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ANALYSIS OF GENERIC INSERTIONS MADE OF TWO SYMMETRIC TRIPLETS

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Abstract

This paper reports on the study undertaken to explore the capabilities of a symmetric triplet to achieve the optics constraints required by the inner triplet of an insertion and more generally of a complete insertion made of two symmetric triplets to match a double focus to a FODO lattice. It is based on analytical treatment formulating a number of constraints equal to the parameters available. This thorough and systematic analysis made it possible to establish for an inner triplet as well as for a complete insertion the existence of solutions and to explicitely find out all the solutions, without resorting to unguided numerical searches. As a by-product, a lattice transformer, made of a single triplet, that matches two different FODO cells has been singled out and studied in details. The results should be profitable in a number of cases. Here, the method is applied to an insertion of the type of an experimental LHC insertion in order to investigate its domain of validity and tunability.

Geneva, Switzerland
13 May 1998

1 INTRODUCTION

The present analysis relies on an analytical treatment of a schematic thin lens model of generic insertions based on two symmetric triplets as described in Section 2 which are separated by a drift. This makes it possible to explore their capabilities, establish the existence or absence of solutions and unequivocally provide all the existing ones without using a numerical search. These solutions can easily be extended to the equivalent thick lens model. Such generic insertions can be a part of an accelerator ring, a fraction of a beam transfer line, a final focus telescope of a linear collider or an experimental low β insertion of a circular collider. In most cases, they have in common the function of transferring the beam from a given point of a FODO lattice to a double waist or focus, with in general different horizontal and vertical beam dimensions. The building blocks of these insertions are the symmetric triplets, which have been thoroughly studied in the framework of the design of an isochronous cell for their adaptability and flexibility [1, 2]. As a first step, the possibilities offered in this context by a single symmetric triplet are reviewed, making use of appropriately defined constraints on the β-functions such as the existence of secondary waists or of a crossing with equal and opposite slopes (Sections 2–4). It is obvious from this initial study that a generic insertion between a double waist and a FODO lattice, with its inherent geometrical and optical constraints, requires the use of two symmetrical triplets for covering a large enough validity domain. In this scheme, the first triplet near the double focus also referred as the Interaction Point, is often nicknamed the ‘inner triplet’, while the second is termed the ‘outer triplet’. A fully deterministic set of equations can then be established for the parameters of such an insertion by making use of constraints similar to those introduced for a single triplet (Section 5). In fact, the clue consists in asking the inner triplet to provide a condition of β-crossing and using the outer triplet as a ‘FODO transformer’ from β-crossing to β-crossing with different amplitudes. This lattice adapter is a useful by-product which can be profitable in other situations. Coming back to our case of interest, the two-step adjustment mentioned provides the required flexibility in the choice of the overall insertion length and in the distance between the two triplets.

In this report, a particular insertion with geometrical parameters as close as possible to those of an LHC (Version 4) experimental insertion is used to illustrate how to apply the present analysis to a specific case. The domain of existence of solutions is given and the tunability of the insertion in terms of β-amplitude at the Interaction Point is explored, for the particular conditions retained. Once this deterministic evaluation is done for thin lenses, it is shown that the extension to the thick lens model can be easily obtained by slowly increasing the quadrupole length (of the order of 5 cm per step) and using now the non simplified transfer matrices for the quadrupoles.

2 SINGLE SYMMETRIC TRIPLET

The study of a symmetric triplet in the framework of the design of an isochronous cell has suggested its application to the investigation of the global behaviour of an interaction point inner triplet when reduced to a symmetric triplet. It should be noted that the term symmetric applies to the geometry and not to the Twiss parameters. The symmetric triplet is the combination of a doublet and its mirror image. The doublet consists of a first drift space of length L_1 followed by a quadrupole of magnetic gradient G_2 and length l_{q2} followed by a second drift space of length L_3 which separates it from a second quadrupole of magnetic gradient G_4 and length l_{q4}. This lattice is shown in Fig. 1.

The full symmetry being too binding we have added a drift space of variable length l_5 and set $l_{q4} = l_{q2} = l_q$. The chosen model is represented in Fig. 2.

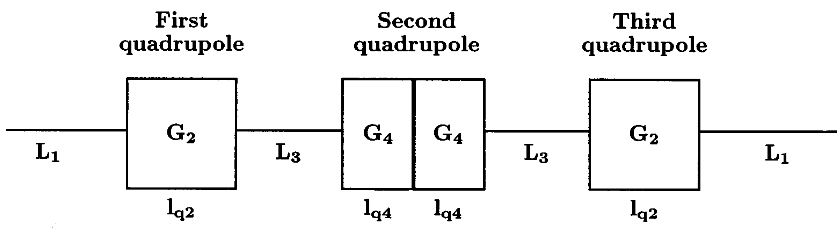

Figure 1: Schematic of a symmetric triplet.

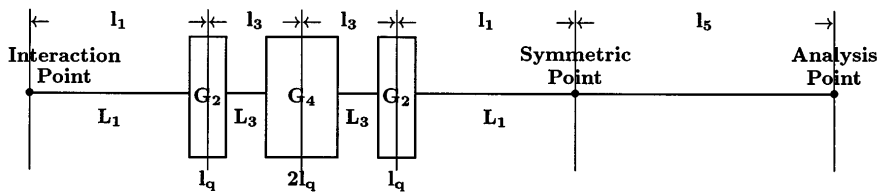

Figure 2: Inner triplet model.

As an example let us take the numerical values and constraints of a low-β insertion of LHC (Version 4). The quadrupole magnetic gradients G_2, G_4 must be less than 225 T/m. The distance L_1 is fixed to 23 m and l_q is equal to 5.5 m. The distance L_3 must be larger than 2 m. The Twiss parameters at the Interaction Point are the same in both planes and should be $\beta^* = 0.5$ m and $\alpha = 0$. At the operating energy of 7 TeV. The free parameters are the strength G_2, the distance L_3 and the ratio $k = G_4/G_2$.

The purpose of the study was to use the expressions derived in [2] to investigate the domains of these free parameters providing Twiss parameters in the Analysis Point which are of interest according to two different criteria. The Twiss parameters at the Analysis Point are related to those in the Symmetric Point (see Fig. 2) by the expressions:

$$\begin{aligned}
\beta_{x,a} &= \beta_{x,s} - 2\alpha_{x,s} l_5 + \gamma_{x,s} l_5^2 \\
\alpha_{x,a} &= \alpha_{x,s} - \gamma_{x,s} l_5 \\
\beta_{y,a} &= \beta_{y,s} - 2\alpha_{y,s} l_5 + \gamma_{y,s} l_5^2 \\
\alpha_{y,a} &= \alpha_{y,s} - \gamma_{y,s} l_5 \ .
\end{aligned} \tag{1}$$

The chosen criteria highlight outstanding features of the β-functions such as minima (waists) or intersections (crossings), which give clues on their overall behaviour (e.g. convergence or divergence, fast or slow slopes). From this analysis we get a feeling of the stability of a family of solutions and we may discover if we are confronted with chaotic sets of solutions.

The first criterion is to obtain the positions of the horizontal and vertical β functions waists for a canonical triplet, i.e. with $G_4 = -G_2 \quad (k = -1)$, the Analysis Point coinciding with the position of the horizontal waist. It is studied in Section 3.

The second criterion is to obtain a symmetric β-crossing ($\beta_x = \beta_y$ and $\alpha_x = -\alpha_y$) at the Analysis Point. In Section 4 we deal with its study based on the thin-lens approximation where the drifts L_1, L_3 are replaced by

$$l_1 = L_1 + l_q/2$$
$$l_3 = L_3 + 3l_q/2 \ .$$

In this approximation the expressions derived in [2] give the Twiss parameters $\beta_{x,s}$, $\alpha_{x,s}$, $\beta_{y,s}$, $\alpha_{y,s}$ at the Symmetric Point.

A set of FORTRAN programs have been written to compute $\beta_{x,a}, \alpha_{x,a}, \beta_{y,a}, \alpha_{y,a}$ as function of the variables G_2, L_3 and of the ratio k. Each one is explained in more details in the two following sections.

3 STUDY OF THE POSITION OF THE HORIZONTAL AND VERTICAL β FUNCTION WAISTS FOR A CANONICAL TRIPLET

In this study we assume the inner triplet to be canonical (i.e. $k = -1$). The requirement is the existence of an horizontal waist at a position called the Analysis Point and of a vertical waist at some other arbitrary point. The Analysis Point position can be easily obtained from the expression (1) where we put $\alpha_{x,a} = 0$:

$$l_5 = \alpha_{x,s}/\gamma_{x,s} \ . \tag{2}$$

The Analysis Point is to the right of the Symmetric Point if $\alpha_{x,s} > 0$. Its distance from the last quadrupole of the triplet is:

$$l_{w,x} = \frac{\alpha_{x,s}}{\gamma_{x,s}} + L_1 \ .$$

The Twiss parameters $\alpha_{x,s}$ and $\gamma_{x,s}$ are related to the value β^* at the Interaction Point by the following expressions where we have used the general transformation of the Twiss parameters through a beam transfer section starting with $\alpha_1 = 0$ and $\beta_1 = \beta^*$ [3]:

$$\alpha_{x,s} = -t_{x,11}(t_{x,21}\beta^* + \frac{t_{x,12}}{\beta^*}) \ ,$$
$$\gamma_{x,s} = t_{x,21}^2\beta^* + \frac{t_{x,11}^2}{\beta^*} \ ,$$

the quantities $t_{x,ij}$ being the elements of the Triplet Transfer Matrix in the horizontal plane which in the thin lens approximation become:

$$t_{x,11} = 1 - 2l_3 g_2^2(l_1 + l_3 - l_1 l_3 g_2)$$
$$t_{x,12} = 2(l_1 + l_3 - l_1 l_3 g_2)(1 + l_3 g_2 - l_1 l_3 g_2^2)$$
$$t_{x,21} = -2l_3 g_2^2(1 - l_3 g_2)$$

where

$$g_2[m^{-1}] = -0.3\frac{G_2[T/m]l_q[m]}{p[\text{GeV}/c]}$$

is the strength of the external quadrupole.

Similarly the distance of the vertical waist from the last quadrupole of the triplet can be obtained from the expressions (1) where we put $\alpha_{y,a} = 0$:

$$l_{w,y} = \frac{\alpha_{y,s}}{\gamma_{y,s}} + L_1 = l_{w,x} + \frac{\alpha_{y,s}}{\gamma_{y,s}} - \frac{\alpha_{x,s}}{\gamma_{x,s}} .$$

The two waist positions (referenced to the Interaction Point) $l_{w,x}$ and $l_{w,y}$ depend on the two parameters L_3 and G_2. A program has been written to explore the values taken by $l_{w,x}$ and $l_{w,y}$ when varying L_3 and G_2 respectively. In the LHC application the domains of variations are 2–12 m and 160–225 T/m. The main result is that the maximum value β_{max} taken by the β-functions inside a canonical triplet cannot be less than 5280 m. In this limiting case there is only one triplet with $G_2 = 224.35$ T/m and $L_3 = 3.25$ m. The waist positions are $l_{w,x} = 1522.9$ m and $l_{w,y} = 26.5$ m. The number of possible cases increase by accepting a larger β_{max}. At the same time the horizontal and vertical waist positions decrease. As an example, Figs. 3 and 4 show $l_{w,x}$ and $l_{w,y}$ when we impose β_{max} to be bounded by 5500 m. It is interesting to observe that the minimum distance of the horizontal waist from the last quadrupole of the canonical triplet is now 410 m. For reasonable values of β_{max} these figures show that β_x outside the canonical triplet drops down with a slope which is too small. At the same time β_y is 18 m. As a consequence β_y rises again and takes very high values at the position of the horizontal waist. Clearly these behaviours of the β-functions cannot be accepted and thus we may conclude that a canonical triplet cannot meet the requirements of the LHC application.

Extending the range of the ratio k, it is interesting to observe that its existence domain for given constraints is quite limited and always less than -1. Figure 5 shows an example where we have imposed that the largest admitted value of the β-functions inside the triplet should not exceed 5000 m and that the distances of the horizontal and vertical waists from the last quadrupole should be less than 200 m. Neglecting scattered points which indicate that the corresponding behaviour of the β-functions is very sensitive to quite small variations of the ratio k around the displayed value, it is important to notice that the largest range of the ratio k for which the given constraints are satisfied is of the order of 5%. Figures 6 and 7 show the positions of the horizontal and vertical waists of the β-functions for one value of k(k $= -0.88$) while still keeping the β maximum inside the symmetric triplet below 5000 m. One can easily observe that the positions of the two waists are much less distant than for the canonical triplet and that it is possible to find symmetric triplet configurations for which the waist positions are less than 200 m from the last quadrupole. We may conclude that, in the given constraints of an LHC-type experimental insertion, the inner triplet can be symmetric but with the gradient of the central quadrupole weaker in absolute value than the gradient of the external quadrupoles.

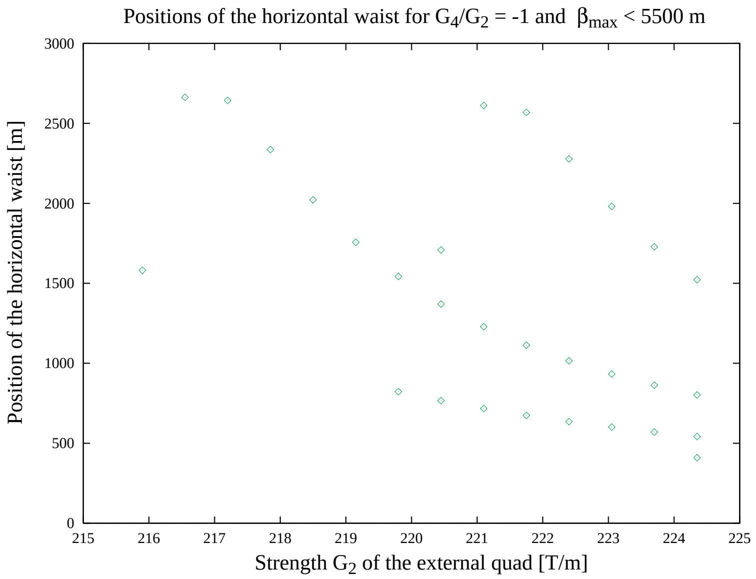

Figure 3: Distance of the horizontal waist from the last quadrupole of a canonical triplet. Each point corresponds to different values of the parameters G_2 and of L_3 satisfying the constraint $\beta_{max} < 5500$ m.

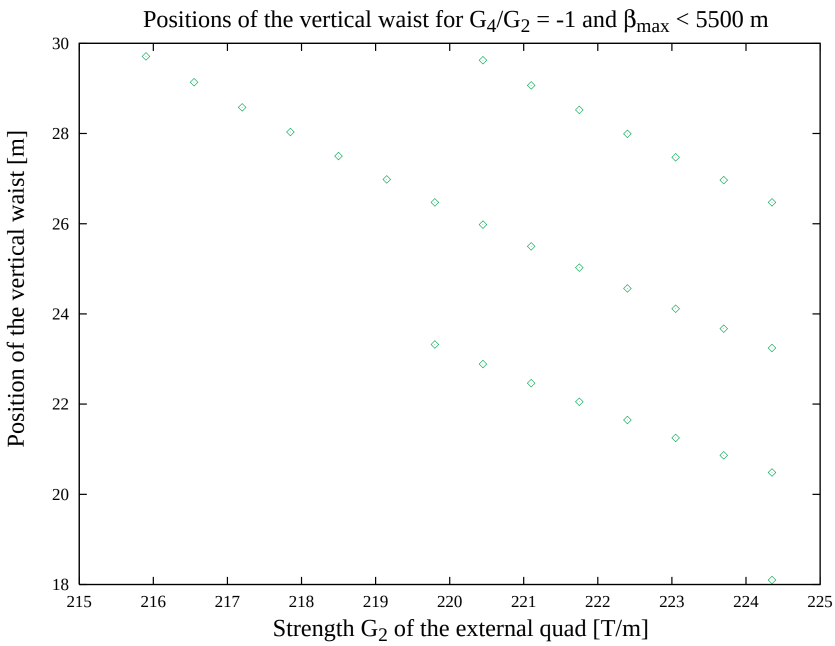

Figure 4: Distance of the vertical waist from the last quadrupole of a canonical triplet. Each point corresponds to different values of the parameters G_2 and of L_3 satisfying the constraint $\beta_{max} < 5500$ m.

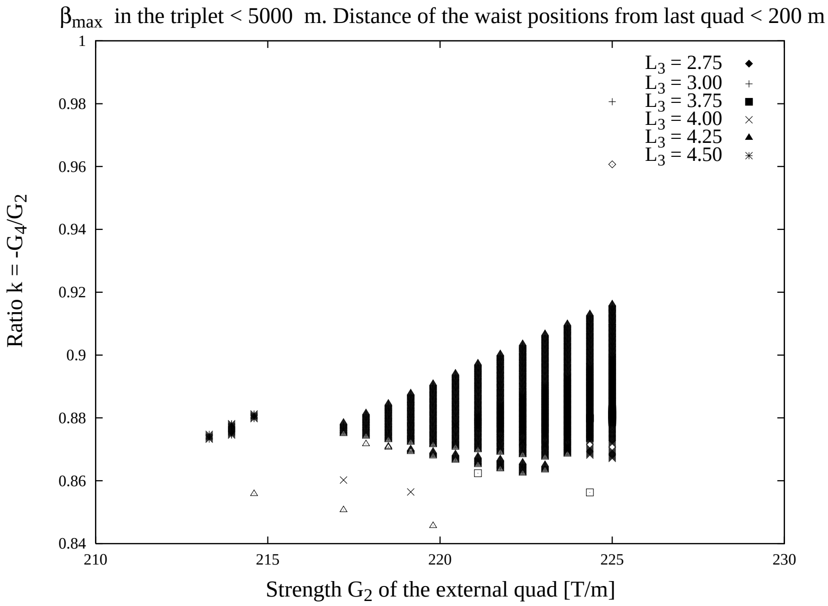

Figure 5: Existence domain of $k = G_4/G_2$ satisfying given constraints.

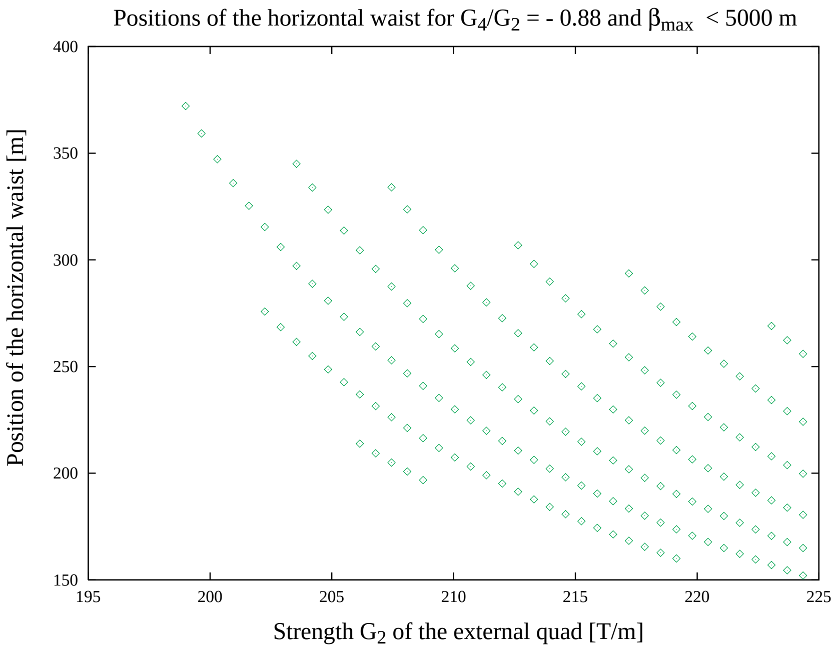

Figure 6: Distances of the horizontal waist from the last quadrupole of the symmetrical triplet for $k = -0.88$. Each point corresponds to different values of the parameters G_2 and of L_3 satisfying the constraint $\beta_{max} < 5000$ m.

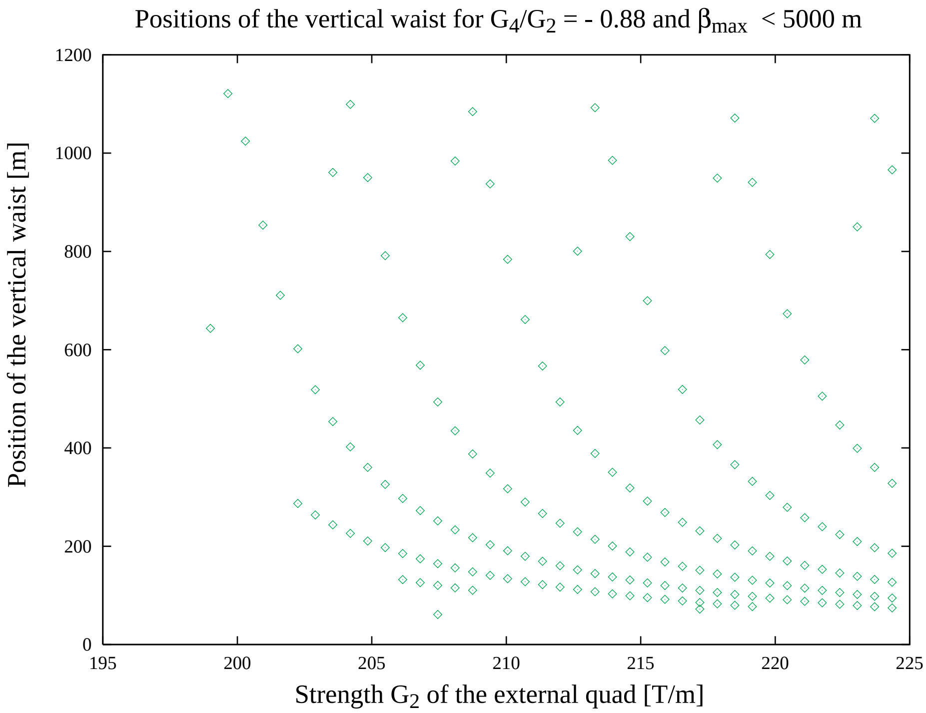

Figure 7: Distances of the vertical waist from the last quadrupole of the symmetrical triplet for $k = -0.88$. Each point corresponds to different values of the parameters G_2 and of L_3 satisfying the constraint $\beta_{max} < 5000$ m.

4 STUDY OF THE SYMMETRIC CROSSING OF THE HORIZONTAL AND VERTICAL β FUNCTIONS

Retaining the lesson of the previous section we would not treat anymore in this report the canonical triplet. We will enlarge our analysis to a symmetric triplet with a predetermined value of L_1 for which analytical tools are available in the thin lens approximation. Three free parameters specify the symmetric triplet: l_3, g_2 and the ratio $k = g_4/g_2$. It is difficult to get a clear insight of the behaviour of the horizontal and vertical β functions by exploring the full space of these parameters. Experience has shown that valuable information can be obtained by studying the properties of the special features of these two functions such as the waists and the crossing with opposite slopes. At the end of the previous section we have shown some results of the analysis of the two waists. In this section we intend to deal with the crossing. We introduce a further simplification by imposing a symmetric crossing, i.e. not only opposite slopes but also equal in absolute value. This choice is suggested by the analogy with the FODO structure and it has the advantage to reduce the range of the ratio k to only two values as we will prove shortly. Moreover it allows for an easy matching to another symmetrical triplet as it will be shown in the next section. Thus we assume that the β-functions will cross with equal and opposite slopes at a position we call Crossing Point (a more specific name for the Analysis

Point defined in Fig. 2). By definition the Twiss parameters at the Crossing Point are:

$$\begin{aligned} \beta_{x,c} &= \beta_{y,c} = \beta_c \\ \alpha_{x,c} &= -\alpha_{y,c} = \alpha_c \; . \end{aligned} \tag{3}$$

We need the expressions for the distance l_5 of the Crossing Point from the Symmetric Point, for β_c and α_c in terms of the Twiss parameters at the Symmetric Point in order to investigate their behaviour depending on the variation of the free parameters, the motivation being that such β functions can be relatively easily matched to a standard FODO lattice. We obtain from the expressions (1):

$$\begin{aligned} \alpha_c &= \alpha_{x,s} - \gamma_{x,s} l_5 \\ -\alpha_c &= \alpha_{y,s} - \gamma_{y,s} l_5 \; . \end{aligned} \tag{4}$$

It follows from the expressions (3):

$$\gamma_{x,c} = \gamma_{y,c} = \gamma_c \; .$$

Observing that a drift does not modify γ, we have also:

$$\gamma_{x,s} = \gamma_{y,s} = \gamma_s \; . \tag{5}$$

From the expressions (4) we obtain:

$$\begin{aligned} l_5 &= \frac{\alpha_{x,s} + \alpha_{y,s}}{2\gamma_s} \\ \alpha_c &= \frac{\alpha_{x,s} - \alpha_{y,s}}{2} \; . \end{aligned} \tag{6}$$

The expressions (1) become:

$$\begin{aligned} \beta_c &= \beta_{x,s} - 2\alpha_{x,s} l_5 + \gamma_s l_5^2 \\ \beta_c &= \beta_{y,s} - 2\alpha_{y,s} l_5 + \gamma_s l_5^2 \; . \end{aligned}$$

Adding them and using expression (6) we get

$$\beta_c = \frac{1}{2}[\beta_{x,s} + \beta_{y,s} - l_5(\alpha_{x,s} + \alpha_{y,s})] \; . \tag{7}$$

The expression (5) imposes a constraint on the initial free parameters which are thus reduced from three to two. Actually in Appendix A it is shown that k is a solution of the following equation, when it exists:

$$k^2(a_2 g_2^4 + b_2 g_2^2) + k(a_1 g_2^4 + b_1 g_2^2 + c_1) + b_0 g_2^2 + c_0 = 0 \tag{8}$$

where a_2,b_2,a_1,b_1,c_1,b_0,c_0 are functions of l_1,l_3 and β^*. Both solutions of (8) when they exist are negative as one would expect. In the following discussion we will refer to the smallest or largest root in the real sense, i.e. including the sign.

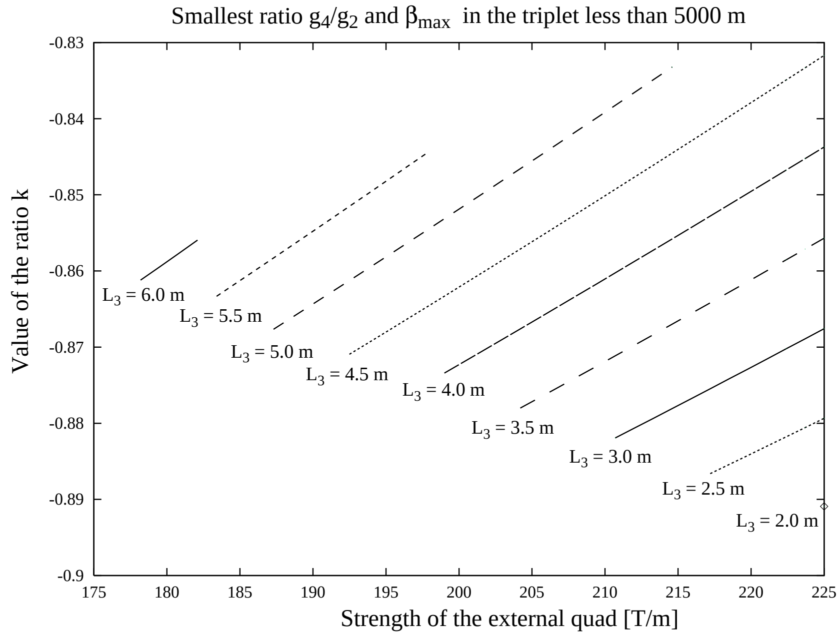

Figure 8: Smallest solution of the second order equation (8).

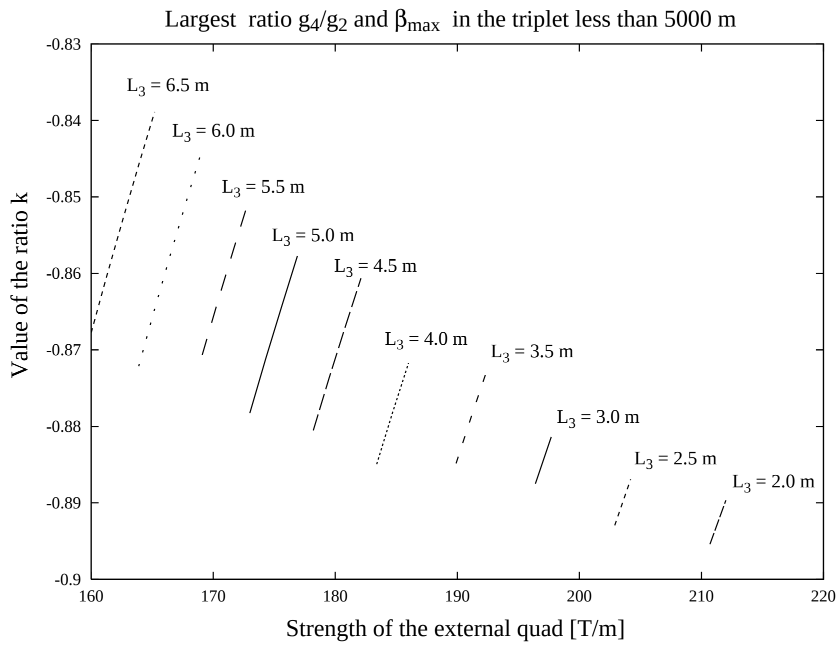

Figure 9: Largest solution of the second order equation (8).

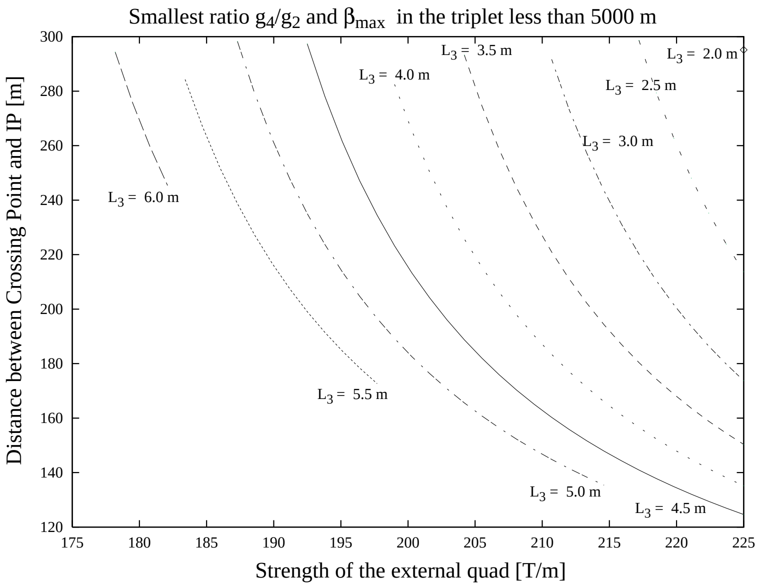

Figure 10: Distance from the Crossing Point to the IP for the smallest solution of the second order equation (8).

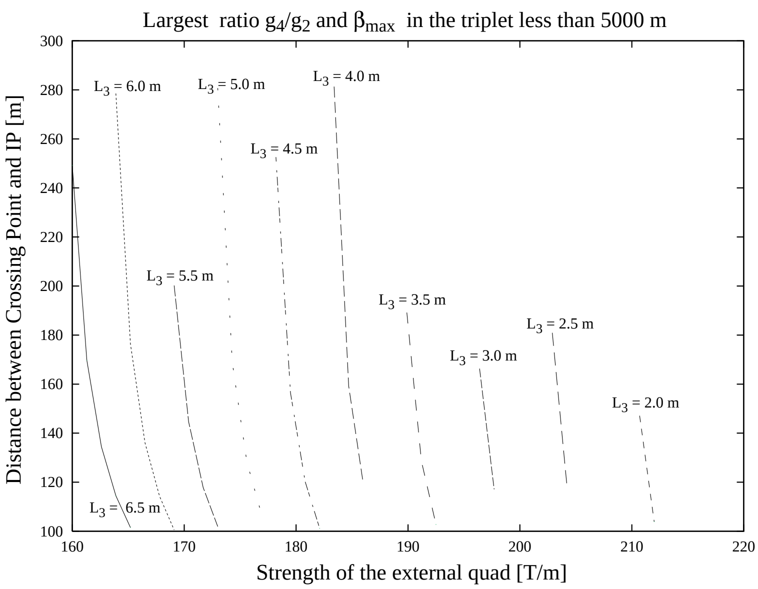

Figure 11: Distance from the Crossing Point to the IP for the largest solution of the second order equation (8).

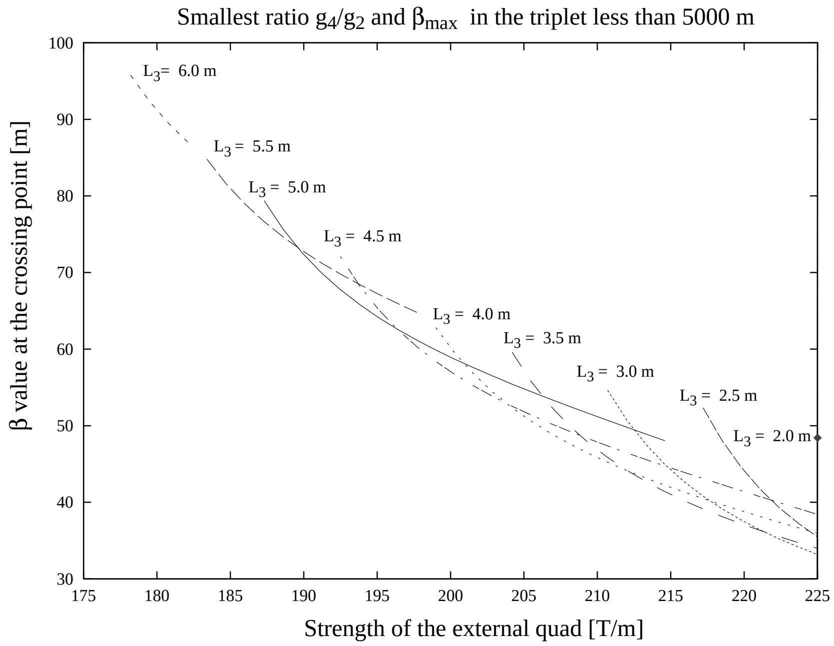

Figure 12: Value of the beta function at the Crossing Point for the smallest solution of the second order equation (8).

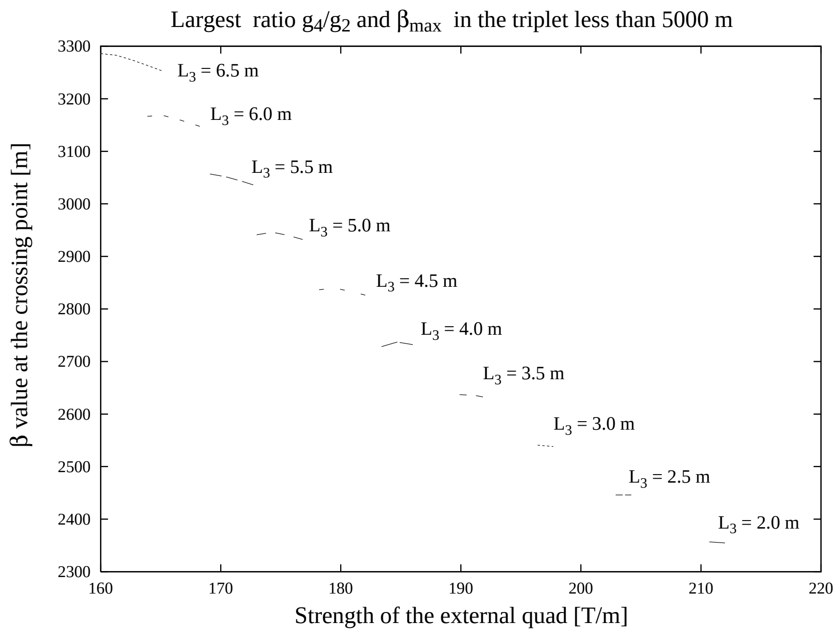

Figure 13: Value of the beta function at the Crossing Point for the largest solution of the second order equation (8).

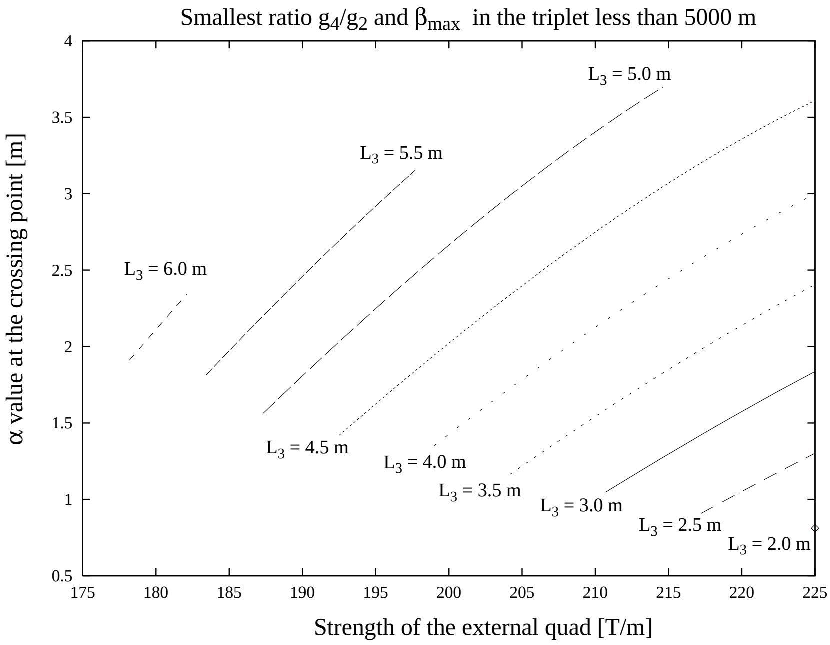

Figure 14: Value of the alpha function at the Crossing Point for the smallest solution of the second order equation (8).

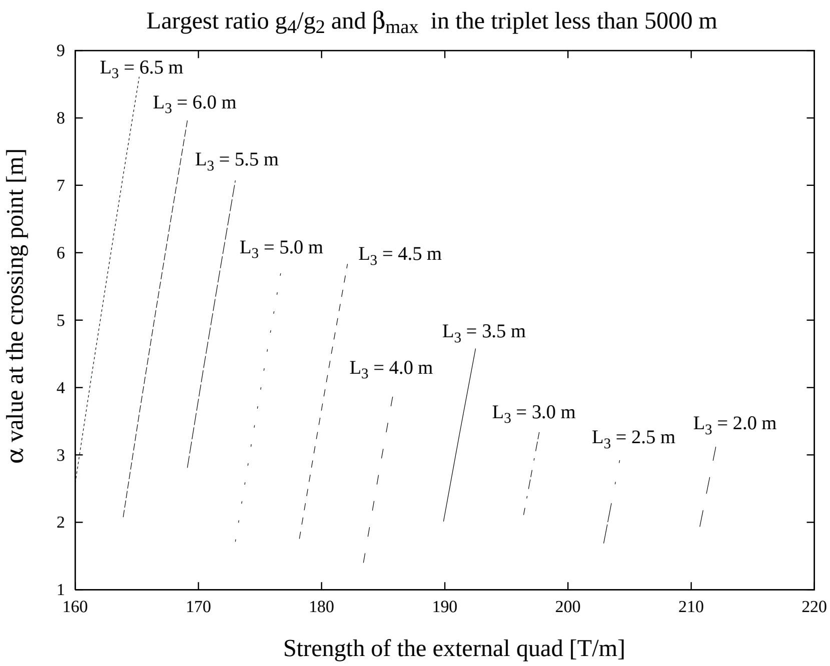

Figure 15: Value of the alpha function at the Crossing Point for the largest solution of the second order equation (8).

A program has been written to explore the values taken by k, $l_c = l_5 + 2(l_1 + l_3)$, β_c and α_c when varying L_3 and G_2 respectively. In the LHC application the domains of variations are 2–8 m and 160–225 T/m. The results are reported in Figs. 8 to 15, where only values are displayed for which the maximum value taken by the β-functions inside the triplet is less than 5000 m. Figures 11, 13 and 15 show that the largest solution of the second order equation (8) is to be avoided because it is associated with very high values of the alpha and beta functions at the Crossing Point. The corresponding distance from the IP is very sensitive to small changes of the gradient G_2.

It is interesting to have a feeling of how l_c, β_c and α_c behave when we vary the value β^* of the β-function at IP. We have neglected the case of the largest solution for the ratio k because the tuning range is very small when it exist (between 0.5 m and about 4 m) as one would expect from the analysis of Figs. 11, 13 and 15. To reduce the number of figures we have chosen three values of the gradient G_2: 180, 200, 220 T/m. Figures 16 to 18, 19 to 21, 22 to 24 show l_c, β_c and α_c for $G_2 = 180$ T/m, $G_2 = 200$ T/m and $G_2 = 220$ T/m respectively. We remark that the tuning range extends up to 20 m. This observation was the main motivation for extending the study to a lattice composed of two symmetric triplets. It will be presented in the next section.

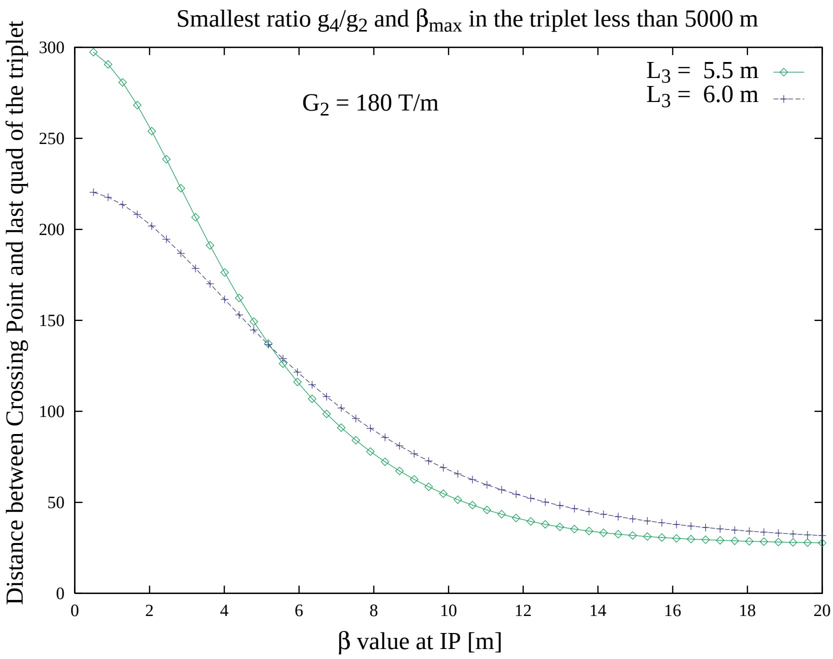

Figure 16: Tuning range of the distance of the Crossing Point from the last quadrupole of the symmetric triplet for $G_2 = 180$ T/m. We have taken the smallest solution of (8).

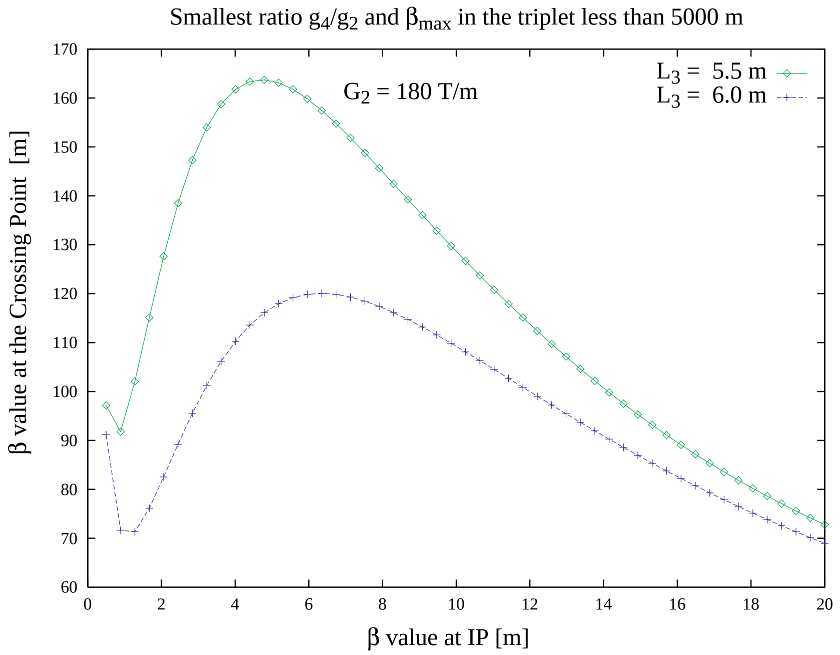

Figure 17: Tuning range of the β-function at the Crossing Point for $G_2 = 180$ T/m. We have taken the smallest solution of (8).

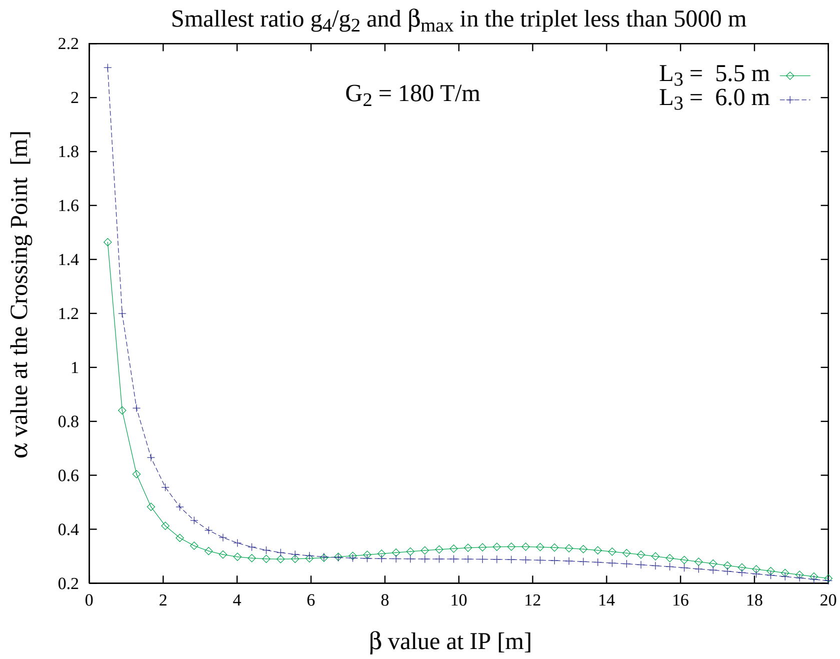

Figure 18: Tuning range of the α-function at the Crossing Point for $G_2 = 180$ T/m. We have taken the smallest solution of (8).

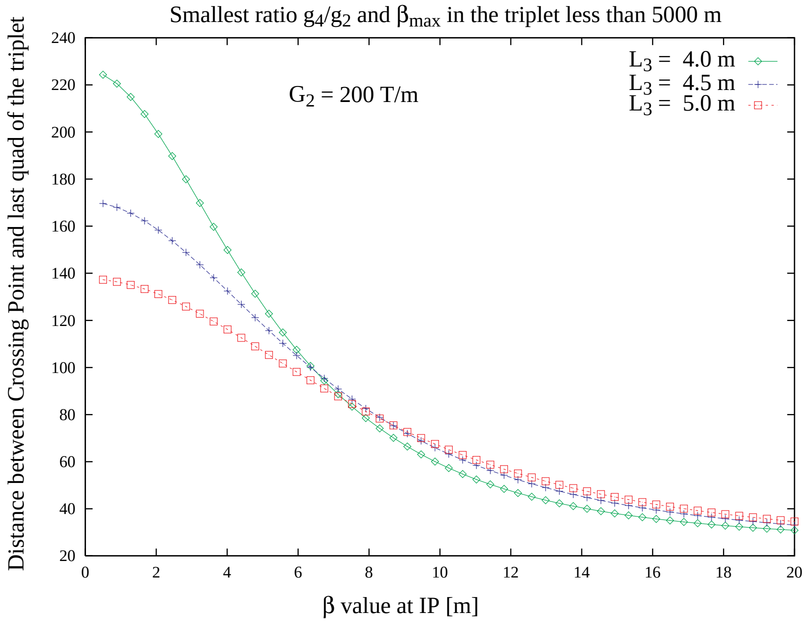

Figure 19: Tuning range of the distance of the Crossing Point from the last quadrupole of the symmetric triplet for $G_2 = 200$ T/m. We have taken the smallest solution of (8).

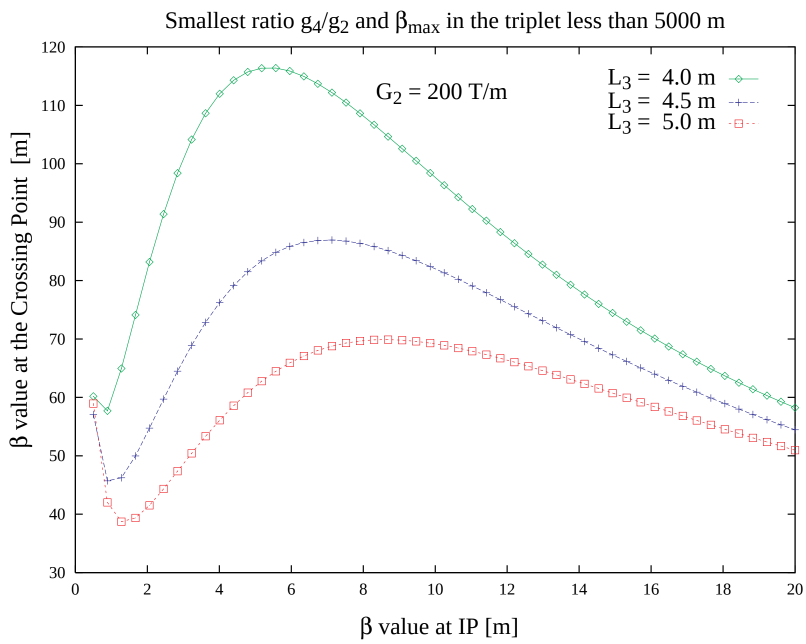

Figure 20: Tuning range of the β-function at the Crossing Point for $G_2 = 200$ T/m. We have taken the smallest solution of (8).

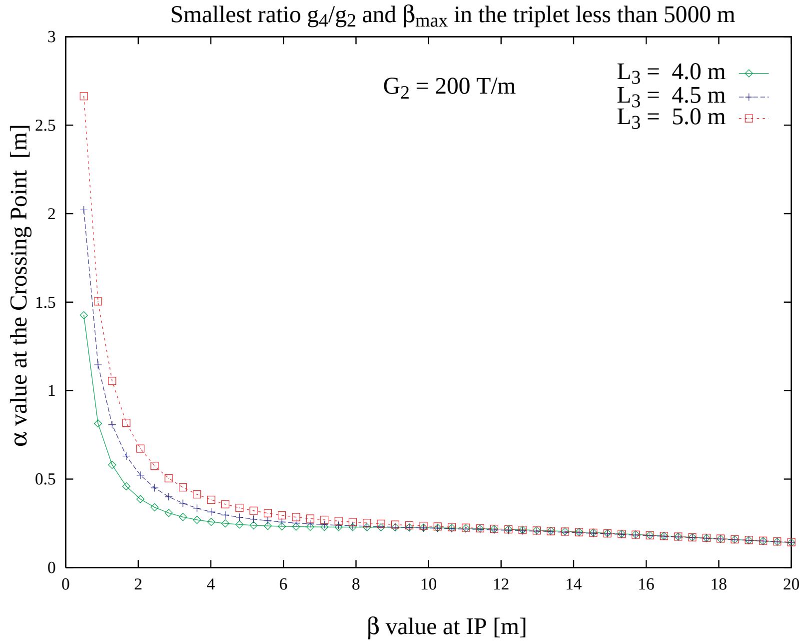

Figure 21: Tuning range of the α-function at the Crossing Point for $G_2 = 200$ T/m. We have taken the smallest solution of (8).

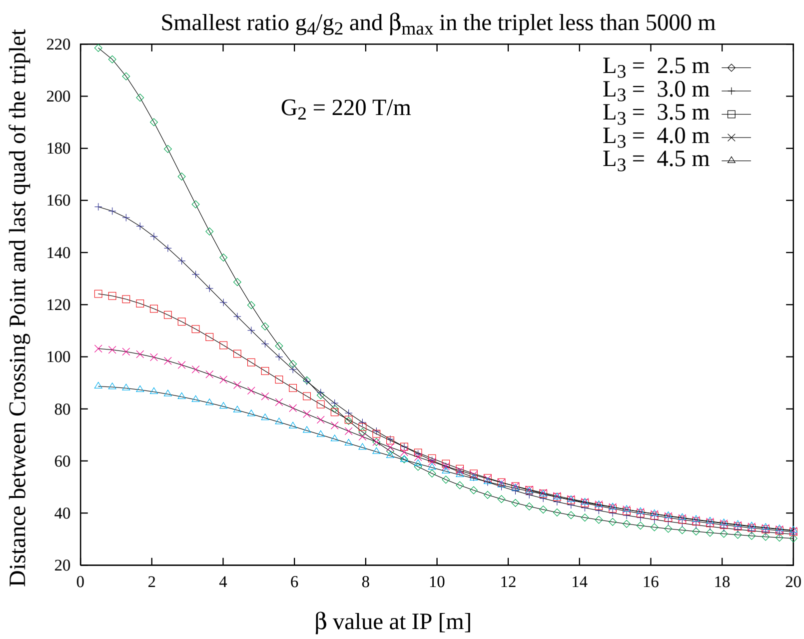

Figure 22: Tuning range of the distance of the Crossing Point from the last quadrupole of the symmetric triplet for $G_2 = 220$ T/m. We have taken the smallest solution of (8).

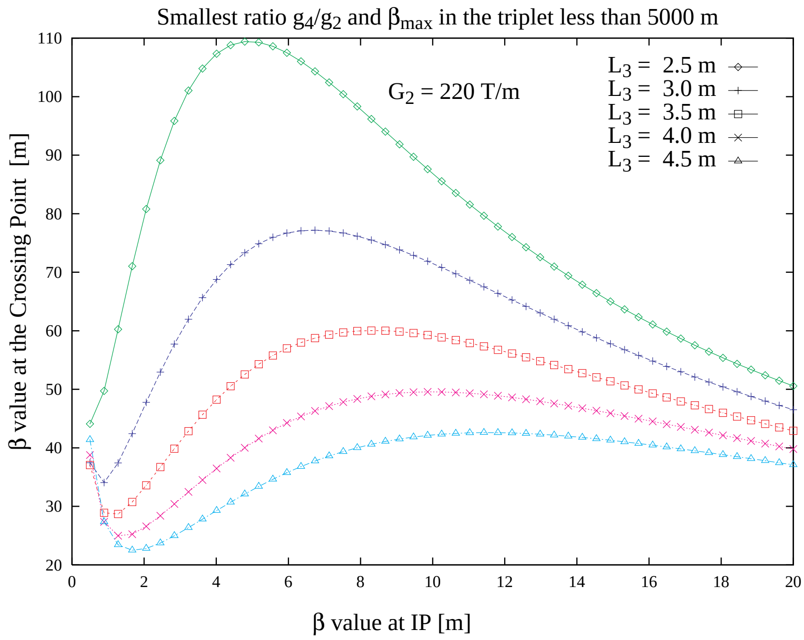

Figure 23: Tuning range of the β-function at the Crossing Point for G_2 = 220 T/m. We have taken the smallest solution of (8).

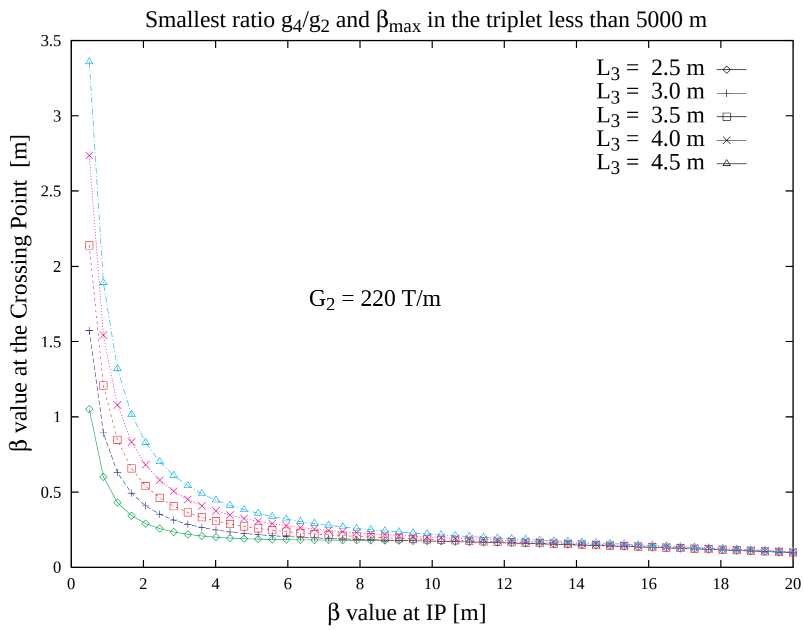

Figure 24: Tuning range of the α-function at the Crossing Point for G_2 = 220 T/m. We have taken the smallest solution of (8).

5 STUDY OF AN INSERTION, WHICH MATCHES A DOUBLE WAIST OF A FODO LATTICE

In the previous section we stated that one motivation for considering symmetric crossing was that it eased the matching to a standard FODO lattice. In this section we will show how this can be done. Let us consider the insertion shown in Fig. 25 which is composed of two symmetric triplets (inner and outer) separated by a drift l_x. The inner triplet has the role studied in the previous section, transforming the double waist at the IP into a symmetric $\beta-$crossing (at the Crossing Point in Fig. 25). The outer triplet matches from this symmetric $\beta-$crossing to the following symmetric crossing corresponding to the FODO mid-cell Point.

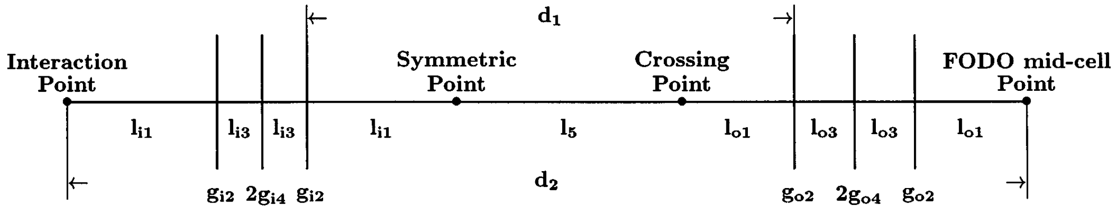

Figure 25: Two triplets model.

By definition we have at the Crossing Point:

$$\begin{aligned} \beta_{c,x} &= \beta_{c,y} = \beta_c \\ \alpha_{c,x} &= -\alpha_{c,y} = \alpha_c \ . \end{aligned} \tag{9}$$

And at the FODO mid-cell Point:

$$\begin{aligned} \beta_{FODO,x} &= \beta_{FODO,y} \\ \alpha_{FODO,x} &= -\alpha_{FODO,y} \ . \end{aligned} \tag{10}$$

In the previous section we have seen that there are in principle two inner triplets which satisfy the expressions 9 depending on the two parameters l_3 and g_2. Actually we discarded one of them for practical reasons.

Using the general transformation of the Twiss parameters through a beam transfer section starting with $\alpha_1 = 0$ and $\beta_1 = \beta^*$ [3], we get:

$$\begin{aligned} \alpha_{x,s} &= -t_{x,11}(\beta^* t_{x,21} + t_{x,12}/\beta^*) \\ \beta_{x,s} &= \beta^* t_{x,11}^2 + t_{x,12}^2/\beta^* \\ \alpha_{y,s} &= -t_{y,11}(\beta^* t_{y,21} + t_{y,12}/\beta^*) \\ \beta_{y,s} &= \beta^* t_{y,11}^2 + t_{y,12}^2/\beta^* \\ \gamma_s = \gamma_{x,s} = \gamma_{y,s} &= t_{x,21}^2 \beta^* + \frac{t_{x,11}^2}{\beta^*} \ . \end{aligned}$$

Replacing these expressions in (6) and (7) we obtain the following expressions for α_c,β_c and l_c:

$$\begin{aligned}
\alpha_c &= \frac{1}{2}[(t_{y,11}t_{y,21} - t_{x,11}t_{x,21})\beta^* + (t_{y,11}t_{y,12} - t_{x,11}t_{x,12})/\beta^*] \\
\beta_c &= \frac{4 + [(t_{y,11}t_{y,21} - t_{x,11}t_{x,21})\beta^* + (t_{y,11}t_{y,12} - t_{x,11}t_{x,12})/\beta^*]^2]}{t_{x,21}^2\beta_* + t_{x,11}^2/\beta^*} \\
l_c &= -\frac{\beta^{*2}(t_{x,11}t_{x,21} + t_{y,11}t_{y,21}) + (t_{x,11}t_{x,12} + t_{y,11}t_{y,12})}{\beta^{*2}t_{x,21}^2 + t_{x,11}^2} + 2(l_1 + l_3) \ .
\end{aligned}$$

Eventually l_5, α_c and β_c depend upon g_2 and l_3.

In Appendix B it is shown that we can obtain analytic expressions for a symmetric triplet which matches the values of the β-functions of (9) to those of (10). It is also proved that there are always two solutions if we do not impose any condition on the phase advances. Thus their parameters are also functions of l_3 and g_2. The whole insertion depends on these two free variables which can be determined by imposing two supplementary conditions. In the following development we have chosen to impose the values for the distance d_1 between the inner and the outer triplets and the overall insertion length d_2:

$$\begin{aligned}
d_1 &= l_{i1} + l_5 + l_{o1} \\
d_2 &= 2(l_{i1} + l_{i3} + l_{o1} + l_{o3}) + l_5
\end{aligned}$$

because they are generally fixed by the geometry (even if more loosely for d_1).

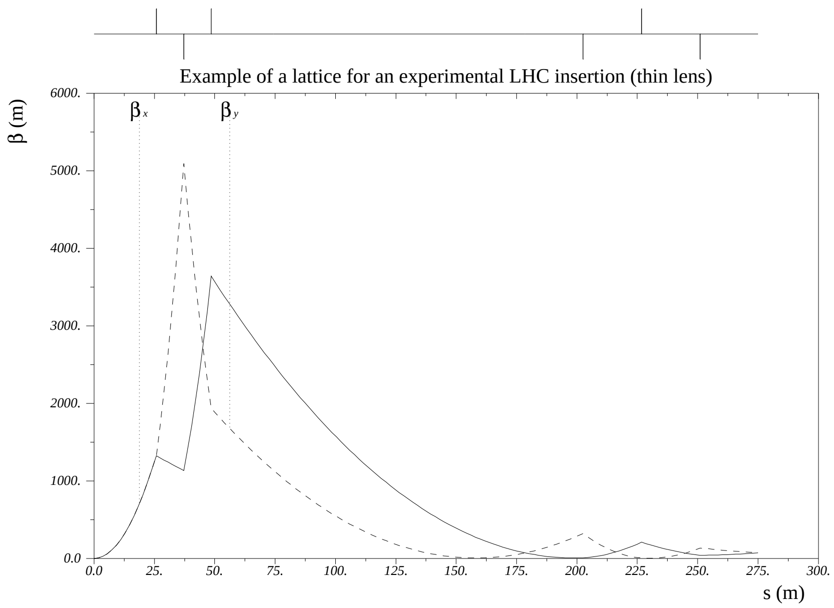

Figure 26: Horizontal and vertical β-functions of an insertion matching the Interaction Point to the Mid-cell Point with $\beta = 72$ m and $\alpha = -1$ (thin lens approximation).

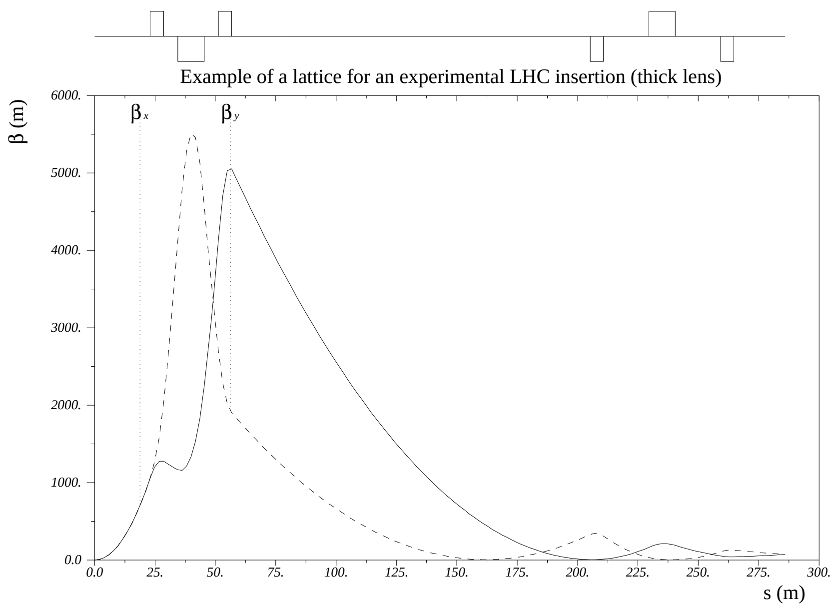

Figure 27: Horizontal and vertical β-functions of an insertion matching the Interaction Point to the Mid-cell Point with $\beta = 72$ m and $\alpha = -1$ (thick lens model).

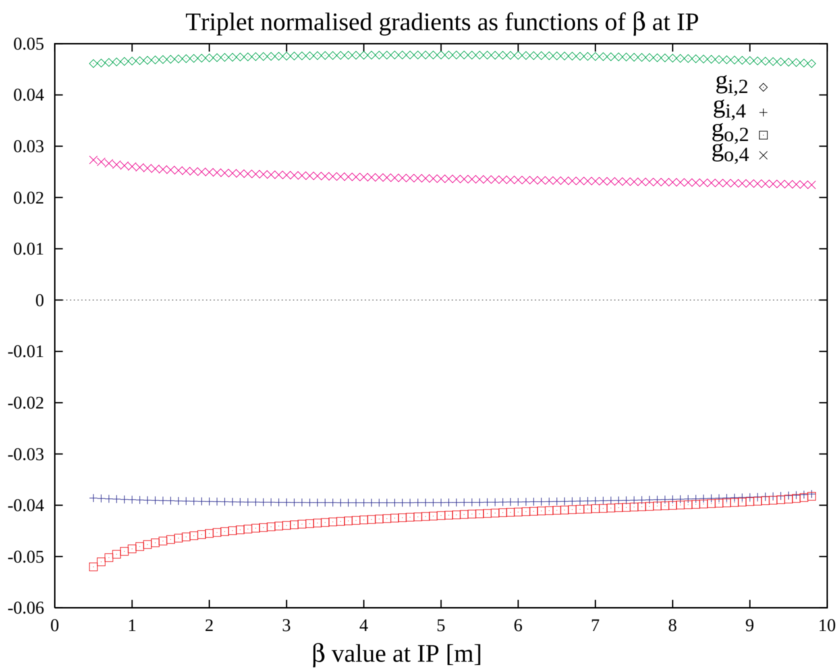

Figure 28: Normalized gradients of the two triplets as function of the β-function value at the Interaction Point ($\beta^* = 0.5$–9.8 m).

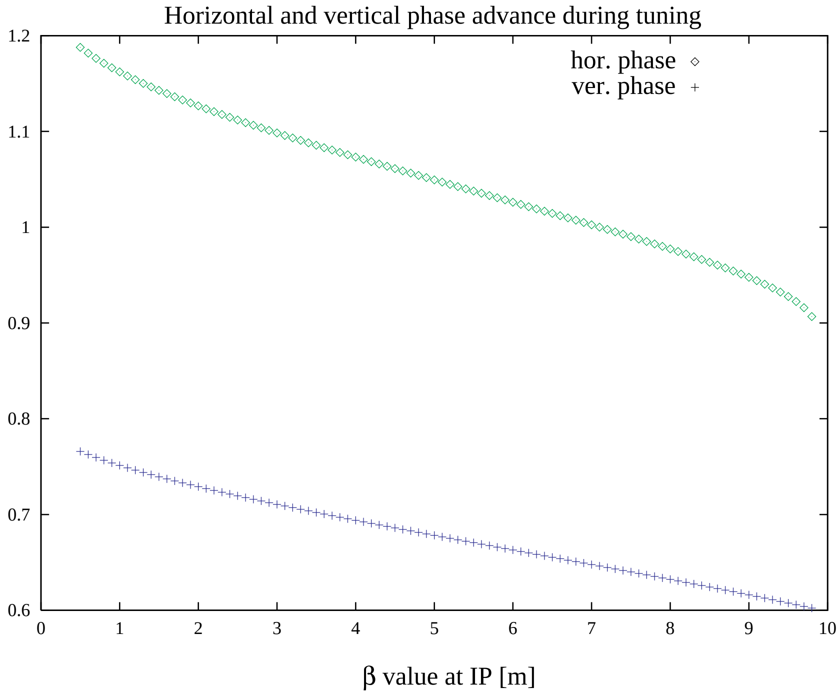

Figure 29: Horizontal and vertical phase advances as function of the β-function value at the Interaction Point ($\beta^* = 0.5$–9.8 m).

The routine DSNLEQ of the standard CERN library is able to compute the parameters of the two triplets and the drift length l_5 which satisfy these constraints if possible. A further constraint is given by the maximum acceptable value of the β-function inside the inner triplet which can be respected by replacing the condition of strict equality for d_1 with a condition on its range only. The following two figures show an application to a schematic configuration of one LHC (Version 4) low-β insertion. The distance between the Interaction Point and the Mid-cell Point of the FODO cell is 275 m while the distance between the last quadrupole of the inner triplet and the first quadrupole of the outer triplet is 154 m in the thin lens approximation. The values of the β-function and of the α-function at the Mid-cell Point have been selected equal to 72 m and -1 m respectively. Figure 26 shows the thin lens approximation while Fig. 27 shows the extension to the thick lens case. A program has also been written to study the behaviour of the quadrupole strengths when the value of the β-function at the Interaction Point increases ('tuning'). Figure 28 shows that their change are quite smooth for values of the β-function at IP up to very near 10 m. Figure 29 shows the corresponding variation of the horizontal and vertical phase advances.

6 CONCLUSIONS

The first result concerns the inner triplet of an insertion, which cannot be canonical because it does not provide enough focussing of the β-functions (see Fig. 3). It can also be proved that no symmetric crossing can be obtained with this kind of triplet if we want to satisfy the constraints imposed by the limitations on the gradient and on the maximum value of the β-function inside the triplet.

Thus the gradient of the outer quadrupole must be about 10–15% stronger than that of the centre one. A second result is that a symmetric inner triplet can indeed achieve waists in the horizontal and vertical planes at acceptable distances but the validity domains of l_3 and G_2 are small and at the limit of the largest acceptable β_x inside the triplet (see Fig. 4). More generally there is a high sensitivity of the insertion to quadrupole field variations when l_1 is large as in the case of the LHC low-β insertion. Relative variations of order of few percents modify considerably the positions of the waists and the slopes at the crossing. In such application an additional quadrupole may be required to give the extra free parameters which can be used to enlarge the acceptable domain of the triplet parameters.

The third result is that symmetric crossing is more robust with respect to variations of the triplet parameters and has the advantage to match quite easily to a standard FODO lattice.

All these results bring us to an insertion based on two symmetric triplets, the inner triplet being used to generate a β-function symmetric crossing. The final matching between this crossing and the crossing associated with the FODO lattice which follows is ensured by the outer symmetric triplet. Three conditions can be imposed simultaneously, keeping fixed the drift preceding the inner triplet, the total length of the insertion and the distance between the triplets in a given range. They provide a set of equations with an equal number of parameters which although non linear can be solved in a deterministic way, avoiding the use of numerical matching. The outer triplet is a convenient specific sub-system which allows to match two FODO cells which differ by their β-function, i.e. also phase advance and cell length, and which can certainly be used in other applications.

Explicit solutions are given for an insertion of LHC type. At first inner and outer triplet parameters are determined as explained for a fixed drift in front of the inner triplet, a given insertion total length, a given distance between the last inner triplet quadrupole and the first outer triplet quadrupole, and β^*-amplitude equal to the nominal value in collision at maximum energy. They typically match the β-crossing of the FODO cell of the dispersion suppressor. Once this is done, the geometry is frozen and only the gradients of both symmetric triplets are varied in order to match detuned β^* values to the same dispersion suppressor cell. The triplet gradients could be adjusted for a detuning reaching about a factor 20, the phase advances of course changing accordingly. Detuning is bounded by the mentioned constraints and by the limitations on the maximum value of the β-function inside the triplet. The corresponding thick lens model can be easily obtained by numerical adjustment to finite quadrupole lengths. This model can be refined to include control on the maximum value of the β-functions inside the inner triplet, on the tunability range and on the phase advance changes during tuning by relaxing the two conditions on the distance between the triplets and on the overall length. Hence it is demonstrated that the proposed method makes it possible to determine unequivocally inside the complete given parameter space the existing solutions of a realistic lattice for a tunable LHC-type experimental insertion, only based on symmetric triplets.

Acknowledgements

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References

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[2] T.E. d'Amico, 'General treatment of a matching quadrupole triplet which is symmetric around its median plane', CLIC Note 322, January 1997.

[3] Philip J. Bryant and Kjell Johnsen, 'The Principles of Circular Accelerators and Storage Rings', Cambridge University Press, 1993.

APPENDIX A: RATIO $k = g_4/g_2$ OF A SYMMETRIC TRIPLET MATCHING A DOUBLE WAIST TO A SYMMETRIC POINT WITH $\gamma_x = \gamma_y$

The condition $\gamma_{x,s} = \gamma_{y,s}$ at the exit of a symmetric triplet implies a relation between the parameters k, g_2 and l_3. It is the purpose of this appendix to derive it. Let us assume that the beam at the entrance of the symmetric triplet is a double waist with β^* being the common value of the two β-functions. There is no lack of generality in making this assumption which greatly simplifies the computations. The form of the final expression remains the same and only the coefficients are more complex. By making explicit the dependence of the γ-functions on the transfer matrices of each plane and on the value β^*, we may write using the general transformation of the Twiss parameters through a beam transfer section starting with $\alpha_1 = 0$ and $\beta_1 = \beta^*$ [3]:

$$t_{x,21}^2\beta^* + \frac{t_{x,11}^2}{\beta^*} = t_{y,21}^2\beta^* + \frac{t_{y,11}^2}{\beta^*}$$

or

$$(t_{x,21}^2 - t_{y,21}^2)\beta^{*2} = (t_{y,11}^2 - t_{x,11}^2) \ . \tag{A.1}$$

Let us obtain the following quantities [2]:

$$\begin{aligned}
t_{x,21} - t_{y,21} &= 2(1 + l_3g_2)(g_2 + g_4 + l_3g_2g_4) - 2(1 - l_3g_2)(-g_2 - g_4 + l_3g_2g_4) \\
&= 4(g_2 + g_4) + 4l_3^2g_2^2g_4 \\
&= 4g_2[(1 + l_3^2g_2^2)k + 1]
\end{aligned}$$

$$\begin{aligned}
t_{x,21} + t_{y,21} &= 2(1 + l_3g_2)(g_2 + g_4 + l_3g_2g_4) + 2(1 - l_3g_2)(-g_2 - g_4 + l_3g_2g_4) \\
&= 4l_3g_2(g_2 + 2g_4) \\
&= 4l_3g_2^2(2k + 1)
\end{aligned}$$

$$\begin{aligned}
t_{y,11} - t_{x,11} &= 1 + 2(l_1 + l_3 - l_1l_3g_2)(-g_2 - g_4 + l_3g_2g_4) - \\
&\quad 1 - 2(l_1 + l_3 + l_1l_3g_2)(g_2 + g_4 + l_3g_2g_4) \\
&= -4(l_1 + l_3)(g_2 + g_4) - 4l_1l_3^2g_2^2g_4 \\
&= -4g_2[(l_1 + l_3 + l_1l_3^2g_2^2)k + l_1 + l_3]
\end{aligned}$$

$$\begin{aligned}
t_{y,11} + t_{x,11} &= 1 + 2(l_1 + l_3 - l_1l_3g_2)(-g_2 - g_4 + l_3g_2g_4) + \\
&\quad 1 + 2(l_1 + l_3 + l_1l_3g_2)(g_2 + g_4 + l_3g_2g_4) \\
&= 2 + 4l_3(2l_1 + l_3)g_2g_4 + 4l_1l_3g_2^2 \\
&= 2[2l_3g_2^2(2l_1 + l_3)k + 1 + 2l_1l_3g_2^2] \ .
\end{aligned}$$

Inserting these expressions into (A.1), we get:

$$\begin{aligned}
&16\beta^{*2}l_3g_2^3[(1 + l_3^2g_2^2)k + 1](2k + 1) = \\
&\qquad - 8g_2[(l_1 + l_3 + l_1l_3^2g_2^2)k + l_1 + l_3][2l_3g_2^2(2l_1 + l_3)k + 1 + 2l_1l_3g_2^2] \ .
\end{aligned}$$

Expanding and simplifying:

$$
\begin{aligned}
2\beta^{*2} l_3 g_2^2 [2(l_3^2 g_2^2 + 1)k^2 + (l_3^2 g_2^2 + 3)k + 1] = \\
& - 2(l_1 l_3^2 g_2^2 + l_1 + l_3)(2l_1 l_3 g_2^2 + l_3^2 g_2^2)k^2 \\
& - [(l_1 l_3^2 g_2^2 + l_1 + l_3)(1 + 2l_1 l_3 g_2^2) + 2(l_1 + l_3)(2l_1 l_3 g_2^2 + l_3^2 g_2^2)]k \\
& - (l_1 + l_3)(1 + 2l_1 l_3 g_2^2) \ .
\end{aligned}
$$

Grouping together the same powers of k we obtain the second order equation

$$k^2(a_2 g_2^4 + b_2 g_2^2) + k(a_1 g_2^4 + b_1 g_2^2 + c_1) + b_0 g_2^2 + c_0 = 0$$

where the coefficients

$$
\begin{aligned}
a_2 &= 2l_3^3[2\beta^{*2} + l_1(2l_1 + l_3)] \\
b_2 &= 2l_3[2\beta^{*2} + (l_1 + l_3)(2l_1 + l_3)] \\
a_1 &= 2l_3^3(\beta^{*2} + l_1^2) \\
b_1 &= l_3[6\beta^{*2} + 2(l_1 + l_3)(3l_1 + l_3) + l_1 l_3] \\
c_1 &= (l_1 + l_3) \\
b_0 &= 2l_3[\beta^{*2} + l_1(l_1 + l_3)] \\
c_0 &= (l_1 + l_3)
\end{aligned}
$$

are functions only of the drift lengths and of the value of the β function at the entrance of the symmetric triplet.

APPENDIX B: STUDY OF A SYMMETRIC TRIPLET MATCHING TWO ROUND BEAMS EACH ONE WITH OPPOSITE TRANSVERSE DIVERGENCES

Let us assume that the betatron functions at the entrance and at the exit of a symmetric triplet are respectively:

$$\begin{aligned} \beta_{x,1} &= \beta_{y,1} &&= \beta_1 \\ \alpha_{x,1} &= -\alpha_{y,1} &&= \alpha_1 \end{aligned}$$

and

$$\begin{aligned} \beta_{x,2} &= \beta_{y,2} &&= \beta_2 \\ \alpha_{x,2} &= -\alpha_{y,2} &&= \alpha_2 \end{aligned} \tag{B.1}$$

and that in addition they satisfy the inequality $\alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$. Then the horizontal and vertical matrices may be written under the form [2]:

$$T_h = \pm \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{11} \end{bmatrix}$$

$$T_v = \pm \begin{bmatrix} -T_{11} & T_{12} \\ T_{21} & -T_{11} \end{bmatrix}$$

where

$$\begin{aligned} T_{11} &= (\alpha_1 + \alpha_2)/\alpha_0 \\ T_{12} &= (\beta_1 - \beta_2)/\alpha_0 \\ T_{21} &= (\gamma_1 - \gamma_2)/\alpha_0 \end{aligned} \tag{B.2}$$

$$\alpha_0 = \frac{\sqrt{(\beta_2 - \beta_1)^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)^2}}{\sqrt{\beta_1\beta_2}}$$

$$\gamma_1 = \frac{1 + \alpha_1^2}{\beta_1} \quad , \quad \gamma_2 = \frac{1 + \alpha_2^2}{\beta_2}$$

It should be noted that T_{11} and T_{21} have the same signs as $\alpha_1 + \alpha_2$ and $\gamma_1 - \gamma_2$ respectively.

The phase advance corresponding to the sign + in front of the horizontal transfer matrix is given by [2]:

$$\mu_h^+ = \pi + \text{sign}\left[\frac{\beta_1 - \beta_2}{\beta_2\alpha_1 + \beta_1\alpha_2}\right] \arctan\left|\frac{\beta_1 - \beta_2}{\beta_2\alpha_1 + \beta_1\alpha_2}\right| \tag{B.3}$$

The other phase advances are:

$$\begin{aligned} \mu_h^- &= \mu_h^+ + \pi \\ \mu_v^+ &= \mu_h^- = \mu_h^+ + \pi \\ \mu_v^- &= \mu_h^+ \end{aligned}$$

If we accept any of these phase advances, there are four sets of transfer matrices which transform the entrance betatron functions into the exit betatron functions:

$$
\begin{aligned}
T_h &= \begin{bmatrix} -T_{11} & -T_{12} \\ -T_{21} & -T_{11} \end{bmatrix}, \quad T_v = \begin{bmatrix} T_{11} & -T_{12} \\ -T_{21} & T_{11} \end{bmatrix} \\
T_h &= \begin{bmatrix} -T_{11} & -T_{12} \\ -T_{21} & -T_{11} \end{bmatrix}, \quad T_v = \begin{bmatrix} -T_{11} & T_{12} \\ T_{21} & -T_{11} \end{bmatrix} \\
T_h &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{11} \end{bmatrix}, \qquad T_v = \begin{bmatrix} T_{11} & -T_{12} \\ -T_{21} & T_{11} \end{bmatrix} \\
T_h &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{11} \end{bmatrix}, \qquad T_v = \begin{bmatrix} -T_{11} & T_{12} \\ T_{21} & -T_{11} \end{bmatrix}
\end{aligned}
$$

Let us compute the quantities a, b, c, d for each set [2]:

$$a = -1/2, \quad b = -T_{11}/2, \quad c = -T_{21}/2, \quad d = 0 \tag{B.4}$$

$$a = -\frac{T_{11}+1}{2}, \quad b = 0, \quad c = 0, \quad d = -T_{21}/2 \tag{B.5}$$

$$a = \frac{T_{11}-1}{2}, \quad b = 0, \quad c = 0, \quad d = T_{21}/2 \tag{B.6}$$

$$a = -1/2, \quad b = T_{11}/2, \quad c = T_{21}/2, \quad d = 0 \tag{B.7}$$

The sets (B.4) and (B.7) can be treated at the same time. Solutions exist if $bc - ad \neq 0$ (eighth case of [2]). This implies $T_{11} \neq 0$ or $T_{21} \neq 0$ that is:

$$
\begin{aligned}
\alpha_1 + \alpha_2 &\neq 0 \\
\gamma_1 - \gamma_2 &\neq 0
\end{aligned} \tag{B.8}
$$

The corresponding triplet parameters are given by

$$l_1 = \frac{bz - 1/2}{c} \tag{B.9}$$

$$g_2 = \frac{c}{b(1-z^2)} \tag{B.10}$$

$$l_3 = \frac{b(1-z^2)}{cz} \tag{B.11}$$

$$g_4 = -\frac{cz^3}{1-z^2} \tag{B.12}$$

where z is a solution of the cubic equation $z^3 + z - 1/b = 0$. The discriminant $\Delta = 1/27 + 1/4b^2$ being positive, there is only one real solution given by

$$z_r = \sqrt[3]{1/2b + \sqrt{\Delta}} + \sqrt[3]{1/2b - \sqrt{\Delta}}$$

From the cubic equation we have

$$1/b = z_r^3 + z_r$$

or

$$bz_r = \frac{1}{z_r^2 + 1} \tag{B.13}$$

Inserting this expression in (B.9) and in (B.11) we get for the two sets (B.4) and (B.5):

$$\begin{aligned} l_1 &= \frac{1 - z_r^2}{2c(1 + z_r^2)} && \text{(B.14)} \\ l_3 &= \frac{1 - z_r^2}{cz_r^2(1 + z_r^2)} = 2l_1/z_r^2 && \text{(B.15)} \end{aligned}$$

Thus if l_1 is positive also l_3 is positive.

Now dealing with the set (B.4) we obtain from the equations (B.9) to (B.12) the following triplet parameters:

$$\begin{aligned} l_1 &= \frac{T_{11} z_{r,1} + 1}{T_{21}} \\ g_2 &= \frac{T_{21}}{T_{11}} \frac{1}{1 - z_{r,1}^2} \\ l_3 &= 2l_1/z_{r,1}^2 \\ g_4 &= -T_{11} z_{r,1}^3 g_2/2 \end{aligned}$$

Where $z_{r,1}$ is the (only) real solution of the cubic equation $z^3 + z + 2/T_{11} = 0$.

Similarly dealing with the set (B.7) we get the following triplet parameters :

$$\begin{aligned} l_1 &= \frac{T_{11} z_{r,2} - 1}{T_{21}} \\ g_2 &= \frac{T_{21}}{T_{11}} \frac{1}{1 - z_{r,2}^2} \\ l_3 &= 2l_1/z_{r,2}^2 \\ g_4 &= T_{11} z_{r,2}^3 g_2/2 \end{aligned}$$

Where $z_{r,2}$ is the (only) real solution of the cubic equation $z^3 + z - 2/T_{11} = 0$.

It is easy to show that $z_{r,2} = -z_{r,1}$. Observing that $T_{11}z_{r,2} - 1 = -(T_{11}z_{r,1} + 1)$, the only acceptable solution corresponding to positive l_1 as well as to positive l_3 because of (B.15) is either $z_{r,1}$ or $z_{r,2}$ depending upon the sign of T_{21}. Thus the two sets B.4 and B.7 provide only one solution given by:

$$\begin{aligned} l_1 &= \left| \frac{T_{11} z_{r,1} + 1}{T_{21}} \right| \\ g_2 &= \frac{T_{21}}{T_{11}} \frac{1}{1 - z_{r,1}^2} \\ l_3 &= 2l_1/z_{r,1}^2 \\ g_4 &= T_{11} z_{r,1}^3 g_2/2 && \text{(B.16)} \end{aligned}$$

Where the betatron functions must satisfy the two conditions (B.8):

$$\alpha_2 \neq -\alpha_1$$
$$\gamma_2 \neq \gamma_1$$

The set (B.5) has two solutions if $T_{11} > -1$ and $T_{11} \neq 0$ (third case of [2]). The first solution is:

$$\begin{aligned} l_1 &= \frac{\sqrt{1+T_{11}}}{T_{21}} \\ g_2 &= \frac{T_{21}}{T_{11}} \\ l_3 &= l_1 T_{11} \\ g_4 &= -g_2/2 \end{aligned}$$

l_1 is positive if $T_{21} > 0$ and l_3 is positive if $T_{11} > 0$. These conditions may be written under the form (B.2):

$$\alpha_1 + \alpha_2 > 0 \qquad \text{and} \qquad \gamma_1 > \gamma_2$$

The second solution is:

$$\begin{aligned} l_1 &= -\frac{\sqrt{1+T_{11}}}{T_{21}} \\ g_2 &= \frac{T_{21}}{T_{11}} \\ l_3 &= l_1 T_{11} \\ g_4 &= -g_2/2 \end{aligned}$$

l_1 is positive if $T_{21} < 0$ and l_3 is positive if $T_{11} > 0$. These conditions are equivalent to (B.2):

$$\alpha_1 + \alpha_2 > 0 \qquad \text{and} \qquad \gamma_1 < \gamma_2$$

The set (B.6) has also two solutions (third case of [2]). The first one is:

$$\begin{aligned} l_1 &= -\frac{\sqrt{1-T_{11}}}{T_{21}} \\ g_2 &= \frac{T_{21}}{T_{11}} \\ l_3 &= -l_1 T_{11} \\ g_4 &= -g_2/2 \end{aligned}$$

l_1 is positive if $T_{21} < 0$ and l_3 is positive if $T_{11} < 0$. These conditions may be written under the form (B.2):

$$\alpha_1 + \alpha_2 < 0 \qquad \text{and} \qquad \gamma_1 < \gamma_2$$

The second solution is

$$
\begin{aligned}
l_1 &= \frac{\sqrt{1+T_{11}}}{T_{21}} \\
g_2 &= \frac{T_{21}}{T_{11}} \\
l_3 &= -l_1 T_{11} \\
g_4 &= -g_2/2
\end{aligned}
$$

l_1 is positive if $T_{21} > 0$ and l_3 is positive if $T_{11} < 0$. These conditions are equivalent to (B.2):

$$\alpha_1 + \alpha_2 < 0 \qquad \text{and} \qquad \gamma_1 > \gamma_2$$

Inspecting the four solutions it is possible to give a unique representation for the results of the two sets (B.5) and (B.6):

$$
\begin{aligned}
l_1 &= \frac{\sqrt{1+|T_{11}|}}{|T_{21}|} \\
g_2 &= \frac{T_{21}}{T_{11}} \\
l_3 &= l_1 |T_{11}| \\
g_4 &= -g_2/2
\end{aligned}
\tag{B.17}
$$

Where the only conditions on the betatron function are

$$\alpha_1 + \alpha_2 \neq 0 \qquad \text{and} \qquad \gamma_2 \neq \gamma_1$$

Summarizing we may say that there are always two symmetric triplets which match the betatron functions at its entrance and exit as specified in (6) and (B.1) with the additional conditions:

$$
\begin{gathered}
\alpha_2 \neq -\alpha_1 \\
\gamma_2 \neq \gamma_1 \\
\alpha_1 \beta_2 + \alpha_2 \beta_1 \neq 0
\end{gathered}
$$

The parameters of the two symmetric triplets (B.16) and (B.17) can be obtained by replacing T_{11} and T_{21} with the expressions (B.2):

First solution:

$$
\begin{aligned}
l_1 &= \left| \frac{(\alpha_1+\alpha_2) z_{r,1} + \alpha_0}{\gamma_1 - \gamma_2} \right| \\
g_2 &= \frac{\gamma_1 - \gamma_2}{(\alpha_1+\alpha_2)} \frac{1}{1 - z_{r,1}^2} \\
l_3 &= 2 l_1 / z_{r,1}^2 \\
g_4 &= \frac{(\alpha_1+\alpha_2) z_{r,1}^3 g_2}{2\alpha_0}
\end{aligned}
$$

Where $z_{r,1}$ is the (only) real solution of the cubic equation $z^3 + z + 2\alpha_0/(\alpha_1 + \alpha_2) = 0$, i.e.

$$z_{r,1} = \frac{\alpha_0}{\alpha_1 + \alpha_2}\left[\sqrt[3]{\sqrt{1 + \frac{(\alpha_1 + \alpha_2)^2}{27\alpha_0^2}} - 1} - \sqrt[3]{\sqrt{1 + \frac{(\alpha_1 + \alpha_2)^2}{27\alpha_0^2}} + 1}\right]$$

Second solution:

$$\begin{aligned}
f &= 1/g_2 = \frac{\alpha_1 + \alpha_2}{\gamma_1 - \gamma_2} \\
l_1 &= |f/(\alpha_1 + \alpha_2)|\sqrt{\alpha_0(\alpha_0 + |\alpha_1 + \alpha_2|)} \\
l_3 &= |f|\sqrt{1 + |\alpha_1 + \alpha_2|/\alpha_0} \\
g_4 &= -g_2/2
\end{aligned}$$