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The Operator Product Expansion, Non-perturbative Couplings and the Landau Pole: Lessons from the $O(N)$ σ -Model

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Abstract:

We obtain the operator product expansion of the self-energy in the $O(N)$ non-linear σ -model to all orders in the coupling and the large momentum, and to next-to-leading order in $1/N$. In the light of this result we discuss recent suggestions that there may be additional power corrections from short distances, associated with defining the coupling constant non-perturbatively. The non-linear σ -model provides no evidence for such ‘non-standard’ power corrections. We also find that the OPE converges for sufficiently large external momentum, presumably because there are no multi-particle thresholds at arbitrarily high energies in the $1/N$ expansion.

1. Short-distance observables in QCD, characterized by a large scale $Q \gg \Lambda_{\text{QCD}}$, can be predicted perturbatively in $\alpha_s(Q) \ll 1$. In some cases non-perturbative power corrections in Λ_{QCD}/Q can be incorporated as well, using the operator product expansion (OPE) [1, 2]. Alternatively, one can deduce the structure of non-perturbative effects by investigating the infrared (IR) contributions to loop integrals in perturbation theory. This is the basis of the so-called renormalon method [3]. Either way, for example for current two-point functions $i \int dx e^{iqx} \langle 0 | T(j(x)j(0)) | 0 \rangle$, where j is a bilinear quark current, one finds that the leading power correction scales as $(\Lambda_{\text{QCD}}/Q)^4$ and is related to the gluon condensate. Both methods parametrize power corrections that arise from long-distance contributions to the observable. In general one cannot exclude non-perturbative effects from short distances. The only known dynamical mechanism for such corrections is due to instantons of small size $\rho^{-1} \gg \Lambda_{\text{QCD}}$ [4]. They scale as $(\Lambda_{\text{QCD}}/Q)^{11-2N_f/3}$ and are strongly suppressed at large Q .

In recent work Grunberg [5] and Akhoury and Zakharov [6] suggested that one should expect quite generically power corrections of order $(\Lambda_{\text{QCD}}/Q)^2$ from short distances on grounds that the perturbative QCD coupling is usually not specified to an accuracy better than this. To make their point let us consider the integral

$$\int dk^2 F(k, Q) \frac{\alpha_s(k)}{k^2} \quad (1)$$

which is typical of renormalon calculations of diagrams with one gluon line. When the loop momentum k is small, and if $F(k, Q)$ behaves as $(k/Q)^p$ for small k , one obtains a contribution of order $(\Lambda/Q)^p$ to the integral. This term stands for the power correction that is taken into account by the OPE or the renormalon method. In this context the definition of the coupling α_s has played no role. It is usually considered to be the one-loop perturbative running coupling, which has a Landau pole at $k^2 = \Lambda_{\text{QCD}}^2$. However, physical quantities do not have a Landau pole and a non-perturbative definition of $\alpha_s(k)$, valid at all k , should not have a Landau pole either. Such a non-perturbative coupling can itself be expanded in a power series, when k is large:

$$\alpha_s(k) = \frac{1}{-\beta_0 \ln(k^2/\Lambda_{\text{QCD}}^2)} + \dots + \text{const} \cdot \frac{\Lambda_{\text{QCD}}^2}{k^2} + \dots \quad (2)$$

The dots denote further perturbative and power correction terms. If the second term is present, and if one insists on extending (1), derived in the context of perturbation theory, to a non-perturbative running coupling, one obtains a power correction of order $\Lambda_{\text{QCD}}^2/Q^2$ to the integral in (1) from the region of large $k \sim Q$. In [5, 6] these additional power corrections are interpreted as a potential new source of non-perturbative effects, not accounted for by the OPE, but also not in contradiction with the OPE, because they arise from short distances (large k). The concept of an effective coupling is not unique and the existence of a particular $1/k^2$ term in (2) is not compelling. Independent of the precise form of power corrections, however, the argument given in [5, 6] suggests the existence of unconventional power corrections at some order.

The present work is motivated by our attempt to understand this argument, and its relation to the discussion of a related issue in [7], in the context of the two-dimensional non-linear $O(N)$ σ -model. We calculate the asymptotic expansion for the self-energy of the σ particle to next-to-leading order in $1/N$ and to all powers in the large momentum. The approximation contains all non-trivial elements of the OPE construction in QCD, except for the absence of multi-particle production. In particular, it allows us to address the question of non-perturbative corrections to the effective coupling and potential power corrections from short distances. Our conclusions can be summarized as follows:

(a) There are power corrections from the region $k \sim Q$, but they are included in the standard OPE construction and correspond to soft subgraphs of the graphs that define the effective coupling non-perturbatively. Despite the fact that $k \sim Q$, these power corrections are therefore of infrared origin.

(b) The fact that the perturbative coupling has a Landau pole does not give rise to additional power corrections. The Landau pole disappears and the correct analyticity structure is recovered only after summation of the OPE. The issue of defining an effective coupling non-perturbatively is not related to the existence of power corrections to particular physical observables. In fact, it turns out that the OPE is naturally expressed in terms of the perturbative coupling (the first term on the right-hand side of (2)).

(c) The OPE for the self-energy is convergent for $|Q^2| > 9m^2$ in the complex Q^2 plane, where m is the dynamically generated mass of the σ particle, analogous to Λ_{QCD} in QCD. The radius of convergence is determined by the onset of the 3-particle cut. The convergence of the OPE is probably a consequence of the $1/N$ expansion, because to the order we are working at most three particles can be produced. More generally, we expect that the OPE of the self-energy is convergent for $|Q^2| > (2k+1)^2 m^2$ if the $1/N$ expansion is truncated at the order k . Thus, for finite N , we expect the OPE to be divergent for any Q^2 . Because of the restriction on the number of produced particles, the $1/N$ expansion in the σ -model is not suited for studies of the analytic continuation of the OPE to Minkowskian momenta and the validity of “parton-hadron duality”, and we do not address both questions in this work.

We emphasize that while the present calculation clarifies the argument of [5, 6] it does not preclude the existence of power corrections from short-distances due to a yet unknown dynamical mechanism other than small instantons, see [8] for a particular suggestion. Up to this date, however, all (known) short-distance power corrections are of semiclassical origin and unrelated to integrals of the type (1) and the issue of defining the coupling constant.

2. The nonlinear $O(N)$ σ -model was considered repeatedly as a toy model for the OPE [9, 10, 11]. In particular, the non-perturbative separation of long-distance and short-distance contributions is well understood in this model, both in cut-off schemes [10] and in dimensional regularization [9]. In the second factorization scheme, the coefficient functions have non-Borel-summable perturbative expansions due to infrared (IR) renor-

malons. The prescription dependence that follows from defining the series is exactly compensated by the prescription dependence of defining the power-divergent matrix elements of higher-dimension operators. In the subsequent calculation we observe this compensation to all orders in the OPE. However, in this Letter we are mainly concerned with power corrections not related to IR renormalons, in particular the effects of a coupling redefinition on the power series expansion, and with analytic properties of the power series truncated to finite order.

The construction of the $1/N$ expansion and the Feynman rules are detailed in [9, 10]. The N σ -particles have mass¹

$$m^2 = \mu^2 e^{-1/g(\mu)}, \quad (3)$$

where μ is a renormalization scale. In addition to the standard massive propagator for the scalar σ -particles, the $1/N$ expansion involves the propagator of the auxiliary “ α -particle”

$$D(k) = -4\pi (k^2 + 2m^2) g_{\text{eff}}(k) = -4\pi \sqrt{k^2(k^2 + 4m^2)} \ln^{-1} \left[\frac{\sqrt{k^2 + 4m^2} + \sqrt{k^2}}{\sqrt{k^2 + 4m^2} - \sqrt{k^2}} \right]. \quad (4)$$

In order to establish a connection with (1), we interpret $g_{\text{eff}}(k)$ defined through the α -propagator as above as a non-perturbative effective coupling. The factor $k^2 + 2m^2$ is separated such that $g_{\text{eff}}(k)$ can be viewed as the sum of the chain of σ -particle bubble graphs with momentum-dependent vertices

$$g_{\text{eff}}(k) = \frac{1}{1/g_0 - I(k, m)} = g_0 + g_0^2 \text{ (bubble)} + g_0^3 \text{ (two bubbles)} + \dots,$$

where an UV regularization is implied, g_0 is the bare coupling and

$$I(k, m) = \int d^2q \frac{q(q+k)}{[q^2 + m^2][(q+k)^2 + m^2]}. \quad (5)$$

The coupling $g_{\text{eff}}(k)$ can be expanded at large momenta generating a power series

$$g_{\text{eff}}(k) = g(k) - 2g^2(k) \frac{m^2}{k^2} + \dots \quad (6)$$

Following the terminology of [5], $g_{\text{eff}}(k)$ naturally splits into a perturbative and non-perturbative part, $g_{\text{eff}}(k) = g(k) + \delta g(k)$, where the perturbative part $g(k)$ is given by the first term on the right-hand side alone.

¹To avoid factors of 4π , we rescale the coupling by a factor 4π . With this convention g satisfies

$$\beta(g) = \mu^2 \frac{\partial g}{\partial \mu^2} = -g^2.$$

Note that this β -function is exact in leading order of the $1/N$ expansion. In particular, $g(k)$ has a Landau pole at $k^2 = m^2$.



Figure 1: σ -self-energy diagrams at order $1/N$.

It is important to note that the power corrections $\delta g(k)$ to the running coupling are of IR origin. Diagrammatically, the reason for this is that even if the momentum that flows into the chain of bubbles is large, one of the lines in any one or more of the bubbles can still be soft.

$D(k)$ does not have a Landau pole, but a cut at $-k^2 > 4m^2$. Because $k^2 + 2m^2$ is factored out, $g_{\text{eff}}(k)$ has a ‘kinematic’ singularity at $-k^2 = 2m^2$ in addition. Note, however, that whenever the power expansion of the effective coupling is truncated, there is an additional Landau singularity at $k^2 = m^2$.

3. Consider the propagator of a σ -particle with momentum $p^2 \gg m^2$. To leading order in $1/N$ it is equal to $(p^2 + m^2)^{-1}$ and we can interpret its expansion at large momenta as the operator product expansion

$$(p^2 + m^2)^{-1} = \frac{1}{p^2} \sum_{n=0}^{\infty} \left(\frac{-m^2}{p^2} \right)^n \equiv \frac{1}{p^2} \sum_{n=0}^{\infty} C_n^{(0)}(p^2) \langle O_n \rangle^{(0)}, \quad (7)$$

with $C_n^{(0)}(p^2) = -1/p^{2n}$ and $\langle O_n \rangle^{(0)} = m^{2n}$. The OPE interpretation implies that the terms identified as operator matrix elements originate from large distances. To see this, one has to start from the perturbative phase of the σ model and construct the OPE explicitly; details can be found in [10]. We shall take the OPE interpretation of the large- N result for granted, and concentrate on the first subleading order in the large N -expansion, formulated in terms of massive σ -fields already.

To $1/N$ accuracy the σ -propagator involves the self-energy correction given by the two diagrams shown in Fig. 1, where wavy lines denote the α -propagator (4). The self-energy is quadratically ultraviolet divergent. We define the renormalized self-energy by zero-momentum subtractions

$$\Sigma^{\text{ren}}(p) = \Sigma(p) - \Sigma(0) - p^2 \frac{\partial}{\partial p^2} \Sigma(p)|_{p^2=0} \quad (8)$$

and drop the superscript “ren” in what follows. The tadpole diagram in Fig. 1b is then subtracted completely, and the self-energy is given by

$$\Sigma(p) = \frac{1}{\pi N} \int d^2 k \frac{k^2 + 2m^2}{(p+k)^2 + m^2} g_{\text{eff}}(k) + \text{subtractions}. \quad (9)$$

We have to verify that the asymptotic expansion of the propagator can be written in the factorized form

$$\begin{aligned} & (p^2 + m^2)^{-1} - (p^2 + m^2)^{-2}\Sigma(p) = \\ & = \frac{1}{p^2} \sum_{n=0}^{\infty} [C_n^{(0)}(p^2) + (1/N)C_n^{(1)}(p^2)] [\langle O_n \rangle^{(0)} + (1/N)\langle O_n \rangle^{(1)}] + O(1/N^2) \end{aligned} \quad (10)$$

where $C_n^{(1)}(p^2)$ and $\langle O_n \rangle^{(1)}$ correspond to $1/N$ corrections to coefficient functions and matrix elements, respectively. Note that the explicit construction of the operators is not necessary and is complicated in the case at hand by the existence of infinitely many operators of the same dimension. The OPE statement means that the contributions from small and large distances to (9) can be factorized. In the present case this also means that all contributions from large distances are associated with operator matrix elements.

Returning to the integral (9), long-distance contributions originate from the IR sensitive regions $k^2 \sim m^2$ and $(p+k)^2 \sim m^2$. The OPE is obtained by expanding the integrand of (9) in small quantities given a particular loop momentum region. The following three loop momentum regions have to be considered separately:

- (i) $k \gg m, p+k \gg m$: Both lines are far off-shell and both the σ -propagator and the effective coupling can be expanded in m^2 . The diagram effectively contracts to a point and the result can be interpreted as a $1/N$ correction to the short-distance coefficients times operator matrix elements evaluated at leading order in $1/N$.
- (ii) $k \sim m, p+k \gg m$: The σ -propagator can be expanded, but the effective coupling cannot. Thus the σ propagator is contracted, but the α -propagator is not; the resulting diagram can be viewed as a $1/N$ -correction to the vacuum expectation value of the product of two α -fields $\langle 0|\alpha(x)\alpha(0)|0\rangle$ with small separation $x \sim 1/p$. Expansion in x yields as the leading contribution a $1/N$ -correction to the matrix element $\langle 0|\alpha^2|0\rangle$ times a leading-order short-distance coefficient. (This matrix element was evaluated explicitly in [9, 10].) Note that from power counting it follows that this region contributes to (9) first at order $p^2 \cdot m^4/p^4$, i.e. with a $1/p^4$ -suppression similar to the gluon condensate in QCD. (The argument does not hold for the contribution of the subtraction terms.)
- (iii) $k \gg m, p+k \sim m$: The effective coupling can be expanded, but the σ -propagator cannot be expanded. This case looks symmetric to the previous one but the interpretation is different: contracting the α -propagator one gets a *leading* contribution at large N to the vacuum expectation value for the product of two σ -fields $\langle 0|\sigma^a(x)\sigma^a(0)|0\rangle = 1/g(x^{-1}) \cdot [1 + m^2x^2/4 + \dots]$ (no sum over 'a'). Note the inverse coupling as a consequence of the normalization of the "length" of the σ -field. The result can therefore be interpreted as a $1/N$ -correction to the short-distance coefficient times a leading large N operator matrix elements of σ fields. Note that

a contribution of this type is usually not considered in the renormalon method. However, it could be accounted for as well. In the QCD analogue of our problem, this would correspond to renormalon contributions to the coefficient function of a quark condensate.

We emphasize that all three contributions are part of the usual OPE construction and that (i,iii) include the “short-distance” power corrections discussed in [5, 6]. As already mentioned, these power corrections are in fact also of long-distance origin.

For the actual calculation of the expansion of (9) one could follow the above procedure and calculate the contribution from each region separately. It is more convenient not to perform this split-up and to calculate the expansion directly. To this end we write the inverse logarithm in $D(k)$ as

$$\int_0^\infty dt \left[\frac{A-1}{A+1} \right]^t, \quad (11)$$

where $A = (1 + 4m^2/k^2)^{1/2}$. Subsequently, we use the Mellin-Barnes representation

$$A \left[\frac{A-1}{A+1} \right]^t = \int \frac{ds}{2\pi i} K(s, t) \left(\frac{m^2}{k^2} \right)^{-s}, \quad (12)$$

where the kernel $K(s, t)$ is given by

$$K(s, t) = \frac{\Gamma(t+s)\Gamma(-1-2s)}{\Gamma(t-1-s)} \left[1 + \frac{2(t+s)}{t-s-1} + \frac{(t+s)(t+s+1)}{(t-s-1)(t-s)} \right]. \quad (13)$$

After these manipulations all integrations can be done and we arrive at the following “Borel representation” for the expansion in powers (and logarithms) of m^2/p^2 :

$$\Sigma(p) = \frac{p^2}{N} \int_0^\infty dt \sum_{n=0}^\infty \left(-\frac{m^2}{p^2} \right)^n \left\{ e^{-t/g(p)} \left[F_p^{(n)}[t] \frac{1}{g(p)} + G_p^{(n)}[t] \right] - H_{np}^{(n)}[t] \right\}. \quad (14)$$

Explicit expressions for the functions $F_p^{(n)}[t]$, $G_p^{(n)}[t]$, $H_{np}^{(n)}[t]$ are given in the appendix. All dependence on m^2/p^2 is either explicit or implicit in $g(p) = 1/\ln(p^2/m^2)$.

We first explain the origin of the various terms in (14).

The functions $F_p^{(n)}[t]$ and $G_p^{(n)}[t]$ correspond to the contributions of regions (i,iii) discussed above. Because they are multiplied by the factor $e^{-t/g(p)}$, they can be interpreted as Borel transforms of the perturbative expansions of short-distance coefficients times matrix elements that do not depend on $g(p)$. The separation of regions (i) and (iii) is not unique. In region (i) the expansion of the σ -propagator in $m^2/(p+k)^2$ leads to IR divergences, which require an intermediate regularization. The result from region (i) then takes the form

$$F_p^{(n)}[t] \ln \frac{p^2}{\mu^2} + G_p^{(n,i)}[t], \quad (15)$$

where μ is the intermediate factorization scale. On the other hand, the expansion of g_{eff} in region (iii) leads to increasingly UV divergent integrals, for which the same intermediate regularization is required. The result from region (iii) then takes the form

$$F_p^{(n)}[t] \ln \frac{\mu^2}{m^2} + G_p^{(n,iii)}[t]. \quad (16)$$

Both contributions combine to the square bracket in (14). The intermediate (and arbitrary) scale μ drops out and the logarithms combine to the inverse running coupling. Recall that region (iii) is a long-distance contribution to the integral (9), but it is multiplied by a perturbative expansion, because large momentum flows through the effective coupling. After combining (i) with (iii) as in (14), both $F_p^{(n)}[t]/g(p)$ and $G_p^{(n)}[t]$ have long- and short-distance contributions.

The function $H_{np}^{(n)}[t]$ originates from region (ii) discussed above and therefore should be identified with the $1/N$ correction to operator matrix elements. It cannot be written as a series expansion in $g(p)$ and has only long-distance contributions. No extra regularization is needed to separate (ii) from (i,iii).

Nevertheless, the separation of (ii) from (i,iii) is not unique. The functions $F_p^{(n)}[t]$, $G_p^{(n)}[t]$, $H_{np}^{(n)}[t]$ have singularities (typically simple poles) in the complex t plane at integer values $t = \pm k$, $k = 1, 2, \dots$. These are just the usual ultraviolet and IR renormalon singularities. (The poles at $t = 0$ correspond to the usual logarithmic UV divergences and are cancelled by renormalization.) Using the expressions collected in the appendix, one can verify that all IR renormalon singularities at positive values of t cancel. The cancellation of a particular singularity at $t = t_0$ occurs between $G_p^{(n)}[t]$ and $H_{np}^{(n+t_0)}[t]$ and thus involves a cancellation between a short-distance coefficient and an operator matrix element over different orders in the power expansion. This is an explicit all-order verification of the cancellation mechanism discussed in [9]. As a consequence of the singularities in individual terms of the sum over n , the summation and the integration over t cannot be interchanged, unless the integration contour is shifted slightly above (or below) the real axis. Only after such a definition can one truncate the OPE. Obviously the prescription dependence cancels in the end.

As an explicit example, consider the leading asymptotic behaviour of $\Sigma(p)$ as $p \rightarrow \infty$, related to the expansion of $F_p^{(n=0)}[t]$ and $G_p^{(n=0)}[t]$ around $t \rightarrow 0$ and to $H_{np}^{(n=0)}[t]$. We find

$$\Sigma(p) = \frac{p^2}{N} \left[\ln g(p) + 1.887537 - 2g(p) + \sum_{n=1}^{\infty} \sigma_n g^{n+1}(p) \right], \quad (17)$$

with factorially divergent coefficients

$$\sigma_n = n! \{ [1 + (-1)^n] \zeta(n+1) - 2 \}. \quad (18)$$

Because of the Riemann zeta-function this gives rise to an infinite series of renormalon poles.

Returning to (14), we note that the OPE is naturally expressed in terms of the perturbative coupling $g(p)$. The reason is that the asymptotic expansion is mathematically an expansion in powers and logarithms of m^2/p^2 , and absorbing logarithms only into the coupling leaves all powers of m^2/p^2 explicit. This implies that at every finite order in the OPE there is a Landau singularity at $p^2 = m^2$, which disappears, when the expansion is summed.

One can avoid unphysical Landau singularities order by order in the asymptotic expansion, if one eliminates $g(p)$ in favour of $g_{eff}(p)$, which amounts to an all-order rearrangement of the OPE. However, it is important to stress that there is no advantage to doing this: although the Landau singularity never appears, the predictive power is not enlarged. One would still have to sum the entire OPE to be able to predict the correlation function accurately in the region $p^2 \sim m^2$. In general, (but not in the case of $g_{eff}(p)$), an inadequate choice of the non-perturbatively defined coupling may obscure the cancellation of renormalons poles, or even introduce power corrections into the OPE which are not natural for the physical quantity in question, but which appear in order to cancel power corrections implicit in the coupling definition (see the discussion in [7]).

In the present example all power corrections do indeed originate from long distances and can be accounted for in the OPE framework. The calculation clarifies that the fact that the perturbative coupling has a Landau pole, or that a non-perturbative coupling has itself power corrections does not provide sufficient motivation for introducing ad hoc power corrections beyond the OPE. Moreover, all known sources of short-distance power corrections are related to semiclassical effects. Semi-classical effects are connected with multi-particle emission, a phenomenon which is certainly beyond the approximation in which one assumes the emission of one quantum gluon through an effective coupling. It therefore seems that to address the question of whether the OPE is complete, one would have to abandon the effective coupling representation.

A short comment is in order on ultraviolet renormalons, which lead to factorially divergent, sign-alternating series in the σ -model (as in QCD). This divergence limits the accuracy of a purely perturbative approach and the best possible approximation has an error that scales as a power in m^2/p^2 . It is sometimes argued (see for example [6]) that UV renormalons require adding power corrections (of short-distance nature) to the OPE. In our example, the series expansion of the leading term (18) as well as all others lead to UV renormalon singularities at negative t . Being located at negative t , these singularities do not lead to ambiguities in the individual terms of (14). Since the OPE (14) reproduces the exact result (as one can check numerically), we also conclude that UV renormalons do not necessitate introducing additional power corrections, as expected on general grounds for sign-alternating series.

4. Given the OPE to all orders, it is interesting to ask about the convergence/divergence properties of the OPE itself (rather than the perturbative expansion that multiplies each power correction). For example, in [12] a toy heavy-quark correlation function is used

to demonstrate that in this case the heavy quark expansion is factorially divergent.

For the self-energy in the σ -model it is straightforward to derive from the formulae collected in the appendix that the contribution of large $n \gg t$ (for fixed t) to the sum in (14) is equal to

$$\frac{9^{5/4} m^4}{\pi p^2} \sum_{n \gg t} \frac{1}{n^2} \left(-\frac{9m^2}{p^2}\right)^n \left\{ \left(\frac{9m^2}{p^2}\right)^t \left[-\ln(9m^2/p^2) + \psi(t) - \psi(1-t) \right] - \Gamma(t)\Gamma(1-t) \right\}. \quad (19)$$

It follows that the sum converges pointwise in t for $|p^2| > 9m^2$ and uniformly on any interval $[0, t_0]$. The contribution to $\Sigma(p)$ from $t > t_0$ can be made arbitrarily small by increasing t_0 as can be seen from inserting (11) with lower integration limit t_0 into (9). Hence the domain of convergence of the OPE (i.e. the series after t -integration) is given by $|p^2| > 9m^2$.

The radius of convergence seems to be related to the fact that the self-energy graph has a discontinuity for $-p^2 > 9m^2$, related to the fact that any cut contains three σ -particles of mass m . We suggest that the convergence of the OPE is a consequence of the $1/N$ expansion. At every finite order of the $1/N$ expansion, a cut can contain only a finite number of particles. We expect that the OPE of the self-energy is convergent for $|p^2| > (2k+1)^2 m^2$ if the $1/N$ expansion is truncated at the order k . For finite N , when there is no restriction on the number of particles in a cut, we expect that the OPE diverges for any value of p^2 in agreement with the conclusion of [12]. It may be interesting to pursue further the connection between the divergence of the OPE and multi-particle production.

Another consequence of the absence of multi-particle thresholds is that, viewed as a function of g , the self-energy has no other singularities in g than cuts stretching from $g = 0$ to $g = 1/(\ln 9 + in\pi)$ (n a non-zero integer) in the complex coupling plane. This differs from QCD and the σ -model at finite N , where the analyticity domain has zero opening angle [13], because in the presence of multi-particle thresholds up to arbitrarily high energies the singular points accumulate at $g = 0$ on curves with zero opening angle at $g = 0$.

5. In conclusion, the expansion of the self-energy in the σ -model provides the first analytic example of an OPE to all orders in m^2/p^2 in the non-trivial situation that each term in the expansion is multiplied by an infinite series in the coupling that contains UV and IR renormalons. We discussed the cancellation of IR renormalon ambiguities in this expansion. We found that the OPE is convergent in the $1/N$ -expansion and converges to the exact result without the need for additional power corrections.

In [14] an unexpected Λ^2/p^2 -correction (where Λ is the QCD scale) is reported in the OPE of the plaquette expectation value. The present analysis does not offer an explanation of this fact. One difference is that the lattice calculation is done at finite UV cut-off $\Lambda_{UV} = a^{-1}$ with p and Λ_{UV} being identified. One then obtains additional

power corrections from higher-dimension operators to the action. We can mimic this effect by adding the dimension-4 operator $C/\Lambda_{UV}^2 \cdot \sigma^2 \partial^2 \alpha$ to the σ -model action. The new vertex contributes an additional factor k^2/Λ_{UV}^2 to the integral (9). For $\Lambda_{UV} \sim p$ this results in an order 1 contribution to regions (i,iii) and an order m^6/p^6 contribution to region (ii). The analogous argument for Yang-Mills theory shows that one cannot obtain a Λ^2/p^2 -contribution in this way. The only loophole is that for $\Lambda_{UV} \sim p$ one should consider the infinite series of higher-dimension operators. If this series is divergent, this may induce further power correction terms in the lattice theory at finite cut-off not present in the continuum theory.

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Appendix. We collect explicit expressions for the coefficients in the asymptotic expansion of the self-energy in (14). For $n = 0$:

$$\begin{aligned} F_{\text{p}}^{(0)}[t] &= 1, \\ G_{\text{p}}^{(0)}[t] &= \frac{1}{t} + \frac{1}{t-1} - \psi(1+t) - \psi(2-t) - 2\gamma_E, \\ H_{\text{np}}^{(0)}[t] &= \frac{1}{t} + B_1(t). \end{aligned} \tag{20}$$

For $n = 1$:

$$\begin{aligned} F_{\text{p}}^{(1)}[t] &= t^2 - 1, \\ G_{\text{p}}^{(1)}[t] &= -\frac{1}{t} + 2t^2 - 4t + 1 + (1-t^2)[\psi(1+t) + \psi(2-t) + 2\gamma_E], \\ H_{\text{np}}^{(1)}[t] &= -\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t+1} - B_0(t)\right), \end{aligned} \tag{21}$$

where $\psi(x)$ is the digamma-function and B_0 and B_1 are scheme-dependent subtraction constants which are analytic in t :

$$\begin{aligned} B_0(t) &= 3J(t, 1), \quad B_1(t) = 6J(t, 2) - 6J(t, 3) + 6J(1+t, 3) - 7J(1+t, 2), \\ J(t, m) &= \frac{1}{t+1} {}_3F_2 \left[\begin{matrix} 1, m, t+1 \\ 1 + \frac{t}{2}, \frac{3}{2} + \frac{t}{2} \end{matrix} \middle| \frac{1}{4} \right] \end{aligned} \tag{22}$$

For generic $n \geq 2$ we obtain:

$$F_{\text{p}}^{(n)}[t] = \sum_{k=0}^n K_{\text{p}}(n-k, t) \frac{(n-k-1+t)_k^2}{k!k!},$$

$$H_{\text{np}}^{(n)}[t] = \sum_{k=0}^{n-2} K_{\text{np}}(n-k-2, t) \frac{(n-k-1)_k^2}{k!k!}, \quad (23)$$

$$G_{\text{p}}^{(n)}[t] = \sum_{k=0}^n K_{\text{p}}(n-k, t) \frac{(n-k-1+t)_k^2}{k!k!} [\psi(t) - \psi(1-t) + 2\psi(k+1) - 2\psi(n+t-1)]$$

where $(z)_k \equiv \Gamma(z+k)/\Gamma(z)$ and

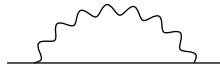
$$K_{\text{p}}(n, t) = \frac{1}{n!} \frac{\Gamma(2t+2n-1)}{\Gamma(2t+n-1)} \left[1 - \frac{2n}{2t+n-1} + \frac{n(n-1)}{(2t+n)(2t+n-1)} \right], \quad (24)$$

$$K_{\text{np}}(n, t) = \frac{\Gamma(-t, 1+t, 2n+3)}{\Gamma(t+n+1, 3+n-t)} \left[1 - 2\frac{n+2-t}{t+n+1} + \frac{(t-2-n)(t-1-n)}{(t+n+1)(t+n+2)} \right].$$

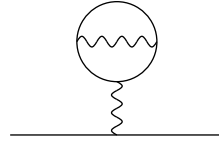
In the last formula we used a shorthand notation $\Gamma(a, b, c, \dots)$ for the product of Γ -functions with the respective arguments.

References

- [1] K.G. Wilson, Phys. Rev. **179** (1969) 1499.
- [2] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. **B147** (1979) 385.
- [3] For reviews see M. Beneke, [hep-ph/9807443]; V.I. Zakharov, [hep-ph/9802416].
- [4] G. 't Hooft, Phys.Rev. **D14** (1976) 3432; N. Andrei and D. Gross, Phys. Rev. **D18** (1978) 468.
- [5] G. Grunberg, [hep-ph/9705290] and *in* Proceedings of the International Europhysics Conference on High-Energy Physics (HEP 97), Jerusalem, Israel, 1997 [hep-ph/9711481] and [hep-ph/9807494].
- [6] R. Akhoury and V.I. Zakharov, *in* Proceedings of the 5th International Conference on Physics Beyond the Standard Model, Balholm, Norway, May 1997 [hep-ph/9705318].
- [7] P. Ball, M. Beneke and V.M. Braun, Nucl. Phys. **B452** (1995) 563.
- [8] R. Akhoury and V.I. Zakharov, [hep-ph/9710457].
- [9] F. David, Nucl. Phys. **B209** (1982) 433; *ibid.* **B234** (1984) 237; *ibid.* **B263** (1986) 637.
- [10] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Phys. Rep. **116** (1984) 104; Nucl. Phys. **B249** (1985) 445.
- [11] M.V. Terent'ev, Sov. J. Nucl. Phys. **45**(2) (1987) 368; Fortschr. Phys. **36** (1988) 117.
- [12] M.A. Shifman, *in* Continuous Advances in QCD 1994, A. Smilga (ed.), p. 249 (World Scientific, Singapore, 1994) [hep-ph/9405246].
- [13] G. 't Hooft, *in* *The Whys of subnuclear physics*, Proc. Int. School, Erice, Italy, 1977, ed. A. Zichichi (Plenum, New York, 1978).
- [14] G. Burgio, F. Di Renzo, G. Marchesini and E. Onofri, Phys. Lett. **B422** (1998) 219.



(a)



(b)