manifestly Weyl-invariant way, by a step-by-step reduction from 10D, as was originally developed in Ref. [218] in the context of 11D supergravity. This procedure leads to a clear identification of "dilatonic" scalars, which appear through exponential factors in the action and include the dilaton g_s and the radii R_i of the torus, versus "Peccei–Quinn" scalars which have constant shift symmetries and are better thought of as 0-forms with a 1-form field strength.

Each field strength $F^{(p)}$ gives rise to field strengths of lower degree $F^{(q)}_{i_1 \dots i_q}$, with internal indices $i_1 \dots i_q$ (given by the exterior derivative of a $(q-1)$-form up to Chern–Simons corrections), while the metric gives rise to Kaluza–Klein two-form field strengths $\mathcal{F}^{(2)i}$ and one-form field strengths $\mathcal{F}^{i(1)}_j$, $i<j$, of the vielbein components in the upper triangular gauge

$$g_{MN} = E^P_M E^Q_N \, \eta_{PQ} \,, \tag{3.24a}$$

$$E^N_M = \left(\begin{array}{ccccc|c} R_1 & & & & & \\ & R_2 & & & & \\ & & \ddots & & & \\ & & & R_{d-1} & & \\ & & & & R_d & \\ \hline & & & & & E^\nu_\mu \end{array} \right) \cdot \left(\begin{array}{ccccc|c} 1 & \mathcal{A}^1_2 & \mathcal{A}^1_3 & \dots & \mathcal{A}^1_d & \mathcal{A}^1_\mu \\ & 1 & \mathcal{A}^2_3 & \dots & \mathcal{A}^2_d & \mathcal{A}^2_\mu \\ & & \ddots & & & \vdots \\ & & & 1 & \mathcal{A}^{d-1}_d & \mathcal{A}^{d-1}_\mu \\ & & & & 1 & \mathcal{A}^d_\mu \\ \hline & & & & & \mathbb{I}_{11-d} \end{array} \right) \,, \tag{3.24b}$$

where E^ν_μ denotes the vielbein in the uncompactified directions. The action (2.7) in the Neveu–Schwarz sector then takes the simple form:

$$\begin{aligned} S_{\mathrm{NS},10-d} = \int \mathrm{d}^{10-d}x \sqrt{-g}\, \frac{V}{g_s^2 l_s^8} \Bigg[& R + (\partial\phi)^2 + \sum_i \left(\frac{\partial R_i}{R_i}\right)^2 + \sum_{i<j} \left(\frac{R_i}{R_j} \mathcal{F}^{i(1)}_j\right)^2 \\ & + \sum_i \left(R_i \mathcal{F}^{(2)i}\right)^2 + \left(l_s^2 F^{(3)}\right)^2 + \sum_i \left(\frac{l_s^2}{R_i} F^{(2)}_i\right)^2 + \sum_{i<j} \left(\frac{l_s^2}{R_i R_j} F^{(1)}_{ij}\right)^2 \Bigg] \,, \end{aligned} \tag{3.25}$$

where the first five terms come from the reduction of the Einstein–Hilbert term and the last three terms from the kinetic term of the two-form.

Putting together the forms of the same degree, we see that their coefficients form the Weyl orbit Φ_s, of the string tension $(\mathcal{F}^{(3)}_\lambda)$, the Weyl orbit Φ_{KK} of the Kaluza–Klein and winding states ($\mathcal{F}^{(2)}_\lambda$), and the set of positive roots $\Phi_+ = \{e_i \pm e_j, i < j\}$ ($\mathcal{F}^{(1)}_\alpha$). We can therefore rewrite the action in the Weyl-invariant form:

$$\begin{aligned} S_{\mathrm{NS},10-d} &= \int \mathrm{d}^{10-d}x\sqrt{-g}\ \tfrac{V}{g_s^2 l_s^8} \Big[R + \partial\varphi\cdot\partial\varphi + \textstyle\sum_{\alpha\in\Phi_+} e^{-2\langle\varphi,\alpha\rangle} \left(\mathcal{F}^{(1)}_\alpha\right)^2 \\ &\quad + \textstyle\sum_{\lambda\in\Phi_{\mathrm{KK}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{F}^{(2)}_\lambda\right)^2 + \textstyle\sum_{\lambda\in\Phi_s} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{F}^{(3)}_\lambda\right)^2 \Big] \ , \end{aligned} \tag{3.26}$$

where $\varphi = (\ln g_s, \ln R_1, \dots, \ln R_d)$ is the vector of dilatonic scalars, $\langle\varphi,\lambda\rangle$ the duality bracket in Eq. (3.13) and $\partial\varphi\cdot\partial\varphi$ the Weyl-invariant kinetic term obtained from the non-diagonal metric (3.15). A diagonal metric on the dilatonic scalars is recovered upon going to the Einstein frame.

The Weyl group acts by permuting the various weights appearing in Eq. (3.26), and the invariance in the gauge sector is therefore manifest. As for the scalars, the set of positive roots Φ_+ is not invariant under Weyl reflections, but the Peccei–Quinn scalars undergo non-linear transformations $\mathcal{A}^{(0)} \to e^{-2\langle\varphi,\alpha\rangle}\mathcal{A}^{(0)}$ that compensate the sign change [220]. The Peccei–Quinn scalars therefore appear as displacements along the positive (non-compact) roots. Together with the dilatonic (non-compact) scalars φ, they generate the solvable Lie subalgebra that forms the tangent space of the moduli space $\mathcal{H}$ [8, 6, 7, 311].

We have so far concentrated on the Neveu–Schwarz sector, but the same reasoning can be applied to the full type II action. The T-duality Weyl symmetry can, however, be exhibited only by dualizing the p-form gauge fields $\mathcal{G}^{(p)} = d\mathcal{R}^{(p-1)}$ into lower rank $(10-d-p)$-form gauge fields when possible, and keeping them together when their dual when the self-duality condition $10-d-p=p$ is satisfied. We then obtain, for the action of the Ramond fields

$$\begin{aligned} S_{\mathrm{RR}} &= \int \mathrm{d}^{10-d}x\sqrt{-g}\ \tfrac{V}{g_s^2 l_s^8} \Big[\textstyle\sum_{\lambda\in\Phi_{\mathrm{DI}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{G}^{(1)}_\lambda\right)^2 + \textstyle\sum_{\lambda\in\Phi_{\mathrm{D0}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{G}^{(2)}_\lambda\right)^2 \\ &\quad + \textstyle\sum_{\lambda\in\Phi_{\mathrm{D1}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{G}^{(3)}_\lambda\right)^2 + \textstyle\sum_{\lambda\in\Phi_{\mathrm{D2}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{G}^{(4)}_\lambda\right)^2 \Big] \ , \end{aligned} \tag{3.27}$$

where Φ_{DI},Φ_{D0},Φ_{D1},Φ_{D2} denote the Weyl orbits with highest weight $1/g_sR_i$, $1/g_sl_s$, $R_i/g_sl_s^2$, $1/g_sl_s^3$ respectively, corresponding in turn to the two spinor representations.

3.7 Spectral flow and Borel generators

Having discussed the structure of the Weyl group we now want to investigate the full $SO(d,d,\mathbb{Z})$ symmetry. For this purpose, it is instructive to go back to the perturbative multiplet of Kaluza–Klein and winding states. The action of the Weyl group on the highest weight $1/R_d$ of the vector representation generates an orbit of $2d$ elements, $1/R_i$ and R_i. However, a particle can have any number of momentum excitation along each axis, and wind along any cycle of the torus T^d. It is therefore described by integer momenta m_i and winding numbers m^i, so that its mass on an arbitrary torus reads

$$\mathcal{M}^2 = m_i g^{ij} m_j + m^i g_{ij} m^j \ , \quad i,j = 1\dots d \ , \tag{3.28}$$

when $B_{ij}=0$. This mass formula is then invariant under modular transformations $\gamma^i \to \gamma^i + \Delta A^i_j \gamma^j$ of the torus, i.e. integer shifts $A^i_j \to A^i_j + \Delta A^i_j$ of the off-diagonal term of the metric (no sum on i)

$$\mathrm{d}s_d^2 = R_i^2 (dx^i + A^i_j dx^j)^2 + g_{jk} dx^j dx^k \ , \tag{3.29}$$

upon transforming the momenta and winding as

$$m_k \to m_k - \Delta A^i_k m_i \ , \ \ m^k \to m^k + \delta^k_i \Delta A^i_j m^j \ . \tag{3.30}$$

This transformation generates a *spectral flow* on the lattice of charges m_i and m^i.

In addition, being charged under the gauge potential $B_{\mu i}$, the momentum of the particle shifts according to $m_i \to \tilde{m}_i = m_i + B_{ij} m^j$, yielding the mass (3.8). From this, we see that the Borel generator $B_{ij} \to B_{ij} + \Delta B_{ij}$ induces a spectral flow

$$m_k \to m_k + \Delta B_{jk} m^j \ , \ \ m^k \to m^k \ . \tag{3.31}$$

The two spectral flows (3.30) and (3.31) can be understood in a unified way as translations on the weight lattice by positive roots. Indeed, the set of all positive roots of $SO(d,d)$ includes the $Sl(d)$ roots $e_j - e_i, i<j$, images of the simple roots $\alpha_i = e_{i+1} - e_i, 1 \le i \le d-1$ under the Weyl group $\mathcal{S}_d$ of $Sl(d)$, as well as the roots $e_i + e_j$, which are images of the T-duality simple root $\alpha_0 = e_1 + e_2$. The translation by a root $e_j - e_i$ generates infinitesimal rotations in the (i,j) plane[‡18]:

$$\Delta | -e_k \rangle = -\Delta A^i_k | -e_i \rangle \ , \ \ \Delta | e_k \rangle = \delta^k_i \Delta A^i_j | e_j \rangle \tag{3.32}$$

equivalent to the spectral flow in Eq. (3.30), whereas translations by a root $e_i + e_j$ generate an infinitesimal B_{ij} shift:

$$\Delta | -e_k \rangle = \Delta B_{jk} | e_j \rangle \ , \ \ \Delta | e_k \rangle = 0 \tag{3.33}$$

as in Eq. (3.31). The moduli A^i_j and B_{ij} can therefore be identified as displacements on the moduli space $\mathcal{H}$ along the positive roots $e_i - e_j$ and $e_i + e_j$. We note that the two displacements do not necessarily commute and that only *integer* shifts are symmetries of the charge lattice.

3.8 D-branes and T-duality invariant mass

In order to study the analogous properties of the D-brane states, we may try to write down the moduli matrix $M_S \in SO(d,d,\mathbb{R})/SO(d)\times SO(d)$ in the spinorial representation and look for the transformations of charges that leave the mass $m^t M_S m$ invariant, when now m is a spinor of D-brane charges. It is in fact much easier to study the D-brane configuration itself and compute its Born–Infeld mass [255, 152].

[‡18]The Borel generators E_α actually either translate the weight vectors λ or annihilate them.

BPS D-brane states are obtained by wrapping Dp-branes on a supersymmetric p-cycle of the compactification manifold. In the case of a torus T^d, this is simply a straight cycle, and in the static gauge the embedding is specified by a set of integer (winding) numbers N^i_α:

$$X^i = N^i_\alpha \sigma^\alpha \ , \quad i = 1 \dots d \ , \quad \alpha = 1 \dots p \ , \tag{3.34}$$

where σ^α and X^i are the space-like world-volume and embedding coordinates respectively. The numbers N^i_α can, however, be changed by a world-volume diffeomorphism, and one should instead look at the invariant

$$m^{ijkl} = \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta N^k_\gamma N^l_\delta \ , \tag{3.35}$$

where we restricted to $p = 4$ for illustrative purposes. m^{ijkl} is a four-form integer charge that specifies the four-cycle in T^d. In addition, the D-brane supports a $U(1)$ gauge field that can be characterized by the invariants

$$m^{ij} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta F_{\gamma\delta} \quad , \quad m = \frac{1}{8} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \ , \tag{3.36}$$

which are again integer-valued, because of the flux and instanton-number integrality. The charges $\mathcal{N} = \{m, m^{ij}, m^{ijkl}, \dots\}$ constitute precisely the right number to make a spinor representation of $SO(d,d,\mathbb{Z})$ when $p = d$ or $p = d+1$ (depending on the type of theory and dimensionality of the torus); indeed, the spinor representation of $SO(d,d)$ decomposes under $Sl(d)$ as a sum of even or odd forms, depending on the chirality of the spinor. The Chern–Simons coupling (2.38) can be rewritten in terms of these charges (up to corrections when $B \neq 0$) as

$$\int e^{\hat{B} + \alpha' F} \mathcal{R} = m \mathcal{R}_0 + \frac{1}{2} m^{ij} \mathcal{R}_{0ij} + \frac{1}{4!} m^{ijkl} \mathcal{R}_{0ijkl} + \dots \tag{3.37}$$

so that (for $p = 4$) the instanton number m can be identified as the D0-brane charge, the flux m^{ij} as the D2-brane charge and m^{ijkl} as the D4-brane charge. Configurations with $m \neq 0$ exist in SYM theory on a torus, even for a $U(1)$ gauge group, and correspond to torons [303, 150, 151].

The mass of the wrapped D-brane can be evaluated by using the Born–Infeld action (2.37), and depends only on the parametrization-independent integer charges $m, m^{ij}, m^{ijkl}, \dots$ Explicitly, we obtain, for $p = d$, the T-duality invariant mass formula:[‡19]

$$\mathcal{M}^2 = \frac{1}{g_s^2 l_s^2} \tilde{m}^2 + \frac{1}{2 \ g_s^2 l_s^6} (\tilde{m}^{ij})^2 + \frac{1}{4! \ g_s^2 l_s^{10}} (\tilde{m}^{ijkl})^2 + \dots \tag{3.38a}$$

$$\tilde{m} = m + \frac{1}{2} m^{ij} B_{ij} + \frac{1}{8} m^{ijkl} B_{ij} B_{kl} + \dots \tag{3.38b}$$

$$\tilde{m}^{ij} = m^{ij} + \frac{1}{2} m^{klij} B_{kl} + \dots \tag{3.38c}$$

$$\tilde{m}^{ijkl} = m^{ijkl} + \dots \tag{3.38d}$$

[‡19] This expression was originally derived in Ref. [255] by a sequence of T-dualities and covariantizations.

where the dots stand for the obvious extra terms when $d \geq 4$. A similar expression holds for $p = d + 1$ and yields the tension of D-strings:

$$\mathcal{T}_1^2 = \frac{1}{g_s^2 l_s^6}(\tilde{m}^i)^2 + \frac{1}{3!\ g_s^2 l_s^{10}}(\tilde{m}^{ijk})^2 + \frac{1}{5!\ g_s^2 l_s^{14}}(\tilde{m}^{ijkll})^2 + \dots \tag{3.39a}$$

$$\tilde{m}^i = m^i + \frac{1}{2} m^{jki} B_{jk} + \frac{1}{8} m^{jklmi} B_{jk} B_{lm} + \dots \tag{3.39b}$$

$$\tilde{m}^{ijk} = m^{ijk} + \frac{1}{2} m^{lmijk} B_{lm} + \dots \tag{3.39c}$$

$$\tilde{m}^{ijklm} = m^{ijklm} + \dots \tag{3.39d}$$

where the integer charges read, e.g. for $p = 5$,

$$m^{ijklm} = \epsilon^{\alpha\beta\gamma\delta\epsilon} N_\alpha^i N_\beta^j N_\gamma^k N_\delta^l N_\epsilon^m \tag{3.40a}$$

$$m^{ijk} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta\epsilon} N_\alpha^i N_\beta^j N_\gamma^k F_{\delta\epsilon} \tag{3.40b}$$

$$m^i = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\epsilon} N_\alpha^i F_{\beta\gamma} F_{\delta\epsilon}\ . \tag{3.40c}$$

The mass formulae (3.38) and (3.39) hold for 1/2-BPS states only; they are the analogues of Eq. (3.8) for the two spinor representations of $SO(d,d)$. They can be derived by analysing the BPS eigenvalue equation in a similar way as in Subsection 2.2. This analysis is carried out in Appendix A.3, and yields, in addition, the conditions for the state to be 1/2-BPS, as well as the extra contribution to the mass in the 1/4-BPS case. In the $d \leq 6$ case, we find a set of conditions:

$$k^{ijkl} \equiv m^{[ij} m^{kl]} + m\ m^{ijkl} = 0 \tag{3.41a}$$

$$k^{i;jklmn} \equiv m^{i[j} m^{klmn]} + m\ m^{ijklmn} = 0 \tag{3.41b}$$

$$k^{ij;klmnpq} \equiv n^{ij} n^{klmnpq} + n^{ij[kl} n^{mnpq]} = 0 \tag{3.41c}$$

analogous to the level-matching condition $\|m\|^2 = 0$ on the perturbative states. In contrast to the latter, they have a very clear geometric origin, since they can be derived by expressing the charges m in terms of the integer numbers N_α^i (Eq. (3.38)). For $d = 6$, they transform in a $\mathbf{15} + \mathbf{36} + \mathbf{15} = \mathbf{66}$ irrep of the T-duality group $SO(6,6,\mathbb{Z})$. The last line in (3.41) drops when $d = 5$, giving a $\mathbf{5} + \mathbf{5} = \mathbf{10}$ irrep of $SO(5,5,\mathbb{Z})$. When $d = 4$, only the $k^{1234} = m^2 \wedge m^2 + m\ m^4 \equiv 0$ component remains, which is a singlet under $SO(4,4,\mathbb{Z})$.

When the conditions $n = 0$ in (3.41) are not met, the state is at most 1/4-BPS, and its mass receives an extra contribution, e.g. for $d = 5$:

$$\mathcal{M}^2 = \frac{1}{g_s^2 l_s^2}\left[\tilde{m}^2 + \frac{1}{2 l_s^4}\left(\tilde{m}^{ij}\right)^2 + \frac{1}{4! l_s^8}\left(\tilde{m}^{ijkl}\right)^2 + \sqrt{\frac{1}{4! l_s^{12}}\left(\tilde{k}^{ijkl}\right)^2 + \frac{1}{5! l_s^{16}}\left(\tilde{k}^{i;jklmn}\right)^2}\ \right]\ , \tag{3.42}$$

where the shifted charges are given by

$$\tilde{k}^{ijkl} = k^{ijkl} + B_{mn} k^{m;nijkl} \,, \qquad \tilde{k}^{i;jklmnp} = k^{i;jklmnp} \,. \tag{3.43}$$

For $d = 6$, there are still conditions to be imposed in order for the state to be 1/4-BPS instead of simply 1/8-BPS, which are now cubic in the charges m and transform as a **32** of $SO(6,6,\mathbb{Z})$ (see Appendix A and Subsection 5.9).

4 U-duality in toroidal compactifications of M-theory

T-duality is only a small part of the symmetries of toroidally compactified string theory, namely the part visible in perturbation theory. We shall now extend the techniques of Section 3 in order to study the algebraic structure of the non-perturbative symmetries, which go under the name of U-duality. In this section, we focus on the subgroup of the U-duality symmetry that preserves compactifications on rectangular tori with vanishing expectation values of the gauge potentials. The most general case of non-rectangular tori with gauge potentials, for which the full U-duality symmetry can be exhibited, is discussed in the next section.

4.1 Continuous R-symmetries of the superalgebra

As in our presentation of uncompactified M-theory in Section 2, the superalgebra offers a convenient starting point to discuss the symmetries of M-theory compactified on a torus T^d. The $N=1$, 11D supersymmetry algebra is preserved under toroidal compactification: the generators Q_α merely decompose as bispinor representations of the unbroken group $SO(1,10-d)\times SO(d)$, and form an N-extended super-Poincaré algebra in dimensions $D=11-d$. The first factor $SO(1,10-d)$ corresponds to the Lorentz group in the uncompactified dimensions and is actually part of the superalgebra, while the second only acts as an automorphism thereof, and is also known as an R-symmetry[‡20]. There can be automorphisms beyond the obvious $SO(d)$ symmetry, however, and these are expected to be symmetries of the field theory.

This symmetry enhancement can be observed at the level of the Clifford algebra itself [181, 220]. The Gamma matrices $\Gamma_M, M=0,d+1\dots 10$ of eleven-dimensional supersymmetry can be kept to form a (reducible) Clifford algebra of $SO(1,10-d)$, while the matrices Γ_I, $I=1\dots d$ form an internal Clifford algebra. Note that we have chosen here, in contrast to the notation of the rest of the review, the internal indices running from 1 to d. The generators Γ_{IJ} generate the $SO(d)$ R-symmetry, but they can be supplemented by generators Γ_I to form the Lie algebra of a larger R-symmetry group $SO(d+1)$ [‡21]. It was the attempt to exhibit the $SO(8)$ symmetry of 11D SUGRA compactified on T^7 that led to the discovery of hidden symmetries [73].

The R-symmetry group is actually larger still. Consider the algebra generated by $\Gamma_{(2)},\Gamma_{(3)},\Gamma_{(6)},\Gamma_{(7)}$, where the subscripts denote the number of antisymmetric internal indices, and the corresponding generators are dropped when the number of internal directions is insufficient:

- For $d=2$, the only generator $\Gamma_{IJ}=\Gamma_{12}$ generates a $U(1)$ R-symmetry.

[‡20] The R-symmetry is actually part of the *local* supersymmetry, but we are only interested in its global flat limit.

[‡21] This is the basis for the twelve-dimensional S-theory proposal [27]. It is important that these generators commute with the momentum charge $C\Gamma_\mu$.

- For $d = 3$, $\Gamma_{(2)}$ and $\Gamma_{(3)}$ commute, and generate an $SO(3) \times U(1)$ symmetry.
- For $d = 4$, $\Gamma_{(3)} = \Gamma_+ \Gamma_{(1)}$, where Γ_+ is the space-time or internal chirality (see Eq. (A.1)) and, together with $\Gamma_{(2)}$, generates an $SO(5)$ symmetry.
- For $d = 5$, $\Gamma_{(2)} \pm \Gamma_+ \Gamma_{(3)}$ generate two commuting $SO(5)$ subgroups.
- For $d = 6$, $\Gamma_{(6)}$ appears in the commutator $[\Gamma_{(3)}, \Gamma_{(3)}]$ and a $USp(8)$ is generated.
- For $d = 7$ (resp. $d = 8$) the generator $\Gamma_{(7)}$ comes into play and one obtains an $SU(8) \times U(1)$ (resp. $SO(16)$) R-symmetry group.

The various R-symmetry groups are summarized in the right column of Table 4.1, which furthermore gives the decomposition of the 528 central charges on the right-hand side of Eq. (2.13a) under the Lorentz group $SO(1, 10-d)$ in the uncompact directions and the R-symmetry group. The various columns correspond to distinct $SO(1, 10-d)$ representations, after dualizing (moving) central charges into charges with less indices when possible. In all these cases, the superalgebra can be recast in a form manifestly invariant under the R-symmetry. Here we collect the cases $D = 4, 5, 6$, including the central charges, which transform linearly under the R-symmetry:

- For $D = 4$ ($d = 7$), the 32 supercharges split into 8 complex Weyl spinors transforming as an $\mathbf{8} \oplus \bar{\mathbf{8}}$ of $SU(8)$:

$$\left\{ Q_{\alpha A}, Q_{\dot\beta \bar B} \right\} = \sigma^\mu_{\alpha\dot\beta}\, P_\mu\, \delta_{A\bar B} \tag{4.1a}$$

$$\left\{ Q_{\alpha A}, Q_{\beta B} \right\} = \epsilon_{\alpha\beta}\, Z_{AB} \tag{4.1b}$$

$$\left\{ Q_{\dot\alpha \bar A}, Q_{\dot\beta \bar B} \right\} = \epsilon_{\dot\alpha\dot\beta}\, Z^*_{\bar A \bar B}, \tag{4.1c}$$

  where $\mu = 0, 1, 2, 3$ are $SO(3, 1)$ vector indices, $\alpha, \dot\alpha = 1, 2$ are Weyl spinor indices, and $A, \bar A = 1, \cdots, 8$ are $\mathbf{8}$, $\bar{\mathbf{8}}$ indices of $SU(8)$. The central charges are incorporated into a complex antisymmetric matrix Z_{AB}.

- For $D = 5$ ($d = 6$), the 32 supercharges split into 8 Dirac spinors of $SO(4, 1)$, transforming in the fundamental representation of $USp(8)$. The $N = 8$ superalgebra in a $USp(8)$ basis is

$$\{ Q_{\alpha A}, Q_{\beta B} \} = P_\mu\, (C \gamma^\mu)_{\alpha\beta}\, \Omega_{AB} + C_{\alpha\beta}\, Z_{AB} \tag{4.2}$$

  where $\mu = 0, 1, 2, 3, 4$ are $SO(4, 1)$ vector indices, $\alpha = 1, 2, 3, 4$ are Dirac spinor indices, $A = 1, \cdots, 8$ are indices in the $\mathbf{8}$ of $USp(8)$, and Ω_{AB} is the invariant symplectic form and Z_{AB} is the central charge matrix.

- For $D = 6$ ($d = 5$), the 32 supercharges form 4 complex spinors transforming in the $(\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4})$ of $SO(5) \times SO(5)$ and the superalgbra takes the form

$$\left\{ Q^a_\alpha, Q^b_\beta \right\} = \omega^{ab} \gamma^\mu_{\alpha\beta} p_\mu, \tag{4.3}$$

$$\left\{ Q^a_\alpha, \bar Q^b_{\overline\beta} \right\} = \delta_{\alpha\overline\beta} Z^{ab}, \tag{4.4}$$

where $a, b = 1, \ldots, 4$ are $SO(5)$ spinor indices and ω^{ab} is an invariant antisymmetric matrix, from the local isomorphism $SO(5) = USp(4)$. The 16 central charges are incorporated in a matrix Z^{ab} transforming as a bispinor under the R-symmetry $SO(5) \times SO(5)$ and satisfying the reality condition $Z^* = \omega Z \omega^t$.

The R-symmetries that we have discussed here will be of use in the next section to determine the scalar manifold of the compactified 11D SUGRA and hence the global symmetries.

d	Q^a_α	$p=0$ Z^I, Z^{IJ} Z^{IJKLM}	$p=1$ $Z^\mu, Z^{\mu I}$ $Z^{\mu IJKL}$	$p=2$ $Z^{\mu I}$ $Z^{\mu\nu IJK}$	$p=3$ $Z^{\mu\nu\rho IJ}$	$p=4$ $Z^{\mu\nu\rho\sigma I}$	$p=5$ $Z^{\mu\nu\rho\sigma\tau}$	H
1	$(\pm, \mathbf{16})$	$1+0$ $+0$	$1+1$ $+0$	1 $+0$	0	1	1^+ $+1^-$	1
2	$(\mathbf{2}, \mathbf{16})$	$2+1$ $+0$ $= \mathbf{2} + \mathbf{1}$	$1+2$ $+0$ $= \mathbf{2} + \mathbf{1}$	$1+0$ $= \mathbf{1}$	1	$[1]$ $+2$ $= \mathbf{2} + \mathbf{1}$	(1) *move*	$SO(2)$
3	$(\mathbf{2}, \mathbf{8}^+)$ $+(\mathbf{2}, \mathbf{8}^-)$	$3+3$ $+0$ $= \mathbf{3}^+ + \mathbf{3}^-$	$1+3$ $+0$ $= \mathbf{3} + \mathbf{1}$	$1+1$ $= \mathbf{1} + \mathbf{1}$	$3+[1]$ $= \mathbf{3} + \mathbf{1}$	3^+ $+3^-$ $= \mathbf{3}^+ + \mathbf{3}^-$	(1) *move*	$SO(2)$ $\times U(1)$
4	$(\mathbf{4}, \mathbf{8})$	$4+6$ $+0$ $= \mathbf{10}$	$1+4$ $+1$ $= \mathbf{5} + \mathbf{1}$	$1+4$ $+[1]$ $= \mathbf{5} + \mathbf{1}$	$6+[4]$ $= \mathbf{10}$	(4) *move*	(1) *move*	$SO(5)$
5	$(\mathbf{4}, \bar{\mathbf{4}})$ $+(\bar{\mathbf{4}}, \mathbf{4})$	$5+10$ $+1$ $= (\mathbf{4}, \mathbf{4})$	$1+5$ $+5+[1]$ $= (\mathbf{5}, \mathbf{1})$ $+(\mathbf{1}, \mathbf{5})$ $+2(\mathbf{1}, \mathbf{1})$	$1+10$ $+[5]$ $= (\mathbf{4}, \mathbf{4})$	$10^+ + 10^-$ $= (\mathbf{10}, \mathbf{1})$ $+(\mathbf{1}, \mathbf{10})$	(5) *move*	(1) *move*	$SO(5)$ $\times SO(5)$
6	$(\mathbf{8}, \mathbf{4})$	$6+15$ $+6$ $+[1]$ $= \mathbf{27} + \mathbf{1}$	$1+6$ $+15+[6]$ $= \mathbf{27} + \mathbf{1}$	$1+20$ $+[15]$ $= \mathbf{36}$	(15) *move*	(6) *move*	(1) *move*	$USp(8)$
7	$(\mathbf{8}^+, \mathbf{2})$ $+(\mathbf{8}^-, \bar{\mathbf{2}})$	$7+21$ $+21$ $+[7]$ $= \mathbf{28}_c$	$1+7$ $+35+[21]$ $= \mathbf{63} + \mathbf{1}$	$1^\pm + 35^\pm$ $= \mathbf{36}_c$	(21) *move*	(7) *move*	0	$SU(8)$
8	$(\mathbf{16}, \mathbf{2})$	$8+28$ $+56$ $+[28]$ $= \mathbf{120}$	$1+8+70$ $+[1+56]$ $= \mathbf{135} + \mathbf{1}$	$(1+56)$ *move*	(28) *move*	0	0	$SO(16)$

Table 4.1: Classification of the supercharges and central charges w.r.t the Lorentz/R-symmetry group $SO(1, 10-d) \times H$. Irreps of H are in bold face. Charges in parenthesis are Poincaré-dualized (*moved*) into charges in square brackets. Adapted from Ref. [27].

4.2 Continuous symmetries of the effective action

In our discussion of the continuous symmetry of the effective action of the toroidally compactified type IIA theory in Subsection 3.1, we have intentionnally focused our attention on the Neveu–Schwarz sector, and have briefly described how the Ramond fields would transform under the symmetries of the Neveu–Schwarz scalar manifold. The distinction between Neveu–Schwarz and Ramond sectors is however an artefact of perturbation theory and, as we discussed in Section 2, the two sets of fields are unified in the 11D SUGRA description. They mix under the eleven-dimensional Lorentz symmetries unbroken by the compactification on T^d ‡22, namely $Sl(d,\mathbb{R})$. The low-energy effective action therefore admits a continuous symmetry group G_d containing

$$SO(d-1,d-1,\mathbb{R}) \bowtie Sl(d,\mathbb{R}) \ , \tag{4.5}$$

where the symbol $\bowtie$ denotes the group generated by the two non-commuting subgroups. As found by Cremmer and Julia [72, 182], the groups G_d turn out to correspond to the $E_{d(d)}$ series, listed in Table 4.2.

D	d	$G_d = E_{d(d)}$	H_d
10	1	$\mathbb{R}^+$	1
9	2	$Sl(2,\mathbb{R}) \times \mathbb{R}^+$	$U(1)$
8	3	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$SO(3) \times U(1)$
7	4	$Sl(5,\mathbb{R})$	$SO(5)$
6	5	$SO(5,5,\mathbb{R})$	$SO(5) \times SO(5)$
5	6	$E_{6(6)}$	$USp(8)$
4	7	$E_{7(7)}$	$SU(8)$
3	8	$E_{8(8)}$	$SO(16)$

Table 4.2: Cremmer–Julia symmetry groups and their maximal compact subgroups.

The notation $E_{d(d)}$ denotes a particular non-compact form of the exceptional group E_d, namely its normal real form‡23, and from now on this distinction will be omitted. As evident from their Dynkin diagrams shown in Table 4.3, the groups E_d form an increasing family, whose members are related by a process of group disintegration reflecting the decompactification of one compact direction in T^d. This is displayed in Table 4.3, and will be discussed more fully in the next subsection.

The occurrence of these groups can be understood by fitting the number of scalar fields (including the duals of forms of higher degree) to the dimension of a coset space G_d/H_d,

‡22 Note that d has been upgraded by one unit with respect to the previous section.

‡23 The normal real form has all its Cartan generators and positive roots non-compact, and is the maximal non-compact real form of the complex algebra $E_d(\mathbb{C})$ [158, 132].

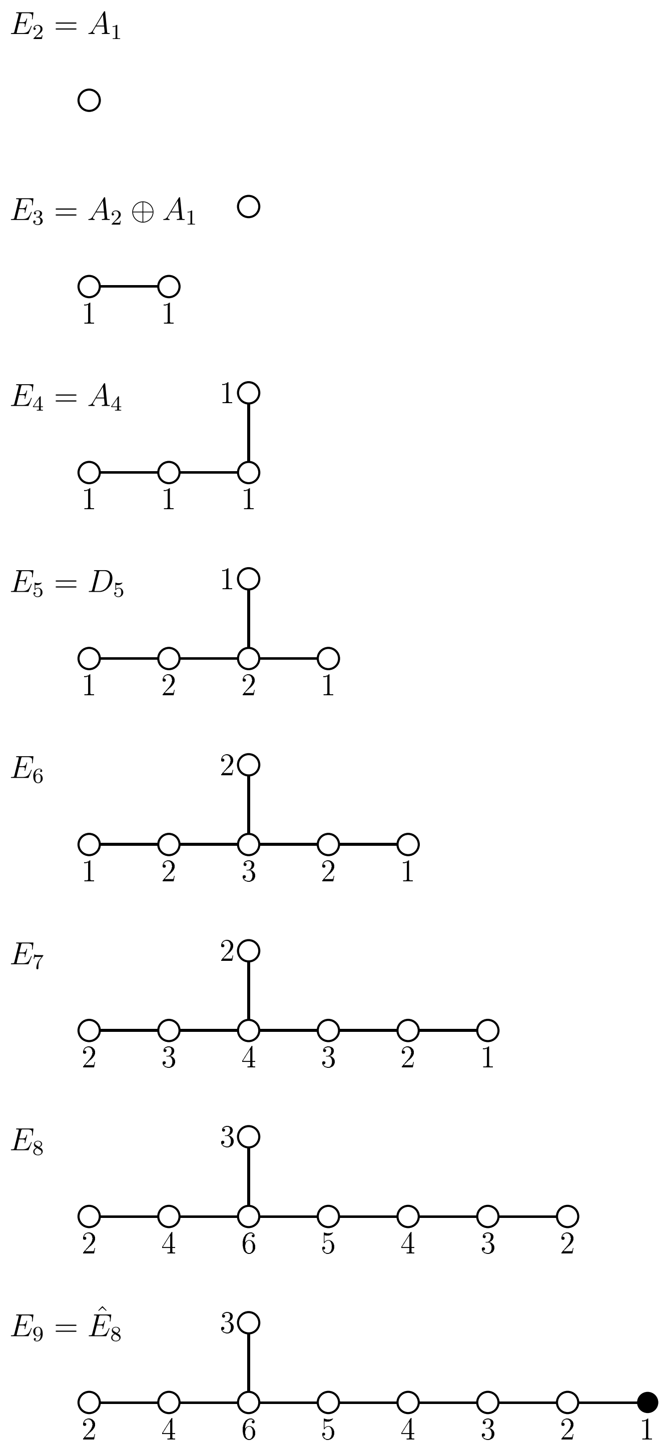

Table 4.3: Dynkin diagrams of the E_d series. The group disintegration proceeds by omitting the rightmost node. The integers shown are the Coxeter labels, that is the coordinates of the highest root on all simple roots.

where H_d is the R-symmetry of the superalgebra described in the previous section. In order to have a positive metric for the scalars, it is necessary that H_d be the maximal compact subgroup of G_d. Together with the dimension of the scalar manifold, this suffices to determine G_d.

Scalar fields arise from the internal components of the metric g_{IJ} of the torus T^d, and from the expectation value of the three-form gauge field $\mathcal{C}_{IJK}$ on T^d; they also arise from the expectation value $\mathcal{E}_{IJKLMN}$ on T^d of the six-form dual to $\mathcal{C}_{MNP}$ in eleven dimensions, or equivalently the expectation value of the scalar dual to the three-form $\mathcal{C}_{\mu\nu\rho}$ in $D=5$, the axion scalar dual to the two-form $\mathcal{C}_{\mu\nu I}$ in $D=4$, or to the one-form $\mathcal{C}_{\mu IJ}$ in $D=3$; similarly, the Kaluza–Klein gauge potentials $g_{\mu I}$ can be dualized in $D=3$ into scalars $\mathcal{K}_I$, which can be interpreted as the expectation value $\mathcal{K}_{I;JKLMNPQR}$ on T^d of the magnetic gauge potential dual to g_{MN} in eleven dimensions. The counting is summarized in Table 4.4. The factor $\mathbb{R}^+$ appearing in $D=10$ and $D=9$ corresponds to the type IIA dilaton, and generates a scaling symmetry of the effective action, called *trombonne symmetry* in Ref. [78]. Note that a quite different U-duality group would be inferred if one did not dualize the Ramond fields into fields with less indices [219, 74], or if one would considerer Euclidean supergravities [174, 77].

An analogous counting has been performed in Tables 4.5 and 4.6 for one-form and two-form potentials, inducing particle and string electric charges, respectively. The latter can be put in one-to-one correspondence to the central charges of the supersymmetry algebra discussed in the previous section, *with two exceptions*. Firstly, the Lorentz-invariant central charge Z^{01234} in five dimensions, where $0\dots 4$ denote the five space-time dimensions, does not correspond to any one-form potential [31, 27][‡24]. This truncation of the superalgebra is consistent with U-duality and is of no concern, except for the twelve-dimensional origin of M-theory. Secondly, there are only 120 Lorentz singlet central charges in $D=3$ for 128 gauge potentials (equivalently, there are only 64 Lorentz vector charges in $D=4$ for 70 two-form gauge fields). As we shall see shortly, U-duality implies that there should in fact be 248 electric charges in $D=3$ (133 string charges in $D=4$), yielding a linear representation of the duality group E_8 (resp. E_7). Of course, the notion of electric charge is ill-defined in $D=3$, where a one-form (or a two-form in $D=4$) is Poincaré-dual to a zero-form and a particle (or a string) to an instanton. Another manifestation of the pathology of the $D=3$ case is the non-asymptotic flatness of the point-like solitons (or string-like in $D=4$), and the logarithmic divergence of the kernel of the Laplacian in the transverse directions. In spite of these difficulties, we shall pursue the algebraic analysis of these cases in the hope that they can be resolved.

If the charges m under the gauge fields can be put in one-to-one correspondence with the central charges Z, they are nevertheless not equal: the gauge charges are integer-quantized, as we will discuss in the next subsection, whereas the central charges are moduli-dependent

[‡24] Equivalently, the central charges Z^{01234},$Z^{02345}\dots$ transform as a vector in six space-time dimensions. These charges could be attributed to a KK6-brane, if only the KK6-brane did not need six compact directions to yield a string, and seven to yield a particle state.

linear combinations of the latter:

$$Z = \mathcal{V} \cdot m \ , \tag{4.6}$$

where $\mathcal{V}$ is an element in the group G_d containing the moduli dependence; it is defined up to the left action of the compact subgroup $K = H_d$, inducing an R-symmetry transformation on Z.

The local H_d gauge invariance can be conveniently gauge-fixed thanks to the Iwasawa decomposition (see for instance [202, 232])

$$\mathcal{V} = k \cdot a \cdot n \in K \cdot A \cdot N \tag{4.7}$$

of G_d into the maximal compact K, Abelian A and nilpotent N. A natural gauge is obtained by taking $K = 1$, in which case the "vielbein" $\mathcal{V}$ becomes a (generalized) upper triangular matrix $\mathcal{V} = a \cdot n$. The Abelian factor A is parametrized by the "dilatonic scalars", namely the radii of the internal torus, whereas the nilpotent factor N incorporates the "gauge scalars", namely the expectation values of the gauge fields (including the off-diagonal metric, three-form and their duals) on the torus. G_d acts on the charges m from the left and on $\mathcal{V}$ from the right. The transformed $\mathcal{V}$ can then be brought back into an upper triangular form by a moduli-dependent R-symmetry compensating transformation on the left. This implies that the central charges Z transform non-linearly under the continuous U-duality group G_d. For the case of T-duality in type II string theory this decomposition is given in Eq. (3.7). In Section 5, we shall obtain an explicit parametrization of $\mathcal{V}$ in terms of the shape of the torus and the various gauge backgrounds.

D	d	g	$\mathcal{C}_3$	$\mathcal{E}_6$	$\mathcal{K}_{1;8}$	total	scalar manifold
10	1	1				1	$\mathbb{R}^+$
9	2	3				3	$Sl(2,\mathbb{R})/U(1) \times \mathbb{R}^+$
8	3	6	1			7	$Sl(3,\mathbb{R})/SO(3) \times Sl(2,\mathbb{R})/U(1)$
7	4	10	4			14	$Sl(5,\mathbb{R})/SO(5)$
6	5	15	10			25	$SO(5,5,\mathbb{R})/SO(5) \times SO(5)$
5	6	21	20	1		42	$E_{6(6)}/USp(8)$
4	7	28	35	7		70	$E_{7(7)}/SU(8)$
3	8	36	56	28	8	128	$E_{8(8)}/SO(16)$

Table 4.4: Scalar counting and scalar manifolds in compactified M-theory.

4.3 Charge quantization and U-duality

As in the case of T-duality, the continuous symmetry $E_{d(d)}(\mathbb{R})$ of the two-derivative effective action cannot be a symmetry of the quantum theory: the gauge potentials transform

D	d	g	$\mathcal{C}_3$	$\mathcal{E}_6$	$\mathcal{K}_{1;8}$	total	charge representation
10	1	1				1	$\mathbf{1}$
9	2	2	1			3	$\mathbf{3}$ of $Sl(2)$
8	3	3	3			6	$(\mathbf{3},\mathbf{2})$ of $Sl(3)\times Sl(2)$
7	4	4	6			10	$\mathbf{10}$ of $Sl(5)$
6	5	5	10	1		16	$\mathbf{16}$ of $SO(5,5)$
5	6	6	15	6		27	$\mathbf{27}$ of $E_{6(6)}$
4	7	7	21	21	7	56	$\mathbf{56}$ of $E_{7(7)}$
3	8	8	28	56	36	128	$\mathbf{248}$ of $E_{8(8)}$

Table 4.5: Vectors and particle charge representations in compactified M-theory.

D	d	g	$\mathcal{C}_3$	$\mathcal{E}_6$	$\mathcal{K}_{1;8}$	total	charge representation
10	1		1			1	$\mathbf{1}$
9	2		2			2	$\mathbf{2}$ of $Sl(2)$
8	3		3			3	$(\mathbf{3},\mathbf{1})$ of $Sl(3)\times Sl(2)$
7	4		4	1		5	$\mathbf{5}$ of $Sl(5)$
6	5		5	5		10	$\mathbf{10}$ of $SO(5,5)$
5	6		6	15	6	27	$\overline{\mathbf{27}}$ of $E_{6(6)}$
4	7		7	35	28	70	$\mathbf{133}$ of $E_{7(7)}$

Table 4.6: Two-forms and string charge representations in compactified M-theory.

non-trivially under E_d, and the continuous symmetry is therefore broken by the existence of states charged under these potentials. At best there can remain a discrete subgroup $E_{d(d)}(\mathbb{Z})$, which leaves the lattice of charges invariant. For one thing, a subset of the charges corresponds to the Kaluza–Klein momentum along the internal torus, and are therefore constrained to lie in the reciprocal lattice of the torus. Another subset of charges corresponds to the wrapping numbers of extended objects around cycles of T^d, and are then constrained to lie in the homology lattice of T^d.

A way to determine the remaining discrete subgroup is to consider M-theory compactified to $D=4$ dimensions, in which case Poincaré duality exchanges gauge one-forms with their magnetic duals [175]. In this dimension, Dirac–Zwanziger charge quantization takes the usual form

$$m^i n'_i - m'^i n_i \in \mathbb{Z} \tag{4.8}$$

for two particles of electric and magnetic charges m^i and n_i respectively, and i runs from

1 to 28, as read off from Table 4.5. This condition is invariant under the electric–magnetic duality $Sp(56,\mathbb{Z})$, under which (m^i, n_i) transforms as a vector. The exact symmetry group is therefore at most

$$E_{7(7)}(\mathbb{Z}) \subset E_{7(7)}(\mathbb{R}) \cap Sp(56,\mathbb{Z}) \ , \tag{4.9}$$

This translates into a condition on $E_{d(d)}(\mathbb{Z})$ for $d \leq 7$ by the embedding $E_{d(d)}(\mathbb{Z}) \subset E_{7(7)}(\mathbb{Z})$. A similar condition can be obtained in $D = 3$, where all one-forms are dual to scalars.

The condition (4.9) requires a precise knowledge of the embedding of $E_{7(7)}(\mathbb{R})$ in $Sp(46,\mathbb{R})$. Instead, we shall take another approach, and *postulate* that the U-duality group of M-theory compactified on a torus T^d is generated by the T-duality $SO(d-1,d-1,\mathbb{Z})$ of type IIA string theory compactified on T^{d-1}, *and* by the modular group $Sl(d,\mathbb{Z})$ of the torus T^d:

$$E_{d(d)}(\mathbb{Z}) = SO(d-1,d-1,\mathbb{Z}) \bowtie Sl(d,\mathbb{Z}) \ . \tag{4.10}$$

The former was argued to be a non-perturbative symmetry of type IIA string theory, as discussed in the previous section, while the latter is the remnant of eleven-dimensional general reparametrization invariance, after compactification on a torus T^d: it is therefore guaranteed to hold, as long as M-theory, whatever its formulation may be, contains the graviton in its spectrum. The above construct is therefore the minimal U-duality group, and since it preserves the symplectic condition (4.8)[‡25] also the maximal one.

In the $d = 2$ case, the U-duality group (4.10) is the modular group $Sl(2,\mathbb{Z})$ of the M-theory torus, which in particular contains the exchange of R_s and R_9; translated in type IIB variables, this is simply the $Sl(2,\mathbb{Z})$ S-duality of type IIB theory (in 9 or 10 dimensions), which contains the strong-weak coupling duality $g_s \to 1/g_s$, as can be seen from Eq. (2.45). Note that we do not expect any quantum symmetry from the trombonne symmetry factor $\mathbb{R}^+$. For $d = 3$, the T-duality group splits into two factors $Sl(2,\mathbb{Z}) \times Sl(2,\mathbb{Z})$, one of which is a subgroup of the modular group $Sl(3,\mathbb{Z})$ of the M-theory torus T^3. The definition (4.10) therefore yields $E_{3(3)}(\mathbb{Z}) = Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ and is the natural discrete group of E_3. For $d = 4$, $SO(3,3,\mathbb{Z})$ is isomorphic to a $Sl(4,\mathbb{Z})$ (in the same way as $SO(6) \sim SU(4)$) which does not commute with the modular group $Sl(4,\mathbb{Z})$ of M-theory on a torus T^4. Altogether, they make the $Sl(5,\mathbb{Z})$ subgroup of $E_{4(4)}(\mathbb{R}) = Sl(5,\mathbb{R})$. For $d = 5$, we obtain the $SO(5,5,\mathbb{Z})$ subgroup of $E_{5(5)}(\mathbb{R}) = SO(5,5,\mathbb{R})$. For $d \geq 6$, this provides a *definition* of the discrete subgroups of the exceptional groups $E_{d(d)}(\mathbb{R})$[‡26]. These groups are summarized in the rather tautological Table 4.7. We note that it is crucial that the groups $E_{d(d)}(\mathbb{R})$ be non-compact in order for an infinite discrete group to exist. The maximal non-compact form is also required in order that all representations be real (i.e. that the mass of a particle and its anti-particle be equal, see Section 4.8).

[‡25] A verification of this statement requires a precise knowledge of the branching functions of $Sp(56)$ into E_7.

[‡26] This is particularly interesting in the $d \geq 9$ case, where we obtain discrete versions of affine and hyperbolic groups, see Section 4.6.

D	d	$E_{d(d)}(\mathbb{R})$	$E_{d(d)}(\mathbb{Z})$
10	1	1	1
9	2	$Sl(2,\mathbb{R})$	$Sl(2,\mathbb{Z})$
8	3	$Sl(3,\mathbb{R}) \times Sl(2,\mathbb{R})$	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$
7	4	$Sl(5,\mathbb{R})$	$Sl(5,\mathbb{Z})$
6	5	$SO(5,5,\mathbb{R})$	$SO(5,5,\mathbb{Z})$
5	6	$E_{6(6)}(\mathbb{R})$	$E_{6(6)}(\mathbb{Z})$
4	7	$E_{7(7)}(\mathbb{R})$	$E_{7(7)}(\mathbb{Z})$
3	8	$E_{8(8)}(\mathbb{R})$	$E_{8(8)}(\mathbb{Z})$

Table 4.7: Discrete subgroups of E_d.

4.4 Weyl and Borel generators

A set of generators of the U-duality group can easily be obtained by conjugating the T-duality generators under $Sl(d,\mathbb{Z})$. The Weyl generators now include the exchange of the eleven-dimensional radius R_s with any radius of the string-theory torus T^{d-1}, in addition to the exchange of the string-theory torus directions among themselves and T-duality on two directions thereof. It is interesting to rephrase the latter in M-theory variables, using relations (2.1) and (3.11):

$$T_{ij} : R_i \to \frac{l_p^3}{R_j R_s} , \quad R_j \to \frac{l_p^3}{R_s R_i} , \quad R_s \to \frac{l_p^3}{R_i R_j} , \quad l_p^3 \to \frac{l_p^6}{R_i R_j R_s} \tag{4.11}$$

These relations are symmetric under permutation of i, j, s indices, and using an $R_k \leftrightarrow R_s$ transformation, we are free to choose i, j, s along any direction of the M-theory torus T^d. The M-theory T-duality therefore reads

$$T_{IJK} : R_I \to \frac{l_p^3}{R_J R_K} , \quad R_J \to \frac{l_p^3}{R_K R_I} , \quad R_K \to \frac{l_p^3}{R_I R_J} , \quad l_p^3 \to \frac{l_p^6}{R_I R_J R_K} \tag{4.12}$$

and in particular involves *three* directions, contrary to the naive expectation. We emphasize that the above equation summarizes the non-trivial part of U-duality, and arises as a mixture of T-duality and S-duality transformations. It can in particular be used to derive [Antoniadis:1999rm] the duality between the heterotic string compactified on T^4 and type IIA compactified on K_3 in the Horava-Witten picture [Horava:1996ma], and thus unify all vacua with 16 supersymmetries. We however restrict ourselves to the maximally supersymmetric case in this review.

The Weyl group can be written in a way, similar to Eq. (3.12)[‡27]:

$$\mathcal{W}(E_d) = \mathbb{Z}_2 \bowtie \mathcal{S}_d \tag{4.13}$$

[‡27] This equation holds for $d \geq 3$ only; when $d < 3$ the $\mathbb{Z}_2$ symmetry (4.12) collapses and only the permutation group $\mathcal{S}_d$ remains.

but it should be borne in mind that the algebraic relations between the $\mathbb{Z}_2$ symmetry T_{123} and the permutations S_{IJ} are different from those of the T-duality generators T_{12} and S_{ij}; in addition d differs by one unit from the one we used there. We also note that the transformations T_{IJK} and S_{IJ} preserve the Newton's constant

$$\frac{1}{\kappa_d^2} = \frac{\prod R_I}{l_p^9} = \frac{V_R}{l_p^9} \ , \tag{4.14}$$

where we have defined V_R to be the volume of the M-theory compactification torus.

On the other hand, the Borel generators now include a generator $\gamma_i \to \gamma_i + \gamma_s$ that mixes the eleven-dimensional direction with the other ones, as well as the T-duality spectral flow $B_{ij} \to B_{ij} + 1$, from which, by an $R_s \leftrightarrow R_i$ conjugation, we can reach the more general *M-theory spectral flow*[‡28]

$$C_{IJK} : \mathcal{C}_{IJK} \to \mathcal{C}_{IJK} + 1 \ . \tag{4.15}$$

We should also include a set of generators shifting the other scalars from the dual gauge potentials, as explained in Section 4.2:

$$E_{IJKLMN} \ : \ \mathcal{E}_{IJKLMN} \to \mathcal{E}_{IJKLMN} + 1 \tag{4.16a}$$

$$K_{I;JKLMNPQR} \ : \ \mathcal{K}_{I;JKLMNPQR} \to \mathcal{K}_{I;JKLMNPQR} + 1 \ . \tag{4.16b}$$

These scalars and corresponding shifts are needed for $d \geq 6$ and $d \geq 8$ respectively. For $d \geq 9$, as will become clear in Section 4.6, the enlargement of the symmetry group to an affine or Kac-Moody symmetry requires an infinite number of such Borel generators. As we shall see in Subsection 5.4, the Borel generators (4.16) can be obtained from commutators of $\mathcal{C}_{IJK}$ transformations.

4.5 Type IIB BPS states and S-duality

Before studying the structure of the U-duality group, we shall pause and briefly discuss the action of the extra Weyl generator $R_s \leftrightarrow R_9$ on the type IIB side. Using the identification (2.45) to convert to type IIB variables, this action inverts the coupling constant and rescales the string length as

$$g_s \leftrightarrow \frac{1}{g_s} \ , \ \ l_s^2 \leftrightarrow l_s^2 g_s \ , \tag{4.17}$$

in such a way that Newton's constant $1/(g_s^2 l_s^8)$ is invariant. Its action on the BPS spectrum can be straightforwardly obtained by working out the action on the masses or tensions, and is summarized in Table 4.8.

In this table, we have displayed the action of the $\mathbb{Z}_2$ Weyl element only. Under more general duality transformations, the fundamental string and the NS5-brane generate orbits

[‡28]As discussed in Subsection 5.4, the $\mathcal{C}$ shift actually has to be accompanied by $\mathcal{E}$ and $\mathcal{K}$ shifts to be a symmetry of the equations of motion.

state	tension	S-dual	dual state
D1-brane	$\frac{1}{g_s l_s^2}$	$\frac{1}{l_s^2}$	F-string
D3-brane	$\frac{1}{g_s l_s^4}$	$\frac{1}{g_s l_s^4}$	D3-brane
D5-brane	$\frac{1}{g_s l_s^6}$	$\frac{1}{g_s^2 l_s^6}$	NS5-brane
KK5-brane	$\frac{R^2}{g_s^2 l_s^8}$	$\frac{R^2}{g_s^2 l_s^8}$	KK5-brane
D7-brane	$\frac{1}{g_s l_s^8}$	$\frac{1}{g_s^3 l_s^8}$	7_3-brane
D9-brane	$\frac{1}{g_s l_s^{10}}$	$\frac{1}{g_s^4 l_s^{10}}$	9_4-brane

Table 4.8: S-dual type IIB BPS states.

of so called (p,q) strings and (p,q) five-branes. The former can be seen as a bound state of p fundamental strings and q D1-branes, or (in the Euclidean case) as a coherent superposition of q D1-branes with p instantons [200]. The (p,q) five-branes similarly correspond to bound states of p NS5-branes and q D5-branes.

On the other hand, the action of S-duality on the D7 and D9-brane yields states with tension $1/g_s^3$ and $1/g_s^4$ respectively. These exotic states will be discussed in Subsection 4.9, where our nomenclature will be explained as well. Again, such states have less than three transverse dimensions, and do not preserve the asymptotic flatness of space-time and the asymptotic constant value of the scalar fields. In particular, the D7-brane generates a monodromy $\tau \to \tau + 1$ in the complex scalar τ at infinity. Its images under S-duality then generate a more general $Sl(2,\mathbb{Z})_B$ monodromy

$$M = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix} \tag{4.18}$$

ascribable to a (p,q) 7-brane[‡29]. We finally remark that the relations in Table 4.8 can also be verified directly using the $R_s \leftrightarrow R_9$ flip and the M-theory/IIB identifications as (un)wrapped M-theory branes, given in Tables 2.5 and 2.6.

4.6 Weyl generators and Weyl reflections

In order to understand the occurrence of the $E_{d(d)}$ U-duality group, we shall now apply the same technique as in the T-duality case and investigate the group generated by the Weyl generators. We choose as a minimal set of Weyl generators the exchange of the M-theory torus directions $S_I : R_I \leftrightarrow R_{I+1}$, where $I = 1 \dots d-1$, as well as the T-duality $T = T_{123}$ on directions 1,2,3 of the M-theory torus. Adapting the construction of Ref.

[‡29]It has also been proposed that the IIB 7-branes transform as a triplet of $Sl(2,\mathbb{Z})$ [230].

[110][‡30]and Subsection 3.4, we represent the monomials $\varphi = (\ln l_p^3, \ln R_1, \ln R_2, \dots, \ln R_d)$ as a form on a vector space V_{d+1} with basis $e_0, e_1, e_2, \dots, e_d$, and associate to any weight vector $\lambda = x^0 e_0 + x^1 e_1 + \dots + x^d e_d$ its "tension"[‡31]

$$\mathcal{T} = e^{\langle \varphi, \lambda \rangle} = l_p^{3x^0} R_1^{x^1} R_2^{x^2} \dots R_d^{x^d} \ . \tag{4.19}$$

The generators S_I and T can then be implemented as linear operators on V_{d+1}, with matrix

$$S_I = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & \mathbb{I}_{d-3} \end{pmatrix} , \quad T = \begin{pmatrix} 2 & 1 & 1 & 1 & \\ -1 & & -1 & -1 & \\ -1 & -1 & & -1 & \\ -1 & -1 & -1 & & \\ & & & & \mathbb{I}_{d-3} \end{pmatrix} \ . \tag{4.20}$$

The operators S_I and T in (4.20) are easily seen to be orthogonal with respect to the Lorentz metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^I)^2 \ , \tag{4.21}$$

and correspond to Weyl reflections

$$\lambda \to \rho_\alpha(\lambda) = \lambda - 2\frac{\alpha \cdot \lambda}{\alpha \cdot \alpha}\alpha \tag{4.22}$$

along planes orthogonal to the vectors

$$\alpha_I = e_{I+1} - e_I \ , I = 1 \dots d-1 \ , \quad \alpha_0 = e_1 + e_2 + e_3 - e_0 \ . \tag{4.23}$$

It is very striking that l_p^3 appears on the same footing as the other radii R_I, but with a minus sign in the metric: it can be interpreted as the radius of an extra time-like direction, much in the spirit of certain proposals about F-theory [313, 27]. The only non-vanishing (Lorentzian) scalar products of these roots turn out to be

$$(\alpha_I)^2 = (\alpha_0)^2 = 2 \ , \quad \alpha_I \cdot \alpha_{I+1} = \alpha_3 \cdot \alpha_0 = -1 \tag{4.24}$$

summarized in the Dynkin diagram:

$$\begin{array}{ccccccccc} & & & & \bigcirc_0 & & & & \\ & & & & | & & & & \\ +_1 & - & \oplus_2 & - & \oplus_3 & - & \oplus_4 & - \cdots - & \oplus_{d-1} \end{array} \tag{4.25}$$

[‡30]In Ref. [110], the discussion was carried out from the gauge theory side, and the U-duality invariant (4.14) was used to eliminate the vector e_0, except when $d = 9$. This vector can, however, be kept for any d, and, as we shall momentarily see, appears as an extra time-like direction.

[‡31]$\mathcal{T}$ actually has the dimension of a p-brane tension $\mathcal{T}_p$, with $p = -(3x^0 + x^1 + \dots + x^d + 1)$.

This is precisely the Dynkin diagram of E_d as shown in Table 4.3, in agreement with the analysis based on moduli counting.

In Eq. (4.25) it is easy to recognize the diagrams of the $SO(d-1,d-1,\mathbb{Z})$ (denoted by $\bigcirc$'s) and $Sl(d,\mathbb{Z})$ (denoted by $+$'s) subgroups. The branching of the $Sl(d)$ diagram on the third root reflects the action of T-duality on three directions. The full diagram can be built from the M-theory Lorentz group $Sl(d,\mathbb{Z})$ denoted by $+$'s, and from the type IIB Lorentz group $Sl(d-1,\mathbb{Z})$ generated by the roots $\alpha_0,\alpha_3,\dots,\alpha_{d-1}$ ‡32. Under decompactification, the rightmost root has to be dropped, so that E_d disintegrates into E_{d-1} ‡33. When the root at the intersection is reached, the diagram falls into two pieces, corresponding to the two $Sl(2)$ and $Sl(3)$ subgroups in $D=8$. The root α_0 itself disappears for $d=2$, leaving only the root α_1 of $Sl(2,\mathbb{R})$.

Again, the action of the Weyl group on V_{d+1} is reducible, at least for $d\leq 8$. Indeed, the invariance of Newton's constant $\prod R_I/l_p^9$ implies that the roots are all orthogonal to the vector

$$\delta = e_1+\cdots+e_d-3e_0\ , \tag{4.26}$$

with proper length $\delta^2=d-9$, so that the reflections actually restrict to the hyperplane V_d normal to δ:

$$x^1+\cdots+x^d+3x^0=0\ . \tag{4.27}$$

The Lorentz metric on V_{d+1} restricts to a metric $g_{IJ}=\delta_{IJ}-1/9$ on V_d, which is positive-definite for $d\leq 8$, so that S_I and T indeed generate the Weyl group of the Lie algebra $E_d(\mathbb{R})$. The order and number of roots of these groups are recalled in Table 4.9 [177].

When $d=9$, however, the invariant vector δ becomes null, so that V_{d+1} no longer splits into δ and its orthogonal space; the generators act on the entire Lorentzian vector space V_{d+1}, and the generators S_I and T no longer span a finite group. Instead, they correspond to the Weyl group of the *affine* Lie algebra $E_9=\hat{E}_8$. This is in agreement with the occurrence of infinitely many conserved currents in $D=2$ space-time dimensions. This case requires a specific treatment and will be discussed in Subsection 4.12. For $d>9$, that is compactification to a line or a point, the situation is even more dramatic, with the occurrence of the hyperbolic Kac–Moody algebras E_{10} and E_{11}, about which very little is known. The reader should go to [184, 183, 239, 129] for further discussion and references.

4.7 BPS spectrum and highest weights

Pursuing the parallel with our presentation on T-duality, we now discuss the representations of the U-duality Weyl group. The fundamental weights dual to the roots $\alpha_1,\dots,\alpha_{d-1},\alpha_0$

‡32 From this point of view, the U-duality is a consequence of general coordinate invariance in M and type IIB theories [209].

‡33 There is a notable exception for $d=8$, where E_8 disintegrates into $E_7\times Sl(2)$. This is because the extended Dynkin diagram of E_8 has an extra root connected to α_8. Only $Sl(2)$ singlets remain in the spectrum, however. The same happens in $d=4$, where $E_3=Sl(3)\times Sl(2)$ in $E_4=Sl(5)$ is not a maximal embedding.

d	2	3	4	5	6	7	8	9
E_d	A_1	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8	$\hat{E}_8$
order	2	6×2	$5!$	$2^4 5!$	$2^7 3^4 5$	$2^{10} 3^4 5\ 7$	$2^{14} 3^5 5^2 7$	∞
roots	2	$6+2$	20	40	72	126	240	∞

Table 4.9: Order and number of roots of E_d Weyl groups.

are easily computed:

$$\lambda^{(1)} = e_1 - e_0 \to \quad \mathcal{T}_1 = \quad \frac{R_1}{l_p^3} \tag{4.28a}$$

$$\lambda^{(2)} = e_1 + e_2 - 2e_0 \to \quad \mathcal{T}_3 = \quad \frac{R_1 R_2}{l_p^6} \tag{4.28b}$$

$$\lambda^{(3)} = e_1 + e_2 + e_3 - 3e_0 \to \quad \mathcal{T}'_5 = \quad \frac{R_1 R_2 R_3}{l_p^9} \tag{4.28c}$$

$$\lambda^{(4)} = e_1 + \cdots + e_4 - 3e_0 \to \quad \mathcal{T}'_4 = \quad \frac{R_1 R_2 R_3 R_4}{l_p^9} \tag{4.28d}$$

$$\dots$$

$$\lambda^{(d-2)} = e_1 + \cdots + e_{d-2} - 3e_0 \to \quad \mathcal{T}'_{10-d} = \quad \frac{R_1 \dots R_{d-2}}{l_p^9} \tag{4.28e}$$

$$\lambda^{(d-1)} = e_1 + \cdots + e_{d-1} - 3e_0 \doteq -e_d \to \quad \mathcal{M} = \quad \frac{1}{R_d} \tag{4.28f}$$

$$\lambda^{(0)} = -e_0 \to \quad \mathcal{T}_2 = \quad \frac{1}{l_p^3} \tag{4.28g}$$

where the symbol $\doteq$ in Eq. (4.28f) denotes equality modulo δ, that is up to a power of the invariant Planck length. In the above equations, we have translated the weight vectors into monomials, and interpreted it as the tension $\mathcal{T}_{p+1}$ of a p-brane:

- The weight $\lambda^{(d-1)}$ corresponds to the Kaluza–Klein states, with mass $1/R_I$, as well as its U-duality descendants. We shall name its orbit the *particle multiplet*, or *flux multiplet*, for reasons that will become apparent in Subsection 6.9.
- The weight $\lambda^{(1)}$ on the other hand has dimension $1/L^2$, and corresponds to the tension of a membrane wrapped on the direction 1: it will go under the name of *string multiplet*, or *momentum multiplet*. The latter name will also become clear in Subsection 6.9.
- The weight $\lambda^{(0)}$ is the highest weight of the *membrane multiplet* containing the fundamental membrane with tension $1/l_p^3$, together with its descendants.

- The weights $\lambda^{(2)}$ and $\lambda^{(5)}$ both correspond to threebrane tensions $\mathcal{T}_3$ and $\mathcal{T}_3'$. Even though they are inequivalent under the Weyl group, it turns out that $\lambda^{(5)}$ is a descendant of $\lambda^{(3)}$ under the full U-duality group. The U-duality orbit of the state with tension $\mathcal{T}_3'$ is therefore a subset of the orbit of the state with tension $\mathcal{T}_3$, and $\lambda^{(3)}$ is the true highest-weight vector of the *threebrane multiplet*.

- The same holds for $\lambda^{(6)}$ associated to a membrane tension $\mathcal{T}_2'$ and descendant of the highest weight $\lambda^{(0)}$ of the membrane multiplet under U-duality, as well as for $\lambda^{(7)}$ and $\lambda^{(1)}$.

- The weight $\lambda^{(3)}$ corresponds to a fivebrane tension $\mathcal{T}_5'$, but is again not the highest weight of the *fivebrane multiplet*, which is instead a non-fundamental weight:

$$\mathcal{T}_5 = \frac{1}{l_p^6} \to \lambda = -2e_6 = 2\lambda^{(0)} \ . \tag{4.29}$$

  Similarly, the weight $\lambda^{(4)}$ corresponds to a fourbrane tension $\mathcal{T}_4'$, and is not the highest weight of the *fourbrane multiplet*, which is instead a non-fundamental weight:

$$\mathcal{T}_4 = \frac{R_1}{l_p^6} \to \lambda = e_1 - 2e_6 = \lambda^{(1)} + \lambda^{(0)} \ . \tag{4.30}$$

- Finally, the instanton multiplet does not appear in Eq. (4.28). An instanton configuration can be obtained by wrapping a membrane on a three-cycle[‡34], and corresponds to a weight vector

$$\mathcal{T}_{-1} = \frac{R_1 R_2 R_3}{l_p^3} \to \lambda = \alpha_0 \ . \tag{4.31}$$

  Since this vector is a simple root, it corresponds to a multiplet in the adjoint representation. It is, however, *not* the highest weight of the U-duality multiplet, which is instead the highest root ψ whose expansion coefficients on the base of the simple roots are given by the Coxeter labels in Table 4.3. An explicit computation gives

$$d=4: \ \psi = \delta - \lambda^{(1)} - \lambda^{(0)} \tag{4.32a}$$

$$d=5: \ \psi = \delta - \lambda^{(2)} \tag{4.32b}$$

$$d=6: \ \psi = \delta - \lambda^{(0)} \tag{4.32c}$$

$$d=7: \ \psi = \delta - \lambda^{(1)} \tag{4.32d}$$

$$d=8: \ \psi = \delta - \lambda^{(7)} \ . \tag{4.32e}$$

  Since the fundamental weights $\lambda^{(i)}$ are dual to the simple roots α_I, it is clear that $\psi \cdot \alpha_I = \delta_{i,I}$, where I is the index appearing on λ in Eq. (4.32) at a given d, and moreover it can be easily checked that $\psi^2 = 2$. The highest root can therefore be added as an extra root in the Dynkin diagrams in Table 4.3, and turns them into extended Dynkin diagrams.

[‡34] We should, however, warn the reader that it is not the representation arising in non-perturbative couplings, as we shall discuss in Subsection 5.8.

The previous considerations are summarized in the diagram

$$\begin{array}{ccccccccc} & & & & \frac{1}{l_p^3} & & & & \\ & & & & | & & & & \\ \frac{R_1}{l_p^3} & - & \frac{R_1R_2}{l_p^6} & - & \frac{R_1R_2R_3}{l_p^9} & - & \frac{R_1R_2R_3R_4}{l_p^9} & -\ldots- & \frac{1}{R_d} \end{array} \tag{4.33}$$

where we have indicated the highest weight associated to each node of the Dynkin diagram. For simplicity, we shall henceforth focus our attention on the *particle* and *string* multiplets, corresponding to the rightmost node with weight $\lambda^{(d-1)}$ and leftmost node with weight $\lambda^{(1)}$ respectively.

4.8 The particle alias flux multiplet

The full particle multiplet can be obtained by acting with Weyl and Borel transformations on the Kaluza–Klein state with mass $1/R_I$. Instead of working out the precise transformation of the supergravity configurations[‡35], we can restrict ourselves to considering the masses of the various states in the multiplet. We note that the action of S_{IJ} and T_{IJK} on the dilatonic scalars R_I is independent of the dimension d of the torus, so that we can work out the maximally compactified case $D=3$, and obtain the higher-dimensional cases by simply deleting states that require too many different directions on T^d to exist.

The results are displayed in Table 4.10, where distinct letters stand for distinct indices. The states are organized in representations of the $Sl(8,\mathbb{Z})$ modular group of the torus T^8. These representations arrange themselves in shells with increasing power of l_p^3; since l_p is invariant under $Sl(8,\mathbb{Z})$, this corresponds to the grading with respect to the simple root α_0. Generalized T-duality T_{IJK} may move from one shell to the next or previous one, whereas S_{IJ} acts within each shell. Eight states with mass V_R/l_p^9 have been added in the middle line, corresponding to zero-length weights that cannot be reached from the length-2 highest state. These states are, however, necessary in order to get a complete representation of the modular group $Sl(8,\mathbb{Z})$, and can be reached by a Borel transformation in $Sl(8,\mathbb{Z})$. They can be thought of as the eight ways to resolve the radius that appears square in the mass of the other states on the same line, into a product of two distinct radii. This is not required for the other lines, since all squares can be absorbed with a power of Newton's constant.

In the last column of Table 4.10, we have indicated the representation of $Sl(8,\mathbb{Z})$ that yields the same dimension. The superscripts denote the number of antisymmetric indices, and no symmetry property is assumed across a semicolon. In other words, $m^{1;7}$ correspond to the $V\otimes\wedge^7 V$ where V is the defining representation of $Sl(d)$. These representations are precisely dual to those under which the various gauge vectors transform (see Table 4.5); they actually correspond to the charges of the BPS state under these $U(1)$ gauge symmetries (see also Subsection 4.2). They generalize the D-brane charges we discussed in Section 3. Altogether, these states sum up to **248**, the adjoint representation of E_8, which

[‡35]See Ref. [221] for the construction of U-duality multiplets of p-brane solutions, and Ref. [115] for a discussion of the continuous U-duality orbits of p-brane solutions.

indeed decomposes in the indicated way under the branching $Sl(8) \subset E_8$. The occurrence of the adjoint representation simply follows from the last equality of Eq. (4.32) identifying the fundamental weight $\lambda^{(7)}$ with the highest root of E_8.

mass $\mathcal{M}$	$Sl(8)$ irrep	charge
$\frac{1}{R_I}$	**8**	m_1
$\frac{R_I R_J}{l_p^3}$	**28**	m^2
$\frac{R_I R_J R_K R_L R_M}{l_p^6}$	**56**	m^5
$\frac{R_I^2 R_J R_K R_L R_M R_N R_P}{l_p^9}$, $8\ \frac{V_R}{l_p^9}$	**1**+**63**	$m^{1;7}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P R_Q}{l_p^{12}}$	**56**	$m^{3;8}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P R_Q}{l_p^{15}}$	**28**	$m^{6;8}$
$\frac{R_I^3 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P^2 R_Q^2}{l_p^{18}}$	**8**	$m^{1;8;8}$

Table 4.10: Particle/flux multiplet **248** of E_8.

The first three lines in Table 4.10 have an obvious interpretation. The state with mass $\frac{1}{R_I}$ is simply the Kaluza–Klein excitation on the dimension I, and m_I denotes the vector of integer momentum charges. The state with mass $R_I R_J / l_p^3$ is the membrane wrapped on a two-cycle T^2 of the compactification torus T^d, and the two-form m^{IJ} labels the precise two-cycle, just as in the D-brane case of the previous section. The third line corresponds to the fivebrane wrapped on the five-cycle labelled by m^{IJKLM}. The states on the fourth line are more interesting. The first of them involves one square radius, and therefore does not exist in uncompactified eleven dimensions. It is simply the KK6-brane with Taub–NUT direction along R_I and wrapped along the directions J to P. The second state with mass V_R/l_p^9, however, does exist in eleven uncompactified directions, and has the tension of a would-be 8-brane. Its asymptotic space-time is however not flat, but logarithmically divergent. The status of this solution is unclear at present, together with that of the following lines of the table. These states only appear as particles in $D = 3$, with the peculiarities that we have already mentioned.

Upon decompactification, the last two lines in Table 4.10 disappear since they require eight distinct radii, and the particle multiplet reduces to a representation of the corresponding U-duality group, as indicated in Table 4.11. When $d \geq 4$, the representation remains the one dual to the rightmost root. For $d = 3$, the U-duality group disconnects into $Sl(3)$ and $Sl(2)$, and $-e_d$ becomes equal to $\lambda^{(2)} + \lambda^{(3)}$ instead of being equal to $\lambda^{(3)}$, as in other cases. Consequently, the particle multiplet transforms as a $(\mathbf{3}, \mathbf{2})$ representation of U-duality.

The full particle multiplet on T^d can be easily decomposed in representations of the U-duality group $E_{d-1}(\mathbb{Z})$ in one dimension higher by separating the states in Table 4.10 according to their dependence on the decompactified radius R_d (which gives a gradation with respect to the simple root α_{d-1}). We obtain the general decomposition[‡36]

$$\mathcal{M}^{(d)} = 1|_{-1} \oplus \mathcal{M}|_0 \oplus (\mathcal{T}_1 \oplus \mathcal{T}_1')|_1 \oplus \mathcal{T}_2'|_2 \oplus (\mathcal{T}_1')^2|_3 \ , \tag{4.34}$$

where we have denoted the multiplets as in Eq. (4.28) and specified the power of R_d in subscript. The notation $(\mathcal{T}_1')^2$ means twice the fundamental weight associated to $\mathcal{T}_1'$. The multiplets on the right-hand side of (4.34) become empty as d decreases. In particular, we note that the particle multiplet on T^d decomposes into a singlet, corresponding to the Kaluza–Klein excitation around the decompactified direction x^d, as well as a particle and a string multiplet on T^{d-1}, depending on whether the state was wrapped around x^d. There are also a number of additional states that appear for $d \geq 6$, to which we shall come back in Subsection 7.7.

D	d	U-duality group	irrep	$Sl(d)$ content
10	1	1	$\mathbf{1}$	$\mathbf{1}$
9	2	$Sl(2,\mathbb{Z})$	$\mathbf{3}$	$\mathbf{2}+\mathbf{1}$
8	3	$Sl(3,\mathbb{Z})\times Sl(2,\mathbb{Z})$	$(\mathbf{3},\mathbf{2})$	$\mathbf{3}+\mathbf{3}$
7	4	$Sl(5,\mathbb{Z})$	$\mathbf{10}$	$\mathbf{4}+\mathbf{6}$
6	5	$SO(5,5,\mathbb{Z})$	$\mathbf{16}$	$\mathbf{5}+\mathbf{10}+\mathbf{1}$
5	6	$E_{6(6)}(\mathbb{Z})$	$\mathbf{27}$	$\mathbf{6}+\mathbf{15}+\mathbf{6}$
4	7	$E_{7(7)}(\mathbb{Z})$	$\mathbf{56}$	$\mathbf{7}+\mathbf{21}+\mathbf{21}+\mathbf{7}$
3	8	$E_{8(8)}(\mathbb{Z})$	$\mathbf{248}$	$2(\mathbf{8}+\mathbf{28}+\mathbf{56})+\mathbf{63}+\mathbf{1}$

Table 4.11: Particle/flux multiplets of E_d.

As a side remark, we note that Table 4.10 is symmetric under reflection with respect to the middle line: for each state with mass $\mathcal{M}$ there is a state with mass $\mathcal{M}'$ satisfying

$$\mathcal{M}\mathcal{M}' = \left(\frac{V_R}{l_p^9}\right)^2 \ , \tag{4.35}$$

where V_R is the volume of the eight-torus. In particular, the lowest weight is equal to minus the highest weight, modulo the invariant vector δ. This is a general property of real representations of compact group, and indeed **248** is the adjoint representation of E_8, therefore real. The same also holds for the **56** representation of E_7 in the $d=7$ case. However, whether real or not with respect to the *compact* real form of the group $E_d(\mathbb{C})$, all the representations appearing in Table 4.11 are real *as representations of the non-compact group* $E_{d(d)}$, as is required by the existence of an anti-particle for each particle. This is obvious for $d \leq 4$; for $d=5$, it is equivalent to the statement that the spinor of $SO(8)$ is real, since the reality properties of spinors of $SO(p,q)$ depend only on $p-q \mod 8$. This property is a characteristic feature of the representations of the maximally non compact real form.

[‡36] For $d=8$, this is $\mathbf{248} = \mathbf{1} \oplus \mathbf{56} \oplus (\mathbf{133} \oplus \mathbf{1}) \oplus \mathbf{56} \oplus \mathbf{1}$.

4.9 T-duality decomposition and exotic states

In order to make contact with string theory solutions, it is useful to decompose the particle multiplet into irreducible representations of the T-duality group $SO(d-1,d-1,\mathbb{Z})$. This can be simply carried out by distinguishing whether the indices lie along the eleventh dimension or not, and substituting the matching relations (2.1). Since T-duality commutes with the grading in powers of the string coupling g_s, the various irreps are then sorted out according to the dependence of the mass of the states on g_s. Table 4.12 summarizes the decomposition of the particle/flux multiplet for M-theory on T^8 into irreducible representations of the $SO(7,7)$ T-duality symmetry group of type IIA string theory on T^7, as well as the $Sl(8)$ (resp. $Sl(7)$) modular group of the M-theory (resp. string theory) torus.

The masses of the states in the a-th column depend on the string coupling constant as $1/g_s^{a-1}$, and are given by

$$V: \quad \frac{1}{R_i}, \frac{R_i}{l_s^2} \tag{4.36a}$$

$$S_A: \quad \frac{1}{g_s}\left(\frac{1}{l_s}, \frac{R_iR_j}{l_s^3}, \frac{R_iR_jR_kR_l}{l_s^5}, \frac{R_iR_jR_kR_lR_mR_n}{l_s^7}\right) \tag{4.36b}$$

$$S+AS: \quad \frac{1}{g_s^2}\left(\frac{R_iR_jR_kR_lR_m}{l_s^6}, \frac{V_R'}{l_s^8}, \frac{R_i^2R_jR_kR_lR_mR_n}{l_s^8}, \frac{V_R'R_iR_j}{l_s^{10}}\right) \tag{4.36c}$$

$$S_B: \quad \frac{V_R'}{g_s^3l_s^8}\left(\frac{R_i}{l_s}, \frac{R_iR_jR_k}{l_s^3}, \frac{R_iR_jR_kR_lR_m}{l_s^5}, \frac{V_R'}{l_s^7}\right) \tag{4.36d}$$

$$V': \quad \left(\frac{V_R'}{g_s^2l_s^8}\right)^2\left(\frac{l_s^2}{R_i}, R_i\right) \tag{4.36e}$$

where V_R' denotes the volume of the string-theory seven-torus. At level $1/g_s^0$ we observe the usual KK and winding states of the string and the level $1/g_s$ reproduces the D0-,D2-,D4- and D6-branes. At level $1/g_s^2$, the NS5 and KK5-brane appear together with two new types of state, a 7_2-brane and a 5_2^2-brane. Our nomenclature displays on-line the number of spatial world-volume directions, i.e. the number of radii appearing linearly in the mass; the superscript specifies the number of directions (if non-zero) that appear quadratic, cubic, etc., listed from the right to the left. the subscript denotes the inverse power of the string coupling appearing in the mass formula; for example, in this convention the KK5-brane is a 5_2^1-brane. According to this notation, we find at level $1/g_s^3$ a 6_3^1-, 4_3^3-, 2_3^5- and 0_3^7-brane. Their masses are related to those of the even Dp-branes, by the type IIA (on T^7) mirror symmetry

$$\mathcal{M}\mathcal{M}' = \left(\frac{V_R'}{g_s^2l_s^8}\right)^2 , \tag{4.37}$$

which follows from the M-theory mirror symmetry relation (4.35). Finally, at level $1/g_s^4$, a 1_4^6- and a $0_4^{(1,6)}$-brane are obtained, whose masses are related to those of the KK and winding states by (4.37).

At this point a few remarks are in order about the new type IIA states that appear in (4.36). The 7_2- and 5_2^2-brane, with mass proportional to $1/g_s^2$ have a conventional dependence on the string coupling, but no supergravity solutions are known for these states. In addition, a variety of states with exotic dependence on the string coupling, $1/g_s^3$ and $1/g_s^4$, are observed. They arise from M-theory states with mass diverging as $1/l_p^9$ or faster. It is not clear what the meaning of these new states in M-theory and type IIA string theory is. These states cannot be accommodated in weakly coupled string theory where the most singular behaviour is expected to be $1/g_s^2$, corresponding to Neveu–Schwarz solitons. A higher power would imply a contribution of a Riemann surface with Euler characteristic $\chi > 2$. Another way to see this is by considering the gravitational field created by these objects, which scales as $\mathcal{M}\kappa_{10}^2$: since $\kappa_{10}^2 \sim g_s^2$, states whose mass goes like g_s^n, $n \leq 2$ create a vanishing or at most finite gravitational field in the weak coupling limit, allowing for a flat space description in the spirit of D-branes. On the other hand, when $n > 2$, the surrounding space becomes infinitely curved at weak coupling, and these states do not correspond to solitons anymore. In fact, the simplest of these states, namely 6_3^1, can be obtained by constructing an array of Kaluza-Klein along a non-compact direction of the Taub-NUT space [‡37], and wrapping the worldvolume directions on the string theory torus T^6 [54]. The summation of the poles in the harmonic function is logarithmically divergent, implying that the asymptotic space-time is logarithmically divergent as well. This is the rule and not the exception for a pointlike state in 3 space-time dimensions (since the Laplacian in the two transverse coordinates has a logarithmic kernel), and the conventional states with an asymptotically flat space-time are simply configurations with a vanishing charge. The same issue arises for p-branes in $p+3$ dimensions (or less). We emphasize, though, that our present purpose is to examine the consequences at the algebraic level of the presence of the conjectured U-duality, which does require these exotic states. The supergravity solutions describing these states can in principle by computed using the known duality relations, which indeed do not preserve the asymptotic flatness of the metric.

$\mathbf{248}(E_8) \supset SO(7,7)$ $\cup$ $Sl(8)$	$\mathbf{14}$ (V)	$\mathbf{64}$ (S_A)	$\mathbf{1}+\mathbf{91}$ $(S \oplus AS)$	$\mathbf{64}$ (S_B)	$\mathbf{14}$ (V')
$\mathbf{8}$ (m_1)	$\mathbf{7}$ (m_1)	$\mathbf{1}$ (m_s)			
$\mathbf{28}$ (m^2)	$\mathbf{7}$ (m^{s1})	$\mathbf{21}$ (m^2)			
$\mathbf{56}$ (m^5)		$\mathbf{35}$ (m^{s4})	$\mathbf{21}$ (m^5)		
$\mathbf{1}+\mathbf{63}$ $(m^{1;7})$		$\mathbf{7}$ $(m^{s;s6})$	$\mathbf{1}+\mathbf{1}+\mathbf{48}$ $(m^{s;7}, m^{1;s6})$	$\mathbf{7}$ $(m^{1;7})$	
$\mathbf{56}$ $(m^{3;8})$			$\mathbf{21}$ $(m^{s2;s7})$	$\mathbf{35}$ $(m^{3;s7})$	
$\mathbf{28}$ $(m^{6;8})$				$\mathbf{21}$ $(m^{s5;s7})$	$\mathbf{7}$ $(m^{6;s7})$
$\mathbf{8}$ $(m^{1;8;8})$				$\mathbf{1}$ $(m^{s;s7;s7})$	$\mathbf{7}$ $(m^{1;s7;s7})$

Table 4.12: Branching of the $d = 8$ particle multiplet into irreps of $Sl(8)$ and $SO(7,7)$. The entries in the table denote the irreps under the common $Sl(7)$ subgroup of $Sl(8)$ and $SO(7,7)$.

[‡37] This construction first appeared in the context of the conifold singularity in the hypermultiplet moduli space [246].

4.10 The string alias momentum multiplet

The same analysis can be carried out for the string multiplet, by applying a sequence of Weyl reflections on the highest weight R_I/l_p^3 describing the wrapped membrane. After adding a multiplet of length 2 and 35 zero-weights for $Sl(8,\mathbb{Z})$ invariance, we obtain a **3875** representation of $E_{8(8)}$. The precise content of this representation is displayed in Appendix B; instead, we display in Table 4.13 the more manageable result for the $d=7$ case, where the string multiplet transforms as a **133** adjoint representation of $E_{7(7)}$. The occurrence of the adjoint representation is again understood from Eq. (4.32) relating the fundamental weight $\lambda^{(1)}$ to the highest root ψ.

tension $\mathcal{T}$ mass	$Sl(7)$ irrep	charge
$\frac{R_I}{l_p^3}$	**7**	n^1
$\frac{R_I R_J R_K R_L}{l_p^6}$	**35**	n^4
$\frac{R_I^2 R_J R_K R_L R_M R_N}{l_p^9}$, $7\frac{V_R}{l_p^9}$	**1**+**48**	$n^{1;6}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P}{l_p^{12}}$	**35**	$n^{3;7}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P}{l_p^{15}}$	**7**	$n^{6;7}$

Table 4.13: String/momentum multiplet **133** of E_7.

D	d	U-duality group	irrep	$Sl(d)$ content
10	1	1	**1**	**1**
9	2	$Sl(2,\mathbb{Z})$	**2**	**2**
8	3	$Sl(3,\mathbb{Z})\times Sl(2,\mathbb{Z})$	**(3, 1)**	**3**
7	4	$Sl(5,\mathbb{Z})$	**5**	**4** + **1**
6	5	$SO(5,5,\mathbb{Z})$	**10**	**5** + **5**
5	6	$E_{6(6)}(\mathbb{Z})$	$\mathbf{\overline{27}}$	**6** + **15** + **6**
4	7	$E_{7(7)}(\mathbb{Z})$	**133**	**7** + **35** + **49** + **35** + **7**
3	8	$E_{8(8)}(\mathbb{Z})$	**3875**	**8** + **70** + ...

Table 4.14: String/momentum multiplets of E_d.

These states have the same interpretation as the states in the particle multiplet, but for wrapping one dimension less of the world-volume. In other words, the states in the particle multiplet can be obtained by wrapping strings on one dimension more– except for the Kaluza–Klein state, which is a genuine point-like (or wave-like, rather) object. We note again that Table 4.13 is symmetric under reflection with respect to its middle line, in agreement with the reality of the **133** adjoint representation of E_7.

The string multiplet in higher dimensions is simply obtained by dropping the states that require too many different radii, as displayed in Table 4.14; in all cases, it corresponds to the representation dual to the leftmost root α_1. We note that in $d=6$ the $\bar{\mathbf{27}}$ string multiplet is distinct from the **27** particle multiplet, but is related to it by an outer automorphism of E_6 corresponding to the $\mathbb{Z}_2$ symmetry of its Dynkin diagram. We also note, for later use, that in all cases the string multiplet representation arises in the symmetric tensor product of two particle multiplets, i.e. $(\mathcal{M} \otimes_s \mathcal{M}) \otimes \mathcal{T}$ always contains a singlet.

Like the particle multiplet, the full string multiplet on T^d can be easily decomposed in representations of the U-duality group $E_{d-1(d-1)}(\mathbb{Z})$ in one dimension higher by using the gradation in powers of the decompactified radius R_d [‡38]

$$\mathcal{T}_1^{(d)} = \mathcal{T}_1|_0 \oplus (\mathcal{T}_2 \oplus \mathcal{T}_2')|_1 \oplus \mathcal{T}_3'|_2 \ , \tag{4.38}$$

where we have denoted the multiplets as in Eq. (4.28) and again specified the power of R_d in subscripts. In particular, we note that the string multiplet on T^d decomposes into a string and a membrane multiplet on T^{d-1}, depending whether the state was wrapped around x^d. There are also a number of additional states that disappear for $d \leq 6$.

$\mathbf{133}(E_7) \supset SO(6,6)$ $\cup$ $Sl(7)$	$\mathbf{1}$ (S)	$\mathbf{32}$ (S_B)	$\mathbf{1}+\mathbf{66}$ $(S' \oplus AS)$	$\mathbf{32}$ (S_B')	$\mathbf{1}$ (S'')
$\mathbf{7}$ (n^1)	$\mathbf{1}$ (n^s)	$\mathbf{6}$ (n^1)			
$\mathbf{35}$ (n^4)		$\mathbf{20}$ (n^{s3})	$\mathbf{15}$ (n^4)		
$\mathbf{1}+\mathbf{48}$ $(n^{1;6})$		$\mathbf{6}$ $(n^{s;s5})$	$\mathbf{1}+\mathbf{1}+\mathbf{35}$ $(n^{s;6}, n^{1;s5})$	$\mathbf{6}$ $(n^{1;6})$	
$\mathbf{35}$ $(n^{3;7})$			$\mathbf{15}$ $(n^{s2;s6})$	$\mathbf{20}$ $(n^{3;s6})$	
$\mathbf{7}$ $(n^{6;7})$				$\mathbf{6}$ $(n^{s5;s6})$	$\mathbf{1}$ $(n^{6;s6})$

Table 4.15: Branching of the $d=7$ string multiplet into irreps of $Sl(7)$ and $SO(6,6)$. The entries in the table denote the irreps under the common $Sl(6)$ subgroup of $Sl(7)$ and $SO(6,6)$.

As in the previous subsection, we give the branching of the $d=7$ string multiplet in terms of irreps of the T-duality $SO(6,6,\mathbb{Z})$ as well as the modular groups $Sl(7,\mathbb{Z})$ and $Sl(6,\mathbb{Z})$ of the M-theory and string theory tori in Table 4.15.

4.11 Weyl-invariant effective action

As in our discussion of T-duality, we would now like to write the supergravity action (2.2) in a manifestly Weyl-invariant form. This has been carried out in Refs. [218, 220], a

[‡38] For $d=7$, this is $\mathbf{133} = \mathbf{27} \oplus (\mathbf{78} \oplus 1) \oplus \bar{\mathbf{27}}$.

simplified version of which will be presented here. As in Eq. (3.25), we decompose the eleven-dimensional field strength $F^{(4)}$ and metric in lower-degree forms. The action then takes the simple form:

$$S_{11-d} = \int \mathrm{d}^{11-d}x\sqrt{-g}\,\frac{V}{l_p^9}\left[R + \sum_i \left(\frac{\partial R_i}{R_i}\right)^2 + \sum_{i<j}\left(\frac{R_i}{R_j}\mathcal{F}_j^{i(1)}\right)^2 + \sum_i \left(R_i \mathcal{F}^{(2)i}\right)^2 \right.$$
$$\left. + \left(l_p^3 F^{(4)}\right)^2 + \sum_i \left(\frac{l_p^3}{R_i}F_i^{(3)}\right)^2 + \sum_{i<j}\left(\frac{l_p^3}{R_iR_j}F_{ij}^{(2)}\right)^2 + \sum_{i<j<k}\left(\frac{l_p^3}{R_iR_jR_k}F_i^{(1)}\right)^2\right] , \quad (4.39)$$

where the first line comes from the reduction of the Einstein–Hilbert term and the second from the kinetic term of the three-form.

In Eq. (4.39) we again recognize in front of the one-form field strength $F^{(1)}$ and $\mathcal{F}^{(1)}$ the positive roots $e_i - e_j$ and $e_i + e_j + e_k - e_0$, in front of the two-form field strength $F^{(2)}$ the weights $-e_i$ of the particle multiplet, in front of the three-form field strength $F^{(3)}$ the weights $e_i - e_0$ of the string multiplet, and in front of the four-form field strength $F^{(4)}$ the weight $-e_0$ of the membrane multiplet. However, these weights do not form complete orbits: it is necessary to dualize the field strengths $F^{(p)}$ into lower-degree field strengths $F^{(11-d-p)}$ so as to display the Weyl symmetry. In the $2p = 11 - d$ case, both the field strength and its dual should be kept. Alternatively, all field strengths may be doubled with their duals, and display an even larger symmetry [74, 75].

We then obtain a manifestly Weyl-invariant action:

$$S_{11-d} = \int \mathrm{d}^{11-d}x\sqrt{-g}\,\frac{V}{l_p^9}\left[R + \partial\varphi\cdot\partial\varphi\right.$$
$$+ \sum_{\alpha\in\Phi_+} e^{-2\langle\varphi,\alpha\rangle}\left(\mathcal{F}_\alpha^{(1)}\right)^2 + \sum_{\lambda\in\Phi_{\mathrm{part}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{F}_\lambda^{(2)}\right)^2$$
$$\left. + \sum_{\lambda\in\Phi_{\mathrm{string}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{F}_\lambda^{(3)}\right)^2 + \sum_{\lambda\in\Phi_{\mathrm{membrane}}} e^{-2\langle\varphi,\lambda\rangle}\left(\mathcal{F}_\lambda^{(4)}\right)^2 + \dots\right] , \quad (4.40)$$

where $\varphi = (\ln l_p^3, \ln R_1, \dots, \ln R_d)$ is the vector of dilatonic scalars (whose first component is non-dynamical), $\langle\varphi,\lambda\rangle = x^0 \ln l_p^3 + x^1 \ln R_1 + \dots$ is the duality bracket (4.19) and $\partial\varphi\cdot\partial\varphi$ the Weyl-invariant kinetic term ($\partial l_p^3 = 0$) obtained from the metric (4.21). In addition to the equations of motion from (4.40), the duality equations $F^{(p)} = *F^{(11-d-p)}$ should also be imposed. As in the case of T-duality, the set of positive roots Φ_+ is not invariant under Weyl reflections, but the Peccei–Quinn scalars undergo non-linear transformations $A^{(0)} \to e^{-2\langle\varphi,\alpha\rangle}A^{(0)}$ that compensate for the sign change [220].

4.12 Compactification on T^9 and affine $\hat{E}_8$ symmetry

As we pointed out in Subsection 4.6, the compactification on a nine-torus T^9 to two space-time dimensions gives rise to a qualitative change in the U-duality group: the invariant vector δ in (4.26) corresponding to the dimensionless Newton constant becomes light-like in the Lorentzian metric $-(dx^0)^2+(dx^1)^2+\cdots+(dx^9)^2$, so that the action of the U-duality group generated by S_I and T in Eq. (4.20) cannot be restricted to its orthogonal subspace. Instead, it generates the Weyl group of the $\hat{E}_8$ affine algebra, as was shown in Ref. [110]; we shall recast their construction in the notation of this review, at the same time settling several issues.

In order to see the affine symmetry $\hat{E}_8$ arise, we simply note that the Dynkin diagram of E_9 (see Table 4.3) is nothing but the extended Dynkin diagram of E_8, where the additional root with Coxeter label 1 corresponds to $\alpha_8 = e_9 - e_8$. The roots $\alpha_0, \alpha_1, \ldots, \alpha_7$ generate the E_8 horizontal Lie algebra, whereas α_8 and $\delta = \sum_{I=1}^{9} e_I - 3e_0$ are the extra dimensions needed to represent the central charge K and degree D generators of the standard construction of affine Lie algebras (see e.g. Ref. [121]). To make the identification precise, we recall that the simple roots of an affine Lie algebra $\hat{G}$ can be chosen as [‡39]

$$\hat{\alpha}_I = (\alpha_I, 0, 0)\ , \quad I = 1\ldots r\ , \quad \hat{\alpha}_0 = (-\psi, 0, 1) \tag{4.41}$$

in the basis (μ, k, d) of the Minkovskian weight space $V_{r+2} = \mathbb{R}^r + \mathbb{R}^{1,1}$ with norm $\mu^2 + 2kd$. Here, ψ is the highest root of G, r is the rank of G, k is the affine level, and d the L_0 eigenvalue. In the case at hand, we have $G = E_8$ so $r = 8$ and want to find the change of basis between the roots $\alpha_I, I = 0\ldots 8$ and null vector δ of our formalism and the standard roots $\hat{\alpha}_I, I = 0\ldots 8$ and vectors $\gamma = (0,0,1), \kappa = (0,1,0)$. From Eq. (4.32) we have,

$$\psi = e_1 + \cdots + e_7 + 2e_8 - 3e_0 = \delta - \alpha_8\ , \tag{4.42}$$

so that, comparing with Eq. (4.41), we can identify δ with $\gamma = (0,0,1)$ and

$$\hat{\alpha}_I = \alpha_I\ , I = 1\ldots 7\ , \tag{4.43a}$$

$$\hat{\alpha}_8 = \alpha_0\ , \tag{4.43b}$$

$$\hat{\alpha}_0 = \alpha_8\ . \tag{4.43c}$$

The vector $\kappa = (0,1,0)$ can be easily calculated from the requirements that $\kappa^2 = \kappa \cdot \hat{\alpha}_I = 0, I = 1\ldots 8$ and $\kappa \cdot \delta = 1$:

$$\kappa = \frac{1}{2}(-e_1 - \cdots - e_8 + e_9 + 3e_0) = e_9 - \frac{\delta}{2}\ . \tag{4.44}$$

The level k and degree d of any weight vector $\lambda \in V_{10}$ can now be obtained from the products $\delta \cdot \lambda$ and $\delta \cdot \kappa$ respectively, and they both have a simple interpretation:

$$k = \delta \cdot \lambda = x^1 + \cdots + x^9 + 3x^0 \tag{4.45}$$

[‡39] In order to keep with the standard notation, the simple roots of the Lie algebra are now labelled by subscripts ranging from 1 to r, as opposed to our notation for the simple roots of the U-duality groups E_r, which carry labels 0 to $r-1$.

is simply the length dimension of the associated monomial $\prod R_I^{x^I} l_p^{3x^0}$, and

$$d = \kappa \cdot \lambda = x^9 - k/2 \tag{4.46}$$

counts the power of R_9 appearing in the same monomial, up to a shift $k/2$. This was expected, since the horizontal subalgebra $E_8 \subset \hat{E}_8$ does not affect R_9 and by definition commutes with L_0. L_{-n} generators, on the other hand, bring additional powers of R_9 and increase the degree d. In particular, the L_0 eigenvalues are integer-spaced, as they should.

We proceed by considering the particle/flux and string/momentum multiplets introduced in Subsections 4.8 and 4.10, with highest weights $\lambda^{(d-1)} = -e_9$ and $\lambda^{(1)} = e_1 - e_0$ respectively (see Eq. (4.28)). The particle multiplet is therefore a level -1 representation with trivial ground state $\mu = 0$ (that is, in the chiral block of the identity). A bit of experimentation reveals the first $Sl(9)$ representations occurring in the particle multiplet:

$$m^{1;1;9},\ m^{1;4;9},\ m^{2;6;9},\ m^{4;7;9},\ m^{7;7;9},\ m^{2;3;9;9}, \dots \tag{4.47}$$

with tensions scaling from $1/l_p^{12}$ to $1/l_p^{24}$, in addition to the representations already present in $d = 8$, given in Table 4.10. However, the full orbit is infinite. On the other hand, the string multiplet is a level -2 representation with ground state in the **3875** of E_8. In both cases, the representations are infinite-dimensional, and need to be supplemented with weights of smaller length as in the E_7 and E_8 cases. The instanton multiplet, on the other hand, is a *level-0* representation of $\hat{E}_8$, with a *non-singlet* ground state in the adjoint of E_8, making it obvious that the usual unitarity restrictions for compact affine Lie algebras do not apply in our case. This concludes our analysis of the $d = 9$ case, and we now restrict ourselves to the better understood $d \leq 8$ case.

5 Mass formulae on skew tori with gauge backgrounds

We would now like to generalize the mass formulae of the U-duality multiplets obtained so far for rectangular tori and vanishing gauge potentials to the more general case of skew tori and arbitrary gauge potentials, which will exhibit the full U-duality group. This will also allow a better understanding of the action of Borel generators on the BPS spectrum. We will concentrate on the $d=7$ flux multiplet, but the same method applies to the other multiplets.

5.1 Skew tori and $Sl(d,\mathbb{Z})$ invariance

We have already argued that BPS states could be labelled by a set of tensors of integer charges describing their various momenta and wrappings. In particular, for the case of the $d=7$ flux multiplet, the charges

$$m_1, m^2, m^5, m^{1;7} \tag{5.1}$$

describe the Kaluza–Klein momentum, membrane, fivebrane and KK6-brane wrappings. The position of the index has been chosen in such a way that we obtain the correct mass by contracting each of them with the vector of radii R^I or inverse radii $1/R_I$. Note that for $d=7$ the tensor $m^{1;7}$ is really a tensor m^1, but the extra seven indices account for an extra factor of the volume in the tension. Of course, a BPS state with generic charges m will not be 1/2-BPS state in general (for $d \geq 5$): some quadratic conditions on m have to be imposed, as already discussed in Subsections 2.2 and 3.8. We shall henceforth assume these conditions fulfilled, deferring the study of the latter to Subsection 5.9.

The 1/2-BPS state mass formula for a non-diagonal metric g_{IJ} can be straightforwardly obtained by replacing contractions with the vector of radii by contractions with the metric, and inserting the proper symmetry factor and power of the Planck length on dimensional grounds:

$$\begin{aligned}\mathcal{M}^2 \ &= (m_1)^2 + (m^2)^2 + (m^5)^2 + (m^{1;7})^2 \\ &= m_I g^{IJ} m_J + \tfrac{1}{2!\ l_p^6} m^{IJ} g_{IK} g_{JL} m^{KL} \\ &\quad + \tfrac{1}{5!\ l_p^{12}} m^{IJKLM} g_{IN} g_{JP} g_{KQ} g_{LR} g_{MS} m^{NPQRS} + \dots\end{aligned} \tag{5.2}$$

This formula is invariant under $Sl(d,\mathbb{Z})$, but *not* yet under the T-duality subgroup $SO(d-1,d-1,\mathbb{Z})$ of the U-duality group. It only holds when the expectation value of the various gauge fields on the torus vanish. To reinstate the dependence on the three-form $\mathcal{C}_{IJK}$, we apply the following strategy.

- Decompose the flux multiplet as a sum of T-duality irreps.
- Include the correct coupling to the NS two-form field B_{ij} using the T-duality invariant mass formulae.

- Study the T-duality spectral flow $B \to B + \Delta B$.
- Covariantize this flow under $Sl(d,\mathbb{Z})$ into a $\mathcal{C} \to C + \Delta\mathcal{C}$ flow.
- Integrate the $\mathcal{C} \to \mathcal{C} + \Delta\mathcal{C}$ flow to obtain the U-duality invariant mass formula.

5.2 T-duality decomposition and invariant mass formula

We have already discussed the first step in Subsection 4.9, and we only need to restrict ourselves to the case $d = 7$. Table 4.12 then truncates to its upper left-hand corner displayed in Table 5.1, as can be read from the $d = 7$ particle multiplet mass formula (5.2) written with s and i indices:

$$\begin{aligned} \mathcal{M}^2 \quad &= \left[\tfrac{m_s^2}{g_s^2} + (m_1)^2\right] + \left[(m^{s1})^2 + \tfrac{(m^2)^2}{g_s^2}\right] \\ &+ \left[\tfrac{(m^{s4})^2}{g_s^2} + \tfrac{(m^5)^2}{g_s^4}\right] + \left[\tfrac{(m^{s;s6})^2}{g_s^2} + \tfrac{m^{1;s6})^2}{g_s^4}\right] \end{aligned} \tag{5.3}$$

corresponding to three $SO(6,6)$ irreps,

$$V = \quad (m_1, m^{s1}) \quad \text{momentum and winding} \tag{5.4a}$$

$$S = \quad (m_s, m^2, m^{s4}, m^{s;s6}) \quad \text{D0-,D2-,D4-,D6-brane} \tag{5.4b}$$

$$V' = \quad (m^5, m^{1;s6}) \quad \text{NS5-brane and KK5-brane} \tag{5.4c}$$

$\mathbf{56}(E_7) \supset SO(6,6)$ $\cup$ $Sl(7)$	$\mathbf{12}$ (V)	$\mathbf{32}$ (S_A)	$\mathbf{12}$ (V')
$\mathbf{7}$ (m_1)	$\mathbf{6}$ (m_1)	$\mathbf{1}$ (m_s)	
$\mathbf{21}$ (m^2)	$\mathbf{6}$ (m^{s1})	$\mathbf{15}$ (m^2)	
$\mathbf{21}$ (m^5)		$\mathbf{15}$ (m^{s4})	$\mathbf{6}$ (m^5)
$\mathbf{7}$ $(m^{1;7})$		$\mathbf{1}$ $(m^{s;s6})$	$\mathbf{6}$ $(m^{1;s6})$

Table 5.1: Branching of the $d = 7$ particle multiplet into irreps of $Sl(7)$ and $SO(6,6)$. The entries in the table denote the irreps under the common $Sl(6)$ subgroup of $Sl(7)$ and $SO(6,6)$.

We can now use the T-duality invariant mass formulae for the T-duality irreps that we obtained in Section 3. In terms of the present charges, they schematically read (in units of

l_s)

$$\mathcal{M}_V^2 = \left(m_i + B_{ij}m^{sj}\right) g^{ik} \left(m_k + B_{kl}m^{sl}\right) + m^{si}g_{ij}m^{sj} \tag{5.5a}$$

$$\begin{aligned}\mathcal{M}_{S_B}^2 &= \frac{1}{g_s^2}\Big[\left(m_s + m^2 B_2 + m^{s4}B_2^2 + m^{s;s6}B_2^3\right)^2 \\ &\quad + \left(m^2 + m^{s4}B_2 + m^{s;s6}B_2^2\right)^2 \\ &\quad + \left(m^{s4} + m^{s;s6}B_2\right)^2 + \left(m^{s;s6}\right)^2\Big]\end{aligned} \tag{5.5b}$$

$$\mathcal{M}_{V'}^2 = \frac{1}{g_s^4}\left[\left(m^5 + m^{1;s6}B_2\right)^2 + \left(m^{1;s6}\right)^2\right] , \tag{5.5c}$$

where we used the vector and spinor representation mass formulae (3.8) and (3.38). Adding the three contributions $\mathcal{M}^2_{\{V,S_B,V'\}}$ together, we now obtain the flux multiplet mass formula for vanishing values of the Ramond fields and arbitrary B-field:

$$\mathcal{M}^2 = \left[\tfrac{\tilde{m}_s^2}{g_s^2} + (\tilde{m}_1)^2\right] + \left[(\tilde{m}^{s1})^2 + \tfrac{(\tilde{m}^2)^2}{g_s^2}\right] \tag{5.6a}$$

$$+ \left[\tfrac{(\tilde{m}^{s4})^2}{g_s^2} + \tfrac{(\tilde{m}^5)^2}{g_s^4}\right] + \left[\tfrac{(\tilde{m}^{s;s6})^2}{g_s^2} + \tfrac{\tilde{m}^{1;s6})^2}{g_s^4}\right] , \tag{5.6b}$$

where the tilded charges are shifted to incorporate the effect of the two-form as in (5.5), so that for instance

$$\tilde{m}_s = m_s + \frac{1}{2}B_2 m^2 + \frac{1}{8}B_2^2 m^{s4} + \frac{1}{48}B_2^3 m^{s;s6} \tag{5.7}$$

is the shift in the D0-brane charge.

5.3 T-duality spectral flow

In Subsections 3.7 and 3.8 we have already discussed the spectral flow $B_{ij} \to B_{ij} + \Delta B_{ij}$ in the vectorial and spinorial representations. We only need to rephrase this flow in terms of the present charges:

$$\begin{aligned}
V:&\quad m_i \to m_i + \Delta B_{ji}m^{sj} , \quad m^{si} \to m^{si} \\
S_B:&\quad m_s \to m_s + \tfrac{1}{2}\Delta B_{ij}m^{ij} , \quad m^{ij} \to m^{ij} + \tfrac{1}{2}\Delta B_{kl}m^{sklij} \\
&\quad m^{sijkl} \to m^{sijkl} + \tfrac{1}{2}\Delta B_{mn}m^{s;smnijkl} , \quad m^{s;s6} \to m^{s;s6} \\
V':&\quad m^{ijklm} \to m^{ijklm} - \Delta B_{np}m^{n;spijklm} \\
&\quad m^{1;s6} \to m^{1;s6} .
\end{aligned} \tag{5.8}$$

The flow indeed acts as an automorphism on the charge lattice, and in particular the charges cannot be restricted to positive integers (except for $m^{1;7}$). This fact will be of use in Subsection 7.8.

Alternatively, the above spectral flow can be recast into a system of differential equations for the shifted charges $\tilde{m}$, e.g. for the spinor representation we have

$$S_B: \begin{array}{rclrcl} \frac{\partial \tilde{m}_s}{\partial B_{ij}} &=& \frac{1}{2}\tilde{m}^{ij} \;, & \frac{\partial \tilde{m}^{ij}}{\partial B_{kl}} &=& \frac{1}{2}\tilde{m}^{sijkl} \\ \frac{\partial \tilde{m}^{sijkl}}{\partial B_{mn}} &=& \frac{1}{2}\tilde{m}^{s;sijklmn} \;, & \frac{\partial \tilde{m}^{s;sijklmn}}{\partial B_{pq}} &=& 0 \;. \end{array} \tag{5.9}$$

This system can be integrated to yield the spinor representation mass formula; the constants of integration correspond to the integer charges m. The integrability of this system of differential equations follows from the commutativity of the spectral flow.

5.4 U-duality spectral flows

The mass formula (5.6) obtained so far is invariant under T-duality and holds for vanishing values of Ramond gauge backgrounds. In order to obtain a U-duality invariant mass formula, we have to allow expectation values of the M-theory gauge three-form $\mathcal{C}_{IJK}$, which extends the Neveu-Schwarz two-form $B_{ij} = \mathcal{C}_{sij}$; the expectation value of the Ramond one-form is already incorporated as the off-diagonal metric component $\mathcal{A}_i = g_{si}/R_s^2 \neq 0$. For $d \geq 6$, one should also allow expectation values of the six-form $\mathcal{E}_{IJKLMN}$ (Poincaré-dual to $\mathcal{C}_{IJK}$ in eleven dimensions). In string-theory language, this corresponds to the Ramond five-form $\mathcal{E}_{s5}$ and the Neveu-Schwarz six-form dual to $B_{\mu\nu}$ in ten dimensions [‡40].

In order to reinstate the $\mathcal{C}_{IJK}$ dependence in mass formula we covariantize the $B_{ij} = \mathcal{C}_{sij}$ spectral flow (5.8) under $Sl(d,\mathbb{Z})$, with the result that

$$\begin{array}{lcll} m_I & \to & m_I & +\ \frac{1}{2}\Delta\mathcal{C}_{JKI}\ m^{JK} \\ m^{IJ} & \to & m^{IJ} & +\ \frac{1}{6}\Delta\mathcal{C}_{KLM}\ m^{KLMIJ} \\ m^{IJKLM} & \to & m^{IJKLM} & +\ \frac{1}{2}\Delta\mathcal{C}_{NPQ}\ m^{N;PQIJKLM} \\ m^{1;7} & \to & m^{1;7}\ . & \end{array} \tag{5.10}$$

Here, however, the $\mathcal{C}$ spectral flow turns out to be *non-integrable*. Defining ∇^{IJK} as the flow induced by the shift $\mathcal{C}_{IJK} \to \mathcal{C}_{IJK} + \Delta\mathcal{C}_{IJK}$, we have the commutator

$$\left[\nabla^{IJK},\ \nabla^{LMN}\right] = 20\nabla^{IJKLMN}\ , \tag{5.11}$$

where ∇^{IJKLMN} is the flow induced by the shift $\mathcal{E}_{IJKLMN} \to \mathcal{E}_{IJKLMN} + \Delta\mathcal{E}_{IJKLMN}$:

$$\begin{array}{lcll} m_I & \to & m_I & +\frac{1}{5!}\Delta\mathcal{E}_{JKLMNI}\ m^{JKLMN} \\ m^{IJ} & \to & m^{IJ} & +\frac{1}{5!}\Delta\mathcal{E}_{KLMNPQ}\ m^{K;LMNPQIJ} \\ m^5 & \to & m^5 & \\ m^{1;7} & \to & m^{1;7}\ . & \end{array} \tag{5.12}$$

[‡40] For $d = 8$, we also need to include the form $\mathcal{K}_{1;8}$, which in string-theory language includes the Ramond seven-form $\mathcal{K}_{s;s7}$, along with a $\mathcal{K}_{1;s7}$ form.

The non-integrability (5.11) of the $\mathcal{C}$-flow can be understood as a consequence of the Chern–Simons interaction in the 11D supergravity action Eq. (2.2) [74, 75]: the equation of motion for $\mathcal{C}$ reads

$$d * F_4 + \frac{1}{2} F_4 \wedge F_4 = 0 \ , \tag{5.13}$$

so that the dual field strength of F_4 has a Chern–Simons term

$$F_7 \equiv *F_4 = d\mathcal{E}_6 - \frac{1}{2}\mathcal{C}_3 \wedge F_4 \ . \tag{5.14}$$

The equation of motion (5.14) is invariant under the gauge transformations

$$\delta\mathcal{C}_3 = \Lambda_3 \ , \quad \delta\mathcal{E}_6 = \Lambda_6 - \frac{1}{2}\Lambda_3 \wedge \mathcal{C}_3 \ , \tag{5.15}$$

for closed Λ_3 and Λ_6. Restricting to constant shifts, this reproduces the commutation relations (5.11). An equivalent statement holds in $D = 3$, where the $\mathcal{C}_3$ and $\mathcal{E}_6$ shifts close on a $\mathcal{K}_{1;8}$ shift.

The non-integrability of the system (5.10) can therefore be evaded by combining the $\Delta\mathcal{C}_3$ shift with a $\Delta\mathcal{E}_6$ shift

$$\frac{1}{5!}\Delta\mathcal{E}_{IJKLMN} = \frac{1}{12}\mathcal{C}_{[IJK}\Delta\mathcal{C}_{LMN]} \ , \tag{5.16}$$

upon which the resulting flow

$$\nabla'^{IJK} = \nabla^{IJK} - 10 C_{KLM}\nabla^{KLMIJK} \tag{5.17}$$

becomes integrable [‡41]. The extra shift is invisible in the type IIA picture for zero Ramond potentials since it does not contribute to the T-duality spectral flow. We emphasize again that these extra terms are generated as a consequence of the integrability of the flow, which we take as a guiding principle for reconstructing the invariant mass formula. The explicit form of the resulting flow equations that follow from (5.17) is then given by [244]

$$\nabla'^{JKL}\tilde{m}_I = \frac{1}{2}\tilde{m}^{JK}\delta^L_I \tag{5.18a}$$

$$\nabla'^{KLM}\tilde{m}^{IJ} = \frac{1}{6}\tilde{m}^{KLMIJ} \tag{5.18b}$$

$$\nabla'^{NPQ}\tilde{m}^{IJKLM} = \frac{1}{2}\tilde{m}^{N;PQIJKLM} \tag{5.18c}$$

$$\nabla'^{RST}\tilde{m}^{I;JKLMNPQ} = 0 \ , \tag{5.18d}$$

[‡41] For $d = 8$ there is also a non-trivial commutator [74] between the $\mathcal{C}_3$ and $\mathcal{E}_6$ flow, closing onto the $\mathcal{K}_{1;8}$ flow, which induces further shifts.

$$\nabla^{JKLMNP}\tilde{m}_I = \frac{1}{5!}\tilde{m}^{JKLMN}\delta_I^P \quad (5.19a)$$

$$\nabla^{KLMNPQ}\tilde{m}^{IJ} = \frac{1}{5!}\tilde{m}^{K;LMNPQIJ} \quad (5.19b)$$

$$\nabla^{NPQRST}\tilde{m}^{IJKLM} = 0 \quad (5.19c)$$

$$\nabla^{RSTUVW}\tilde{m}^{I;JKLMNPQ} = 0 \ , \quad (5.19d)$$

which now can be integrated, as will be shown in Subsection 5.6.

5.5 A digression on Iwasawa decomposition

In order to understand the non-commutativity of the spectral flow from another perspective, it is worthwhile coming back to a simpler example of a non-compact group, namely the prototypical $G(\mathbb{R}) = Sl(n,\mathbb{R})$ group. The Iwasawa decomposition (4.7) then takes the form

$$g = k \cdot a \cdot n \in K \cdot A \cdot N \ , \quad (5.20)$$

where $K = SO(n,\mathbb{R})$ is the maximal compact subgroup of $G(\mathbb{R})$, A is the Abelian group of diagonal matrices with determinant 1 and N is the nilpotent group of upper triangular matrices. The factor k is absorbed in the coset $G(\mathbb{R})/K$, and the coset space is really parametrized by $A \cdot N$.

Now the subgroup of $G(\mathbb{Z})$ leaving A invariant is nothing but the Weyl group $\mathcal{S}_n$ of permutations of entries of A, whereas that leaving N invariant is the Borel group of integer-valued upper triangular matrices with 1's on the diagonal. The latter is graded by the distance away from the diagonal, in the sense that

$$[B_p, B_{p'}] \subset B_{p+p'} \ , \quad (5.21)$$

where B_p is the subset of upper triangular matrices with 1's on the diagonal and other non zero entries on the p-th diagonal only. In particular, B_p is a non-compact Abelian subgroup when $p > n/2$.

Returning to the case at hand, we see that ∇^3, ∇^6 (and $\nabla^{1;8}$ in the $d=8$ case) are analogous to the B_1, B_2 (and B_3) Borel generators of $Sl(3)$ (or $Sl(4)$). More precisely, they correspond to the grading of the root lattice of E_d with respect to the simple root α_0 extending the $Sl(d,\mathbb{Z})$ Lorentz subgroup to the full $E_{d(d)}(\mathbb{Z})$ subgroup, or in other words the grading of the adjoint representation in powers of l_p^3. This can be seen from Table 4.10 for $d=8$ since, in this case, the particle multiplet happens to be in the adjoint representation **248** of E_8. For $d<8$, this can also be seen from the Coxeter label a_0 of α_0 in Table 4.3, i.e. the α_0 component of the highest root of E_d: the degree p of all the positive roots then runs from 0 (corresponding to the g_{IJ} Borel generators) to a_0, with intermediate values 1 for the $\mathcal{C}_3$ flow, 2 for $\mathcal{E}_6$ and 3 for $\mathcal{K}_{1;8}$.

We finally note that, in the notation of Eq. (5.20), the mass formula we are seeking takes the form,

$$\mathcal{M}^2 = m^t \ R^t(a\cdot n) R(a\cdot n)\ m \ , \tag{5.22}$$

where m is the vector of integer charges transforming in the appropriate linear representation R of $E_{d(d)}(\mathbb{R})$.

5.6 Particle multiplet and U-duality invariant mass formula

The flow (5.18) can be integrated to obtain the $E_{7(7)}(\mathbb{Z})$-invariant mass formula for the particle multiplet of M-theory compactified on a torus T^7 with arbitrary shape and gauge background. The result is:

$$\mathcal{M}^2 = \left(\tilde{m}_1\right)^2 + \frac{1}{2!\ l_p^6}\left(\tilde{m}^2\right)^2 + \frac{1}{5!\ l_p^{12}}\left(\tilde{m}^5\right)^2 + \frac{1}{7!\ l_p^{18}}\left(\tilde{m}^{1;7}\right)^2 \ , \tag{5.23}$$

where the shifted charges depend on the gauge potentials as

$$\begin{aligned}
\tilde{m}_I &= m_I + \frac{1}{2}\mathcal{C}_{JKI}m^{JK} + \left(\frac{1}{4!}\mathcal{C}_{JKL}\mathcal{C}_{MNI} + \frac{1}{5!}\mathcal{E}_{JKLMNI}\right) m^{JKLMN} \\
&\quad + \left(\frac{1}{3!4!}\mathcal{C}_{JKL}\mathcal{C}_{MNP}\mathcal{C}_{QRI} + \frac{1}{2\cdot 5!}\mathcal{C}_{JKL}\mathcal{E}_{MNPQRI}\right) m^{J;KLMNPQR}
\end{aligned} \tag{5.24a}$$

$$\begin{aligned}
\tilde{m}^{IJ} &= m^{IJ} + \frac{1}{3!}\mathcal{C}_{KLM}m^{KLMIJ} \\
&\quad + \left(\frac{1}{4!}\mathcal{C}_{KLM}\mathcal{C}_{NPQ} + \frac{1}{5!}\mathcal{E}_{KLMNPQ}\right) m^{K;LMNPQIJ}
\end{aligned} \tag{5.24b}$$

$$\tilde{m}^{IJKLM} = m^{IJKLM} + \frac{1}{2}\mathcal{C}_{NPQ}m^{N;PQIJKLM} \tag{5.24c}$$

$$\tilde{m}^{I;JKLMNPQ} = m^{I;JKLMNPQ} \ . \tag{5.24d}$$

The shifts induced by the expectation values of $\mathcal{C}_3$ and $\mathcal{E}_6$ give an explicit parametrization of the upper triangular[‡42] vielbein $\mathcal{V}$ in terms of the physical compactification parameters (see Eq. (4.6)). The mass formula (5.23) is now invariant under T-duality, besides the manifest $Sl(d,\mathbb{Z})$ symmetry.

As an illustration, we can look at the shift in T-duality vector charge m^{s1} implied by the above equation:

$$\begin{aligned}
\tilde{m}^{s1} + \mathcal{A}_1\tilde{m}^2 = m^{s1}& \\
&+ \mathcal{A}_1 m^2 + \left(\mathcal{C}_3 + \mathcal{A}_1 B_2\right) m^{s4} + \left(\mathcal{E}_{s5} + \mathcal{C}_3 B_2 + \mathcal{A}_1 B_2 B_2\right) m^{s;s6} \\
&+ \left(\mathcal{A}_1\mathcal{C}_3\right) m^5 + \left(\mathcal{E}_6 + \mathcal{C}_3^3 + \mathcal{A}_1\mathcal{E}_{s5} + \mathcal{A}_1 B_2\mathcal{C}_3\right) m^{1;s6} \ .
\end{aligned} \tag{5.25}$$

[‡42] $\mathcal{V}$ is actually upper triangular in blocks, because we did not decompose the metric g_{IJ} in a product of upper triangular vielbeins.

The second line precisely involves the tensor product of the *charge* spinor representation S with the spinor representation made up by the Ramond moduli. In fact, to see that the set $(\mathcal{A}_1,\mathcal{C}_3+\mathcal{A}_1 B_2,\ \mathcal{E}_{s5}+\mathcal{C}_3 B_2+\mathcal{A}_1 B_2 B_2)$ transforms as a spinor, one may simply note that it is precisely the combination that appears in the expansion in powers of F of the T-duality invariant D-brane coupling $\int e^{B+l_s^2 F}\wedge\mathcal{R}$. Formula (5.23) reduces to the $d=5$ result of Ref. [91] for vanishing expectation values of the gauge backgrounds (see also [298]).

5.7 String multiplet and U-duality invariant tension formula

Exactly the same analysis can be done for the momentum multiplet. We give here the result for $d=6$. The contributing charges $n^1, n^4, n^{1;6}$ decompose into $SO(6,6)$ T-duality multiplets

$$I=(n^s)\ , \quad S'=(n^1,n^{s3},n^{s;s5})\ , \quad V=(n^4,n^{1;s5})\ , \tag{5.26}$$

and we obtain the $E_{6(6)}(\mathbb{Z})$-invariant tension formula for the $d=6$ string multiplet:

$$\mathcal{T}^2=\left[\frac{1}{l_p^6}\left(\tilde{n}^1\right)^2+\frac{1}{l_p^{12}}\left(\tilde{n}^4\right)^2+\frac{1}{l_p^{18}}\left(\tilde{n}^{1;6}\right)^2\right]\ , \tag{5.27}$$

where the shifted charges are

$$\begin{array}{rcllll} \tilde{n}^1 & = & n^1 & + \ \mathcal{C}_3 n^4 & + & (\mathcal{C}_3\mathcal{C}_3+\mathcal{E}_6)\, n^{1;6} \\ \tilde{n}^4 & = & n^4 & + \ \mathcal{C}_3 n^{1;6} & & \\ \tilde{n}^{1;6} & = & n^{1;6} & & & \end{array} \tag{5.28}$$

The combinatorial factors and explicit index contractions are easily reinstated in this equation by comparison with (5.24a). This yields the parametrization of the vielbein $\mathcal{V}$ of Eq. (4.6) in the representation appropriate to the string multiplet.

5.8 Application to R^4 couplings

As an illustration of the result (5.27), we display the $d\le 5$ string multiplet invariant tension formula. Because of antisymmetry, only the charges n^1 and n^4 contribute, so that the tension of the string multiplet is given by

$$\mathcal{T}^2=\frac{1}{l_p^6}\left(n^I+\frac{1}{3!}n^{IJKL}C_{JKL}\right)g_{IM}\left(n^M+\frac{1}{3!}n^{MNPQ}C_{NPQ}\right) \\ +\frac{1}{4!\ l_p^{12}}n^{IJKL}g_{IM}g_{JN}g_{KP}g_{LQ}n^{MNPQ}\ . \tag{5.29}$$

This is precisely the U-duality invariant quantity that was obtained in the study of instanton corrections to R^4 corrections in type II theories in Ref. [200], where it was conjectured that the coupling for $d=5$ is given by the $SO(5,5,\mathbb{Z})$ Eisenstein series

$$A = \frac{V}{l_p^9} \hat{\sum}_{n^I, n^{IJKL}} \left[\mathcal{T}^2\right]^{-3/2} = \frac{2\pi V}{l_p^9} \int_0^\infty \frac{\mathrm{d}t}{t^{5/2}} \hat{\sum}_{n^I, n^{IJKL}} e^{-\pi \mathcal{T}^2/t} \ , \tag{5.30}$$

where $\mathcal{T}$ is given by the tension formula (5.29), and V denotes the volume of the M-theory torus T^d. As will become clear in the next Subsection (see Eq. (5.38a)), the sum has to be restricted to integers such that $n^{[I} n^{JKLM]} = 0$, in order to pick up the contribution of half-BPS states only. The generalization of this construction to compactifications of M-theory to lower dimensions was addressed in Ref. [243].

Under Poisson resummation on the charge n^s, the U-duality invariant function (5.30) exhibits a sum of instanton effects of order e^{-1/g_s}, corresponding to the D0-branes (with charge n^1) and D2-branes (with charge n^{s3}), but there is also a contribution of the extra charge n^4 superficially of order e^{-1/g_s^2}. The NS5-brane does not yield any instanton on T^4, so these effects seem rather mysterious. On the other hand, we may interpret Eq. (5.30) as a sum of loops from all perturbative and non-perturbative strings. The occurrence of the NS5-brane of the string multiplet is then no longer surprising. This soliton loop interpretation should, however, be taken with care, since in any case we have not succeeded yet in recovering the one-loop R^4 coupling from the $SO(5,5,\mathbb{Z})$ Eisenstein series.

5.9 Half-BPS conditions and Quarter-BPS states

The U-duality mass formulae (5.23) and (5.27) that we have obtained only hold for 1/2-BPS states, and require particular conditions on the various integer charges. These conditions can be obtained from a precise analysis of the BPS eigenvalue equation, as in Subsection 2.2, or from a sequence of U-dualities from the perturbative level-matching condition $\|m\|^2 = 0$ in Eq. (3.8). In analogy to the latter condition, they should be quadratic in the integer charges, be moduli-independent, and constitute a representation of the U-duality group $E_{d(d)}(\mathbb{Z})$, appearing in the symmetric tensor product of two charge multiplets.

We have already noticed in Subsection 4.10 that the string multiplet always appears in the symmetric product of two particle multiplets, and indeed all the computations in Appendix A point to the fact ‡43 that *the 1/2-BPS condition on the particle multiplet is the string multiplet constructed out of the particle charges.* This has also been observed in Ref. [115], where it was shown that for $d=7$ the 1/2-BPS conditions on the **56** particle multiplet were transforming in a **133** adjoint representation of E_7, which is the corresponding string multiplet.

In order to extract the precise conditions, it is convenient to consider the branching

‡43 The naive inclusion of the KK6-brane as an extra $\Gamma_{IJKLMN} Z^{0IJKLMN}$ term does not seem, however, to yield a U-duality invariant mass formula by this method.

under the ST-duality group:

$$\begin{aligned} E_{7(7)} &\supset SO(6,6)\times Sl(2) \\ \mathbf{56} &= (\mathbf{12},\mathbf{2})+(\mathbf{32},\mathbf{1}) \\ \mathbf{133} &= (\mathbf{1},\mathbf{3})+(\mathbf{\bar{32}},\mathbf{2})+(\mathbf{66},\mathbf{1}), \end{aligned} \tag{5.31}$$

where the **32** correspond to the D-brane charges $m_s, m^2, m^{s4}, m^{s;s6}$ and the two **12**'s to the Kaluza–Klein and winding charges m_1, m^{1s} and the NS5-brane/KK5-brane charges $m^5, m^{1;s6}$ respectively (see also Tables 5.1 and 4.15). The **133** in the symmetric tensor product $\mathbf{56}\otimes_s\mathbf{56}$ of two particle multiplets is therefore

$$(\mathbf{1}\in\mathbf{12}\otimes_s\mathbf{12},\mathbf{3})+(\mathbf{\bar{32}}\in\mathbf{12}\otimes\mathbf{32},\mathbf{2})+(\mathbf{66}\in\mathbf{32}\otimes_s\mathbf{32}+\mathbf{12}\wedge\mathbf{12},\mathbf{1})\ . \tag{5.32}$$

So as to work out the tensor products in Eq. (5.32), it is advisable to consider the further branching

$$\begin{aligned} SO(6,6) &\supset Sl(6)\times O(1,1) \\ \mathbf{12} &= \mathbf{6}_1+\mathbf{\bar{6}}_{-1} \\ \mathbf{32} &= \mathbf{1}_3+\mathbf{\bar{15}}_1+\mathbf{15}_{-1}+\mathbf{1}_{-3} \\ \mathbf{66} &= \mathbf{15}_2+\mathbf{1}_0+\mathbf{35}_0+\mathbf{\bar{15}}_{-2}\ . \end{aligned} \tag{5.33}$$

The decomposition of the **133** conditions in terms of the various $Sl(6)\subset SO(6,6)$ charges is therefore

$$\mathbf{1}_2\ :\ k^s \equiv m_1 m^{s1} \tag{5.34a}$$

$$\mathbf{32}_1\ :\ \begin{cases} k^1 &\equiv m_1 m^2 + m_s m^{s1} \\ k^{3s} &\equiv m_1 m^{s4} + m^{s1} m^2 \\ k^{s;s5} &\equiv m_1 m^{s;s6} + m^{s1} m^{s4} \end{cases} \tag{5.34b}$$

$$\mathbf{66}_0\ :\ \begin{cases} k^4 &\equiv m_s m^{s4} + m^2 m^2 + m_1 m^5 \\ k^{1;s5} &\equiv m^2 m^{s4} + m_s m^{s;s6} + m^{s1} m^5 + m_1 m^{1;s6} \\ k^{s2;s6} &\equiv m^2 m^{s;s6} + m^{s4} m^{s4} + m^{s1} m^{1;s6} \end{cases} \tag{5.34c}$$

$$\mathbf{1}_0\ :\ k^{s;6} \equiv m^{s1} m^5 + m_s m^{s;s6} \tag{5.34d}$$

$$\mathbf{32}_{-1}\ :\ \begin{cases} k^{1;6} &\equiv m^5 m^2 + m^{1;s6} m_s \\ k^{3;s6} &\equiv m^5 m^{s4} + m^{1;s6} m^2 \\ k^{s5;s6} &\equiv m^5 m^{s;s6} + m^{1;s6} m^{s4} \end{cases} \tag{5.34e}$$

$$\mathbf{1}_{-2}\ :\ k^{6;s6} \equiv m^5 m^{1;s6} \tag{5.34f}$$

where the subindex denotes the $SO(1,1)\subset Sl(2)$ charge, and the contractions are the obvious ones. In particular, for D-brane charges only, the condition $\mathbf{66}_0$ reduces to the one introduced in Subsection 3.8. The condition $\mathbf{1}_2$ is the familiar perturbative level-matching condition, whereas $\mathbf{1}_{-2}$ is the analogous condition on NS5–KK5 bound states. The other conditions mix different T-duality multiplets. For example, the spinor constraints (5.34b) and (5.34e) are composed of products of D-brane charges with either KK- and winding charges or NS5- and KK5-brane charges.

As suggested by the index structure of the conditions k in (5.34), the constraints combine in a string or momentum multiplet as[‡44]

$$k^1 = m_1 m^2 \tag{5.35a}$$
$$k^4 = m_1 m^5 + m^2 m^2 \tag{5.35b}$$
$$k^{1;6} = m_1 m^{1;7} + m^2 m^5 \tag{5.35c}$$
$$k^{3;7} = m^2 m^{1;7} + m^5 m^5 \tag{5.35d}$$
$$k^{6;7} = m^5 m^{1;7} \tag{5.35e}$$

If these composite charges do not vanish, the state is at most 1/4-BPS, in which case its mass formula is given by

$$\mathcal{M}^2 = \mathcal{M}_0^2(m) + \sqrt{[\mathcal{T}(k)]^2}\ , \tag{5.36}$$

where $\mathcal{M}_0(m)$ and $\mathcal{T}(k)$ are given by the half-BPS mass and tension formulae (5.23) and (5.27).

Noting from Eq. (4.34) that the string multiplet $\mathcal{T}_1$ appears in the decompactification of the particle multiplet $\mathcal{M}$, we can obtain the half-BPS condition on the string multiplet by allowing non-zero $m^{s1}, m^{s4}, m^{1;s6}$ charges only, where s denotes a fixed direction on the torus:

$$k^{s;s5} = m^{s1} m^{s4} \tag{5.37a}$$
$$k^{s2;s6} = m^{s1} m^{1;s6} + m^{s4} m^{s4} \tag{5.37b}$$
$$k^{s5;s6} = m^{s4} m^{1;s6} \tag{5.37c}$$

and identifying these charges with the $n^1, n^4, n^{1;6}$ charges of the string multiplet in one dimension lower. We therefore obtain a multiplet of half-BPS conditions

$$k^5 = n^1 n^4 \tag{5.38a}$$
$$k^{2;6} = n^1 n^{1;6} + n^4 n^4 \tag{5.38b}$$
$$k^{5;6} = n^4 n^{1;6} \tag{5.38c}$$

This is easily seen to transform as a $\mathcal{T}_3'$ multiplet, as can also be inferred from the decomposition (4.38) at level 2 of the string multiplet under decompactification. For $d = 6$, this is

[‡44] One could have alternatively derived these conditions from the branching $E_7 \supset Sl(7)$, but the one we used is more constrained and more convenient.

a $\bar{\mathbf{27}}$ quadratic condition on the $\bar{\mathbf{27}}$ string multiplet of E_6, whereas for $d = 5$ only the first condition remains, giving a singlet condition on the $\mathbf{10}$ multiplet of $SO(5,5)$. For $d < 5$, a BPS string state is automatically 1/2-BPS, while for $d = 7$ the $\mathcal{T}_3'$ condition transform as a $\mathbf{1539}$ of E_7. The tension of a 1/4-BPS string can also be obtained by decompactifying one direction in Eq. (5.36), and has an analogous structure

$$\mathcal{T}^2 = \mathcal{T}_0^2(n) + \sqrt{[\mathcal{T}_3'(k)]^2} \tag{5.39}$$

where $\mathcal{T}_0(n)$ and $\mathcal{T}_3'(k)$ are given by the half-BPS tension (5.27) and the half-BPS 3-brane tension, which can be worked out easily.

For $d \geq 6$ (resp $d \geq 5$), there still remain conditions to be imposed on the particle multiplet (resp. string multiplet) in order for the state to be 1/4-BPS and not 1/8. In the $d = 7$ case, it should be required that the $\mathbf{56}$ in the third symmetric tensor power of the $\mathbf{56}$ particle charges vanishes [115]. For $d = 6$, this reduces to the statement that the singlet in $\mathbf{27}^3$ should vanish. This condition is empty for $d \leq 5$. We shall however not investigate the 1/8-BPS case any further, and refer to Appendix A.4 for the 1/8-BPS mass formula of a NS5–KK-winding bound state in $d = 6$ ($D = 5$). In contrast to 1/2-BPS states, 1/4-BPS and 1/8-BPS states in general have a non-trivial degeneracy and therefore entropy, which still has to be a U-duality invariant quantity depending on the charges m [167, 189, 79, 92, 5]. This allows non-trivial checks on U-duality and predictions on BPS bound states, which we shall only mention here [314, 315, 282].

6 Matrix gauge theory

The definition of M-theory as the strong-coupling limit of type IIA string theory and the finite energy extension of the eleven-dimensional SUGRA does not allow the systematic computation of S-matrix elements, since type IIA theory is only defined through its perturbative expansion and 11D SUGRA is severely non-renormalizable. In Ref. [24], Banks, Fischler, Susskind and Shenker (BFSS) formulated a proposal for a non-perturbative definition of M-theory, in which M-theory in the infinite momentum frame (IMF) with IMF momentum $P = N/R$, is related to the supersymmetric quantum mechanics [‡45] of $N \times N$ Hermitian matrices in the large-N limit, the same as the one describing the interactions of N D0-branes induced by fluctuations of open strings. Despite the powerful constraints of supersymmetry, it is still a formidable problem to solve this quantum mechanics in the large-N limit.

As was argued by Susskind [300], sense can however be made of the finite-N Matrix gauge theory, as describing the Discrete Light-Cone Quantization (DLCQ) of M-theory, that is quantization on a light-like circle. This stronger conjecture has been further motivated in Ref. [278], relating through an infinite Lorentz boost the compactification of M-theory on a light-like circle to compactification on a vanishing space-like circle, i.e. to type IIA string theory in the presence of D0-branes. This argument gives a general prescription for compactification of M-theory (see also Sen's argument Ref. [283]), and we shall briefly go through it in this Section.

Upon toroidal compactification on T^d, the extra degrees of freedom brought in by the wrapping modes of the open strings extend this quantum mechanics to a quantum field theory, namely a $U(N)$ Yang–Mills theory with 16 supersymmetries on the T-dual torus $\tilde{T}^d$ in the large-N limit [127, 305]. This prescription is consistent up to $d \leq 3$, but breaks down for compactification on higher-dimensional tori, owing to the ill-definition of SYM theory at short distances. Several proposals have been made as to how to supplement the SYM theory with additional degrees of freedom while still avoiding the coupling to gravity, which will be briefly discussed in this section. Besides their relevance for M-theory compactification, these theories are also interesting theories in their own right, as non-trivial interacting field theories in higher dimensions.

Our aim is to provide the background to discuss in Section 7 the implications of U-duality for the Matrix gauge theory describing toroidal compactification of M-theory. The relation between the M-theory compactification moduli, including gauge backgrounds, and Matrix gauge-theory parameters will be obtained, as well as the spectrum of excitations that Matrix gauge theory should exhibit in order to describe compactified M-theory. This will leave open the issue of what is the correct Matrix gauge theory reproducing these features.

[‡45] This model was first introduced in Ref. [68, 118, 17].

6.1 Discrete Light-Cone Quantization

The finite-N conjecture of Ref. [300] is formulated in the framework of the DLCQ, the essentials of which we review first. In field theory, it is customary to use equal-time ($t = x^0$) quantization, which breaks Poincaré invariance, but preserves invariance under the kinematical generators consisting of spatial rotations and translations. However, an alternative quantization procedure exists, in which the theory is quantized with respect to the proper time $x^+ = (x^0 + x^1)/\sqrt{2}$, which is referred to as light-cone quantization. In this case, the transverse translations P^i and rotations L^{ij}, as well as the longitudinal momentum P^+ and the boosts L^{-i}, L^{+-} do not depend on the dynamics, while the generator P^- generates the translations in the x^+ direction and plays the role of the Hamiltonian. The usual dispersion relation $H = \sqrt{P^i P_i + \mathcal{M}^2}$ in equal-time quantization, is replaced in the light-cone quantization by

$$P^- = \frac{P^i P_i + \mathcal{M}^2}{2P^+} \ , \tag{6.1}$$

exhibiting Galilean invariance on the transverse space. Particles, with positive energy $P^- > 0$, necessarily have positive longitudinal momentum P^+, while antiparticles will have negative P^+. The vacuum of P^- is hence reduced to the Fock-space state $|0\rangle$, and the negative-norm ghost states are decoupled as well. This simplification of the theory is at the expense of instantaneous non-local interactions due to the $P^+ = 0$ pole in (6.1).

Discrete light-cone quantization proceeds by compactifying the longitudinal direction x^- on a circle of radius R_l:

$$x_- \simeq x_- + 2\pi R_l \ . \tag{6.2}$$

This results into a quantization of the longitudinal momentum of any particle i according to

$$P_i^+ = \frac{n_i}{R_l} \ . \tag{6.3}$$

Because the total momentum is conserved, the Hilbert space decomposes into finite-dimensional superselection sectors labelled by $N = \sum n_i$. Note that the finite dimension does *not* require imposing any ultraviolet cut-off on the eigenvalues n_i, but follows from the condition $n_i > 0$.

It is important to note that, because the x^- direction is a light-like direction, the length R_l of the radius is not invariant, but can be modified by a Lorentz boost L^{+-},

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \to \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \ , \tag{6.4a}$$

which amounts to

$$R_l \to e^{\beta} R_l \ , \quad P^- \to e^{\beta} P^- \ , \quad P^+ \to e^{-\beta} P^+ \ . \tag{6.5}$$

This implies that the Hamiltonian P^+ depends on the radius R_l through an over-all factor

$$P^+ = R_l H_N \ , \tag{6.6}$$

so that the mass $\mathcal{M}^2 = 2P^+P^-$ is independent of R_l.

6.2 Why is Matrix theory correct ?

Following Ref. [278], we will now derive the Hamiltonian H_N describing the DLCQ of M-theory, and obtain the BFSS Matrix-theory conjecture. The basic idea is to consider the compactification on the light-like circle as Lorentz-equivalent to a limit of a compactification on a space-like circle. Acting with a boost (6.4) on an ordinary space-like circle, we find

$$\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} 0 \\ R_s \end{pmatrix} = \frac{R_l}{\sqrt{2}} \begin{pmatrix} -1+e^{-2\beta} \\ 1+e^{2\beta} \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} -R_l \\ R_l \end{pmatrix} \tag{6.7}$$

where $R_s = R_l e^{-\beta}$. Sending $\beta \to \infty$ while keeping R_l finite, we see that the light-like circle is Lorentz-equivalent to a space-like circle of radius $R_s \to 0$.

In order to keep the energy finite, which from Eq. (6.6) and on dimensional ground scales as R_l/l_p^2, we should also rescale the Planck length (and any other length) as $l_{p,s} = e^{-\beta/2} l_p$. Altogether, M-theory with Planck length l_p on the light-like circle of radius R_l in the momentum $P^+ = \frac{N}{R_l}$ sector is equivalent to M-theory with Planck length $l_{p,s}$ on the space-like circle of radius R_s in the momentum $P = \frac{N}{R_s}$ sector, with

$$R_s = R_l e^{-\beta} \ , \ \ l_{p,s} = e^{-\beta/2} l_p \tag{6.8}$$

in the limit $\beta \to \infty$. Eliminating β, we obtain the following scaling limit:

$$R_s \to 0 \ , \ \ M = \frac{R_s}{l_{p,s}^2} = \frac{R_l}{l_p^2} = \text{fixed} \ . \tag{6.9}$$

Following Ref. [278], we shall denote the latter theory as $\tilde{M}$ theory.

Since the space-like circle R_s shrinks to zero in $l_{p,s}$ units, this relates the DLCQ of M-theory to weakly coupled type IIA string theory in the presence of N D0-branes carrying the momentum along the vanishing compact dimension. Using Eq. (2.1), the scaling limit becomes

$$g_s = (R_s M)^{3/4} \ , \ \ \alpha' = l_s^2 = \frac{R_s^{1/2}}{M^{3/2}} \ , \ \ R_s \to 0 \ , \ \ M = \text{fixed} \ . \tag{6.10}$$

In particular, g_s and α' go to zero, so that the bulk degrees of freedom decouple, and only the leading-order Yang–Mills interactions between D0-branes remain. This validates the BFSS conjecture, up to the possible ambiguities in the light-like limit $\beta \to \infty$ [159, 52]. Several difficulties have also been shown to arise for compactification on curved manifolds [103, 102], but since we are only concerned with toroidal compactifications, we will ignore these issues.

6.3 Compactification and Matrix gauge theory

For toroidal compactifications of M-theory, we consider the same scaling limit as in (6.9), and keep the torus size constant in Planck length units, that is

$$R_I = r_I \left(\frac{R_s}{M} \right)^{1/2} \ , \ \ r_I = \frac{R_I}{l_{p,s}} = \text{fixed} \ . \tag{6.11}$$

However, comparing (6.9) and (6.11), we find that the size of the torus goes to zero in the scaling limit. To avoid this it is convenient to consider the theory on the T-dual torus $\tilde{T}^d$, obtained by a maximal T-duality in all d directions. From Eq. (2.5), this has the effect that,

$$\text{IIA with } N \text{ D0-branes} \rightarrow \begin{cases} \text{IIA with } N \text{ D}d\text{-branes} & d = \text{even} \\ \text{IIB with } N \text{ D}d\text{-branes} & d = \text{odd} \end{cases} \tag{6.12}$$

Using the maximal T-duality transformation $\prod_{I=1}^{d} T_I$, with T_I given in (2.5), the type II parameters then become

$$g_s = \frac{(R_s M)^{(3-d)/4}}{\prod r_I} \ , \quad \alpha' = l_s^2 = \frac{R_s^{1/2}}{M^{3/2}} \ , \quad \tilde{R}_I = \frac{1}{r_I M} \ , \tag{6.13a}$$

$$R_s \rightarrow 0 \ , \quad M = \frac{R_s}{l_{p,s}^2} = \text{fixed} \ , \quad r_I = \frac{R_I}{l_{p,s}} = \text{fixed} \ , \tag{6.13b}$$

so that, in particular, the size of the dual torus is fixed in the scaling limit. We will sometimes refer to the type II theory in this T-dual picture as the $\widetilde{\text{II}}$-theory.

The behaviour of the string coupling in the scaling limit is now different according to the dimension of the torus:

$$g_s \rightarrow \begin{cases} 0 & d < 3 \\ \text{finite} & d = 3 \\ \infty & d > 3 \end{cases} \tag{6.14}$$

In particular for $d < 3$ we still have weakly coupled type IIA or IIB string theory in the presence of N Dd-branes, so that M-theory is described by the SYM theory with 16 supercharges living on the world-volume of the N Dd-branes. The gauge coupling constant of this Matrix gauge theory and the radii s_I of the torus on which the D-branes are wrapped read

$$g_{\text{YM}}^2 = g_s l_s^{d-3} = \frac{M^{3-d}}{V_r} \ , \quad V_r \equiv \prod_I r_I \ , \quad s_I = \tilde{R}_I = \frac{1}{r_I M} \tag{6.15}$$

showing, in particular, that g_{YM}^2 is finite in the scaling limit.

The special case of Matrix theory on a circle ($d = 1$) yields (after an S-duality transforming the background D1-strings into fundamental strings) Matrix string theory [234, 93, 94], in which an identification between the large-N limit of two-dimensional N=8 supersymmetric YM theory and type IIA string theory is established. We will not further discuss this topic here, and refer to Ref. [116, 291] for the next case $d = 2$ and its relation to type IIB string theory. Moving on to the case $d = 3$, the same conclusion as in the $d < 3$ case continues to hold, since although the string coupling is finite, the string length goes to zero so that loop corrections are suppressed in the $\alpha' \to 0$ limit. Consequently, the $d = 3$ Matrix gauge theory is N=4 supersymmetric Yang Mills theory.

For $d > 3$, however, the coupling g_s blows up, and the weakly coupled string description of the D-branes is no longer valid. This coincides with the fact that the Yang–Mills theory becomes non-renormalizable and strongly coupled in the UV. Hence, in order to define a consistent quantum theory, one needs to supplement the theory with additional degrees of freedom. In the following we briefly review the proposals for $d = 4$ and $d = 5$, and show the complication that arises for $d = 6$. These proposals follow from the above prescription, using further duality symmetries, which will be examined in more detail in Section 7. Other decoupling limits have been considered in [171].

6.4 Matrix gauge theory on T^4

In the case $d = 4$, it follows from (6.12) that the effective theory is 4+1 SYM coming from the type IIA D4-brane world-volume theory. In the scaling limit the type IIA theory becomes strongly coupled and using the correspondence between strongly coupled IIA theory and M-theory a new eleventh dimension is generated, which plays the role of a fifth space dimension in the gauge theory [265, 47, 45]. Using Eqs. (2.11) and (6.13a), the radius and 11D Planck length are

$$\tilde{R} = g_s l_s = \frac{1}{MV_r} \ , \quad \tilde{l}_p = g_s^{1/3} l_s = R_s^{1/6} M^{-5/6} V_r^{-1/3} \ . \tag{6.16}$$

Moreover, comparing with (6.15) we find that the radius $\tilde{R}$ is in fact equal to the YM coupling constant

$$\tilde{R} = g_{\rm YM}^2 \ . \tag{6.17}$$

Hence, in the scaling limit (6.9), the Planck length $\tilde{l}_p$ goes to zero so that the bulk degrees of freedom decouple, while the radius $\tilde{R}$ remains finite. The N type IIA D4-branes become N M5-branes wrapped around the extra radius $\tilde{R}$, and M-theory on $T^4 \times S^1$ is then described by the (2,0) world-volume theory of N M5-branes, wrapped on T^4 and the extra radius $\tilde{R}$, related to the Yang–Mills coupling constant by Eq. (6.17). The proper formulation of this theory is still unclear, but Matrix light-cone descriptions have been proposed in Refs. [56, 279, 214, 15, 2, 188, 128] and the low-energy formulation studied in Refs. [125, 148]. In particular, at energies of order $1/g_{\rm YM}^2$ the Kaluza–Klein states along the extra circle come into play. They can be identified as instantons of 4D SYM lifted as particles in the (4+1)-dimensional gauge theory. Additional evidence for this conjecture that follows from the U-duality symmetry will be discussed in Section 7.

6.5 Matrix gauge theory on T^5

In the case $d = 5$, we have N type IIB D5-branes at strong string coupling, so that it is useful to perform an S-duality that maps the D5-branes to NS5-branes. Using Eqs. (4.17)) and (6.13a), we find that the string coupling and length become

$$\hat{g}_s = \frac{1}{g_s} = (R_s M)^{1/2} V_r \ , \quad \hat{l}_s^2 = g_s l_s^2 = \frac{1}{M^2 V_r} \ , \quad \hat{R}_I = \tilde{R}_I \ . \tag{6.18}$$

Moreover, comparing with Eq. (6.15), we find that the string tension is related to the gauge coupling constant by

$$\hat{l}_s^2 = g_{\rm YM}^2 \ . \tag{6.19}$$

The string coupling $\hat{g}_s$ goes to zero in the scaling limit, so that the bulk modes are decoupled from those localized on the NS5-branes. However, the string theory on the NS5-branes is still non-trivial, and has a finite string tension in the scaling limit [47]. As a consequence, we find that M-theory on $T^5 \times S^1$ is described by a theory of non-critical strings propagating on the NS5-brane world-volume with a tension related to the gauge coupling by Eq. (6.19). The proper formulation of this theory is still unclear, but light-cone Matrix formulations have been proposed [279, 290]. The string can be identified with a 4D Yang–Mills instanton lifted to 1+5 dimensions. This description is close but not identical to the proposal in Refs. [92, 91] according to which the (1/4-) BPS sector of M-theory should be described by the (1/2-) BPS excitations of the M5-brane, whose dynamics would be described by a (ground-state) non-critical "micro-string" theory on its six-dimensional world-volume. In particular, the theory on the type IIB NS5-brane is non-chiral, whereas that on the M5-brane is chiral. We refer the reader to the work of [94] for a discussion of these two approaches.

6.6 Matrix gauge theory on T^6

Finally, we discuss the problems that arise for $d = 6$, in which case we have N type IIA D6-branes at strong coupling. As in the $d = 4$ case, an eleventh dimension opens up, and we find M-theory compactified on a circle of radius $\tilde{R}$ with

$$\tilde{R} = g_s l_s = \frac{1}{R_s^{1/2} M^{3/2} V_r} \ , \quad \tilde{l}_p = g_s^{1/3} l_s = \frac{1}{M V_r^{1/3}} \ . \tag{6.20}$$

The N D6-branes actually correspond to N coinciding Kaluza–Klein monopoles with Taub–NUT direction along the eleventh direction, and as $\tilde{R} \to \infty$, the monopoles shrink to zero size and reduce to an A_N singularity in the eleven-dimensional metric. It was suggested in Ref. [155] that the bulk dynamics still decouples from the (6+1)-dimensional world-volume, and that the latter can be described in the IMF by the m(atrix) quantum mechanics of N_1 D0-branes inside N ten-dimensional Kaluza–Klein monopole, in the large-N_1 limit. This is very reminiscent to the BFSS description of M-theory, but the quantum mechanics is now a matrix model with eight supersymmetries and corresponds to the Coulomb phase of the quiver gauge theory in 0+1 dimensions associated to the Dynkin diagram A_N [101]. In other words, this is a sigma model with vector multiplets in the adjoint representation of $[U(N_1)]^{\otimes N}$ and hypermultiplets in bifundamental representations $(N_1, \bar{N}_1)$ of $U(N_1)_k \times$

$U(N_1)_{k+1}$ for $k = 1 \ldots N$, with $U(N_1)_k$ denoting the k-th copy of $U(N_1)$ and $U(N_1)_1$ identified with $U(N_1)_N$; this model is restricted to its Coulomb phase, where the hypers have no expectation value. In the low-energy limit, it is expected to reduce to SYM in 1+6 dimensions, with gauge coupling

$$\tilde{l}_p^3 = g_{\mathrm{YM}}^2 \ . \tag{6.21}$$

Other approaches have been proposed in Refs. [57, 126]. We shall come back to the $d = 6$ case in Section 7 when we display the BPS states in terms of the low-energy SYM theory.

6.7 Dictionary between M-theory and Matrix gauge theory

We finally give the dictionary that allows us to go from M-theory on $T^d \times S^1$ (with S^1 a light-like circle) in the sector $P^+ = N/R_l$ and Matrix gauge theory on $\tilde{T}^d$. This can be obtained by solving (6.13a) and (6.15), for the parameters $(s_I, N, g_{\mathrm{YM}})$ of the $U(N)$ Matrix gauge theory in terms of the parameters (R_I, R_l, l_p) of M-theory compactification on $T^d \times S^1$:

$$s_I = \frac{l_p^3}{R_l R_I} \tag{6.22a}$$

$$g_{\mathrm{YM}}^2 = \frac{l_p^{3(d-2)}}{R_l^{d-3} \prod R_I} \ . \tag{6.22b}$$

For completeness we also give the inverse relations

$$R_I = \frac{1}{s_I} \left(\frac{R_l V_s}{g_{\mathrm{YM}}^2} \right)^{1/2} \ , \quad l_p^3 = \left(\frac{R_l^3 V_s}{g_{\mathrm{YM}}^2} \right)^{1/2} \ , \quad P^+ = \frac{N}{R_l} \ , \tag{6.23}$$

where we have defined $V_s = \prod_I s_I$ as the volume of the dual torus on which the Matrix gauge theory lives.

6.8 Comparison of M-theory and Matrix gauge theory SUSY

In order to describe the M-theory BPS states from the point of view of the gauge theory, we need to understand how the space-time supersymmetry translates to the brane world-volume. This is in complete analogy with the perturbative string in the Ramond–Neveu–Schwarz formalism, in which space-time supersymmetry emerges from world-sheet supersymmetry (see [144], Section 5.2), and the case of the M5-brane has been thoroughly discussed in Ref. [91]. We will abstract their argument and discuss the case of a general 1/2-BPS brane, whether D, M, KK or otherwise, referring to that work for computational details.

In the presence of a p-brane, the breaking of the 11D $N = 1$ space-time supersymmetry is only spontaneous. The unbroken SUSY charges generate a superalgebra on the world-volume of the brane, whereas the broken ones generate fermionic zero modes. The fixing of the reparametrization invariance on the world-volume is most easily done in the light-cone gauge. The 32-component supercharge Q_α then decomposes as a[‡46] (spinor,spinor,spinor) of the unbroken Lorentz group $SO(1,1)\times SO(p-1)\times SO(10-p)$. The algebra is graded by the eigenvalue $\pm 1/2$ of the generator of $SO(1,1)$, so that the *unbroken* generators $Q^+_{a\alpha}$ have charge $+1/2$ and the broken ones $Q^-_{a\alpha}$ charge $-1/2$, where a is the spinorial index of the $SO(10-p)$ R-symmetry and α the spinorial index of the $SO(p-1)$ Lorentz world-volume symmetry. The anticommutation relations then take the form

$$\{Q^+,Q^+\} = H+P+Z^{++} \tag{6.24a}$$

$$\{Q^+,Q^-\} = p+Z^0 \tag{6.24b}$$

$$\{Q^-,Q^-\} = Z^{--}\ . \tag{6.24c}$$

In this expression, H and P are the world-volume Hamiltonian and momentum, Z^{++}, Z^0, Z^{--} some possible central charges and p is the transverse momentum. A contraction of the central charges with the appropriate Gamma matrices is also assumed. In the following, we absorb the momentum in the charges Z^{++}, and set $p=0$ by considering a particle at rest in the transverse directions.

The central charges Z^0 and $Z^{\pm\pm}$ are simply a renaming of the Z^{MN}, Z^{MNPQR} central charges of the 11D superalgebra (2.13a). As their indices show, Z^0 is a singlet of $SO(1,1)$, whereas Z^{++} and Z^{--} combine in a vector of $SO(1,1)$; Z^0 is therefore identified with the Z^{IJ}, Z^{IJKLM} charges, whereas $Z^{\pm\pm}$ correspond to the Z^{1I}, Z^{1IJKLM} charges, where as usual $I,J,\dots$, are directions on the torus and 1 is the space-time direction combined with the time direction on the light cone. In other words, Z^0 is identified with the *particle* charges, whereas $Z^{\pm\pm}$ correspond to the *string* charges. In order for the superalgebra (6.24a) to reproduce the space-time superalgebra (2.13a) *with particle charges only*, we therefore need to impose $Z^{\pm\pm}=0$ on the physical states. This is the analogue of the $L_0=\bar{L}_0$ level-matching condition.

The broken generators Q^- and the central charges Z^0 are given by the fermionic and bosonic zero modes only. On the other hand, the unbroken generators as well as the central charges $Z^{\pm\pm}$ have a non-zero-mode contribution:

$$Q^+ = Q^+_0+\hat{Q}^+\ ,\quad Z^{\pm\pm}=Z^{\pm\pm}_0+\hat{Z}^{\pm\pm}\ ,\quad H=H_0+\hat{H}\ . \tag{6.25}$$

The zero-mode part of the generators Q^+_0 is built out of the bosonic and fermionic zero modes Z^0 and Q^-, and anticommutes with the oscillator part $\hat{Q}^+$. It generates the same algebra as in Eq. (6.24a), while the oscillator parts generate the same algebra on their own and anticommute with the zero-mode broken generators $Q^-=Q^-_0$. The level-matching conditions $Z^{\pm\pm}=0$ are achieved through a cancellation of the zero-mode part, quadratic in the particle charges Z^0, and the oscillator parts.

Let us now consider the Hamiltonian H. Because of supersymmetry, both H_0 and $\hat{H}$ are positive operators and for given zero modes Z^0, the supersymmetric ground state is given by the condition $\hat{H}=0$, or $\hat{Q}^+|0\rangle=0$. This state is therefore annihilated by all the

[‡46] or a sum of, depending on the parity of p.

$\hat{Q}^+$ supersymmetries, that is half the space-time supersymmetries, and must have vanishing $\hat{Z}^{\pm\pm}$ charge, that is from the level-matching conditions $Z_0^{\pm\pm} = (Z^0)^2 = 0$. This condition is, in less detail, the 1/2-BPS condition $k = 0$ with k defined as in Eq. (2.21b). The energy of this state is given by the zero-mode part $H_0 = \{Q_0^+, Q_0^+\}$ quadratic in the *particle* charges Z^0. This is equivalent to the mass formula Eq. (2.18) for 1/2-BPS states in space-time.

On the other hand, BPS states preserving 1/4 of the space-time supersymmetry are only annihilated by *half* the world-volume supercharges $\hat{Q}^+$, and their energy is shifted by the non-zero-mode contribution $\hat{H}$. The latter is quadratic in the non-zero-mode part of the *string* charges $\hat{Z}^{\pm\pm} = -Z_0^{\pm\pm} = -(Z^0)^2$, therefore *quartic* in the particle charge. This is precisely what was found in Eq. (2.21a):

$$E = \mathcal{M}^2(Z) + \sqrt{[\mathcal{T}(K)]^2} \; . \tag{6.26}$$

This equation has a simple interpretation: the quadratic term corresponds to the 1/2-BPS bound state between the heavy mass $\mathcal{M}_p$ p-brane and the mass $\mathcal{M}$ particle, with binding energy

$$E = \sqrt{\mathcal{M}_p^2 + \mathcal{M}^2} - \mathcal{M}_p \simeq \frac{\mathcal{M}^2}{\mathcal{M}_p} \; , \tag{6.27}$$

whereas the second corresponds to a 1/4-BPS bound state between the p-brane and the mass $\mathcal{M} = R\mathcal{T}$ of the string with tension $\mathcal{T}$ wrapped on the circle R:

$$E = (\mathcal{M}_p + \mathcal{M}) - \mathcal{M}_p = R\mathcal{T} \; . \tag{6.28}$$

There is therefore a complete identity between i) the space-time supersymmetry algebra and particle spectrum *in the absence of the p-brane*, ii) the p-brane world-volume gauge theory and iii) the bound states of the p-brane with other particles. This also holds at the level of space-time field configurations, which can be seen as configurations on the world-volume [60, 39].

6.9 SYM masses from M-theory masses

We shall now explicit the correspondence of the previous subsection in the D-brane case, relevant for Matrix gauge theory, and relate the energies in the Yang–Mills theory to the masses in space-time. This has been discussed in particular in Refs. [299, 127, 116, 136]. Based on the last interpretation as bound states of the N background D-branes with other particles, we identify $R = R_l$ and $\mathcal{M}_p = P_+ = N/R_l$, where R_l is the radius of the light-like direction, and fix the normalization of the Yang–Mills energies as

$$E_{\rm YM} = \frac{R_l}{N}\mathcal{M}^2(Z) + R_l\sqrt{[\mathcal{T}(K)]^2} \; . \tag{6.29}$$

We then proceed by using the dictionary (6.22) to obtain the Yang–Mills energy of the BPS states we discussed previously.

We now apply these considerations to the highest-weight states of the two U-duality multiplets of Subsections 4.8 and 4.10. The highest-weight state of the particle multiplet is a Kaluza–Klein excitation on the I-th direction, which becomes, after the maximal T-duality, an NS-winding state bound to the background Dd-brane. Hence, it is a bound state with non-zero binding energy, and using Eq. (6.27) we find

$$E_{\rm YM} = \frac{(1/R_I)^2}{N/R_l} = \frac{g_{\rm YM}^2 s_I^2}{N V_s} \ , \tag{6.30}$$

where in the second step we used the dictionary (6.22) to translate to Matrix gauge theory variables. This is the energy of a state in the gauge theory carrying electric flux in the I-th direction. For this reason, the particle multiplet is also called the *flux multiplet*.

Next we turn to the highest-weight state of the string multiplet, wrapped on the light-like direction R_l. The highest weight is a membrane wrapped on R_I and R_l, which becomes, after the maximal T-duality, a Kaluza–Klein state bound to the background Dd-brane. These two states form a bound state at threshold and according to (6.28) we have

$$E_{\rm YM} = \frac{R_l R_I}{l_p^3} = \frac{1}{s_I} \ , \tag{6.31}$$

where Eq. (6.22) was used again in the second step. This is the energy of a massless particle with momentum along the I-th direction in the gauge theory, so that we may alternatively call the string multiplet the *momentum multiplet* from the point of view of Matrix gauge theory.

This translation can be carried out for all other members of the U-duality multiplets, and since U-duality preserves the supersymmetry properties of the bound state, one finds the following general relation between SYM masses and M-theory masses:

$$\text{particle/flux multiplet} \quad : \quad E_{\rm YM} = \frac{R_l}{N} \mathcal{M} \ , \tag{6.32a}$$

$$\text{string/momentum multiplet} \quad : \quad E_{\rm YM} = R_l \mathcal{T}_1 \ . \tag{6.32b}$$

In Subsection 7.2, we will explicitly see for the cases $d = 3, 4, 5$ that indeed all non-zero binding energy and threshold bound states appear in the particle/flux and string/momentum multiplets respectively. Finally, we remark that the equalities in the two equations (6.30) and (6.31) can be solved to yield the dictionary (6.22), so that the comparison of these two types of energy quanta gives a convenient short-cut to (6.22).

7 U-duality symmetry of Matrix gauge theory

If any of the previously discussed Matrix gauge theories purports to describe compactified Matrix gauge theory, it should certainly exhibit U-duality invariance. In this section, we wish to investigate the implications of U-duality on the Matrix gauge theory at the algebraic level, irrespective of its precise realization.

To this end we use the dictionary (6.22) between compactified M-theory and Matrix gauge theory. We first recast the Weyl transformations of the U-duality group (see Subsection 4.4) in the gauge-theory language and interpret them as generalized electric–magnetic dualities of the gauge theory. Then, we translate the U-duality multiplets of Subsections 4.8 and 4.10 in Matrix gauge theory and discuss the interpretation of the states. Finally, we use the results of Subsection 5.4 to discuss the realization of the full U-duality group in Matrix gauge theory and in particular the couplings induced by non-vanishing gauge potentials.

At the end of this section a more speculative aspect of finite-N matrix gauge theory is discussed. By promoting the rank N to an ordinary charge, we show the existence of an $E_{d+1(d+1)}(\mathbb{Z})$ action on the spectrum of BPS states. In this way, we find that the conjectured extended U-duality symmetry of matrix theory on T^d in DLCQ has an implementation as action of $E_{d+1(d+1)}(\mathbb{Z})$ on the BPS spectrum, as demanded by eleven-dimensional Lorentz invariance.

7.1 Weyl transformations in Matrix gauge theory

The discussion of Matrix gauge theory from M-theory in Section 6 has been restricted to rectangular tori with vanishing gauge potentials, so that we first focus on the transformations in the Weyl subgroup of the U-duality group [‡47]

$$\mathcal{W}(E_{d(d)}(\mathbb{Z})) = \mathbb{Z}_2 \bowtie \mathcal{S}_d \ . \tag{7.1}$$

The permutation group S_d that interchanges the radii R_I of the M-theory torus obviously still permutes the radii s_I of the Matrix gauge theory T-dual torus. On the other hand, the generalized T-duality T_{IJK} in (4.12), using the dictionary (6.22), translates into the following transformation of the Matrix gauge theory parameters:

$$S_{IJK}: \quad \left\{ \begin{array}{llll} g^2_{\rm YM} & \to & \frac{g_{\rm YM}^{2(d-4)}}{W^{d-5}} & W \equiv \prod_{a\neq I,J,K} s_a \\ s_\alpha & \to & s_\alpha & \alpha = I,J,K \\ s_a & \to & \frac{g^2_{\rm YM}}{W} s_a & a \neq I,J,K \end{array} \right. \tag{7.2}$$

For $d=3$ the transformation (7.2) is precisely the (Weyl subgroup of) S-duality symmetry of $N=4$ SYM in 3+1 dimensions [299, 127]:

$$g^2_{\rm YM} \to 1/g^2_{\rm YM} \ , \tag{7.3}$$

[‡47] We restrict to the case $d \geq 3$; the case $d=1$ has trivial Weyl group, while for the case $d=2$ there is only the permutation symmetry $\mathcal{S}_2$.

obtained for zero theta angle. The transformation (7.2) generalizes this symmetry to the case $d > 3$, by acting as S-duality in the (3+1)-dimensional theory obtained by reducing the Matrix gauge theory in $d+1$ dimensions to the directions I, J, K and the time only [110]. Indeed, the coupling constant for the effective (3+1)-dimensional gauge theory reads

$$\frac{1}{g_{\rm eff}^2} = \frac{W}{g_{\rm YM}^2} \ , \tag{7.4}$$

and the transformation (7.2) becomes

$$(g_{\rm eff}^2, s_\alpha, s_a) \to (1/g_{\rm eff}^2, s_\alpha, g_{\rm eff}^2 s_a) \ . \tag{7.5}$$

To summarize, we see that from the point of view of the Matrix gauge theory the U-dualities are accounted for by the modular group of the torus on which the gauge theory lives (yielding the $Sl(d,\mathbb{Z})$ subgroup) as well as by generalized electric–magnetic dualities (implementing the T-dualities of type IIA string) [‡48].

We now discuss in more detail the $d = 4, 5, 6$ cases, in order to give more support to the proposals discussed in Section 6. Explicitly, one obtains

$$d=4: \quad S_{IJK} \quad \begin{cases} g_{\rm YM}^2 & \leftrightarrow & s_a & a \neq I,J,K \\ s_\alpha & \to & s_\alpha & \alpha = I,J,K \end{cases} \tag{7.6a}$$

$$d=5: \quad S_{IJK} \quad \begin{cases} g_{\rm YM}^2 & \to & g_{\rm YM}^2 & \\ s_a & \to & \frac{g_{\rm YM}^2}{s_b} & a,b \neq I,J,K \\ s_\alpha & \to & s_\alpha & \alpha = I,J,K \end{cases} \tag{7.6b}$$

$$d=6: \quad S_{IJK} \quad \begin{cases} g_{\rm YM}^2 & \to & \frac{g_{\rm YM}^4}{s_a s_b s_c} & a,b,c \neq I,J,K \\ s_a & \to & \frac{g_{\rm YM}^2}{s_b s_c} & \\ s_\alpha & \to & s_\alpha & \alpha = I,J,K \end{cases} \tag{7.6c}$$

For $d = 4$ we see that (7.6a) induces a permutation of the YM coupling constant with the radii, in accordance with the interpretation (6.17) of the YM coupling constant as an extra radius. For $d = 5$, Eq. (7.6b) takes the form of a T-duality symmetry (2.41) of the non-critical string theory living on the type IIB NS5-brane world-volume with the YM coupling related to the string length as in (6.19). Finally, for $d = 6$, we see by comparing (7.6c) with the U-duality transformation in (4.12) that we recover the symmetry transformation T_{IJK} in M-theory with the YM coupling constant related to the Planck length by (6.21).

At this point, it is also instructive to recall the full U-duality groups for toroidal compactifications of M-theory, as summarized in Table 4.2, and discuss their interpretation in view of the Matrix gauge theories for $d = 3, 4, 5$ (see Table 7.1). For $d = 3$,

[‡48] From the point of view of type IIB theory, it can be shown that the latter also account for the restoration of the transverse Lorentz invariance [291].

D	d	U-duality	origin
8	3	$Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$	S-duality $\times$ symmetry of T^3
7	4	$Sl(5,\mathbb{Z})$	symmetry of T^5 of M5-brane
6	5	$SO(5,5,\mathbb{Z})$	T-duality symmetry on NS5-brane
$D \leq 5$	$d \geq 6$	$E_{d(d)}(\mathbb{Z})$	unclear

Table 7.1: Interpretation of U-duality in Matrix gauge theory.

the $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ U-duality symmetry is the product of the (full) S-duality and the reparametrization group of the three-torus. For $d = 4$, the $Sl(5,\mathbb{Z})$ symmetry is the modular group of the five-torus, corroborating the interpretation of this case as the (2,0) theory on the M5-brane [265]. Finally, for $d = 5$ the $SO(5,5,\mathbb{Z})$ symmetry should be interpreted as the T-duality symmetry of the string theory living on the NS5-brane [91, 47]. The $E_{6(6)}(\mathbb{Z})$ symmetry is by no means obvious in the IMF description discussed in Subsection 6.6, but this is expected since part of it are Lorentz transformations broken by the IMF quantization. The interpretation of the exceptional groups $E_{d(d)}(\mathbb{Z})$, $d = 7, 8$ is not obvious either, since a consistent quantum description for these cases is lacking as well.

In Subsections 7.4–7.6, the precise identification of the full U-duality groups for $d = 3, 4, 5$ will be discussed in further detail. Note also that as we are considering M-theory compactified on a torus times a light-like circle, it has been conjectured that the $E_{d(d)}(\mathbb{Z})$ U-duality symmetry should be extended to $E_{d+1(d+1)}(\mathbb{Z})$, as a consequence of Lorentz invariance. This extended U-duality symmetry will be discussed in Subsection 7.7.

Finally, we can translate the U-duality invariant Newton constant (4.14) in the Matrix gauge theory language. The most convenient form is obtained by writing

$$I_d = \frac{V_s^{d-5}}{g_{\rm YM}^{2(d-3)}} = \left(\frac{V_R}{l_p^9} \right)^2 R_l^{9-d} \ , \tag{7.7}$$

which depends on the invariant D-dimensional Planck length and the radius of the light-like circle, invariant under the $E_{d(d)}(\mathbb{Z})$ transformations acting on the transverse space. Again, in agreement with the Matrix gauge theory descriptions, we see that for $d = 3$ the invariant $I_3 = 1/V_s^2$ is related to the volume V_s of the three-torus; for $d = 4$ the invariant $I_4 = 1/(V_s g_{\rm YM}^2)$ is related to the total volume of the five-torus, constructed from the four-torus and the extra radius $\tilde{R} = g_{\rm YM}^2$; for $d = 5$ the invariant $I_5 = 1/g_{\rm YM}^4$ is related to the finite string tension $T = 1/g_{\rm YM}^2$ of the string theory. Finally, note also that for $d = 6$ the U-duality invariant $I_6 = V_s/g_{\rm YM}^6$ is related to the 5-D Planck length, when using $l_p^3 = g_{\rm YM}^2$.

7.2 U-duality multiplets of Matrix gauge theory

We now turn to the translation of the U-duality multiplets of Subsections 4.8 and 4.10 in the Matrix gauge theory picture. To this end we use the dictionary (6.22) and the mass relations in Eq. (6.32). Equivalently, one may start with the highest-weight states corresponding to electric flux (6.30) and momentum states (6.31) in the Matrix gauge theory and subsequently act with the transformations (7.2) of the Weyl subgroup. Of course these two methods lead to the same result, which are summarized in Tables 7.2 and 7.3, for the particle/flux and string/momentum multiplet respectively. As a compromise between explicitness and complexity, we have chosen to write down the content for $d = 7$ in the first case, and for $d = 6$ in the latter case. The tables list the mass $\mathcal{M}$ in M-theory variables, the corresponding gauge theory energy E_{YM} and their associated charges, obtained from the M-theory charges by raising lower indices or lowering upper indices.

$\mathcal{M}$	E_{YM}	charge
$\frac{1}{R_I}$	$\frac{g_{\mathrm{YM}}^2 s_I^2}{N V_s}$	m^1
$\frac{R_I R_J}{l_p^3}$	$\frac{V_s}{N g_{\mathrm{YM}}^2 (s_I s_J)^2}$	m_2
$\frac{R_I R_J R_K R_L R_M}{l_p^6}$	$\frac{V_s^3}{N g_{\mathrm{YM}}^6 (s_I s_J s_K s_L s_M)^2}$	m_5
$\frac{R_I ; R_J R_K R_L R_M R_N R_P R_Q}{l_p^9}$	$\frac{V_s^5}{N g_{\mathrm{YM}}^{10} (s_I ; s_J s_K s_L s_M s_N s_P s_Q)^2}$	$m_{1;7}$

Table 7.2: Flux multiplet (**56** of E_7) for Matrix gauge theory on T^7.

$\mathcal{M}$	E_{YM}	charge
$\frac{R_l R_I}{l_p^3}$	$\frac{1}{s_I}$	n_1
$\frac{R_l R_I R_J R_K R_L}{l_p^6}$	$\frac{V_s}{g_{\mathrm{YM}}^2 s_I s_J s_K s_L}$	n_4
$\frac{R_l R_I ; R_J R_K R_L R_M R_N R_P}{l_p^9}$	$\frac{V_s^2}{g_{\mathrm{YM}}^4 s_I ; s_J s_K s_L s_M s_N s_P}$	$n_{1;6}$
$\frac{R_l R_I R_J R_K ; R_L R_M R_N R_P R_Q R_R R_S}{l_p^{12}}$	$\frac{V_s^3}{g_{\mathrm{YM}}^6 s_I s_J s_K ; s_L s_M s_N s_P s_Q s_R s_S}$	$n_{3;7}$
$\frac{R_l R_I R_J R_K R_L R_M R_N ; R_P R_Q R_R R_S R_T R_U R_V}{l_p^{15}}$	$\frac{V_s^4}{g_{\mathrm{YM}}^8 s_I s_J s_K s_L s_M s_N ; s_P s_Q s_R s_S s_T s_U s_V}$	$n_{6;7}$

Table 7.3: Momentum multiplet (**133** of E_7) for Matrix gauge theory on T^7.

In Table 7.2, the first entry corresponds to a state with electric flux in the I-th direction, while the second one carries magnetic flux in the I, J direction. The first entry in Table 7.3 is a KK state of the gauge theory, while the second one is a YM instanton in 3+1 dimensions, lifted to $d+1$ dimensions. For $d \geq 5$, new states appear. As a further illustration, we take a closer look at the special cases $d = 3, 4, 5, 6$, which can be obtained from the tables by omitting those states that have too many compactified dimensions. The Tables 7.4–7.7 list the content of each of the two multiplets for these cases [110, 244, 269], including the M-theory mass, the YM energy, the multiplicity of each type of state and its interpretation both in the Matrix gauge theory and as a bound state with the N background type $\tilde{\mathrm{II}}$ Dd-branes. For $d = 4, 5$ we have also added a column giving the bound-state interpretation in the M5- and NS5-brane theories respectively.

$\mathcal{M}$	$E_{\rm YM}$	#	YM state	b.s. of N D3
$\frac{1}{R_I}$	$\frac{g_{\rm YM}^2 s_I^2}{NV_s}$	3	electric flux	NS-w
$\frac{R_I R_J}{l_p^3}$	$\frac{V_s}{N g_{\rm YM}^2 (s_I s_J)^2}$	3	magnetic flux	D1
$\frac{R_l R_I}{l_p^3}$	$\frac{1}{s_I}$	3	momentum	KK

Table 7.4: Flux and momentum multiplet for $d = 3$: $\mathbf{(3,2)}$ and $\mathbf{(3,1)}$ of $Sl(3) \times Sl(2)$.

$\mathcal{M}$	$E_{\rm YM}$	#	YM state	b.s. of N D4	b.s. of N M5
$\frac{1}{R_I}$	$\frac{g_{\rm YM}^2 s_I^2}{NV_s}$	4	electric flux	NS-w	M2
$\frac{R_I R_J}{l_p^3}$	$\frac{V_s}{N g_{\rm YM}^2 (s_I s_J)^2}$	6	magnetic flux	D2	
$\frac{R_l R_I}{l_p^3}$	$\frac{1}{s_I}$	4	momentum	KK	KK
$\frac{R_l V_R}{l_p^6}$	$\frac{1}{g_{\rm YM}^2}$	1	YM particle	D0	

Table 7.5: Flux and momentum multiplet for $d = 4$: **10** and **5** of $Sl(5)$.

$\mathcal{M}$	$E_{\rm YM}$	#	YM state	b.s. of N D6
$\frac{1}{R_I}$	$\frac{g_{\rm YM}^2 s_I^2}{NV_s}$	6	electric flux	NS-w
$\frac{R_I R_J}{l_p^3}$	$\frac{V_s}{N g_{\rm YM}^2 (s_I s_J)^2}$	15	magnetic flux	D4
$\frac{V_R}{R_I l_p^6}$	$\frac{V_s s_I^2}{N g_{\rm YM}^6}$	6	new sector	KK5
$\frac{R_l R_I}{l_p^3}$	$\frac{1}{s_I}$	6	momentum	KK
$\frac{R_l V_R}{R_I R_J l_p^6}$	$\frac{s_I s_J}{g_{\rm YM}^2}$	15	YM membrane	D2
$\frac{R_l R_I V_R}{l_p^9}$	$\frac{V_s}{g_{\rm YM}^4 s_I}$	6	new sector	NS5

Table 7.7: Flux and momentum multiplet for $d = 6$: **27** and $\mathbf{\bar{27}}$ of E_6.

A few comments on these tables are in order.

- A number of states in the Matrix gauge theory have a uniformly valid interpretation as bound states with the background Dd-branes, namely, for the flux multiplet,

$$\text{electric flux} = \text{D}d\text{–NS-winding bound state} \tag{7.8a}$$

$$\text{magnetic flux} = \text{D}d\text{–D}(d-2) \text{ bound state} \tag{7.8b}$$

$\mathcal{M}$	$E_{\rm YM}$	#	YM state	b.s. of N D5	b.s. of N NS5
$\frac{1}{R_I}$	$\frac{g_{\rm YM}^2 s_I^2}{NV_s}$	5	electric flux	NS-w	D1
$\frac{R_I R_J}{l_p^3}$	$\frac{V_s}{Ng_{\rm YM}^2(s_I s_J)^2}$	10	magnetic flux	D3	D3
$\frac{V_R}{l_p^6}$	$\frac{V_s}{Ng_{\rm YM}^6}$	1	new sector	NS5	D5
$\frac{R_l R_I}{l_p^3}$	$\frac{1}{s_I}$	5	momentum	KK	KK
$\frac{R_l V_R}{R_I l_p^6}$	$\frac{s_I}{g_{\rm YM}^2}$	5	YM string	D1	NS-w

Table 7.6: Flux and momentum multiplet for $d=5$: **16** and **10** of $SO(5,5)$.

and for the momentum multiplet,

$$\text{KK momentum} = \text{D}d\text{–KK bound state} \tag{7.9a}$$

$$\text{YM state} = \text{D}d\text{–D}(d-4) \text{ bound state} \tag{7.9b}$$

where the YM state denote the 4D Yang–Mills instanton lifted to $d+1$ dimensions. The correspondences in Eq. (7.8) and (7.9) were noted in Refs. [250, 97, 320].

- In the $d=3$ case, only perturbative states are observed in Table 7.4.

- For $d=4$ one non-perturbative state occurs in Table 7.5, which corresponds precisely to momentum along the dynamically generated fifth direction, i.e. to a Yang–Mills instanton lifted to 4+1 dimensions [265]. From the M5-brane point of view, the flux multiplet describes the M2-brane excitations, while the momentum multiplet comprises the KK states, as indicated in the last column.

- For the case $d=5$ in Table 7.6, we focus on the last column obtained by S-duality from the D5-brane picture of the $\widetilde{\rm II}$ theory. The YM string in the momentum multiplet arises in this case from the wound strings on the NS5-brane. The wrapped transverse fivebrane on T^5 appears as a bound state of D5-branes with the background NS5-branes, with non-zero binding energy (since it is related by electric–magnetic duality to the D1–NS1 bound state). It corresponds to a new sector in the Matrix gauge theory Hilbert space, with energy scaling as $1/g_{\rm YM}^6$. This state does not correspond to any known configuration of the 1+5 gauge theory, but may be understood as a magnetic flux along one ordinary dimension together with the dynamically generated dimension in a 1+4 gauge theory obtained by reducing the original one on a circle [154].

For the $d=6$ case, we see from Table 7.7 that all BPS states of type IIA theory on T^6 are involved in the bound states of the flux and momentum multiplet, except for the D6–D0 "bound state". It has been argued that the latter forms a non-supersymmetric resonance with the unconventional mass relation [88, 304]:

$$\mathcal{M} = \left(\mathcal{M}_{\rm D6}^{2/3} + \mathcal{M}_{\rm D0}^{2/3}\right)^{3/2} \ . \tag{7.10}$$

As a consequence we expect to find a state in the gauge theory with energy

$$E_{\rm YM} = \left(\mathcal{M}_{N\rm D6}^{2/3} + \mathcal{M}_{\rm D0}^{2/3}\right)^{3/2} - \mathcal{M}_{N\rm D6} \simeq \mathcal{M}_{\rm D0}^{2/3}\mathcal{M}_{N\rm D6}^{1/3} \ . \tag{7.11}$$

Using the corresponding D-brane masses and the relation $g_{\rm YM}^2 = g_s l_s^3$ we then obtain

$$E_{\rm YM} = N^{1/3}\frac{V_s^{1/3}}{g_{\rm YM}^2} = N^{1/3} I_6^{1/3} \ . \tag{7.12}$$

In the last step we have expressed the mass in terms of the U-duality invariant (7.7), explicitly showing that this extra state transforms as a singlet under the U-duality group $E_{6(6)}(\mathbb{Z})$. Since the D0-brane is mapped onto a D6-brane under the maximal T-duality, the space-time interpretation of this extra U-duality multiplet follows from the M-theory origin of the D6-brane, i.e. the state is KK6-brane with the TN direction along the light-like direction. The corresponding data of this extra singlet are summarized in Table 7.8. The $d = 7$ case is discussed in Appendix C, and exhibits a number of similar states (with M-theory masses depending on multiple factors of R_l) as the extra singlet in $d = 6$.

$\mathcal{M}$	$E_{\rm YM}$	#	YM state	b.s. of N D6
$\frac{R_l^2 V_R}{l_p^9}$	$\frac{N^{1/3}V_s^{1/3}}{g_{\rm YM}^2}$	1	new sector	D0

Table 7.8: Additional multiplet for $d = 6$: **1** of E_6.

7.3 Gauge backgrounds in Matrix gauge theory

Our discussion of the Matrix gauge theory U-duality symmetries and mass formulae has so far been restricted to the rectangular-torus case, with zero expectation values for the M-theory gauge potentials. However, gauge backgrounds in M-theory yield moduli, and should have a counterpart as couplings in the Matrix gauge theory.

As a simple example, consider first M-theory on T^3, in which case we can switch on an expectation value for the component $\mathcal{C}_{123}$ of the three-form. Together with the volume V of T^3, it forms a complex scalar

$$\tau = \mathcal{C}_{123} + i\,\frac{V}{l_p^3} \ , \tag{7.13}$$

which transforms as a modular parameter under the subgroup $Sl(2,\mathbb{Z})$ of the U-duality group $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$ [146]. On the other hand, according to Eq. (6.22a) the volume is identified in the Matrix gauge theory with $1/g_{\rm YM}^2$, which together with the theta angle forms a complex scalar

$$S = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{\rm YM}^2} \ , \tag{7.14}$$

transforming as a modular parameter under the electric–magnetic duality group $Sl(2,\mathbb{Z})$. One should therefore identify $\mathcal{C}_{123}$ with θ, or in other words the three-form background induces a topological coupling $\int F\wedge F$ on the D3-brane world-volume.

This can be derived more generally for any d by making use of Seiberg's argument and the well-known coupling of Ramond gauge fields to the D-brane world-volume. Details of the derivation can be found in Ref. [244] and we only quote the result which is that the expectation value of the three-form induces the following topological coupling in Matrix gauge theory:

$$S_{\mathcal{C}} = \mathcal{C}^{IJK} \int \mathrm{d}t \int_{\tilde{T}^d} F_{0I} F_{JK} \ . \tag{7.15}$$

This coupling reduces to the θ term (7.14) for $d=3$ and was conjectured in Ref. [35]. As we now show, the coupling (7.15) can also be inferred from the U-duality invariant mass formulae.

To see this, we first translate the general U-duality invariant mass formulae (5.23) into the gauge theory language using (6.22) and (6.32), restricting to $d\leq 6$ for simplicity:

$$\begin{aligned} E_{\rm YM} = \frac{g_{\rm YM}^2}{NV_s}\left[\left(\tilde{m}^1\right)^2 + \left(\frac{V_s}{g_{\rm YM}^2}\right)^2 (\tilde{m}_2)^2 + \left(\frac{V_s}{g_{\rm YM}^2}\right)^4 (\tilde{m}_5)^2\right] \\ + \sqrt{(\tilde{n}_1)^2 + \left(\frac{V_s}{g_{\rm YM}^2}\right)^2 (\tilde{n}_4)^2 + \left(\frac{V_s}{g_{\rm YM}^2}\right)^4 (\tilde{n}_{1;6})^2} \end{aligned} \tag{7.16}$$

in which we have added the flux multiplet and momentum multiplet together, as was argued in Subsection 6.8. Index contractions are performed with the dual metric $\tilde{g}_{IJ} = g^{IJ} l_p^6/R_l^2$, and upper (lower) indices in the M-theory picture have become lower (upper) indices in the Matrix gauge theory picture. We also recall that V_s is the volume of the dual torus $\tilde{T}^d$ on which the Matrix gauge theory lives. The expression of shifted charges is then given by

$$\tilde{m}^1 = m^1 + \mathcal{C}^3 m_2 + \left(\mathcal{C}^3\mathcal{C}^3 + \mathcal{E}^6\right) m_5 \tag{7.17a}$$

$$\tilde{m}_2 = m_2 + \mathcal{C}^3 m_5 \tag{7.17b}$$

$$\tilde{m}_5 = m_5 \tag{7.17c}$$

$$\tilde{n}_1 = n_1 + \mathcal{C}^3 n_4 + \left(\mathcal{C}^3\mathcal{C}^3 + \mathcal{E}^6\right) n_{1;6} \tag{7.17d}$$

$$\tilde{n}_4 = n_4 + \mathcal{C}^3 n_{1;6} \tag{7.17e}$$

$$\tilde{n}_{1;6} = n_{1;6} \ . \tag{7.17f}$$

As we will see below, the linear shift in $\mathcal{C}^3$ is in agreement with the coupling obtained in Eq. (7.15). As a preview, the interpretation of the $\mathcal{C}^3$ coupling in the various Matrix gauge theories is summarized in Table 7.9. We will discuss these formulae in further detail for $d = 3, 4, 5$ below. There is as yet no derivation of the coupling of the $\mathcal{E}_6$ gauge potential to the Matrix gauge theory.

D	d	$\mathcal{C}_{IJK}$	interpretation
8	3	1	θ-parameter
7	4	4	off-diagonal component $\mathcal{A}_I$ of T^5-metric
6	5	10	B_{IJ}-background field of string theory on 5-brane
$D \leq 5$	$d \geq 6$	$\binom{d}{3}$	unclear

Table 7.9: Matrix gauge theory interpretation of three-form potential.

7.4 $Sl(3,\mathbb{Z}) \times Sl(2,\mathbb{Z})$-invariant mass formula for N=4 SYM in 3+1 dimensions

As a first case, we consider the mass formula (7.16) for $d = 3$,

$$E_{\rm YM} = \frac{g_{\rm YM}^2}{NV_s}\left(m^I + \frac{1}{2}\mathcal{C}^{IJK}m_{JK}\right)\tilde{g}_{IL}\left(m^L + \frac{1}{2}\mathcal{C}^{LMN}m_{MN}\right) \\ + \frac{V_s}{Ng_{\rm YM}^2}\left(m_{IJ}\tilde{g}^{IK}\tilde{g}^{JL}m_{KL}\right) + \sqrt{n_I\tilde{g}^{IJ}n_J}\ . \tag{7.18}$$

This includes the energy of the electric flux m^I (i.e. the momentum conjugate to $\int F_{0I}$) and the magnetic flux $m_{IJ} = \int F_{IJ}$ in the diagonal Abelian subgroup of $U(N)$, together with the energy of a massless excitation with quantized momentum n_I. We observe that the effect of the M-theory background value of the three-form $\mathcal{C}_3$ is to shift the electric flux m^I, which is a manifestation of the Witten effect and indicates that the coupling of $\mathcal{C}_3$ to gauge theory occurs through the topological term (7.15). Indeed, the only effect of such a coupling is to shift the momentum conjugate to $\partial_0 A_I$ by a quantity $\mathcal{C}^{IJK}\int F_{JK}$.

Moreover, introducing the dual magnetic charge $m_*^I = \frac{1}{2}\epsilon^{IJK}m_{JK}$ and setting $\mathcal{C}^{IJK} = \theta\epsilon^{IJK}$, the mass formula (7.18) can be written in the alternative form

$$E_{\rm YM} = \frac{1}{NV_s}\left(g_{\rm YM}^2(m^I + \theta m_*^I)\tilde{g}_{IJ}(m^J + \theta m_*^J) + \frac{1}{g_{\rm YM}^2}m_*^I\tilde{g}_{IJ}m_*^J\right) \\ + \sqrt{n_I\tilde{g}^{IJ}n_J}\ , \tag{7.19}$$

which manifestly exhibits the $Sl(2,\mathbb{Z})$ S-duality symmetry as well as the $Sl(3,\mathbb{Z})$ modular group of the three-torus.

7.5 $Sl(5,\mathbb{Z})$-invariant mass formula for (2,0) theory on the M5-brane

Moving on to the case $d = 4$, an extra momentum charge n_4 appears in (7.16), which corresponds to the momentum along the dynamically generated 5th dimension. After some algebra, the total mass (7.16) can be rewritten in a manifestly U-duality ($Sl(5,\mathbb{Z})$)-invariant form:

$$E_{\rm YM} = \frac{1}{NV_5} m^{AB} \tilde{g}_{AC} \tilde{g}_{BD} m^{CD} + \sqrt{n_A \tilde{g}^{AB} n_B} \tag{7.20}$$

where $A, B, \dots = 1 \dots 5$ and $V_5 = V_s g_{\rm YM}^2$ is the volume of the five-dimensional torus. Here, the two-form and vector charges m^{AB}, n_A on the five-torus are related to the original set on the four-torus by

$$m^{I5} = m^I\ , \qquad m^{IJ} = \frac{1}{2}\epsilon^{IJKL} m_{KL} \tag{7.21a}$$

$$n_I = n_I\ , \qquad n_5 = \frac{1}{4!}\epsilon^{IJKL} n_{IJKL} \qquad I, J, \dots = 1, \dots, 4 \tag{7.21b}$$

where the charge m^{AB} is the quantized flux (in the diagonal Abelian group) conjugate to the two-form gauge field that lives on the (5+1)-dimensional world-volume, and n_A is simply the momentum along the direction A. The gauge potential $\mathcal{C}_{IJK}$ combines with the gauge coupling and the T^4 metric to make the metric on T^5:

$$\mathrm{d}s_5^2 = \tilde{R}^2 (\mathrm{d}x^5 + \mathcal{A}^I \mathrm{d}x_I)^2 + \mathrm{d}s_4^2 \tag{7.22a}$$

$$\tilde{R} = g_{\rm YM}^2\ , \quad \mathcal{A}_I = \frac{1}{3!}\epsilon_{IJKL} \mathcal{C}^{JKL}\ . \tag{7.22b}$$

In particular, it is seen that the three-form potential plays the role of the off-diagonal component of the five-dimensional metric relevant to the M5-brane.

As a check, we recall that the bosonic part of the M5-brane action can be written in a non-covariant form by solving the self-duality condition after singling out a special (fifth) space-like direction and integrating the resulting equations of motion [274, 251, 1]. In particular, it contains the coupling

$$\mathcal{L} = -\frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma} \frac{G^{5\lambda}}{G^{55}} \tilde{H}^{\mu\nu} \tilde{H}^{\rho\sigma} \tag{7.23a}$$

$$\tilde{H}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma}\ , \quad \mu, \nu = 0 \dots 4\ , \tag{7.23b}$$

which precisely reproduces, upon the identifications in (7.22), the topological coupling (7.15) in the effective (4+1)-dimensional SYM theory, where the field strength $F_{\mu\nu}$ is identified with the dual field strength $\tilde{H}_{\mu\nu}$. Finally, we note that $E_{\rm YM}$ in (7.20) depends on the volume of T^5 through an over-all factor $V_5^{-1/5}$, in agreement with the scale invariance of the conjectured (5+1)-dimensional (2,0) theory.

7.6 $SO(5,5,\mathbb{Z})$-invariant mass formula for non-critical string theory on the NS5-brane

Finally, we consider the case $d = 5$, for which according to the reasoning in Subsection 6.5 the Matrix gauge theory should correspond to a non-critical string theory on the type IIB NS5-brane with vanishing string coupling $\hat{g}_s$ [‡49] and finite string tension $\hat{l}_s^2 = g_{\mathrm{YM}}^2$. After some algebra, the mass formula (7.16) can be rewritten in the manifestly U-duality ($SO(5,5)$) invariant form

$$E_{\mathrm{YM}} = \frac{1}{N\mathcal{M}_{\mathrm{NS5}}}\mathcal{M}^2(\mathrm{D1,D3,D5}) + \sqrt{\mathcal{M}^2(\mathrm{KK,F1})} \; , \tag{7.24}$$

where $\mathcal{M}_{\mathrm{NS5}} = \frac{V_s}{\hat{g}_s^2 \hat{l}_s^6}$ is the mass of the background NS5-brane.

The second part of (7.24) involves the momentum (n_1) and winding (n^1, dual to n_4) excitations of the strings living on the world-volume, which form the vector representation **10** of the $SO(5,5)$ T-duality group. The corresponding invariant mass

$$\mathcal{M}^2(\mathrm{KK,F1}) = (n_1 + B_2 n^1)^2 + \frac{1}{\hat{l}_s^2}(n^1)^2 \tag{7.25}$$

directly follows from the second part of (7.16), using the identification

$$B_{IJ} = \frac{1}{3!}\epsilon_{IJKLM}\mathcal{C}^{KLM} \tag{7.26}$$

for the background antisymmetric tensor field in terms of the components of the three-form gauge potential on the five-torus.

The first term in (7.24) involves the D-brane excitations arising from the charges (m^1, m_2, m_5) that can be dualized into (m^1, m^3, m^5). It exhibits the correct invariant mass Eq. (3.39) for a spinor representation of $SO(5,5)$:

$$\mathcal{M}^2(\mathrm{D1,D3,D5}) = \left(\frac{\tilde{m}^1}{\hat{g}_s\hat{l}_s^2}\right)^2 + \left(\frac{\tilde{m}^3}{\hat{g}_s\hat{l}_s^4}\right)^2 + \left(\frac{m^5}{\hat{g}_s\hat{l}_s^6}\right)^2 \tag{7.27a}$$

$$\tilde{m}^1 = m^1 + B_2 m^3 + B_2^2 m^5 \; , \quad \tilde{m}^3 = m^3 + B_2 m^5 \tag{7.27b}$$

where we again used the identification (7.26).

As a further check, let us note that the Green–Schwarz term $\int \mathrm{d}^6 x B \wedge F \wedge F$ in the effective action of the six-dimensional string theory, correctly gives the topological term (7.15) after using the relation (7.26) between the background string theory B-field and the vacuum expectation values of the M-theory three-form.

[‡49] $\hat{g}_s$ cancels out in the following formulae.

7.7 Extended U-duality symmetry and Lorentz invariance

M(atrix) theory still lacks a proof of eleven-dimensional Lorentz covariance to shorten its name to M-theory. In the original conjecture, this feature was credited to the large-N infinite-momentum limit. The much stronger Discrete Light Cone (DLC) conjecture, if correct, allows Lorentz invariance to be checked at finite N – or rather at finite N's, since the non-manifest Lorentz generators mix distinct N superselection sectors. In particular, M(atrix) theory on T^d in the DLC should exhibit a U-duality $E_{d+1(d+1)}(\mathbb{Z})$, if it is assumed that U-duality is unaffected by light-like compactifications [172, 54, 244]. In this section, we show that an action of $E_{d+1}(\mathbb{Z})$ on the M-theory BPS spectrum can be defined when we include the light-like circle S^1 on an equal footing with the space-like torus T^d.

particle multiplet	charge	string multiplet	charge	missing charges	ext. part.
$\frac{1}{R_I}$	$m_1(\mathbf{7})$			$N(\mathbf{1})$	$M_1(\mathbf{8})$
$\frac{R_I R_J}{l_p^3}$	$m^2(\mathbf{21})$	$\frac{R_l R_J}{l_p^3}$	$n^1(\mathbf{7})$		$M^2(\mathbf{28})$
$\frac{R_I R_J R_K R_L R_M}{l_p^6}$	$m^5(\mathbf{21})$	$\frac{R_l R_I R_J R_K R_L}{l_p^6}$	$n^4(\mathbf{35})$		$M^5(\mathbf{56})$
$\frac{R_I^2 R_J R_K R_L R_M R_N R_P}{l_p^9}$	$m^{1;7}(\mathbf{7})$	$\frac{R_l R_I^2 R_K R_L R_M R_N R_P}{l_p^9}$	$n^{1;6}(\mathbf{49})$	$N^6(\mathbf{7}), N^7(\mathbf{1})$	$M^{1;7}(\mathbf{64})$
		$\frac{R_l R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P}{l_p^{12}}$	$n^{3;7}(\mathbf{35})$	$N^{2;7}(\mathbf{21})$	$M^{3;8}(\mathbf{56})$
		$\frac{R_l R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P}{l_p^{15}}$	$n^{6;7}(\mathbf{7})$	$N^{5;7}(\mathbf{21})$	$M^{6;8}(\mathbf{28})$
				$N^{1;7;7}(\mathbf{7}), N^{7;7}(\mathbf{1})$	$M^{1;8;8}(\mathbf{8})$

Table 7.10: Particle multiplet and string multiplet wrapped on R_l for $d=7$. Together with the rank multiplets, they form the $d=8$ particle multiplet.

In the presence of an extra (light-like) compact direction of radius R_l, the states from the string multiplet in Table 4.13 can be wrapped to yield extra particles in the spectrum that join the already existing states from the particle multiplet in Table 4.10. We have summarized in Table 7.10 the various particles obtained in the case $d=7$. It clearly appears that altogether, the $d=7$ particle and string multiplets build a particle multiplet of the $d=8$ U-duality group, whose charges M are obtained from the particle m and string n charges through the relations

$$\begin{aligned}
&m_1 = M_1 && m^{1;7} = M^{1;7}\ ,\quad n^{1;6} = M^{1;l6} \\
&m^2 = M^2\ ,\quad n^1 = M^{l1} && m^{3,8} = M^{3;8}\ ,\quad n^{3,7} = M^{3;l7} \\
&m^5 = M^5\ ,\quad n^4 = M^{l4} && m^{6,8} = M^{6,8}\ ,\quad n^{6,7} = M^{6;l7}
\end{aligned} \tag{7.28}$$

where we have denoted the light-cone direction by an index l ‡50. This is not quite correct, however, since in particular there is no candidate for the M_l state, which would correspond

‡50 As usual, the same relations hold for $d < 7$ by dropping the tensors with too many antisymmetric indices.

to a Kaluza–Klein excitation along the light-like direction. Obviously, this missing charge is nothing but the rank of the gauge group

$$N = M_l \ , \tag{7.29}$$

which indeed denotes the momentum along the light-like direction, and should therefore be considered as a charge on the same footing as the others. labelling the vacuum of some M(eta) theory on which the eleven-dimensional Lorentz group is represented. This charge has to be invariant under the U-duality group $E_{d(d)}(\mathbb{Z})$, but it gets mixed with other charges under $E_{d+1(d+1)}(\mathbb{Z})$.

While N is the only missing charge for $d \leq 5$, there is still, for $d \geq 6$, an extra missing U-duality singlet

$$N^6 = M^{l;l6} \tag{7.30}$$

which can be interpreted as the D6–D0 bound state of Eq. (7.12). For $d = 7$, one needs even more extra charges, namely

$$N^{2;7} \equiv M^{l2;l7} \ , \ \ N^6 \equiv M^{l;l6} \ , \ \ N^{5;7} \equiv M^{l5;l7} \ , \ \ N^{1;7;7} \equiv M^{1;l7;l7} \ , \tag{7.31}$$

which form the **56** of E_7, isomorphic to the particle multiplet of Table 5.1, as well as the two singlets

$$N^7 = M^{l;7} \ , \ \ N^{7;7} = M^{l;l7;l7} \ , \tag{7.32}$$

for which Table C.2 gives the bound-state interpretation as well. These extra charges along with N were referred to in [244] as the *rank multiplet*. The results are summarized in the Table 7.11, which lists, for all d's, the dimensions of the particle and string multiplet, as well as the rank multiplets that are needed to complete the first two into the particle multiplet of the $d + 1$ case.

We note that the above discussion follows immediately from the decomposition (4.34) of the particle multiplet of $E_{d(d)}$ into the particle and string multiplet of $E_{d-1(d-1)}$ plus extra irreps for $d \geq 6$. In particular, the extra representations that appear are nothing but the extra charges forming the rank multiplet. If we omit the light-like direction, we indeed see an extra $\mathcal{T}_1'|_1$ for $d = 6$; for $d = 7$ we have the extra representations $\mathcal{T}_1'|_1$, $\mathcal{T}_2'|_2$ and $(\mathcal{T}_1')^2|_3$, whose subscripts are in precise correspondence with the number of times the light-like direction appears in the charges of (7.30) and (7.31),(7.32).

7.8 Nahm-type duality and interpretation of rank

To see the physical significance of the U-duality enhancement, we discuss the extra generators in $E_{d+1(d+1)}(\mathbb{Z})$. First there is the *Weyl* generator, exchanging the light-cone direction with a chosen direction I on T^d:

$$R_l \leftrightarrow R_I \ . \tag{7.33}$$

D	d	U-duality $E_{d(d)}(\mathbb{Z})$	Flux $\{m\}$	Mom. $\{n\}$	Rank $\{N\}$	Total $\{M\}$
10	1	1	**1**	**1**	**1**	**3**
9	2	$Sl(2)$	**3**	**2**	**1**	**6**
8	3	$Sl(3)\times Sl(2)$	$\mathbf{(3,2)}$	$\mathbf{(3,1)}$	**1**	**10**
7	4	$Sl(5)$	**10**	**5**	**1**	**16**
6	5	$SO(5,5)$	**16**	**10**	**1**	**27**
5	6	E_6	**27**	$\bar{\mathbf{27}}$	**1**+**1**	**56**
4	7	E_7	**56**	**133**	**56** + **1** +**1**+**1**	**248**
3	8	E_8	**248**	**3875**	∞	∞

Table 7.11: Flux, momentum and rank multiplets.

The action of this Weyl transformation leaves the other R_J's and l_p invariant. In particular, Newton's constant in $11-(d+1)$ dimensions

$$\frac{1}{\kappa^2} = \frac{V_R R_l}{l_p^9} = R_l^{(d-7)/2} \frac{V_s^{(d-5)/2}}{g^{d-3}} \tag{7.34}$$

is invariant under U-duality. In terms of Matrix gauge theory, this means

$$g^2_{\rm YM} \to \left(\frac{R_l}{R_I}\right)^{d-4} g^2_{\rm YM}\ , \quad s_I \to s_I\ , \quad s_{J\neq I} \to \left(\frac{R_l}{R_I}\right) s_J\ . \tag{7.35}$$

Note that the transformed parameters depend on the original ones *and on* R_l. On the other hand, the only dependence of the gauge theory on R_l should be through a multiplicative factor in the Hamiltonian, since R_l can be rescaled by a Lorentz boost (see Eq. (6.4)). This leaves open the question of how the M(eta) theory itself depends on R_l.

The action on the charges follows from the exchange of the I and l indices, so that restricting to $d=6$ for simplicity, we have

$$\begin{aligned} N &\leftrightarrow m_I \\ n^1 &\leftrightarrow m^{I1} \\ n^4 &\leftrightarrow m^{I4} \\ n^{1;6} &\leftrightarrow m^{1;I6}\ . \end{aligned} \tag{7.36}$$

In particular, the rank N of the gauge group is exchanged with the electric flux m_I, whereas the momenta are exchanged with magnetic fluxes. This is reminiscent of Nahm duality, relating (at the classical level) a $U(N)$ gauge theory on T^2 with background flux m to a $U(m)$ gauge theory on the dual torus with background flux N [236]. In the context of higher-dimensional Yang-Mills theories, this symmetry was first observed at the level of the multiplicities of the BPS spectrum of SYM in $1+3$ dimensions [153], and extended in the context of Matrix theory on T^d in Refs. [173, 54, 244, 85]. Non-commutative geometry may provide the correct framework for this duality [237].

The other generator is the Borel generator,

$$\mathcal{C}_{lJK} \to \mathcal{C}_{lJK} \ , + \Delta\mathcal{C}_{lJK} \tag{7.37}$$

which is obtained from the usual $E_{d(d)}(\mathbb{Z})$ shifts by conjugation under Nahm-type duality. It is therefore not an independent generator, but still gives a spectral flow on the BPS spectrum

$$\begin{aligned} N &\to N + \Delta\mathcal{C}_{l2}\ m^2 \\ m_1 &\to m_1 + \Delta\mathcal{C}_{l2}\ n^1 \\ m^2 &\to m^2 + \Delta\mathcal{C}_{l2}\ n^4 \\ m^5 &\to m^5 + \Delta\mathcal{C}_{l2}\ n^{1;6}\ . \end{aligned} \tag{7.38}$$

In particular, this implies that states with negative N need to be incorporated in the M(eta) theory if it is to be $E_{d+1(d+1)}(\mathbb{Z})$-invariant. This is somewhat surprising since the DLC quantization selects $N > 0$, and it seems to require a revision both of the interpretation of N as the rank of a gauge theory and of the relation between N and the light-cone momentum P^+ [‡51].

Finally, let us comment in some more generality on the occurrence of this extended U-duality group. At least at low energies, the Matrix gauge theory describing the DLCQ of M-theory compactified on T^d is nothing but the gauge theory on the N Dd-brane wrapped on T^d. The latter is certainly invariant under the T-duality $SO(d,d,\mathbb{Z})$, and not only $SO(d-1,d-1,\mathbb{Z}) \bowtie Sl(d)$. Its spectrum of excitations, or equivalently bound states, is therefore invariant under $SO(d,d,\mathbb{Z})$, and very plausibly under the extended duality group $E_{d+1(d+1)}(\mathbb{Z})$. On the other hand, we have expanded the bound-state mass in the limit where the N Dd-branes are much heavier than their bound partners, whereas T-duality can exchange the Dd-branes with some of their excitations. $SO(d,d,\mathbb{Z})$ is therefore explicitly broken, and $E_{d+1(d+1)}(\mathbb{Z})$ is broken to $E_{d(d)}(\mathbb{Z})$. The invariance of the mass spectrum can be restored by using the full non-commutative Born–Infeld dynamics instead of its small α' Yang–Mills limit [70]. While not relevant for M(atrix) theory anymore, interesting insights can certainly be obtained by studying these truly stringy gauge theories.

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[‡51] See Ref. [263] for a discussion of DLCQ with negative light-cone momentum.

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Note added in proof : Boundaries of M-theory moduli space

As we discussed in Section 2, M-theory arises in the strong coupling regime of type II string theory, and reduces at low energy to 11D supergravity. It is important to determine what portion of the M-theory moduli space are covered by these weakly coupled descriptions, and thus what room is left for truly M-theoretic dynamics. The techniques we developed in Section 4 allow us to easily answer this question, first addressed in Ref. [319] for compactifications down to $D \geq 4$, and recently for $D = 2$ in Ref. [23]. We first consider the case $D > 2$, and consider an asymptotic direction in the moduli space, represented by an arbitrarily large weight vector λ in the weight space V_{d+1}, see Section 4.6. Modulo U-duality, λ can be chosen in the fundamental Weyl chamber $\lambda \cdot \alpha > 0$ for all positive roots α. This corresponds to choosing

$$R_1 < R_2 < \cdots < R_d \ , \qquad R_1 R_2 R_3 > l_p^3 \ , \tag{7.39}$$

where the inequalities are understood to be large inequalities, in order to have a maximal degeneration in the moduli space [319]. The 11D supergravity description is valid provided all radii are larger than the Planck length, i.e.

$$11D\ SUGRA\ : \qquad l_p < R_1 \ . \tag{7.40}$$

On the other hand, when the radius R_1 is much smaller than l_p, we can have a type IIA description with weak coupling $g_s^2 = (R_1/l_p)^3$, provided all radii are larger than the string length $l_s^2 = l_p^3/R_1$:

$$IIA\ : \qquad R_1 < l_p \ , \qquad R_1 R_2^2 > l_p^3 \ . \tag{7.41}$$

If this is not the case, then we may instead try a type IIB description with weak coupling $g_s = R_1/R_2$, same string length $l_s^2 = l_p^3/R_1$ and 10-th radius $R_B = l_p^3/(R_1 R_2)$. The IIB radii R_B and $R_3, \dots, R_d$ are larger than the string length provided $R_1 R_2^2 < l_p^3$ and $R_1 R_3^2 > l_p^3$, and it is not difficult to see that, using Eq. (7.39), the first implies $R_1 < l_p$, and the second is automatically satisfied. The type IIB description thus hold in the region

$$IIB\ : \qquad R_1 < l_p \ , \qquad R_1 R_2^2 < l_p^3 \ . \tag{7.42}$$

The weakly coupled 11D supergravity, type IIA and type IIB descriptions therefore cover, up to U-duality, the entire asymptotic moduli space of M-theory on T^d, $d > 2$. Of course, these descriptions fail when any of the large inequalities above become approximate equalities, hence the need for a more fundamental definition of M-theory. On the contrary, when $D \leq 2$, there are asymptotic sectors of the moduli space where no perturbative description is possible. Indeed, the weight space V_{d+1} is now intrinsically Minkovskian, and the light-cone $\lambda \cdot \lambda = -(x^0)^2 + \sum (x^i)^2 = 0$ separates the moduli space into three sectors that can never be related to each other by U-duality. For instance, the 11D supergravity region (7.39) has $x^i > x^0/3$ for all i, so that $\lambda \cdot \lambda > 0$. It therefore sits in the interior of the future light-cone if $x^0 > 0$, or past light-cone if $x^0 < 0$ (one may choose $x^0 = 0$ by working in l_p units). In fact, the time-like region can be shown to have a weakly coupled 11D supergravity, type IIA or type IIB description, whereas the spacelike region can be argued to be cosmologically forbidden by the holographic principle [23].

Appendices

A BPS mass formulae

In this Appendix, we analyse the BPS eigenvalue equation (2.14) for various choices of non-vanishing central charges. This gives a check on the mass formulae obtained on the basis of duality, and yields the conditions on the charges for a state to preserve a given fraction of supersymmetry.

A.1 Gamma Matrix theory

In order to maintain manifest eleven-dimensional Lorentz invariance, we use the 11D Clifford algebra $[\Gamma_M, \Gamma_N] = 2\eta_{MN}$, with signature $(-, +, \dots)$, even after compactification. The matrices Γ_M are then 32×32 real symmetric except for the charge conjugation matrix $C = \Gamma_0$, which is real antisymmetric. All products of Gamma matrices are traceless except for

$$\Gamma_0 \Gamma_1 \dots \Gamma_9 \Gamma_s = 1 \ , \tag{A.1}$$

where we denote by s the eleventh direction. We define $\Gamma_{MN\dots} = \Gamma_M \Gamma_N \dots$ if the p indices $M, N, \dots$ are distinct, zero otherwise, and abbreviate it as $\Gamma_{(p)}$. We have

$$(\Gamma_{(p)})^2 = (-1)^{\left[\frac{p}{2}\right]} \ , \ \ (\Gamma_0 \Gamma_{(p)})^2 = (-1)^{\left[\frac{p-1}{2}\right]} \ , \tag{A.2}$$

where the p indices are non-zero and the square brackets denote the integer part. Furthermore,

$$\Gamma_{(p)}\Gamma_{(q)} + (-)^{pq}\Gamma_{(q)}\Gamma_{(p)} = \sum_{\substack{k=0 \\ p+q-4k\geq|p-q|}}^{\infty} \Gamma_{(p+q-4k)} \tag{A.3a}$$

$$\Gamma_{(p)}\Gamma_{(q)} - (-)^{pq}\Gamma_{(q)}\Gamma_{(p)} = \sum_{\substack{k=0 \\ p+q-2-4k\geq|p-q|}}^{\infty} \Gamma_{(p+q-2-4k)} \ , \tag{A.3b}$$

with no restrictions on the $p+q$ indices. On the right-hand side of Eq. (A.3a) (resp. (A.3b)), a contraction between the first $2k$ (resp. $2k+1$) indices of $\Gamma_{(p)}$ and the first $2k$ (resp. $2k+1$) indices of $\Gamma_{(p)}$ is implied. In particular,

$$[\Gamma_{(2)}, \Gamma_{(p)}] = \Gamma_{(p)} \ , \tag{A.4}$$

since $\Gamma_{(2)}$ generates Lorentz rotations.

A.2 A general configuration of KK-M2-M5 on T^5

Here we consider M-theory compactified on T^5, and allow for non-vanishing central charges Z_I, Z_{IJ}, Z_{IJKLM}, where the indices $I, J, \dots$ are internal indices on T^5. We therefore look for solutions to the eigenvalue equation

$$\begin{aligned} \Gamma\epsilon &= \mathcal{M}\epsilon \\ \Gamma &\equiv Z_I\Gamma^{0I} + Z^{IJ}\Gamma_{0IJ} + Z^{IJKLM}\Gamma_{0IJKLM} \ . \end{aligned} \tag{A.5}$$

Squaring this equation, we obtain

$$\begin{aligned} Z_I Z_J\{\Gamma_I, \Gamma_J\} - Z^{IJ}Z_{KL}\{\Gamma_{IJ}, \Gamma_{KL}\} + Z^{IJKLM}Z^{NPQRS}\{\Gamma_{IJKLM}, \Gamma_{NPQRS}\} \\ + 2Z_I Z^{JK}\left[\Gamma^I, \Gamma_{JK}\right] + 2Z_I Z^{JKLMN}\left\{\Gamma^I, \Gamma_{JKLMN}\right\} \\ - 2Z^{IJ}Z^{KLMNP}\left[\Gamma_{IJ}, \Gamma_{KLMNP}\right] \doteq \mathcal{M}^2 \ , \end{aligned} \tag{A.6}$$

where the symbol $\doteq$ denotes the equality when acting on ϵ. Using the identities (A.3a),(A.3b), this reduces to

$$(Z_I)^2 + (Z^{IJ})^2 + (Z^{IJKLM})^2 + +Z_J Z^{IJ}\Gamma_I + \left(Z_M Z^{MIJKL} + Z^{IJ}Z^{KL}\right)\Gamma_{IJKL} \doteq 1 \tag{A.7}$$

A 1/2-BPS state is obtained under the conditions

$$k^I \equiv Z_J Z^{IJ} = 0 \tag{A.8a}$$

$$k^{IJKL} \equiv Z_M Z^{MIJKL} + Z^{IJ}Z^{KL} = 0 \ , \tag{A.8b}$$

which indeed form a string multiplet (**10**) of $E_5 = SO(5,5)$, and has a mass given by

$$\mathcal{M}_0^2 = (Z_I)^2 + (Z^{IJ})^2 + (Z^{IJKLM})^2 \ . \tag{A.9}$$

If the conditions are not satisfied, we can define $k_I = \epsilon_{IJKLM} k^{IJKL}/4!$, $\Gamma_6 = \Gamma_{12345}$ and rewrite Eq. (A.7) as

$$k^I \Gamma_I + k_I \Gamma_6 \Gamma^I \doteq \mathcal{M}^2 - \mathcal{M}_0^2 \ . \tag{A.10}$$

Note that the $SO(5,5)$ vector (k_I, k^I) is null: $k_I k^I = 0$. Squaring again yields the 1/4-BPS state mass formula

$$\mathcal{M}^2 = (Z_I)^2 + (Z^{IJ})^2 + (Z^{IJKLM})^2 + \sqrt{(k^I)^2 + (k_I)^2} \ . \tag{A.11}$$

This result can be straightforwardly made invariant under the full U-duality group by including the couplings to the gauge potentials through the lower charges as found for the particle and string multiplet in (5.23) and (5.27).

A.3 A general configuration of D0,D2,D4-branes on T^5

We now consider the D-brane sector of M-theory on T^6, that is a general configuration of D0,D2,D4-branes. The eigenvalue equation becomes

$$\Gamma\epsilon = \mathcal{M}\epsilon \tag{A.12a}$$

$$\Gamma \equiv Z\Gamma_{0s} + Z^{ij}\Gamma_{0ij} + Z^{ijkl}\Gamma_{0ijkls} \ , \tag{A.12b}$$

where Z, Z^{ij}, Z^{ijkl} denote the D0,D2,D4-brane charges respectively, and $i, j, \dots$ run from 1 to 5. Squaring this equation, we obtain

$$2Z^2 + Z^{ij}Z^{kl}\{\Gamma_{ij}, \Gamma_{kl}\} + Z^{ijkl}Z^{mnpq}\{\Gamma_{ijkl}\Gamma_{mnpq}\} \\ + 4Z\ Z^{ijkl}\Gamma_{ijkl} + 2Z^{ij}Z^{klmn}\left[\Gamma_{ij}, \Gamma_{klmn}\right]\Gamma_s \doteq \mathcal{M}^2 \ . \tag{A.13}$$

Using identities (A.3a),(A.3b), this becomes

$$Z^2 + (Z^{ij})^2 + (Z^{ijkl})^2 + k^{ijkl}\Gamma_{ijkl} + (k')^{ijkl}\Gamma_{ijkls} \doteq \mathcal{M}^2 \ , \tag{A.14}$$

where we defined

$$k^{ijkl} \equiv Z^{[ij}Z^{kl]} + Z\ Z^{ijkl} \tag{A.15a}$$

$$k'^{ijkl} \equiv Z^{m[i}Z^{jkl]m} \ . \tag{A.15b}$$

The second combination can be rewritten on T^5 as a form $k^{i;jklmn} = Z^{i[j}Z^{klmn]}$. Then, k^4 and $k^{1;5}$ can be dualized into a **10** null vector (k_i, k^i) of the T-duality group $SO(5,5)$. A state with $k = k' = 0$ is 1/2-BPS with mass

$$\mathcal{M}_0^2 = (Z)^2 + (Z^{ij})^2 + (Z^{ijkl})^2 \ . \tag{A.16}$$

If these conditions are not met, we can rewrite Eq. (A.14) as

$$k_i \Gamma^i \Gamma_6 + k'_i \Gamma^i \Gamma_6 \Gamma_s \mathrel{\hat{=}} \mathcal{M}^2 - \mathcal{M}_0^2 \ , \tag{A.17}$$

implying a mass formula

$$\mathcal{M}^2 = (Z)^2 + (Z^{ij})^2 + (Z^{ijkl})^2 + 2\sqrt{(k^i)^2 + (k_i)^2} \tag{A.18}$$

or, in terms of the natural undualized charges,

$$\mathcal{M}^2 = (Z)^2 + (Z^{ij})^2 + (Z^{ijkl})^2 + 2\sqrt{(k^{ijkl})^2 + (k^{i;jklmn})^2} \ . \tag{A.19}$$

A.4 A general configuration of KK–w–NS5 on T^5

Finally, we consider the Neveu–Schwarz sector of the theory considered in the Appendix A.3, namely the bound states of NS5-branes, winding and Kaluza–Klein states. The eigenvalue equation then reads

$$\left(z_i \Gamma^{0i} + z^i \Gamma_{0si} + z^{ijklm} \Gamma_{0ijklm} \right) \epsilon = \mathcal{M} \epsilon \ . \tag{A.20}$$

Taking the square gives

$$z^2 + (z^i)^2 + (z_i)^2 + 2 z z^i \Gamma_{6i} + 2 z z^i \Gamma_{6si} - 2\Gamma_s z^i z_i \mathrel{\hat{=}} \mathcal{M}^2 \ , \tag{A.21}$$

so the 1/2-BPS conditions appear to be

$$z \ z^i = z \ z_i = z^i z_i = 0 \ . \tag{A.22}$$

This agrees with the vanishing of the entropy $z z^i z_i$ and its first derivatives, as obtained in Ref. [115]. We can go further and find the complete 1/8-BPS mass formula: multiply Eq. (A.20) by $z\Gamma_{06}$:

$$-z \ z_i \Gamma_{6i} - z \ z^i \Gamma_{6si} - z^2 \epsilon \mathrel{\hat{=}} z \mathcal{M} \Gamma_{06} \tag{A.23}$$

and combine with Eq. (A.21) to obtain:

$$\left(-z^2 + (z^i)^2 + (z_i)^2 + 2 z \mathcal{M} \Gamma_{60} - 2 z^i z_i \Gamma_s \right) \mathrel{\hat{=}} \mathcal{M}^2 \ . \tag{A.24}$$

Now Γ_s and Γ_{06} commute, are traceless and square to 1, so this is a second-order equation:

$$-z^2 + (z^i)^2 + (z_i)^2 \pm 2 z \mathcal{M} \pm 2 z^i z_i \mathrel{\hat{=}} \mathcal{M}^2 \ , \tag{A.25}$$

with solutions

$$\mathcal{M} = \pm z \pm \sqrt{(z_i \pm z^i)^2} \tag{A.26}$$

or, equivalently:

$$\mathcal{M}^2 = z^2 + (z_i)^2 + (z^i)^2 + 2|z_i z^i| + 2|z| \sqrt{(z_i)^2 + (z^i)^2 + 2|z_i z^i|} \ . \tag{A.27}$$

This reduces to the usual mass formula for perturbative string states ($z = 0$) and for KK–NS5 or w–NS5 bound states. For momentum and winding charges along a single direction, this reduces to $\mathcal{M} = \pm z \pm z_1 \pm z^1$, in agreement with the identification of central charges in Ref. [115]. The U-duality invariant generalization of this mass formula is however unclear.

B The $d = 8$ string/momentum multiplet

For completeness, we give in Table B.1 the content of the string/momentum multiplet for $d = 8$ in the **3875** of $E_{8(8)}$. It comprises the 2160 states in the Weyl orbit of the highest weight R_i/l_p^3 of length 4, together with 7 copies of the 240 weights of length 2 with tension

$$\mathcal{T} = \frac{V_R}{l_p^9} \times (d = 8 \text{ particle multiplet}) , \tag{B.1}$$

as well as 35 zero weights with tension

$$\mathcal{T} = \left(\frac{V_R}{l_p^9} \right)^2 . \tag{B.2}$$

As in $d = 7$, the resulting multiplet exhibits a mirror symmetry, which relates each state with tension R^{3a-2}/l_p^{3a}, $a = 1 \dots 6$ to another state with tension $R^{34-3a}/l_p^{3(12-a)}$ through the relation

$$\mathcal{M}\mathcal{M}' = \left(\frac{V_R}{l_p^9} \right)^4 , \tag{B.3}$$

where V_R is the volume of the eight-torus. For this reason, Table B.1 only gives the explicit form of the tensions for the lower half $a = 1 \dots 5$ and the self-mirror part $a = 6$. The second column gives the $Sl(8)$ irreps at each level graded by $1/l_p^{3a}$, while the last column lists the corresponding charges. Here the notation is as follows: a semicolon denotes an ordinary tensor product as before (so in general contains more than one $Sl(8)$ irrep); two superscripts $(p;q)$ grouped within parentheses and separated by a semicolon denote the irrep, whose Young tableau is formed by juxtaposition of a column with p rows and one with q rows.

As an aid to the reader, we give the charges of the dual states at level $l_p^{3(12-a)}$:

$$\begin{gathered} a = 1 : \quad n^{7;8;8;8} , \qquad a = 2 : \quad n^{4;8;8;8} , \qquad a = 3 : \quad n^{2;7;8;8} \\ a = 4 : \quad n^{1;5;8;8}, n^{(7;7);8;8} , \qquad a = 5 : \quad n^{(1;2);8;8}, n^{4;7;8;8} \end{gathered} \tag{B.4}$$

Finally, we display the decomposition of the $d = 8$ string multiplet under the T-duality subgroup group $SO(7,7,\mathbb{Z})$. Here, we may again restrict to those states with (type IIA) tensions $M \sim 1/g_s^a$, $a = 0 \dots 4$, for each of which there is a dual state with tension $\mathcal{M}'$ related to it by

$$\mathcal{M}\mathcal{M}' = \left(\frac{V_R'}{g_s^2 l_s^8} \right)^4 , \tag{B.5}$$

where V_R' stands for the seven-dimensional type IIA torus. The type IIA mirror symmetry (B.5) easily follows from (B.3) using the M-theory/type IIA connection in (2.11). The results are summarized in Table B.2.

mass	$Sl(8)$ irrep	charge
$\frac{R_I}{l_p^3}$	$\mathbf{8}$	n^1
$\frac{R_I R_J R_K R_L}{l_p^6}$	$\mathbf{70}$	n^4
$\frac{R_I^2 R_J R_K R_L R_M R_N}{l_p^9}$, $7\frac{V_R}{R_I l_p^9}$	$\mathbf{8}+\mathbf{216}$	$n^{1;6}$
$\frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P}{l_p^{12}}$		
$\frac{V_R R_I^2}{l_p^{12}}$, $7\frac{V_R R_I R_J}{l_p^{12}}$	$\mathbf{28}+\mathbf{36}+\mathbf{420}$	$n^{3;7}, n^{(1;1);8}$
$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P}{l_p^{15}}$		
$\frac{V_R R_I^2 R_J R_K R_L}{l_p^{15}}$, $7\frac{V_R R_I R_J R_K R_L R_M}{l_p^{15}}$	$\mathbf{56}+\mathbf{168}+\mathbf{404}$	$n^{(6;7)}$, $n^{1;4;8}$
$\frac{V_R R_I^2 R_J^2 R_K R_L R_M R_N}{l_p^{18}}$, $7\frac{V_R R_I^2 R_J R_K R_L R_M R_N R_P}{l_p^{18}}$, $35\frac{V_R^2}{l_p^{18}}$	$\mathbf{1}+\mathbf{63}+\mathbf{720}$	$n^{(1;7);8}$, $n^{2;6;8}$

Table B.1: String/momentum multiplet $\mathbf{3875}$ of E_8.

Here the type IIA states in the a-th column, have a tension proportional to $1/g_s^{a-1}$. The first column is the singlet irrep formed by the fundamental string. The second column is the spinor irrep consisting of Dp=0-,2-,4-,6-branes, with one unwrapped world-volume direction. The third column can be decomposed into the $SO(7,7)$ irreps $378 = \mathbf{1} \oplus 3 \times \mathbf{91} \oplus \mathbf{104}$, and contains, together with NS5 and KK5 with one unwrapped direction, many non-standard states with tension $\sim 1/g_s^2$. The fourth column contains the representation $896 = \mathbf{14} \otimes \mathbf{64}$ formed by tensor product of vector and spinor representation, and has states with tension $\sim 1/g_s^3$. The fifth column consists of 1197 states with tension $\sim 1/g_s^4$. The set of duals of these states includes states with tension up to $1/g_s^8$, all of which are at present far from understood.

We note that the string state with tension $\frac{V_R R_I^2}{l_p^{12}}$ is presumably related to the conjectured M9-brane [38, 43], which should more properly be called M8-brane. In fact, for $d = 9$ there will be a corresponding particle with mass $\frac{V_R R_I^2}{l_p^{12}}$, where V_R is now the volume of the nine-torus. Taking $R_I = R_s = l_s g_s$, this reduces to the mass of the type IIA D8-brane, while taking R_s in one of the other world-volume directions gives an 8-brane with exotic mass $\frac{V_R' R_i^2}{l_s^{11} g_s^3}$. Vertical reduction, on the other hand, would give a type IIA 9-brane.

C Matrix gauge theory on T^7

In this appendix we discuss in some detail the Matrix gauge theory on T^7, performing the analysis of Subsection 7.2 for the case $d = 7$.

For our discussion, it will be useful to first consider the type IIB states obtained from the set of type IIA states in (4.36), by performing a maximal T-duality on the seven-torus. Using (2.41), we find the following T-duality multiplets for type IIB string theory

$\mathbf{3875}(E_8) \supset SO(7,7)$ $\cup$ $Sl(8)$	1	64	378	896	1197	896	...
8	1	7					
70		35	35				
224		21	154	49			
484		1	154	294	35		
728			35	292	294	7	
847				154	539	154	
728				7	294	292	...
484					35	294	...
⋮						⋮	⋱

Table B.2: Branching of the $d = 8$ string multiplet into representations of $Sl(8)$ and $SO(7,7)$. The entries in the table denote the common $Sl(7)$ reps. The full table can be reconstructed using mirror symmetry in the point with $Sl(7)$ representation 539.

compactified on T^7

$$V: \quad \frac{R_i}{l_s^2}, \frac{1}{R_i} \tag{C.1a}$$

$$S_B: \quad \frac{1}{g_s}\left(\frac{V_R}{l_s^8}, \frac{R_iR_jR_kR_lR_m}{l_s^6}, \frac{R_iR_jR_k}{l_s^4}, \frac{R_i}{l_s^2}\right) \tag{C.1b}$$

$$S+AS: \quad \frac{1}{g_s^2}\left(\frac{V_RR_iR_j}{l_s^{10}}, 8\frac{V_R}{l_s^8}, \frac{R_i^2R_jR_kR_lR_mR_n}{l_s^8}, \frac{R_iR_jR_kR_lR_m}{l_s^6}\right) \tag{C.1c}$$

$$S_A: \quad \frac{V_R}{g_s^3l_s^8}\left(\frac{R_iR_jR_kR_lR_mR_n}{l_s^6}, \frac{R_iR_jR_kR_l}{l_s^4}, \frac{R_iR_j}{l_s^2}, 1\right) \tag{C.1d}$$

$$V': \quad \left(\frac{V_R}{g_s^2l_s^8}\right)^2\left(R_i, \frac{l_s^2}{R_i}\right) \ . \tag{C.1e}$$

Here we have given the states in each multiplet in the order in which they are obtained from the corresponding type IIA states. At the levels $1/g_s^a$, with a even, we obtain the same set of states as in type IIA. At odd level, however, the spinor representations are interchanged, so that at level $1/g_s$ we obtain the odd Dp branes, while at level $1/g_s^3$ we find the set of p_3^{7-p} branes with $p = 1,3,5,7$.

We also give the S-duality structure of the type IIB states (C.1). Using (4.17), the following list of S-duality singlets (appearing at each level) is found:

$$\text{KK}\ ,\ \ \text{D3}\ ,\ \ 7_2\ ,\ \ \text{KK5}\ ,\ \ 3_3^4\ ,\ \ 0_4^{(1,6)}\ . \tag{C.2}$$

The remaining states pair up into S-duality doublets

$$\text{F1–D1}\ ,\ \ \text{D5–NS5}\ ,\ \ \text{D7–}7_3\ ,\ \ 5_2^2\text{–}5_3^2\ ,\ \ 1_3^6\text{–}1_4^6\ . \tag{C.3}$$

The M-theory mass, gauge-theory energy and bound-state interpretation of the flux and momentum multiplets is given in Table C.1.

$\mathcal{M}$	$E_{\rm YM}$	#	YM state	b.s. of N D7
$\frac{1}{R_I}$	$\frac{g_{\rm YM}^2 s_I^2}{N V_s}$	7	electric flux	NS–w
$\frac{R_I R_J}{l_p^3}$	$\frac{V_s}{N g_{\rm YM}^2 (s_I s_J)^2}$	21	magnetic flux	D5
$\frac{V_R}{R_I R_J l_p^6}$	$\frac{V_s (s_I s_J)^2}{N g_{\rm YM}^6}$	21	new sector	5_2^2
$\frac{V_R R_I}{l_p^9}$	$\frac{V_s^3}{N g_{\rm YM}^{10} s_I^2}$	7	new sector	1_3^6
$\frac{R_l R_I}{l_p^3}$	$\frac{1}{s_I}$	7	KK momentum	KK
$\frac{R_l V_R}{R_I R_J R_K l_p^6}$	$\frac{s_I s_J s_K}{g_{\rm YM}^2}$	35	YM threebrane	D3
$\frac{R_l R_I V_R}{R_J l_p^9}$	$\frac{V_s s_J}{g_{\rm YM}^4 s_I}$	49	new sector	KK5, 7_2
$\frac{R_l R_I R_J R_K V_R}{l_p^{12}}$	$\frac{V_s^2}{g_{\rm YM}^6 s_I s_J s_K}$	35	new sector	3_3^4
$\frac{R_l V_R^2}{R_I l_p^{15}}$	$\frac{V_s^2 s_I}{g_{\rm YM}^8}$	7	new sector	$0_4^{(1,6)}$

Table C.1: Flux and momentum multiplet for $d = 7$: **56** and **133** of E_7.

Comparing the states in the last column of this table with the total set of 1/2 BPS states (C.1) for type IIB on T^7, we note that there is a large number of states that do not appear. In analogy with the extra D6–D0 multiplet (a singlet) that appeared for $d = 6$, we can construct in this case an extra multiplet that contains the D7–D1 bound state, for which we conjecture (by T-duality) the same bound-state mass formula as in Eq. (7.10), so that

$$E_{\rm YM} = \mathcal{M}_{D1}^{2/3} \mathcal{M}_{ND7}^{1/3} = N^{1/3} \frac{V_s^{1/3} s_I^{2/3}}{g_{\rm YM}^2} \ , \tag{C.4}$$

where we used $g_{\rm YM}^2 = g_s l_s^4$. The relevant data of the corresponding U-duality multiplet, which forms the **56** of $E_{7(7)}(\mathbb{Z})$, is given in Table C.2. The easiest way to obtain this table, starting with the gauge-theory mass (C.4) obtained for the D7–D1 bound state, is by noticing that this state is, up to a multiplicative U-duality invariant factor $I_7^{1/3}$ (see Eq. (7.7)) and up to a power of 1/3, exactly analogous to the flux multiplet of Table C.1. Note that the 56 states are precisely the S-dual states of those involved in the flux multiplet bound states. The bound states relevant to the momentum multiplet, on the other hand, involve S-duality singlets.

Besides the D7 itself, this leaves two more possible states left in the type IIB, which can form a bound state with the D7, namely the two 7-branes with mass

$$\frac{V_s}{g_s^2 l_s^8} \ , \quad \frac{V_s}{g_s^3 l_s^8} \ . \tag{C.5}$$

For the first one, we know already from the momentum multiplet that the mass relation is

$$E_{\mathrm{YM}} = \mathcal{M} = \frac{V_s}{g_{\mathrm{YM}}^4} = I_7^{1/2} \tag{C.6}$$

and hence a U-duality singlet. For the second state in (C.5), we deduce the mass relation by the requirement that the bound-state energy be such that

$$E_{\mathrm{YM}} = [\mathcal{M}_{ND7}^a + \mathcal{M}^a]^{1/a} - \mathcal{M}_{ND7} \simeq \mathcal{M}^a \mathcal{M}_{ND7}^{1-a} \tag{C.7}$$

i) can be written in gauge-theory variables, and ii) is a U-duality singlet. Either of these requirements yields $a = 1/2$, and we are left with a gauge-theory state with energy

$$E_{\mathrm{YM}} = \mathcal{M}_{7_3}^{1/2} \mathcal{M}_{ND7}^{1/2} = N^{1/2} \frac{V_s}{g_{\mathrm{YM}}^4} = N^{1/2} I_7^{1/2} \ . \tag{C.8}$$

The singlets in Eqs. (C.6) and (C.8) are also given in Table C.2.

$\mathcal{M}$	E_{YM}	#	YM state	b.s. of N D7
$\frac{R_l^2 V_R}{R_I l_p^9}$	$\frac{N^{1/3} V_s^{1/3} s_I^{2/3}}{g_{\mathrm{YM}}^2}$	7	new sector	D1
$\frac{R_l^2 R_I R_J V_R}{l_p^{12}}$	$\frac{N^{1/3} V_s}{g_{\mathrm{YM}}^{8/3} (s_I s_J)^{2/3}}$	21	new sector	NS5
$\frac{R_l^2 V_R^2}{R_I R_J l_p^{15}}$	$\frac{N^{1/3} V_s (s_I s_J)^{2/3}}{g_{\mathrm{YM}}^{14/3}}$	21	new sector	5_3^2
$\frac{R_l^2 R_I V_R^2}{l_p^{18}}$	$\frac{N^{1/3} V_s^{5/3}}{g_{\mathrm{YM}}^6 s_I^{2/3}}$	7	new sector	1_4^6
$\frac{R_l V_R}{l_p^9}$	$\frac{V_s}{g_{\mathrm{YM}}^4}$	1	new sector	7_2
$\frac{R_l^3 V_R^2}{l_p^{18}}$	$\frac{N^{1/2} V_s}{g_{\mathrm{YM}}^4}$	1	new sector	7_3

Table C.2: Additional multiplets for $d = 7$: **56**, **1** and **1** of E_7.

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