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NON-PERTURBATIVE GRAVITATIONAL CORRECTIONS IN A CLASS OF $N = 2$ STRING DUALS [‡]

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Abstract

We investigate the non-perturbative equivalence of some heterotic/type II dual pairs with $N = 2$ supersymmetry. The perturbative heterotic scalar manifolds are respectively $SU(1,1)/U(1) \times SO(2,2+N_V)/SO(2) \times SO(2+N_V)$ and $SO(4,4+N_H)/SO(4) \times SO(4+N_H)$ for moduli in the vector multiplets and hypermultiplets. The models under consideration correspond, on the type II side, to self-mirror Calabi–Yau threefolds with Hodge numbers $h^{1,1} = N_V + 3 = h^{2,1} = N_H + 3$, which are $K3$ fibrations. We consider three classes of dual pairs, with $N_V = N_H = 8$, 4 and 2. The models with $h^{1,1} = 7$ and 5 provide new constructions, while the $h^{1,1} = 11$, already studied in the literature, is reconsidered here. Perturbative R^2-like corrections are computed on the heterotic side by using a universal operator whose amplitude has no singularities in the (T, U) space, and can therefore be compared with the type II side result. We point out several properties connecting $K3$ fibrations and spontaneous breaking of the $N = 4$ supersymmetry to $N = 2$. As a consequence of the reduced S- and T- duality symmetries, the instanton numbers in these three classes are restricted to integers, which are multiples of 2, 2 and 4, for $N_V = 8$, 4 and 2, respectively.

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1 Introduction

Different perturbative string theories with the same number of supersymmetries might be equivalent at the non-perturbative level [1, 2, 3]. There is a conjectured duality between the heterotic string compactified on $T^4 \times T^2$ and the type IIA (IIB) string compactified on $K3 \times T^2$ [1]. Both theories have $N = 4$ supersymmetry and 22 massless vector multiplets in four dimensions. In both theories the space of the moduli-field vacuum expectation values is spanned by 134 physical scalars, which are coordinates of the coset space [1, 4]

$$\left(\frac{SL(2,R)}{U(1)}\right)_S \times \left(\frac{SO(6,6+r)}{SO(6)\times SO(6+r)}\right)_T , \quad r = 16 . \tag{1.1}$$

On the heterotic side, the dilaton $S_{\rm Het} = S$ is in the gravitational multiplet, while on the type II side it is one of the moduli of the vector multiplets: $S_{\rm II} = T^1$, where T^1 is the volume form of the two-torus. Thus the duality relation implies an interchange between the fields of S and T manifolds [1]: the perturbative heterotic states are mapped to non-perturbative states of the type II theory and vice versa. Consequently, perturbative T-duality of the type II strings implies S-duality [5] of the heterotic string and vice versa.

Several arguments support this duality conjecture. For instance, the anomaly cancellation of the six-dimensional heterotic string implies that there should be a one-loop correction to the gravitational R^2 term in the type II theory. Such a term was found by direct calculation in [6]. Its one-loop threshold correction upon compactification to four dimensions [7] implies instanton corrections on the heterotic side due to five-branes wrapped around the six-torus.

Heterotic/type II dual pairs with lower rank (i.e. $r < 16$) and with $N = 4$ supersymmetry, share properties similar to those just mentioned. The study of such theories and the determination of the heterotic non-perturbative corrections to the R^2 terms were considered in [8]. Here we would like to extend this analysis for $N = 2$ heterotic/type II dual theories.

In general (non-freely-acting) symmetric orbifolds still give rise to $N = 2$ heterotic/type II dual pairs in four dimensions [9, 10, 11]. On the heterotic side they can be interpreted as $K3$ plus gauge-bundle compactifications, while on the type II side they are Calabi–Yau compactifications of the ten-dimensional type IIA theory. The heterotic dilaton is in a vector multiplet and the vector moduli space receives both perturbative and non-perturbative corrections. On the other hand, the hypermultiplet moduli space does not receive perturbative corrections; if $N = 2$ is assumed unbroken, it receives no non-perturbative corrections either. On the type II side the dilaton is in a hypermultiplet and the prepotential for the vector multiplets receives only tree-level contributions. The tree-level type II prepotential was computed and shown to give the correct one-loop heterotic result. This provides a quantitative test of the duality [9, 11] and allows us to reach the non-perturbative corrections of the heterotic side.

The purpose of this paper is to provide quantitative tests of $N = 2$ heterotic/type II duality. Quantitative tests of non-perturbative dualities can be obtained by considering the renormalization of certain terms in the effective action of the massless fields. Extended supersymmetry plays an essential role in this since it allows the existence of BPS states that, being short representations of the supersymmetry algebra, are (generically) non-perturbatively stable and provide a reliable window into non-perturbative physics. There are terms in the effective action, the couplings of which can be shown to obtain contributions only from BPS states. The relevant structures for this analysis are helicity supertrace formulae, which distinguish between various BPS and non-BPS states [8, 12, 13, 14]. For ($N = 2$)-supersymmetric models, these supertraces appear in particular in the two-derivative terms R^2 or in a special class of higher-order terms constructed out of the Riemann tensor and the graviphoton field strength [15]. In the four-dimensional heterotic string, these terms are anomaly-related and it can be shown that they receive only tree-level and one-loop corrections. In higher dimensions, they receive no non-perturbative correction [16].

In this paper we investigate the non-perturbative equivalence of some heterotic/type II dual pairs with $N = 2$ supersymmetry. The perturbative heterotic scalar manifolds are

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2+N_V)}{SO(2)\times SO(2+N_V)} \quad \text{and} \quad \frac{SO(4,4+N_H)}{SO(4)\times SO(4+N_H)} \tag{1.2}$$

for moduli in the vector multiplets and hypermultiplets, respectively [1]. On the type II side, the models under consideration correspond to self-mirror Calabi–Yau threefolds with Hodge numbers $h^{1,1} = N_V + 3 = h^{2,1} = N_H + 3$, which are $K3$ fibrations, necessary condition for the existence of heterotic duals [17, 18].

In Section 2 we examine the $N = 2$ type II models with $N_V = N_H = 8$, 4, and 2, which are particular examples of $Z_2 \times Z_2$ symmetric orbifolds. A detailed and complete classification of such orbifolds will appear in [19]. The models with $h^{1,1} = 7$ and 5 provide new constructions, while the one with $h^{1,1} = 11$ was already considered in the literature [10, 20]. It is reconsidered here since, as we will see, our choice of the R^2 terms is not exactly the one taken previously. We derive the perturbative type II R^2 corrections and point out several properties that connect the $K3$ fibrations and the spontaneous breaking of the $N = 4$ to the $N = 2$ supersymmetry.

In Section 3 we construct the heterotic duals. The construction is performed in a unified formalism for all values of N_V. Particular attention is payed to the singularities arising along various lines in the (T, U) moduli space, where extra massless states appear. This analysis serves as a guideline in the determination of an operator on the heterotic side that reproduces the R^2 term already considered in the type II side. This is achieved in Section 4, where it is precisely shown that indeed there exists a universal, holomorphic and modular-covariant operator Q'^2_{grav}; its coupling constant receives perturbative corrections, regular at every point of the two-torus moduli space. This enables us to check the heterotic/type II duality conjecture at the perturbative and non-perturbative level.

Our conclusion and comments are given in Section 5.

[1] In the models under consideration, N_V and N_H are the number of vector multiplets and hypermultiplets, respectively, which on the type II construction are originated from the twisted sectors. The total number of those are $N_V + 2$ and $N_H + 4$ for heterotic constructions, $N_V + 3$ and $N_H + 3$ in the type II cases. These numbers neither include the vector-tensor multiplet present in heterotic models, which is dual to a vector multiplet, nor the tensor multiplet of the type II ground states dual to a hypermultiplet.

2 Type II reduced-rank models and gravitational corrections

2.1 Construction of the reduced-rank models

a) The $N_V = N_H = 8$ model

We start by considering the $N = 2$ type II model with $N_V = N_H = 8$. This model is obtained by compactification of the ten-dimensional superstring on a Calabi–Yau threefold with $h^{1,1} = h^{2,1} = 11$ [10]. It can be constructed in two steps. We start with the type II $N = 4$ supersymmetric model defined by $K3 \times T^2$ compactification. The massless spectrum of this model contains the $N = 4$ supergravity multiplet as well as 22 vector multiplets. For convenience, we will go to the T^4/Z_2 orbifold limit of $K3$, where it can easily be shown that the 6 graviphotons and 6 of the vector multiplets are coming from the untwisted sector, while the remaining 16 come from the twisted sector and are in one-to-one correspondence with the 16 fixed points of the Z_2 action. We then break the $N = 4$ supersymmetry to $N = 2$ by introducing an extra Z_2 projection in which two of the T^4 coordinates are shifted; the remaining two coordinates of T^4 together with the two coordinates of T^2 are Z_2-twisted.

If we indicate by $Z_2^{(\rm o)}$ the projection that defines the $T^4/Z_2^{(\rm o)}$ orbifold limit of $K3$, and by $Z_2^{(\rm f)}$ the second projection defined above, we can summarize the action of the two Z_2's on the three complex (super-)coordinate planes as in Table 1; R indicates the twist, while T is a half-unit lattice shift.

Orbifold	Plane 1	Plane 2	Plane 3
$Z_2^{(\rm o)}$	1	R	R
$Z_2^{(\rm f)}$	R	T	RT
$Z_2^{(\rm o)} \times Z_2^{(\rm f)}$	R	RT	T

Table 1. The action of $Z_2^{(\rm o)} \times Z_2^{(\rm f)}$ on the T^6.

Since the translations on the second and third planes are non-vanishing, the $Z_2^{(\rm f)}$-operation has no fixed points and there are therefore no extra massless states coming from the twisted sectors; the massless spectrum of this model contains the $N = 2$ supergravity multiplet, 11 vector multiplets (3 from the untwisted and 8 from the twisted sector), 1 tensor multiplet and 11 hypermultiplets (3 are from the untwisted sector and 8 from the twisted sector). The tensor multiplet is the type II dilaton supermultiplet, and it is equivalent to an extra hypermultiplet.

The partition function of the model reads:

$$\begin{aligned} Z_{\rm II}^{N_V} \quad = \quad & \frac{1}{{\rm Im}\,\tau\,|\eta|^{24}}\frac{1}{4}\sum_{H^{\rm o},G^{\rm o}}\sum_{H^{\rm f},G^{\rm f}}\Gamma_{6,6}^{N_V}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix} \\ & \times\frac{1}{2}\sum_{a,b}(-)^{a+b+ab}\vartheta\begin{bmatrix}a\\ b\end{bmatrix}\vartheta\begin{bmatrix}a+H^{\rm o}\\ b+G^{\rm o}\end{bmatrix}\vartheta\begin{bmatrix}a+H^{\rm f}\\ b+G^{\rm f}\end{bmatrix}\vartheta\begin{bmatrix}a-H^{\rm o}-H^{\rm f}\\ b-G^{\rm o}-G^{\rm f}\end{bmatrix} \\ & \times\frac{1}{2}\sum_{\bar a,\bar b}(-)^{\bar a+\bar b+\bar a\bar b}\bar\vartheta\begin{bmatrix}\bar a\\ \bar b\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a+H^{\rm o}\\ \bar b+G^{\rm o}\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a+H^{\rm f}\\ \bar b+G^{\rm f}\end{bmatrix}\bar\vartheta\begin{bmatrix}\bar a-H^{\rm o}-H^{\rm f}\\ \bar b-G^{\rm o}-G^{\rm f}\end{bmatrix}, \end{aligned} \qquad (2.1)$$

with

$$\Gamma_{6,6}^{N_V=8}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix}=\Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\rm f}|0\\ G^{\rm f}|0\end{bmatrix}\Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\rm o}|H^{\rm f}\\ G^{\rm o}|G^{\rm f}\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}H^{\rm o}+H^{\rm f}|H^{\rm f}\\ G^{\rm o}+G^{\rm f}|G^{\rm f}\end{bmatrix}, \tag{2.2}$$

the contribution of the six compactified left and right coordinates X^I and $\bar{X}^I$, where $(H^{\rm o},G^{\rm o})$ refer to the boundary conditions introduced by the projection $Z_2^{(\rm o)}$ and $(H^{\rm f},G^{\rm f})$ to the projection $Z_2^{(\rm f)}$. Here we have introduced the twisted and shifted characters of a $c=(2,2)$ block, $\Gamma_{2,2}\left[{h|h' \atop g|g'}\right]$; the first column refers to the twist, the second to the shift. The non-vanishing components are the following:

$$\begin{aligned}\Gamma_{2,2}\begin{bmatrix}h|h'\\ g|g'\end{bmatrix} &= \frac{4\,|\eta|^6}{\left|\vartheta\left[{1+h \atop 1+g}\right]\vartheta\left[{1-h \atop 1-g}\right]\right|}\ ,\quad \text{for } (h',g')=(0,0)\ \text{ or } (h',g')=(h,g)\\ &= \Gamma_{2,2}\begin{bmatrix}h'\\ g'\end{bmatrix}\ ,\quad \text{for } (h,g)=(0,0)\,,\end{aligned} \tag{2.3}$$

where $\Gamma_{2,2}\left[{h' \atop g'}\right]$ is the Z_2 shifted $(2,2)$ lattice sum. The shift has to be specified by the way it acts on the windings and momenta (such technical details can be found in various references, our conventions are those of [8]). All the moduli of the type II compactification at hand are contained in expression (2.2), which depends on the volume forms T^1,T^2,T^3 and the complex-structure forms U^1,U^2,U^3 of the three tori.

b) The $N_V=N_H=4$ model

In order to construct models with lower rank, we start with the $N=4$ ground state defined above as the $K3\times T^2$ compactification and then apply several Z_2 freely-acting projections; each one of these projections removes half of the $Z_2^{(\rm o)}$-twisted vector multiplets without changing the Euler number of the compactification manifold, $\chi\equiv 2(h^{1,1}-h^{2,1})=2(N_V-N_H)=0$, and leads therefore to models with $N_V=N_H=4$ and 2. It is however important to stress that this procedure, already developed in $N=4$ models [8], can be implemented provided we start from an orbifold point of the $K3$, namely from the $N=4$ compactification manifold $\left(T^4/Z_2^{(\rm o)}\right)\times T^2$. This choice is no longer dictated by convenience.

The model with $N_V=N_H=4$ is constructed by modding out an extra Z_2^D symmetry; the compactification manifold is therefore $\mathcal{M}^{N_V=4}=\mathcal{M}^{N_V=8}/Z_2^D$. In order to describe the action of this projection, we choose special coordinates of the compact space, in terms of which the T^4 torus is described as a product of circles. There is then a $(D_4)^4$ symmetry generated by the elements D and $\tilde{D}$, which act on each S^1/Z_2 block as [21]:

$$\begin{aligned}D:&\quad (\sigma_+,\sigma_-,V_{nm})\rightarrow\ (\sigma_-,\sigma_+,(-)^m V_{nm})\\ \tilde{D}:&\quad (\sigma_+,\sigma_-,V_{nm})\rightarrow(-\sigma_+,\sigma_-,(-)^n V_{nm}),\end{aligned} \tag{2.4}$$

where σ_+ and σ_- are the 2 twist fields of S^1/Z_2, and V_{nm} are the untwisted world-sheet instantons labelled by the momentum m and the winding n of the $\Gamma_{1,1}$ lattice. In the untwisted sector of $T^4/Z_2^{(\rm o)}$, D and $\tilde{D}$ act as ordinary shifts on the $\Gamma_{1,1}$ sublattice of the $\Gamma_{4,4}$.

Orbifolding by Z_2^D the original $\left(T^4/Z_2^{(\rm o)}\right)\times T^2$, we obtain an $N=4$ ground state with a reduced number of vector multiplets. The Z_2^D acts on the $T^4/Z_2^{(\rm o)}$ part as a D-operation, while it acts on T^2 as an ordinary shift. In the $\left(\left(T^4/Z_2^{(\mathrm{o})}\right)\times T^2\right)\Big/Z_2^D$ orbifold, the number of the $Z_2^{(\mathrm{o})}$-twisted vector multiplets (16) is reduced to 8 by the extra Z_2^D projection, without breaking the $N=4$ supersymmetry further [8]. To reduce $N=4$ to $N=2$ we perform a further orbifold by using $Z_2^{(\mathrm{f})}$ as in the $N_V=N_H=8$ model above. The $N=2$ ground state obtained in this way is a compactification on the manifold $\mathcal{M}^{N_V=4}=\left(\left(T^4/Z_2^{(\mathrm{o})}\right)\times T^2\right)\Big/\left(Z_2^{(\mathrm{f})}\times Z_2^D\right)$ and has $N_V=N_H=4$. The massless spectrum consists, apart from the gravity and tensor multiplets, which include a graviphoton and a dilaton as before, of 7 vector multiplets (4 coming from the twisted sector) and 7 hypermultiplets (4 twisted).

The partition function based on $\mathcal{M}^{N_V=4}$ is still given in Eq. (2.1), but now with

$$\Gamma_{6,6}^{N_V=4}\begin{bmatrix}H^{\mathrm{o}},H^{\mathrm{f}}\\G^{\mathrm{o}},G^{\mathrm{f}}\end{bmatrix}=\frac{1}{2}\sum_{H,G}\Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\mathrm{f}}|H\\G^{\mathrm{f}}|G\end{bmatrix}\Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\mathrm{o}}|H^{\mathrm{f}},H\\G^{\mathrm{o}}|G^{\mathrm{f}},G\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}H^{\mathrm{o}}+H^{\mathrm{f}}|H^{\mathrm{f}}\\G^{\mathrm{o}}+G^{\mathrm{f}}|G^{\mathrm{f}}\end{bmatrix},\qquad(2.5)$$

where (H,G) correspond to the Z_2^D-projection. We introduced the $(2,2)$ conformal blocks $\Gamma_{2,2}\left[{}^{h|h',h''}_{g|g',g''}\right]$, where, as previously, (h,g) refer to the twist, while (h',g') and (h'',g'') correspond to shifts along two circles of T^2. The non-vanishing components are

$$\Gamma_{2,2}\begin{bmatrix}h|h',h''\\g|g',g''\end{bmatrix}=\frac{4\,|\eta|^6}{\left|\vartheta\left[{}^{1+h}_{1+g}\right]\vartheta\left[{}^{1-h}_{1-g}\right]\right|}$$

for $(h',g')=(h'',g'')=(0,0)$ or $(h',g')=(h,g)$ $(h'',g'')=(0,0)$ or $(h',g')=(0,0)$ $(h'',g'')=(h,g)$ or $(h',g')=(h'',g'')=(h,g)$, and

$$\Gamma_{2,2}\begin{bmatrix}0|h',h''\\0|g',g''\end{bmatrix}=\Gamma_{2,2}\begin{bmatrix}h',h''\\g',g''\end{bmatrix},$$

where $\Gamma_{2,2}\left[{}^{h',h''}_{g',g''}\right]$ is the lattice sum of a $(Z_2\times Z_2)$-twisted torus.

c) The $N_V=N_H=2$ model

The model with $N_V=N_H=2$ is obtained by modding out a further Z_2^D-symmetry, which acts on another circle of the first and the third torus:

$$\Gamma_{6,6}^{N_V=2}\begin{bmatrix}H^{\mathrm{o}},H^{\mathrm{f}}\\G^{\mathrm{o}},G^{\mathrm{f}}\end{bmatrix}=\frac{1}{4}\sum_{H,G}\sum_{H',G'}\Gamma_{2,2}^{(1)}\begin{bmatrix}H^{\mathrm{f}}|H,H'\\G^{\mathrm{f}}|G,G'\end{bmatrix}\Gamma_{2,2}^{(2)}\begin{bmatrix}H^{\mathrm{o}}|H^{\mathrm{f}},H\\G^{\mathrm{o}}|G^{\mathrm{f}},G\end{bmatrix}\Gamma_{2,2}^{(3)}\begin{bmatrix}H^{\mathrm{o}}+H^{\mathrm{f}}|H^{\mathrm{f}},H'\\G^{\mathrm{o}}+G^{\mathrm{f}}|G^{\mathrm{f}},G'\end{bmatrix}\qquad(2.6)$$

(notice that the two Z_2^D-projections commute). The resulting massless spectrum consists of the gravity and tensor multiplets plus 5 vector multiplets (2 from the twisted sector), and 5 hypermultiplets (2 from the twisted sector).

In all the above models, the realization of the $N=2$ supersymmetry plays a key role in the search of heterotic duals, as we will see in Section 3. Indeed, by using the techniques developed in Refs. [13, 22], we can indeed show that these models actually possess a spontaneously broken $N=4$ supersymmetry through a Higgs phenomenon, due to the free action of $Z_2^{(\mathrm{f})}$. The restoration of the 16 supersymmetric charges (2 extra massless gravitinos) takes place in some appropriate limits of the moduli according to the precise shifts in the lattices. It is accompanied by a logarithmic instead of a linear blow-up of various thresholds [13, 22, 23, 24], which is nothing but an infrared artifact due to an accumulation of massless states. These can be lifted by introducing an infra-red cut-off μ larger than the two gravitino masses. The thresholds thus vanish as expected in the limit in which supersymmetry is extended to $N = 4$ as $m_{3/2}/\mu \to 0$.

2.2 Helicity supertraces and the R^2 corrections

Usually string ground states are best described by writing their (four-dimensional) helicity-generating partition functions. Moreover, since our motivation is eventually to compute couplings associated with interactions such as R^2, we need in general to evaluate amplitudes involving operators such as $i\left(X^3 \overset{\leftrightarrow}{\partial} X^4 + 2\psi^3\psi^4\right)\overline{J}^k$ [25], where $\overline{J}^k$ is an appropriate right-moving current and the left-moving factor corresponds to the left-helicity operator (the right-helicity operator is the antiholomorphic counterpart of the latter). We will not expand here on the various operators of this kind, or on the procedures that have been used in order to calculate their correlation functions exactly (i.e. to all orders in α' and without infrared ambiguities [25]); details will be given in Section 4, when analysing some specific R^2 corrections for the heterotic duals of the models under consideration. Here we will restrict ourselves to the helicity-generating partition functions since these allow for a direct computation of perturbative type II R^2 corrections. They are defined as:

$$Z(v,\bar{v}) = \operatorname{Tr}' q^{L_0-\frac{c}{24}}\,\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\, e^{2\pi i\left(vQ-\bar{v}\overline{Q}\right)}\,, \tag{2.7}$$

where the prime over the trace excludes the zero-modes related to the space-time coordinates (consequently $Z(v,\bar{v})|_{v=\bar{v}=0} = \tau_2 Z$, where Z is the vacuum amplitude) and $Q, \overline{Q}$ stand for the left- and right-helicity contributions to the four-dimensional physical helicity. Various helicity supertraces are finally obtained by taking appropriate derivatives of (2.7):

$$B_{2n} = \left\langle \left(Q+\overline{Q}\right)^{2n} \right\rangle_{\text{genus}-\text{one}} \tag{2.8}$$

is obtained by acting on $Z(v,\bar{v})$ with $\frac{1}{(-4\pi^2)^n}\left(\partial_v - \partial_{\bar{v}}\right)^{2n}$ at $v = \bar{v} = 0$.

For the models at hand (see Eq. (2.1)), after some algebra, (2.7) reads:

$$Z_{\text{II}}^{N_V}(v,\bar{v}) = \frac{\xi(v)\,\bar{\xi}(\bar{v})}{|\eta|^4}\frac{1}{4}\sum_{H^{\text{o}},G^{\text{o}}}\sum_{H^{\text{f}},G^{\text{f}}}\frac{\Gamma_{6,6}^{N_V}\left[{H^{\text{o}},H^{\text{f}} \atop G^{\text{o}},G^{\text{f}}}\right]}{|\eta|^{12}}\, Z_{\text{L}}^{\text{F}}\left[{H^{\text{o}},H^{\text{f}} \atop G^{\text{o}},G^{\text{f}}}\right](v)\, Z_{\text{R}}^{\text{F}}\left[{H^{\text{o}},H^{\text{f}} \atop G^{\text{o}},G^{\text{f}}}\right](\bar{v})\,, \tag{2.9}$$

where $Z_{\text{L}}^{\text{F}}\left[{H^{\text{o}},H^{\text{f}} \atop G^{\text{o}},G^{\text{f}}}\right](v)$ and $Z_{\text{R}}^{\text{F}}\left[{H^{\text{o}},H^{\text{f}} \atop G^{\text{o}},G^{\text{f}}}\right](\bar{v})$ denote the contribution of the 8 left- and 8 right-moving world-sheet fermions ψ^μ, Ψ^I and $\bar{\psi}^\mu$, $\bar{\Psi}^I$; Ψ^I and $\bar{\Psi}^I$ are the 6 left- and 6 right-moving fermionic degrees of freedom of the six-dimensional internal space. The arguments v and $\bar{v}$ are due to ψ^μ and $\bar{\psi}^\mu$. By using the Riemann identity of theta functions, one can perform the summation over the spin structures with the result:

$$Z_{\rm L}^{\rm F}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix}(v)=\frac{1}{\eta^4}\,\vartheta\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{v}{2}\right)\,\vartheta\begin{bmatrix}1-H^{\rm o}\\1-G^{\rm o}\end{bmatrix}\left(\frac{v}{2}\right)\,\vartheta\begin{bmatrix}1-H^{\rm f}\\1-G^{\rm f}\end{bmatrix}\left(\frac{v}{2}\right)\,\vartheta\begin{bmatrix}1+H^{\rm o}+H^{\rm f}\\1+G^{\rm o}+G^{\rm f}\end{bmatrix}\left(\frac{v}{2}\right) \tag{2.10}$$

and

$$Z_{\rm R}^{\rm F}\begin{bmatrix}H^{\rm o},H^{\rm f}\\ G^{\rm o},G^{\rm f}\end{bmatrix}(\bar v)=\frac{1}{\bar\eta^4}\,\bar\vartheta\begin{bmatrix}1\\1\end{bmatrix}\left(\frac{\bar v}{2}\right)\,\bar\vartheta\begin{bmatrix}1-H^{\rm o}\\1-G^{\rm o}\end{bmatrix}\left(\frac{\bar v}{2}\right)\,\bar\vartheta\begin{bmatrix}1-H^{\rm f}\\1-G^{\rm f}\end{bmatrix}\left(\frac{\bar v}{2}\right)\,\bar\vartheta\begin{bmatrix}1+H^{\rm o}+H^{\rm f}\\1+G^{\rm o}+G^{\rm f}\end{bmatrix}\left(\frac{\bar v}{2}\right)\,. \tag{2.11}$$

Finally

$$\xi(v)=\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^n e^{2\pi i v})\,(1-q^n e^{-2\pi i v})}=\frac{\sin\pi v}{\pi}\frac{\vartheta_1'(0)}{\vartheta_1(v)}$$

counts the helicity contributions of the space-time bosonic oscillators.

A straightforward computation based on the techniques developed so far shows that $B_2=0$ for all the models under consideration, as expected. Indeed, in all the $N=2$ type II $Z_2\times Z_2$ symmetric orbifolds, B_2 can receive a non-zero contribution only from the $N=(1,1)$ sectors of the orbifold (see Ref. [19]). The internal coordinates in these sectors are twisted; all corrections are therefore moduli-independent and are coming from the massless states only. One finds $B_2=B_2|_{\rm massless}=N_V-N_H$, which vanishes in all models we are considering here.

On the other hand, B_4 receives non-zero contributions from the $N=(2,2)$ sectors of the orbifold. We find [2]:

$$B_4^{N_V=8}=18\,\Gamma_{2,2}^{(1)}+6\sum_{i=2,3}\sum_{(h,g)}{}'\,\Gamma_{2,2}^{(i)}\begin{bmatrix}0|h\\0|g\end{bmatrix}\,, \tag{2.12}$$

$$\begin{aligned}B_4^{N_V=4}&=9\,\Gamma_{2,2}^{(1)}+3\sum_{(h,g)}{}'\,\Gamma_{2,2}^{(1)}\begin{bmatrix}0|h\\0|g\end{bmatrix}+3\sum_{(h,g)}{}'\,\Gamma_{2,2}^{(2)}\begin{bmatrix}0|h,0\\0|g,0\end{bmatrix}\\&\quad+3\sum_{(h,g)}{}'\left(\Gamma_{2,2}^{(2)}\begin{bmatrix}0|h,h\\0|g,g\end{bmatrix}+2\,\Gamma_{2,2}^{(3)}\begin{bmatrix}0|h\\0|g\end{bmatrix}\right)\,,\end{aligned} \tag{2.13}$$

$$\begin{aligned}B_4^{N_V=2}&=\frac{9}{2}\,\Gamma_{2,2}^{(1)}+\frac{3}{2}\sum_{(h,g)}{}'\left(\Gamma_{2,2}^{(1)}\begin{bmatrix}0|h,0\\0|g,0\end{bmatrix}+\Gamma_{2,2}^{(1)}\begin{bmatrix}0|0,h\\0|0,g\end{bmatrix}+\Gamma_{2,2}^{(1)}\begin{bmatrix}0|h,h\\0|g,g\end{bmatrix}\right)\\&\quad+3\sum_{i=2,3}\sum_{(h,g)}{}'\left(\Gamma_{2,2}^{(i)}\begin{bmatrix}0|h,0\\0|g,0\end{bmatrix}+\Gamma_{2,2}^{(i)}\begin{bmatrix}0|h,h\\0|g,g\end{bmatrix}\right)\,.\end{aligned} \tag{2.14}$$

The massless contributions are in agreement with the generic result of the $N=2$ supergravity:

$$B_4|_{\rm massless}=18+\frac{7N_V-N_H}{4}\,. \tag{2.15}$$

[2] The prime summation over (h,g) stands for $(h,g)=\{(0,1),(1,0),(1,1)\}$.

We now turn to the actual computation of R^2 corrections. The four-derivative gravitational corrections we will consider here are precisely those that were analysed in the framework of $N=4$ ground states of Ref. [8]. On the type II side, there is no tree-level contribution to these operators, and the R^2 corrections are related to the insertion of the two-dimensional operator $2Q^2\overline{Q}^2$ in the one-loop partition function. In the models at hand, where supersymmetry is realized symmetrically and where, moreover, $N_V=N_H$, the contribution of $N=(1,1)$ sectors to B_4 vanishes, and therefore $\left\langle 2Q^2\overline{Q}^2\right\rangle$ can be identified with $B_4/3$. The massless contributions of the latter give rise to an infrared logarithmic behaviour $2b_{\rm II}\log\left(M^{({\rm II})\,2}\big/\mu^{({\rm II})\,2}\right)$ [25, 26], where $M^{({\rm II})}\equiv\frac{1}{\sqrt{\alpha'_{\rm II}}}$ is the type II string scale and $\mu^{({\rm II})}$ is the infrared cut-off. Besides this running, the one-loop correction contains, as usual, the thresholds $\Delta_{\rm II}$, which account for the infinite tower of string modes.

The one-loop corrections of the R^2 term are related to the infrared-regularized genus-one integral of $B_4/3$. There is however a subtlety: in the type IIA string, these R^2 corrections depend on the Kähler moduli (spanning the vector manifold), and are independent of the complex-structure moduli (spanning the scalar manifold):

$$\partial_{T^i}\Delta_{\rm IIA}=\frac{1}{3}\int_{\mathcal F}\frac{{\rm d}^2\tau}{{\rm Im}\,\tau}\partial_{T^i}B_4\ ,\quad \partial_{U^i}\Delta_{\rm IIA}=0\,. \tag{2.16}$$

In the type IIB string, the roles of T^i and U^i are interchanged:

$$\partial_{U^i}\Delta_{\rm IIB}=\frac{1}{3}\int_{\mathcal F}\frac{{\rm d}^2\tau}{{\rm Im}\,\tau}\partial_{U^i}B_4\ ,\quad \partial_{T^i}\Delta_{\rm IIB}=0\,. \tag{2.17}$$

The above properties of $\Delta_{\rm IIA,B}$ generalize the results of Ref. [8], which are valid for the R^2 corrections in $N=4$ models with reduced-rank gauge group. They also account for the massless contribution $b_{\rm II}\log\left(M^{({\rm II})\,2}\big/\mu^{({\rm II})\,2}\right)$ being half the infrared-regularized massless contribution of the integral of $B_4/3$ over the fundamental domain.

In order to perform the integration, we have to specify what the translations induced by $Z_2^{({\rm f})}$, Z_2^{D} and $Z_2^{D'}$ are. For definiteness we will choose the same half-unit shifts on momenta for all three complex planes: the first shift ($Z_2^{({\rm f})}$) will act on the first momentum (insertion of $(-1)^{m_1^i}$ in the ith plane); the second shift (Z_2^{D} or $Z_2^{D'}$, when necessary, as in models $(4,4)$ and $(2,2)$) will act on the second momentum (insertion of $(-1)^{m_2^i}$). Given this choice, the final result for the running now reads:

$$\begin{aligned}\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{({\rm II})})} \quad = \quad & -\frac{3N_V}{4}\log\frac{\mu^{({\rm II})}\,{\rm Im}\,T^1}{M^{({\rm II})}}\,\left|\eta\left(T^1\right)\right|^4-\left(2-\frac{N_V}{4}\right)\log\frac{\mu^{({\rm II})}\,{\rm Im}\,T^1}{M^{({\rm II})}}\,\left|\vartheta_4\left(T^1\right)\right|^4\\ & -2\log\frac{\mu^{({\rm II})}\,{\rm Im}\,T^2}{M^{({\rm II})}}\,\left|\vartheta_4\left(T^2\right)\right|^4-2\log\frac{\mu^{({\rm II})}\,{\rm Im}\,T^3}{M^{({\rm II})}}\,\left|\vartheta_4\left(T^3\right)\right|^4+\ {\rm const.}\end{aligned} \tag{2.18}$$

Expression (2.18) deserves several comments:

(i) The shifts on the $\Gamma_{2,2}^{(i)}$ lattices break the $SL(2,Z)_{T^i}$ duality groups. The residual subgroup depends in fact on the kind of shifts performed (see Refs. [8, 22, 24]).

(ii) The $SL(2,Z)_{T^1}$ remains unbroken only in the model with $N_V=N_H=8$; in the other situations all $SL(2,Z)_{T^i}$ are necessarily broken for all $i=1,2,3$.

(iii) The $N=4$ restoration limit corresponds to $T^2, T^3 \to \infty$. For the specific choice of translations we are considering, the mass of the 2 extra gravitinos is given by:

$$m_{3/2}^2 = \frac{1}{4\,\mathrm{Im}\,T^2\,\mathrm{Im}\,U^2} + \frac{1}{4\,\mathrm{Im}\,T^3\,\mathrm{Im}\,U^3}\ , \quad \text{for } N_V = 8\,, \tag{2.19}$$

$$m_{3/2}^2 = \frac{1}{4\,\mathrm{Im}\,T^2\,\mathrm{Im}\,U^2} + \frac{\mathrm{Im}\,U^2}{4\,\mathrm{Im}\,T^2} + \frac{1}{4\,\mathrm{Im}\,T^3\,\mathrm{Im}\,U^3}\ , \quad \text{for } N_V = 4 \tag{2.20}$$

and

$$m_{3/2}^2 = \frac{1}{4\,\mathrm{Im}\,T^2\,\mathrm{Im}\,U^2} + \frac{\mathrm{Im}\,U^2}{4\,\mathrm{Im}\,T^2} + \frac{1}{4\,\mathrm{Im}\,T^3\,\mathrm{Im}\,U^3} + \frac{\mathrm{Im}\,U^3}{4\,\mathrm{Im}\,T^3}\ , \quad \text{for } N_V = 2\,. \tag{2.21}$$

Owing to the effective restoration of $N=4$ supersymmetry in this limit, there is no linear behaviour either in $\mathrm{Im}\,T^2$ or in $\mathrm{Im}\,T^3$; the remaining contribution is logarithmic:

$$\frac{16\,\pi^2}{g^2_{\mathrm{grav}}(\mu^{(\mathrm{II})})} \underset{\mathrm{Im}\,T^2,\,\mathrm{Im}\,T^3\to\infty}{\longrightarrow} -2\log\,\mathrm{Im}\,T^2 - 2\log\,\mathrm{Im}\,T^3\,. \tag{2.22}$$

(iv) The threshold corrections diverge linearly in the large $\mathrm{Im}\,T^1$ limit.

$$\frac{16\,\pi^2}{g^2_{\mathrm{grav}}\left(\mu^{(\mathrm{II})}\right)} \underset{\mathrm{Im}\,T^1\to\infty}{\longrightarrow} \frac{\pi N_V}{4}\,\mathrm{Im}\,T^1 - \left(2+\frac{N_V}{2}\right)\log\,\mathrm{Im}\,T^1 + \left(6+\frac{N_V}{2}\right)\log\frac{M^{(\mathrm{II})}}{\mu^{(\mathrm{II})}}\,. \tag{2.23}$$

Observe that the coefficients of the linear and logarithmic term have a different dependence on N_V. The coefficient of the third term, $6+\frac{N_V}{2}$, is actually $B_4/3$, while that of the second term, $2+\frac{N_V}{2}$, is $\left(B_4 - 6b_{\mathrm{grav}}\right)/3$, where by B_4 we actually intend the massless contribution to this quantity: $B_4|_{\mathrm{massless}}$ given in Eq. (2.15); b_{grav} is the "gravitational beta-function", $b_{\mathrm{grav}} = \left(24 - N_V + N_H\right)/12$, as computed in field theory. Notice that in general, when $N_V \neq N_H$, the relevant quantities that appear in the above corrections are:

$$\frac{B_4 - B_2}{3} - 2b_{\mathrm{grav}} = 2 + \frac{5N_V + N_H}{12} \qquad \text{and} \qquad \frac{B_4 - B_2}{3} = 6 + \frac{N_V + N_H}{4}, \tag{2.24}$$

where by B_4 and B_2 we intend, as before, the massless contribution to these quantities ($B_2|_{\mathrm{massless}} = N_V - N_H$).

(v) Under $T^i \leftrightarrow U^i$ interchange, we obtain the results of a mirror type IIB model.

3 Heterotic duals

3.1 Outline

Our scope is now to determine the heterotic duals of the type II ground states with $N_V = 8, 4, 2$ discussed in the previous section. In view of this construction, some basic properties and requirements have to be settled, namely:

(i) The $N=2$ heterotic models must have the same massless spectrum as their type II duals.

(ii) The type II dual of the heterotic dilaton $S_{\rm Het}$ is one of the type IIA(B) moduli, T_D, and belongs to a vector multiplet. Thus, the perturbative heterotic limit ($S_{\rm Het}$ large) corresponds in type II to the limit of large T_D. This implies the identification of T_D with T^1 in our type IIA constructions.

(iii) The $N=2$ heterotic ground state must describe a spontaneously broken phase of an $N=4$ model. The limit in which the $N=4$ is restored corresponds to large perturbative type IIA vector-multiplet moduli T^2, T^3 (see Eqs. (2.19)–(2.21) and (2.22)). Therefore, we must identify the T^2 and T^3 of type IIA with two perturbative moduli of the heterotic duals, $T_{\rm Het}$ and $U_{\rm Het}$.

(iv) On the type II side, the T-duality group is a *subgroup* of

$$SL(2,Z)_{T^1} \times SL(2,Z)_{T^2} \times SL(2,Z)_{T^3}\,. \tag{3.1}$$

Moreover, the duality symmetries of $T^1 \equiv S_{\rm Het}$ translate into the non-perturbative instanton-duality properties on the heterotic side, while the symmetries of the moduli $T^2 \equiv T_{\rm Het}$ and $T^3 \equiv U_{\rm Het}$ appear in the heterotic duals as perturbative T-duality symmetries. These properties can be summarized as follows:

$$\begin{array}{lll} \text{type IIA:} & SL(2,Z)_{T^1}\;, & SL(2,Z)_{T^2}\times SL(2,Z)_{T^3} \\ & \quad\updownarrow & \qquad\qquad\updownarrow \\ \text{heterotic:} & SL(2,Z)_{S}\;, & SL(2,Z)_{T}\times SL(2,Z)_{U}\,, \end{array} \tag{3.2}$$

where the $SL(2,Z)$'s can be broken to some $\Gamma(2)$ subgroups.

On the heterotic side, the general expression for the helicity-generating function of Z_2-orbifold models is

$$Z^{N_V}_{\rm Het}(v,\bar v) = \frac{\xi(v)\,\bar\xi(\bar v)}{|\eta|^4}\,\frac{1}{2}\sum_{H^{\rm f},G^{\rm f}} Z^{N_V}_{6,22}\left[{H^{\rm f} \atop G^{\rm f}}\right] Z^{\rm F}_{\rm L}\left[{H^{\rm f} \atop G^{\rm f}}\right](v)\,, \tag{3.3}$$

where $Z^{\rm F}_{\rm L}\left[{H^{\rm f} \atop G^{\rm f}}\right](v)$ is the contribution of the 8 left-moving world-sheet fermions ψ^μ, Ψ^I in the light-cone gauge (see (2.10) with $H^{\rm o}=G^{\rm o}=0$) and $Z^{N_V}_{6,22}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ accounts for the (6,22) compactified coordinates. In order to allow for a comparison with the type II orbifolds, the $\Gamma_{2,2}(T,U)$ shifted lattice needs to be separated. It is also necessary to choose special values of the remaining Wilson-line moduli, Y_I, $I=1,\dots,N_V$ of the $\Gamma_{2,2+N_V}$ lattice, which break the gauge group to $U(1)$ factors. At such points, $Z^{N_V}_{6,22}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ takes the following form:

$$Z^{N_V}_{6,22}\left[{H^{\rm f} \atop G^{\rm f}}\right] = \frac{1}{2^{n+1}}\sum_{\vec h,\vec g}\frac{\Gamma_{2,2}\left[{H^{\rm f},\vec h \atop G^{\rm f},\vec g}\right]}{|\eta|^4}\,\frac{\Gamma_{4,4}\left[{H^{\rm f}|\vec h \atop G^{\rm f}|\vec g}\right]}{|\eta|^8}\sum_{\gamma,\delta}\overline{\Phi}^V\left[{H^{\rm f},\gamma,\vec h \atop G^{\rm f},\delta,\vec g}\right]\overline{\Phi}^H\left[{H^{\rm f},\gamma,\vec h \atop G^{\rm f},\delta,\vec g}\right], \tag{3.4}$$

where $(\vec h,\vec g)$ denote either the values of the Wilson lines or the heterotic duals of the Z_2^D operations; n is the number of projections (dimension of the vectors $(\vec h,\vec g)$) needed to reach the correct N_V. The specific lattice shifts define the modular-transformation properties of the multi-shifted two-torus lattice sum $\Gamma_{2,2}\left[{H^{\rm f},\vec h \atop G^{\rm f},\vec g}\right]$, and must fit with the transformation properties of the $\overline{\Phi}$'s in order to lead to the proper modular-covariance properties of $Z^{N_V}_{6,22}\left[{H^{\rm f} \atop G^{\rm f}}\right]$.

Taking into account the above considerations, we will now construct the heterotic duals of the various reduced-rank type II models.

3.2 Construction of the heterotic reduced-rank models

a) The $N_V = N_H = 8$ model

Our starting point is the $N = 4$ heterotic ground state obtained by compactification on $T^4 \times T^2$ and its type II dual on $K3 \times T^2$. In order to reduce the $N = 4$ supersymmetry to $N = 2$ we must define an appropriate $Z_2^{(\mathrm{f})}$ freely-acting orbifold, which simultaneously reduces by a factor of 2 the rank of the gauge group.

To properly define the $Z_2^{(\mathrm{f})}$ action, we work at a point of the moduli space where the $E_8 \times E_8$ gauge group is broken to $\left(SU(2)^{(1)}_{k=1} \times SU(2)^{(2)}_{k=1}\right)^8 \simeq SO(4)^8_{k=1}$. This point is reached by switching on appropriate Wilson lines of the $(6,22)$ lattice. The action of $Z_2^{(\mathrm{f})}$ then amounts to (i) a translation on T^2, which produces a half-unit-vector shift in the $(2,2)$ lattice; (ii) a $Z_2^{(\mathrm{f})}$ symmetric twist on T^4, which breaks the $N = 4$ supersymmetry to $N = 2$; (iii) a pair-wise interchange of $SU(2)^{(1)}$ with $SU(2)^{(2)}$.

We can give an explicit expression for the partition function of this orbifold by using the following $SO(4)$ twisted characters:

$$F_1\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \equiv \frac{1}{\eta^2}\,\vartheta^{1/2}\begin{bmatrix}\gamma+h_1\\ \delta+g_1\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_2\\ \delta+g_2\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_3\\ \delta+g_3\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma-h_1-h_2-h_3\\ \delta-g_1-g_2-g_3\end{bmatrix} \tag{3.5}$$

and

$$F_2\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \equiv \frac{1}{\eta^2}\,\vartheta^{1/2}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_1-h_2\\ \delta+g_1-g_2\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_2-h_3\\ \delta+g_2-g_3\end{bmatrix}\vartheta^{1/2}\begin{bmatrix}\gamma+h_3-h_1\\ \delta+g_3-g_1\end{bmatrix}, \tag{3.6}$$

where we introduced the notation (valid all over the paper) $h \equiv (h_1, h_2, h_3)$ and similarly for g. Under $\tau \to \tau + 1$, F_I transform as:

$$\begin{aligned} F_1\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \;&\to\; F_1\begin{bmatrix}\gamma, h\\ \gamma+\delta+1, h+g\end{bmatrix}\\ &\times \exp -\frac{i\pi}{4}\left(\frac{2}{3} - 4\gamma + 2\gamma^2 + h_1^2 + h_2^2 + h_3^2 + h_1h_2 + h_2h_3 + h_3h_1\right) \qquad (3.7)\\ F_2\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \;&\to\; F_2\begin{bmatrix}\gamma, h\\ \gamma+\delta+1, h+g\end{bmatrix}\\ &\times \exp -\frac{i\pi}{4}\left(\frac{2}{3} - 4\gamma + 2\gamma^2 + h_1^2 + h_2^2 + h_3^2 - h_1h_2 - h_2h_3 - h_3h_1\right). \qquad (3.8)\end{aligned}$$

Notice that $\overline{F}_I$ are $c = (0, 2)$ conformal characters of 4 different right-moving Isings. In the fermionic language [27] this is a system of 4 right-moving real fermions with different boundary conditions. All currents $\bar{J}^{IJ} = \bar{\Psi}^I\bar{\Psi}^J$ are twisted and therefore the initial $SO(4)$ is broken. On the other hand, when $\overline{F}_I$ is raised to a power n, the gauge group becomes $SO(n)^4$:

$$\overline{F}_I^{\,n}\begin{bmatrix}\gamma, h\\ \delta, g\end{bmatrix} \to SO(n)^4\,. \tag{3.9}$$

In this framework, the partition function of the model with $N_V = N_H = 8$ is given by (3.3) and (3.4), with Φ^V and Φ^H expressed in terms of combinations of $F_1\left[{\gamma,h \atop \delta,g}\right]$ and $F_2\left[{\gamma,h \atop \delta,g}\right]$:

$$\Phi^V\left[{\gamma,\vec{h} \atop \delta,\vec{g}}\right] = F_1^2\left[{\gamma,h \atop \delta,g}\right] F_2^2\left[{\gamma,h \atop \delta,g}\right] \tag{3.10}$$

(no dependence on $(H^{\rm f},G^{\rm f})$), which leads to a group [3] $G = U(1)^8$ and therefore $N_V = 8$, whereas

$$\Phi^H\left[{H^{\rm f},\gamma,\vec{h} \atop G^{\rm f},\delta,\vec{g}}\right] = F_1\left[{\gamma',h \atop \delta',g}\right] F_1\left[{\gamma'+H^{\rm f},h \atop \delta'+G^{\rm f},g}\right] F_2\left[{\gamma',h \atop \delta',g}\right] F_2\left[{\gamma'+H^{\rm f},h \atop \delta'+G^{\rm f},g}\right], \tag{3.11}$$

leading to $N_H = 8$ (here $\vec{h} \equiv (h_1,h_2,h_3,h_4) = (h,h_4)$, $\vec{g} \equiv (g_1,g_2,g_3,g_4) = (g,g_4)$ and $(\gamma',\delta') = (\gamma+h_4,\delta+g_4)$). For this model, we must use the simply-shifted $(2,2)$ lattice sum $\Gamma_{2,2}\left[{H^{\rm f} \atop G^{\rm f}}\right]$, where the shift is asymmetric on one circle S^1, projection $(-)^{(m_2+n_2)G^{\rm f}}$, in order to cancel the phase of $Z_{\rm L}^{\rm F}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ under modular transformations (this projection was referred to as "XIII" in Ref. [24], where the various lattice shifts were discussed in detail). On the other hand, the h_i shifts in $\Gamma_{4,4}$ must be symmetric: $(-)^{M_i g_i}$.

An alternative construction is obtained by shifting with $(H^{\rm f},G^{\rm f})$ two $U(1)$ factors in G:

$$\begin{aligned}\tilde{\Phi}^V\left[{H^{\rm f},\gamma,\vec{h} \atop G^{\rm f},\,\delta,\vec{g}}\right] &= \frac{1}{\eta^8}\,\vartheta\left[{\gamma+h_1+H^{\rm f} \atop \delta+g_1+G^{\rm f}}\right]\vartheta\left[{\gamma+h_2 \atop \delta+g_2}\right]\vartheta\left[{\gamma+h_3 \atop \delta+g_3}\right]\vartheta\left[{\gamma-h_1-h_2-h_3 \atop \delta-g_1-g_2-g_3}\right] \\ &\quad\times\vartheta\left[{\gamma-H^{\rm f} \atop \delta-G^{\rm f}}\right]\vartheta\left[{\gamma+h_1-h_2 \atop \delta+g_1-g_2}\right]\vartheta\left[{\gamma+h_2-h_3 \atop \delta+g_2-g_3}\right]\vartheta\left[{\gamma+h_3-h_1 \atop \delta+g_3-g_1}\right].\end{aligned} \tag{3.12}$$

The $(2,2)$ lattice is now double-shifted, and the lattice sum $\Gamma_{2,2}\left[{H^{\rm f},h_1 \atop G^{\rm f},g_1}\right]$ is given with the insertion $(-)^{m_2 G^{\rm f}+n_2 g_1}$. Both models, with Φ^V and $\tilde{\Phi}^V$, have the same $N=4$ sector (defined by $(H^{\rm f},G^{\rm f}) = (0,0)$, whose contribution is half the partition function of an $N=4$ model in which the gauge group $E_8\times E_8$ is broken to $U(1)^{16}$. In the $N=2$ sectors, $(H^{\rm f},G^{\rm f}) \neq (0,0)$ and (h_i,g_i) are either $(0,0)$ or $(H^{\rm f},G^{\rm f})$. This constraint comes from the $(H^{\rm f},G^{\rm f})$-twisted sector of $\Gamma_{4,4}\left[{H^{\rm f}|\vec{h} \atop G^{\rm f}|\vec{g}}\right]$.

As a consistency check, we can now proceed to the computation of the helicity supertrace B_2. The $(N=4)$-sector contribution to this quantity vanishes. For the model constructed with Φ^V (Eq. (3.10)) we find:

$$B_2\left(\Phi^V\right) = \frac{1}{\bar{\eta}^{24}}\sum_{(H^{\rm f},G^{\rm f})}{}' \,\Gamma_{2,2}^{\lambda=1}\left[{H^{\rm f} \atop G^{\rm f}}\right]\overline{\Omega}\left[{H^{\rm f} \atop G^{\rm f}}\right], \tag{3.13}$$

[3] Observe that the right moving gauge group is systematically $U(1)^2 \times G$.

where $\Omega\left[{H^{\rm f} \atop G^{\rm f}}\right]$ are the following analytic functions [4]:

$$
\begin{aligned}
\Omega\begin{bmatrix}0\\1\end{bmatrix} &= \frac{1}{16}\left(\vartheta_3^8+\vartheta_4^8+14\,\vartheta_3^4\,\vartheta_4^4\right)\vartheta_3^6\,\vartheta_4^6\\
\Omega\begin{bmatrix}0\\1\end{bmatrix} &= -\frac{1}{16}\left(\vartheta_2^8+\vartheta_3^8+14\,\vartheta_2^4\,\vartheta_3^4\right)\vartheta_2^6\,\vartheta_3^6\\
\Omega\begin{bmatrix}0\\1\end{bmatrix} &= \frac{1}{16}\left(\vartheta_2^8+\vartheta_4^8-14\,\vartheta_2^4\,\vartheta_4^4\right)\vartheta_2^6\,\vartheta_4^6\,;
\end{aligned}
\tag{3.14}
$$

the lattice sum $\Gamma_{2,2}^{\lambda=1}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ corresponds to the projection $(-)^{(m_2+n_2)G^{\rm f}}$ (lattice "XIII").

For the model constructed with $\tilde{\Phi}^V$ (Eq. (3.12)) we find instead:

$$
B_2\left(\tilde{\Phi}^V\right)=\frac{1}{\bar\eta^{24}}\sum_{(H^{\rm f},G^{\rm f})}{}'\,\frac{1}{2}\left(\Gamma_{2,2}^{\lambda=0}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\overline{\Omega}^{(0)}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}+\Gamma_{2,2}^{\lambda=1}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\overline{\Omega}^{(1)}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\right)\,, \tag{3.15}
$$

where in this case

$$
\begin{aligned}
\Omega^{(0)}\begin{bmatrix}0\\1\end{bmatrix} &= \frac{1}{2}\left(\vartheta_3^4+\vartheta_4^4\right)\vartheta_3^8\,\vartheta_4^8\\
\Omega^{(0)}\begin{bmatrix}1\\0\end{bmatrix} &= -\frac{1}{2}\left(\vartheta_2^4+\vartheta_3^4\right)\vartheta_2^8\,\vartheta_3^8\\
\Omega^{(0)}\begin{bmatrix}1\\1\end{bmatrix} &= \frac{1}{2}\left(\vartheta_2^4-\vartheta_4^4\right)\vartheta_2^8\,\vartheta_4^8
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
\Omega^{(1)}\begin{bmatrix}0\\1\end{bmatrix} &= \vartheta_3^{10}\,\vartheta_4^{10}\\
\Omega^{(1)}\begin{bmatrix}1\\0\end{bmatrix} &= -\vartheta_2^{10}\,\vartheta_3^{10}\\
\Omega^{(1)}\begin{bmatrix}1\\1\end{bmatrix} &= -\vartheta_2^{10}\,\vartheta_4^{10}\,.
\end{aligned}
\tag{3.17}
$$

The lattice sums $\Gamma_{2,2}^{\lambda=0}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ and $\Gamma_{2,2}^{\lambda=1}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ correspond to simply-shifted lattices with projections respectively $(-)^{m_2 G^{\rm f}}$ and $(-)^{(m_2+n_2)G^{\rm f}}$ (in [24] they are referred to respectively as "II" and "XIII").

It is easy to check that, in both constructions, the massless contribution to the B_2 vanishes, as it should for models where $N_V = N_H$.

[4]The parameter λ, which takes the values 0 or 1, determines the phases appearing in the modular transformations of the shifted lattice sums. Under modular transformations, the functions Ω acquire phases that are complementary to those coming from the lattice $\Gamma_{2,2}^{\lambda=1}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ that appears in (3.13). This ensures that the spin connection is correctly embedded. Similarly the $\Omega^{(0)}$'s and $\Omega^{(1)}$'s in (3.15) are respectively of type $\lambda=0$ and $\lambda=1$.

b) The $N_V = N_H = 4$ heterotic model

This model is realized by using the following functions:

$$\Phi^V \begin{bmatrix} \gamma, \vec{h} \\ \delta, \vec{g} \end{bmatrix} = \frac{1}{\eta^4}\, \vartheta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_1 - h_2 \\ \delta + g_1 - g_2 \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_2 - h_3 \\ \delta + g_2 - g_3 \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_3 - h_1 \\ \delta + g_3 - g_1 \end{bmatrix} \tag{3.18}$$

(no dependence on $(H^{\rm f}, G^{\rm f})$) and

$$\begin{aligned} \Phi^H \begin{bmatrix} H^{\rm f}, \gamma, \vec{h} \\ G^{\rm f}, \delta, \vec{g} \end{bmatrix} &= F_1 \begin{bmatrix} \gamma + h_4 + h_5, h \\ \delta + g_4 + g_5, g \end{bmatrix} F_1 \begin{bmatrix} \gamma + h_4 + h_5 + H^{\rm f}, h \\ \delta + g_4 + g_5 + G^{\rm f}, g \end{bmatrix} \\ &\times F_1 \begin{bmatrix} \gamma + h_5, h \\ \delta + g_5, g \end{bmatrix} F_1 \begin{bmatrix} \gamma + h_4 + H^{\rm f}, h \\ \delta + g_4 + G^{\rm f}, g \end{bmatrix} \\ &\times F_2 \begin{bmatrix} \gamma + h_5, h \\ \delta + g_5, g \end{bmatrix} F_2 \begin{bmatrix} \gamma + h_4, h \\ \delta + g_4, g \end{bmatrix}. \end{aligned} \tag{3.19}$$

Now $\vec{h} \equiv (h_1, h_2, h_3, h_4, h_5) = (h, h_4, h_5)$ and similarly for $\vec{g}$. The shift in the $(2,2)$ lattice is asymmetric and along a single circle with projection $(-)^{(m_2+n_2)G^{\rm f}}$, while it is symmetric in the $(4,4)$ block: $(-)^{M_i g_i}$, $i = 1, \dots, 4$.

Again an alternative construction exists for Φ^V; it is given by

$$\tilde{\Phi}^V \begin{bmatrix} H^{\rm f}, \gamma, \vec{h} \\ G^{\rm f},\, \delta, \vec{g} \end{bmatrix} = \frac{1}{\eta^4}\, \vartheta \begin{bmatrix} \gamma - H^{\rm f} \\ \delta - G^{\rm f} \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_1 - h_2 + H^{\rm f} \\ \delta + g_1 - g_2 + G^{\rm f} \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_2 - h_3 \\ \delta + g_2 - g_3 \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_3 - h_1 \\ \delta + g_3 - g_1 \end{bmatrix}. \tag{3.20}$$

In this case the $(2,2)$ lattice sum is $\Gamma_{2,2}\left[{}^{H^{\rm f}, h_1 - h_2}_{G^{\rm f}, g_1 - g_2}\right]$; the shift corresponds to the projection $(-)^{m_2 G^{\rm f} + n_2 (g_1 - g_2)}$.

Finally, the computation of B_2 can be performed, and we obtain the same results as in the $N_V = N_H = 8$ model, summarized in Eqs. (3.13) and (3.15).

c) The $N_V = N_H = 2$ heterotic model

Here, we introduce the characters

$$\widehat{F}_1 \begin{bmatrix} \gamma, h \\ \delta, g \end{bmatrix} \equiv \frac{1}{\eta^2}\, \vartheta^{1/2} \begin{bmatrix} \gamma - h_1 - h_2 - h_3 \\ \delta - g_1 - g_2 - g_3 \end{bmatrix} \vartheta^{1/2} \begin{bmatrix} \gamma + h_3 \\ \delta + g_3 \end{bmatrix} \vartheta^{1/2} \begin{bmatrix} \gamma + h_3 - h_1 \\ \delta + g_3 - g_1 \end{bmatrix} \vartheta^{1/2} \begin{bmatrix} \gamma + h_2 - h_3 \\ \delta + g_2 - g_3 \end{bmatrix} \tag{3.21}$$

and

$$\widehat{F}_2 \begin{bmatrix} \gamma, h \\ \delta, g \end{bmatrix} \equiv \frac{1}{\eta^2}\, \vartheta^{1/2} \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \vartheta^{1/2} \begin{bmatrix} \gamma + h_1 - h_2 \\ \delta + g_1 - g_2 \end{bmatrix} \vartheta^{1/2} \begin{bmatrix} \gamma + h_1 \\ \delta + g_1 \end{bmatrix} \vartheta^{1/2} \begin{bmatrix} \gamma + h_2 \\ \delta + g_2 \end{bmatrix}. \tag{3.22}$$

Note that the product $\widehat{F}_1\left[{}^{\gamma,h}_{\delta,g}\right] \widehat{F}_2\left[{}^{\gamma,h}_{\delta,g}\right]$ has the same modular properties as $F_1\left[{}^{\gamma,h}_{\delta,g}\right] F_2\left[{}^{\gamma,h}_{\delta,g}\right]$. For the model at hand,

$$\Phi^V \begin{bmatrix} \gamma, \vec{h} \\ \delta, \vec{g} \end{bmatrix} = \frac{1}{\eta^2}\, \vartheta \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \vartheta \begin{bmatrix} \gamma + h_1 - h_2 \\ \delta + g_1 - g_2 \end{bmatrix} \tag{3.23}$$

(there is no dependence on $(H^{\rm f},G^{\rm f})$ and $\vec h,\vec g$ stand again for $(h_1,\dots,h_5),(g_1,\dots,g_5)$), and

$$\begin{aligned}\Phi^V\begin{bmatrix}\gamma,\vec h\\ \delta,\vec g\end{bmatrix}\Phi^H\begin{bmatrix}H^{\rm f},\gamma,\vec h\\ G^{\rm f},\ \delta,\vec g\end{bmatrix} \;=\;& \widehat F_1\begin{bmatrix}\gamma+h_4+h_5,h\\ \delta+g_4+g_5,g\end{bmatrix}F_1\begin{bmatrix}\gamma+h_4+h_5+H^{\rm f},h\\ \delta+g_4+g_5+G^{\rm f},g\end{bmatrix}\\ &\times\widehat F_2\begin{bmatrix}\gamma,h\\ \delta,g\end{bmatrix}F_2\begin{bmatrix}\gamma,h\\ \delta,g\end{bmatrix}\\ &\times F_1\begin{bmatrix}\gamma+h_5,h\\ \delta+g_5,g\end{bmatrix}F_1\begin{bmatrix}\gamma+h_4+H^{\rm f},h\\ \delta+g_4+G^{\rm f},g\end{bmatrix}\\ &\times F_2\begin{bmatrix}\gamma+h_5,h\\ \delta+g_5,g\end{bmatrix}F_2\begin{bmatrix}\gamma+h_4,h\\ \delta+g_4,g\end{bmatrix}. \end{aligned} \tag{3.24}$$

The structure of Φ^V now shows that $G=U(1)^2$ and therefore $N_V=2$. The shifts in the $(2,2)$ and $(4,4)$ lattices are the same as those in the $N_V=4$ model.

An alternative embedding of $(H^{\rm f},G^{\rm f})$ is realized as follows:

$$\tilde\Phi^V\begin{bmatrix}H^{\rm f},\gamma,\vec h\\ G^{\rm f},\ \delta,\vec g\end{bmatrix}=\frac{1}{\eta^2}\,\vartheta\begin{bmatrix}\gamma+H^{\rm f}\\ \delta+G^{\rm f}\end{bmatrix}\vartheta\begin{bmatrix}\gamma+h_1-h_2+H^{\rm f}\\ \delta+g_1-g_2+G^{\rm f}\end{bmatrix} \tag{3.25}$$

and

$$\begin{aligned}\tilde\Phi^V\begin{bmatrix}H^{\rm f},\gamma,\vec h\\ G^{\rm f},\ \delta,\vec g\end{bmatrix}\tilde\Phi^H\begin{bmatrix}H^{\rm f},\gamma,\vec h\\ G^{\rm f},\ \delta,\vec g\end{bmatrix} \;=\;& \widehat F_1\begin{bmatrix}\gamma+h_4+h_5,h\\ \delta+g_4+g_5,g\end{bmatrix}F_1\begin{bmatrix}\gamma+h_4+h_5+H^{\rm f},h\\ \delta+g_4+g_5+G^{\rm f},g\end{bmatrix}\\ &\times\widehat F_2\begin{bmatrix}\gamma+H^{\rm f},h\\ \delta+G^{\rm f},g\end{bmatrix}F_2\begin{bmatrix}\gamma+H^{\rm f},h\\ \delta+G^{\rm f},g\end{bmatrix}\\ &\times F_1\begin{bmatrix}\gamma+h_5,h\\ \delta+g_5,g\end{bmatrix}F_1\begin{bmatrix}\gamma+h_4,h\\ \delta+g_4,g\end{bmatrix}\\ &\times F_2\begin{bmatrix}\gamma+h_5,h\\ \delta+g_5,g\end{bmatrix}F_2\begin{bmatrix}\gamma+h_4,h\\ \delta+g_4,g\end{bmatrix}; \end{aligned} \tag{3.26}$$

this also leads to $N_V=N_H=2$. As for the $N_V=4$ model defined through (3.19) and (3.20), it is necessary to perform a double shift in the $(2,2)$ lattice with the projection $(-)^{m_2G^{\rm f}+n_2(g_1-g_2)}$. The shift in the $(4,4)$ lattice is identical to the one used together with the above Φ^V defined in (3.23).

Here also the second helicity supertrace is given in (3.13) or (3.15).

3.3 Some general comments on the heterotic duals

Our first observation is that all models with $N_V=N_H$ defined above have identical B_2 helicity supertrace, which is given either by (3.13) or by (3.15), depending on the embedding of the $Z_2^{(\rm f)}$. This universality follows from (i) the modular-transformation properties of $\Gamma^{\lambda}_{2,2}\left[{}^{H^{\rm f}}_{G^{\rm f}}\right]$, (ii) the condition $N_V=N_H$, and (iii) the spontaneous breaking of $N=4$ supersymmetry to $N=2$. These three requirements fix uniquely the functions Ω. We thus expect that this universality of B_2 will remain valid for all $N_V = N_H$ models in which the $U(1)^r$ gauge group is extended to a larger gauge group G_r with the same rank.

This can be checked explicitly in several examples, for instance the model with a gauge group

$$G_r = SU(2)^8_{k=2} \tag{3.27}$$

defined with the choice ($\vec{h} \equiv (h_1, h_2, h_3) = h$, and similarly for $\vec{g}$)

$$\Phi^V \begin{bmatrix} \gamma, \vec{h} \\ \delta, \vec{g} \end{bmatrix} = F_1^3 \begin{bmatrix} \gamma, h \\ \delta, g \end{bmatrix} F_2^3 \begin{bmatrix} \gamma, h \\ \delta, g \end{bmatrix} \tag{3.28}$$

$$\Phi^H \begin{bmatrix} H^{\rm f}, \gamma, \vec{h} \\ G^{\rm f},\, \delta, \vec{g} \end{bmatrix} = F_1 \begin{bmatrix} \gamma + H^{\rm f}, h \\ \delta + G^{\rm f}, g \end{bmatrix} F_2 \begin{bmatrix} \gamma + H^{\rm f}, h \\ \delta + G^{\rm f}, g \end{bmatrix} . \tag{3.29}$$

From this model one obtains the model $N_V = 8$ by switching on discrete Wilson lines, which break the $SU(2)_{k=2}$ factors to eight $U(1)$'s. Actually, when (h_4, g_4) is turned on in Eq. (3.11), we obtain the model $N_V = 8$; when it is turned off we obtain instead (3.27). Switching on a non-zero (h_4, g_4) is equivalent to switching on a Wilson line defined as $Y_I = (2p_I + 1)h_4 + 2q_I$, with (p_I, q_I) integers. Another model with $r = 8$ is the one constructed in [10], based on E_8 at level two. In that case h defines discrete Wilson lines that break $(E_8)_{k=2}$ to $SU(2)^8_{k=2}$.

The connection of our previous construction to $(E_8)_{k=2}$ and $SU(2)^8_{k=2}$ is useful and describes well the action of $Z_2^{(\rm f)}$ on the heterotic and type II duals. At the orbifold point of $K3 \sim T^4/Z_2^{(\rm o)}$, the intersection matrix of $H^2(K3)$ is given by the following elements:

$$L_{IJ} = \left(\Gamma_8^{(\rm o)} \oplus \Gamma_8^{(\rm o)} \oplus \sigma^1 \oplus \sigma^1 \oplus \sigma^1\right)_{IJ} \ , \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{3.30}$$

with

$$\Gamma_8^{(\rm o)} \oplus \Gamma_8^{(\rm o)} = (-A_1) \oplus \cdots \oplus (-A_1) \, , \tag{3.31}$$

A_1 being the Cartan matrix of $SU(2)$. On type IIA, $Z_2^{(\rm f)}$ has 8 positive and 8 negative eigenvalues in the submatrix (3.31), and thus removes 8 of the 16 fixed points.

On the heterotic side, $Z_2^{(\rm f)}$ interchanges the eight $SU(2)^{(1)}$ with the eight $SU(2)^{(2)}$. This results in a level-two realization of $SU(2)^8$. Furthermore, on type IIA, $Z_2^{(\rm f)}$ has 2 positive and 4 negative eigenvalues on the torus intersection matrix $\sigma^1 \oplus \sigma^1 \oplus \sigma^1$; in our constructions, it rotates by π two σ^1 matrices ($\sigma^1 \to -\sigma^1$) and leaves invariant the third σ^1. On the heterotic side, $Z_2^{(\rm f)}$ acts as a π rotation in T^4 coordinates and as a translation in T^2 ones. The 4 additional negative eigenvalues required by modular invariance on the heterotic side can be embedded either in the gauge group (leading therefore to $\Gamma_{2,2}^{\lambda=0}$) or directly in the $\Gamma_{2,2}$, where they produce an asymmetric shift ($\Gamma_{2,2}^{\lambda=1}$).

In the models $N_V = 4$ and 2, also the actions of Z_2^D on heterotic side and on the intersection matrix of T^4/Z_2 correspond. On type IIA, the Z_2^D projection has 8 positive and 8 negative eigenvalues on the 16 twist fields; it acts by exchanging eight A_1 matrices with eight others. Four such exchanges are common also to $Z_2^{(\rm f)}$, and also four are the exchanges that are common to Z_2^D and $Z_2^{D'}$.

On the heterotic side, the corresponding operations act as a lattice exchange on the $SU(2)^{16}$ gauge group and as a translation in the six-dimensional internal space. Since for each one of them the orbifold twist has 8 negative eigenvalues, modular invariance forces the shift to be symmetric. On the compact space, therefore, the action of the Z_2^D projections is analogous to that of the Z_2^i, $i = 1, \ldots, 5$, which break the gauge group. When combined with the shift due to $Z_2^{(\mathrm{f})}$, we therefore obtain the same two different embeddings of the shift in $\Gamma_{2,2}$ as for the $N_V = 8$ model.

It is interesting to observe that the construction of Ref. [10] is based on a different separation of $\Gamma_{2,2+8}$:

$$\Gamma_{2,2+8}(T, U, \vec{Y}) = \Gamma_{1,1}^{\lambda=1}(R) \begin{bmatrix} H^{\mathrm{f}} \\ G^{\mathrm{f}} \end{bmatrix} \Gamma_{1,1+8}^{+} \begin{bmatrix} H^{\mathrm{f}} \\ G^{\mathrm{f}} \end{bmatrix}, \tag{3.32}$$

where $\Gamma_{1,1+8}^{+}$ is a lattice invariant under the interchange of $\Gamma_{1,1+8}^{1}$ with $\Gamma_{1,1+8}^{2}$ induced by $Z_2^{(\mathrm{f})}$. In this case, not only a rank 8 part but also one of the $U(1)$'s of the compact space can be enhanced to $SU(2)_{k=2}$. This separation, in which only one circle is singled out, is not useful for our purpose. The separation of a shifted $\Gamma_{2,2}$ lattice is necessary because we want to identify the perturbative heterotic moduli T, U with the moduli T^2, T^3 of type IIA. This has led us to the two heterotic constructions, based on Φ^V and $\tilde{\Phi}^V$, where the part of the gauge group that comes from the separated two-torus is always realized at the level one.

The two classes of $N_V = N_H$ heterotic orbifolds, Φ^V and $\tilde{\Phi}^V$, correspond to different regions in the moduli space of the lattice $\Gamma_{2,2+r}\left(T, U, \vec{Y}\right)$. Here, we would like to argue that the perturbative constructions based on $\tilde{\Phi}^V$ are the most suitable for comparison with the type IIA orbifolds presented in Section 2. In fact, in models based on Φ^V, even though the $Z_2^{(\mathrm{f})}$ translation in the two-torus produces a spontaneous breaking of the $N = 4$ to $N = 2$ supersymmetry, *it does not reproduce the situation of the perturbative type IIA orbifold*. On type IIA, the restoration of the $N = 4$ supersymmetry is achieved by taking only appropriate limits in the perturbative moduli: for the specific lattice shifts we considered, the $N = 4$ supersymmetry is restored when the moduli T^1 and T^2 are large, while in the opposite limit supersymmetry is broken to $N = 2$ as in a non-freely-acting orbifold. In the heterotic construction based on Φ^V, however, the $N = 4$ supersymmetry is always restored in any decompactification limit, because, for any choice of direction of the shift, a ($\lambda = 1$)-shifted lattice sum vanishes both for large and for small moduli. It is therefore impossible to find a translation that reproduces the perturbative properties of the type IIA duals. In this case the map between the heterotic moduli T and U and the type IIA moduli T^2 and T^3 is non-linear. On the other hand, in the constructions with $\tilde{\Phi}^V$, the $N = 4$ supersymmetry is again spontaneously broken because of the $Z_2^{(\mathrm{f})}$ translation on the T^2. Here, however, the appropriate limit of $N = 4$ supersymmetry restoration is determined by the choice of the shifts in the $\Gamma_{2,2}^{\lambda=0}$ lattice since the term with $\Gamma_{2,2}^{\lambda=1}$ becomes irrelevant: this kind of shifted lattice sum vanishes in any decompactification limit. For the particular $Z_2^{(\mathrm{f})}$ shift we have considered, $(-)^{m_2 G^{\mathrm{f}}}$, in all the models based on $\tilde{\Phi}^V$, the mass of the two extra gravitinos is

$$m_{3/2}^2 = \frac{1}{4\,\mathrm{Im}\,T\,\mathrm{Im}\,U}\,. \tag{3.33}$$

The $N = 4$ supersymmetry is restored only when $R_2 = \sqrt{\mathrm{Im}\, T\, \mathrm{Im}\, U}$ is large, whereas for small values of the (T, U) moduli we recover a genuine $N = 2$ non-freely-acting orbifold. This is precisely the perturbative behaviour of the above type IIA dual orbifolds.

Finally, we would like to comment on another important issue, namely on the appearance of extra massless states along submanifolds of the moduli space. In particular we are interested in "$N = 2$ singularities" corresponding to enhancements where $\Delta N_V - \Delta N_H \neq 0$. Perturbatively, in the type IIA models of Section 2, there are no such singularities in the (T^2, T^3) plane (as we already pointed out, $B_2 = B_2|_{\text{massless}} = N_V - N_H \equiv 0$).

On the heterotic side, however, already at the perturbative level, the massless part of the helicity supertrace $B_2(T, U)$ jumps across several submanifolds of the moduli space (T, U). This is a straightforward consequence of the appearance of specific powers of $\bar{q}$ in the $\Gamma^{\lambda}_{2,2}\left[{h \atop g}\right]$'s. A detailed analysis of the rational behaviours that appear in shifted lattice sums is given in [24]. In the constructions based on Φ^V there are lines where the $U(1)^2$ gauge symmetry corresponding to the vectors originating from the two-torus is enhanced to $SU(2) \times U(1)$, $SU(2) \times SU(2)$, or $SU(3)$. Furthermore, along some lines, 16 or 22 extra hypermultiplets appear; most of these new hypermultiplets are charged under the gauge group of the two-torus. On the other hand, the only $N = 4$ enhancements come from the rank-8 part of the gauge group, while the part of the gauge group that comes from the two-torus is always realized at the level one. The models based on $\tilde{\Phi}^V$ have a similar behaviour, although, depending on the choice of shift vectors, some of the would-be $N = 2$ singularities occurring along several submanifolds of the (T, U) plane cancel between the $\lambda = 0$ and $\lambda = 1$ term. In fact, a careful analysis of the helicity supertrace $B_2\left(\tilde{\Phi}^V\right)$ (Eq. (3.15)) shows that the $N = 2$ singularities present in this class of models are $SU(2)$ enhancements of one or both of the $U(1)$'s of the torus and the appearance of new hypermultiplets, all charged under the gauge group of the two-torus. Also in these constructions, the part of the gauge group that comes from the two-torus is always realized at the level one.

Despite the presence of the above $N = 2$ singularities, we expect that the heterotic amplitude corresponding to the R^2 coupling analysed in the type II dual models should not be sensitive to the perturbative enhancement of the massless spectrum. In the next section we will see how this heterotic amplitude can indeed be computed, by demanding regularity in the (T, U) space, making it therefore possible to establish the mapping between the heterotic and the type II moduli.

4 Heterotic gravitational corrections and a test of duality

Having the explicit expression of $B_2(\tilde{\Phi}^V)$ in terms of $\Gamma^{\lambda}_{2,2}\left[{H^{\mathrm{f}} \atop G^{\mathrm{f}}}\right]$ and using the techniques developed in Refs. [8, 24, 28, 29, 30, 31], we can calculate, on the heterotic side, the perturbative gravitational and gauge corrections in terms of the moduli T_{Het} and U_{Het} of the $(2, 2)$ lattice. These must be identical to the analogous corrections in type IIA given in terms of the type II moduli T^2 and T^3 (see Eqs. (2.18) and (2.23) for the gravitational coupling).

In order to compare the perturbative heterotic and type II results, it is necessary to look for an operator on the heterotic side that allows for the computation of the same physical quantity as on the type II side. The usual gravitational operator is given by

$$Q^2_{\rm grav} \equiv Q^2\, \overline{P}^2_{\rm grav}\,, \tag{4.1}$$

where Q stands again for the left-helicity operator, and $\overline{P}^2_{\rm grav}$, when inserted inside the one-loop vacuum amplitude, acts as $\frac{-1}{2\pi i}\frac{\partial}{\partial\bar\tau}$ on $\frac{1}{{\rm Im}\,\tau\,\bar\eta^2}$; namely, it acts on the contribution of the two right-moving transverse space-time coordinates $\overline{X}^{\mu=3,4}$, *including their zero-modes.* This latter fact is responsible for the appearance of a non-holomorphic gravity-backreaction contribution, which ensures modular covariance but has no type II counterpart. Indeed, the one-loop amplitude of the above operator is [25, 26, 31, 32, 33] [5]

$$F_{\rm grav} \equiv \left\langle Q^2_{\rm grav}\right\rangle_{\rm genus-one} = -\frac{1}{12}\left(\overline{E}_2 - \frac{3}{\pi\,{\rm Im}\,\tau}\right) B_2\,, \tag{4.2}$$

where, in the class of models under consideration, B_2 is given in Eq. (3.15) (or (3.13) in models constructed with Φ^V). The massless contribution to $F_{\rm grav}$ is precisely the gravitational anomaly, $b_{\rm grav} = \frac{24-N_V+N_H}{12}$, which in all heterotic models at hand equals 2 ($B_2|_{\rm massless}$ vanishes), at generic points of the (T,U) moduli space.

The operator $Q^2_{\rm grav}$ is not suitable for comparison with the type II result (i) because of the non-holomorphic term it generates; (ii) because its amplitude is sensitive to the $N=2$ singularities occurring in the (T,U) plane: the gravitational anomaly jumps along several rational lines where $N_V - N_H$ is no longer zero. We must therefore replace $Q^2_{\rm grav}$ with an appropriate operator $Q'^2_{\rm grav} = Q^2\,\overline{P}'^2_{\rm grav}$ such that (i) $\overline{P}'^2_{\rm grav}$ is essentially a universal combination of $\overline{P}^2_{\rm grav}$ and some appropriate Cartan generators of the gauge group; (ii) the massless contribution of the corresponding amplitude should remain the gravitational anomaly, and (iii) this amplitude should be regular everywhere in (T,U), at least in the models constructed with $\tilde\Phi^V$.

To this purpose, we introduce the following combinations of gravitational and gauge operators (we only display their right-moving factor):

$$\overline{P}^2_1 \;\equiv\; 12\,\overline{P}^2_{\rm grav} + \overline{P}^2_{2,2}\,; \tag{4.3}$$

$$\overline{P}^2_2 \;\equiv\; 12\,\overline{P}^2_{\rm grav} + \overline{P}^2_{2,2} + \frac{8}{N_V}\overline{P}^2_{\rm gauge}\,. \tag{4.4}$$

After insertion into the one-loop heterotic vacuum amplitude, $\overline{P}^2_{2,2}$ acts as $\frac{-1}{2\pi i}\frac{\partial}{\partial\bar\tau}$ on the modular-covariant factor of weight zero ${\rm Im}\,\tau\,\Gamma^{\lambda}_{2,2}\left[{H^{\rm f}\atop G^{\rm f}}\right]$. This amounts to inserting the sum of the two right-moving lattice momenta $\bar p_1^2 + \bar p_2^2$ of T^2, which correspond to the Cartan of the $U(1)$ factor. The amplitude $\left\langle Q^2\,\overline{P}^2_{2,2}\right\rangle$ therefore generates the corresponding gauge-coupling

[5] In general the heterotic one-loop amplitude of an operator of the form $Q^2\,\overline{P}^2$ reads:

$$\left\langle Q^2\,\overline{P}^2\right\rangle_{\rm genus-one} = \overline{P}^2\, B_2\,,$$

where, in the l.h.s., $\overline{P}^2$ acts as a differential operator on some specific factor of B_2.

correction. To be more precise, we must in fact consider the integral $\int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau} \left\langle Q^2\, \overline{P}^2_{2,2} \right\rangle \gamma(\tau,\bar\tau)$ ($\gamma(\tau,\bar\tau)$ is an appropriate modular-invariant infrared-regularizing function [25]). An integration by parts can be performed which leads to vanishing boundary terms *all over the* (T,U) *plane*, irrespectively of the specific behaviour of the lattice sums across rational lines. This allows us to recast the above amplitude as:

$$\overline{P}^2_{2,2}\, B_2\left(\tilde\Phi^V\right) = {\sum_{(H^{\rm f},G^{\rm f})}}' \frac{1}{2} \sum_{\lambda=0,1} \left(\Gamma^\lambda_{2,2}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \left[\frac{1}{2\pi i} \frac{\partial}{\partial\bar\tau} - \frac{1}{2\pi \operatorname{Im}\tau} \right] \frac{\overline{\Omega}^{(\lambda)}\left[{H^{\rm f} \atop G^{\rm f}}\right]}{\bar\eta^{24}} \right) \tag{4.5}$$

(in models constructed with Φ^V, only a $\lambda=1$ term appears). The differential operator inside the brackets is covariant since $\overline{\Omega}^{(\lambda)}\left[{H^{\rm f} \atop G^{\rm f}}\right] \Big/ \bar\eta^{24}$ has modular weight -2, but non-holomorphic as in the case of $\overline{P}^2_{\rm grav}$.

Finally, $\overline{P}^2_{\rm gauge}$ acts as $\frac{-1}{2\pi i}\frac{\partial}{\partial\bar\tau}$ on the ($\bar\tau$-covariantized) charge lattice of the remaining right-moving gauge group G_r of rank r, namely on $(\operatorname{Im}\tau)^{N_V/2}\, \bar\eta^{N_V}\, \overline{\Phi}^V$ (or $\overline{\tilde\Phi}^V$). It is modular-covariant since the Φ^V's are of weight zero, and contains a non-holomorphic $1/\operatorname{Im}\tau$ term. Notice that in the $U(1)^{N_V}$ models,

$$\overline{P}^2_{\rm gauge} = \sum_{I=1}^{N_V} \overline{P}^2_I \,, \tag{4.6}$$

where $\overline{P}_I$ are the $U(1)$ zero-modes. After some straightforward algebra we obtain:

$$\begin{aligned} \overline{P}^2_{\rm gauge}\, B_2\left(\tilde\Phi^V\right) &= \frac{N_V}{48} {\sum_{(H^{\rm f},G^{\rm f})}}' \Gamma^{\lambda=0}_{2,2}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \left(\left(\overline{E}_2 - \frac{3}{\pi \operatorname{Im}\tau} - \frac{1}{2}\overline{H}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \right) \frac{\overline{\Omega}^{(0)}\left[{H^{\rm f} \atop G^{\rm f}}\right]}{\bar\eta^{24}} + 96 \right) \\ &\quad + \frac{N_V}{48} {\sum_{(H^{\rm f},G^{\rm f})}}' \Gamma^{\lambda=1}_{2,2}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \left(\overline{E}_2 - \frac{3}{\pi \operatorname{Im}\tau} - \frac{1}{2}\overline{H}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \right) \frac{\overline{\Omega}^{(1)}\left[{H^{\rm f} \atop G^{\rm f}}\right]}{\bar\eta^{24}} \,, \end{aligned} \tag{4.7}$$

where we have introduced the modular-covariant functions

$$H\begin{bmatrix} h \\ g \end{bmatrix} = \frac{12}{\pi i}\partial_\tau \log \frac{\vartheta\left[{1-h \atop 1-g}\right]}{\eta} = \begin{cases} \vartheta_3^4 + \vartheta_4^4 \,, & (h,g) = (0,1) \\ -\vartheta_2^4 - \vartheta_3^4 \,, & (h,g) = (1,0) \\ \vartheta_2^4 - \vartheta_4^4 \,, & (h,g) = (1,1) \end{cases} \tag{4.8}$$

of weight 2. Similarly, for the models based on Φ^V,

$$\overline{P}^2_{\rm gauge}\, B_2\left(\Phi^V\right) = \frac{N_V}{24} {\sum_{(H^{\rm f},G^{\rm f})}}' \Gamma^{\lambda=1}_{2,2}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \left(\left(\overline{E}_2 - \frac{3}{\pi \operatorname{Im}\tau} - \frac{1}{2}\overline{H}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \right) \frac{\overline{\Omega}\left[{H^{\rm f} \atop G^{\rm f}}\right]}{\bar\eta^{24}} + 48\, \overline{\chi}\begin{bmatrix} H^{\rm f} \\ G^{\rm f} \end{bmatrix} \right) \,, \tag{4.9}$$

where $\chi\left[{h \atop g}\right]$ is a modular-covariant form of weight zero:

$$\chi\begin{bmatrix} h \\ g \end{bmatrix} = (-1)^{hg} \frac{\Omega^{(0)}\left[{h \atop g}\right]}{\Omega^{(1)}\left[{h \atop g}\right]} = \begin{cases} \frac{1}{2}\frac{\vartheta_3^4+\vartheta_4^4}{\vartheta_3^2\,\vartheta_4^2} \,, & (h,g) = (0,1) \\ \frac{1}{2}\frac{\vartheta_2^4+\vartheta_3^4}{\vartheta_2^2\,\vartheta_3^2} \,, & (h,g) = (1,0) \\ \frac{1}{2}\frac{\vartheta_2^4-\vartheta_4^4}{\vartheta_2^2\,\vartheta_4^2} \,, & (h,g) = (1,1) \,. \end{cases} \tag{4.10}$$

Here, we would like to pause and analyse several properties of the various operators P_A introduced so far ($P_{\rm grav},\ P_1,\ P_2,\ P_{2,2},\ P_{\rm gauge},\ P_I$). The beta-function coefficients of such operators are the constant term in the large- $\mathrm{Im}\,\tau$ expansion of the corresponding genus-one amplitude:

$$\overline{P}^2_A\, B_2 = b_A + \frac{1}{\bar q} \sum_{0<p(\ell)\neq 1} c_\ell\, \bar q^{p(\ell)} \,. \tag{4.11}$$

Let us concentrate on the constructions relevant to our duality purposes, namely those based on $\tilde\Phi^V$. At generic points in the (T,U) plane, $b\,(P_{2,2})$ vanishes, and therefore $b\,(P_1) = 12\, b_{\rm grav} = 24$. When a line of enhanced massless spectrum is approached, the beta-function coefficients $b\,(P_{2,2})$ and $b_{\rm grav}$ jump in the opposite way: $\Delta b\,(P_{2,2}) = -12\,\Delta b_{\rm grav}$. This implies that the combination (4.3) is smooth along the entire (T,U) moduli space and we have $b\,(P_1) = 24$ everywhere. On the other hand, in the models under consideration, $b\,(P_{\rm gauge})$ is the diagonal sum of higher-level gauge beta-functions, and thus vanishes without any discontinuity. As a consequence, $b\,(P_2) = 12\, b_{\rm grav} = 24$ in the whole (T,U) plane.

Both operators $\overline{P}^2_1$ and $\overline{P}^2_2$ lead to the insertion of a covariant derivative that contains a non-holomorphic $1/\,\mathrm{Im}\,\tau$ term. There is a unique linear combination of these, which is purely holomorphic and whose beta-function coefficient is $b'_{\rm grav} \equiv b\left(P'_{\rm grav}\right) = b_{\rm grav} = 2$ *everywhere in the* (T,U) *plane.* This operator is

$$\begin{aligned} \overline{P}'^2_{\rm grav} &\equiv -\frac{1}{8}\overline{P}^2_1 + \frac{5}{24}\overline{P}^2_2 \\ &= \overline{P}^2_{\rm grav} + \frac{1}{12}\overline{P}^2_{2,2} + \frac{5}{3N_V}\overline{P}^2_{\rm gauge}\,, \end{aligned} \tag{4.12}$$

and satisfies all the requirements we have demanded at the beginning of the section. It defines the modified gravitational operator for which the one-loop amplitude $\left\langle Q'^2_{\rm grav}\right\rangle$ reads:

$$\begin{aligned} \overline{P}'^2_{\rm grav}\, B_2\left(\tilde\Phi^V\right) &= -\frac{1}{24}\sum_{(H^{\rm f},G^{\rm f})}{}' \Gamma^{\lambda=0}_{2,2}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix} \left(\left(\frac{1}{6}\overline{E}_2 - \frac{1}{2\pi i}\frac{\partial}{\partial\bar\tau} + \frac{5}{12}\overline{H}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\right)\frac{\overline{\Omega}^{(0)}\left[{H^{\rm f}\atop G^{\rm f}}\right]}{\bar\eta^{24}} - 80\right) \\ &\quad -\frac{1}{24}\sum_{(H^{\rm f},G^{\rm f})}{}' \Gamma^{\lambda=1}_{2,2}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix} \left(\frac{1}{6}\overline{E}_2 - \frac{1}{2\pi i}\frac{\partial}{\partial\bar\tau} + \frac{5}{12}\overline{H}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}\right)\frac{\overline{\Omega}^{(1)}\left[{H^{\rm f}\atop G^{\rm f}}\right]}{\bar\eta^{24}}\,. \end{aligned} \tag{4.13}$$

In the above expression, the contribution of the $\lambda = 1$ part vanishes identically, while the action of the new holomorphic covariant derivative on $\overline{\Omega}^{(0)}\left[{H^{\rm f}\atop G^{\rm f}}\right]\Big/\bar\eta^{24}$ gives a constant. This fact is a direct consequence of the identity

$$\frac{1}{2\pi i}\partial_\tau \chi\begin{bmatrix}h\\g\end{bmatrix} = 32\,(-1)^{hg}\frac{\eta^{24}}{\Omega^{(1)}\left[{h\atop g}\right]}\,. \tag{4.14}$$

We therefore obtain:

$$\left\langle Q'^2_{\rm grav}\right\rangle_{\rm genus-one} = 2\sum_{(H^{\rm f},G^{\rm f})}{}' \Gamma^{\lambda=0}_{2,2}\begin{bmatrix}H^{\rm f}\\G^{\rm f}\end{bmatrix}. \tag{4.15}$$

Finally, the perturbative heterotic gravitational corrections of the models constructed with $\tilde{\Phi}^V$ are

$$\Delta_{\rm Het}^{N_V=N_H} = 2\int_{\mathcal{F}} \frac{{\rm d}^2\tau}{{\rm Im}\,\tau}\left(\sum_{(H^{\rm f},G^{\rm f})}{}' \Gamma_{2,2}^{\lambda=0}\begin{bmatrix}H^{\rm f}\\ G^{\rm f}\end{bmatrix} - 1\right) . \tag{4.16}$$

For the specific case in which the shift $(H^{\rm f}, G^{\rm f})$ in $\Gamma_{2,2}$ is due to a translation of momenta, $(-1)^{m_2 G^{\rm f}}$, we get:

$$\Delta_{\rm Het}^{N_V=N_H} = -2\log\,{\rm Im}\,T\left|\vartheta_4\left(T\right)\right|^4 - 2\log\,{\rm Im}\,U\left|\vartheta_4\left(U\right)\right|^4 + \text{ const.} \tag{4.17}$$

The heterotic models under consideration, based on $\tilde{\Phi}^V$, were advertised to be dual to the type II ground states presented in Section 2. It is therefore important to observe that on the type II side, the replacement of $Q^2_{\rm grav}$ by $Q'^2_{\rm grav}$ does not modify the perturbative results, Eq. (2.18). Indeed, owing to the absence of perturbative Ramond–Ramond charges, the contribution of the duals of $\overline{P}^2_{2,2}$ and $\overline{P}^2_{\rm gauge}$ is always zero, and the one-loop amplitude $\left\langle Q'^2_{\rm grav}\right\rangle_{\rm II}$ therefore reduces to $\left\langle Q^2_{\rm grav}\right\rangle_{\rm II}$. We can now compare the perturbative gravitational corrections in the heterotic and type II dual orbifolds. On type II, the full result is computed at one loop and given by (2.18). Since in the large-moduli limit a linear term comes only from the field T^1 (see Eq. (2.23)), this modulus has to be identified with the heterotic dilaton: the heterotic coupling receives in fact a tree-level contribution, linear in the dilaton expectation value. Including the one-loop computation (4.17), we obtain:

$$\begin{aligned}\frac{16\,\pi^2}{g^2_{\rm grav}(\mu^{\rm (Het)})} \;=\;& 16\,\pi^2\,{\rm Im}\,S - 2\log\,{\rm Im}\,T\left|\vartheta_4\left(T\right)\right|^4 - 2\log\,{\rm Im}\,U\left|\vartheta_4\left(U\right)\right|^4 \\ & +4\log\frac{M^{\rm (Het)}}{\mu^{\rm (Het)}} + \text{ const.}\,,\end{aligned} \tag{4.18}$$

where S is the heterotic axion–dilaton field and

$$ {\rm Im}\,S = \frac{1}{g^2_{\rm Het}}\,. \tag{4.19}$$

In (4.18) we used the string scale $M^{\rm (Het)} \equiv \frac{1}{\sqrt{\alpha'_{\rm Het}}}$ and the infrared cut-off of the heterotic string.

In order to match the heterotic and type II results (2.23) and (4.18) in the heterotic weak coupling limit, ${\rm Im}\,S \to \infty$, it is necessary to identify T^1 with $16\tau_S/N_V$ [6]. This map and its normalization are consistent with the interpretation of the type II vacuum as an orbifold limit of a $K3$ fibration. In the $N_V = 8$ case the base of the fibration is $T^2/Z_2 = \mathbf{P}^1$, with, as volume form, half of that of the torus T^2; the heterotic dilaton then corresponds to the volume form of the base of the fibration, $\tau_S = T^1/2$. In the case of $N_V = 4$, the base of the fibration is instead $\mathbf{P}^1/Z_2^D$, where Z_2^D acts on the sphere $\mathbf{P}^1$ as a half-circumference translation, and $\tau_S = T^1/4$. Finally, when $N_V = 2$ the base of the fibration is $\mathbf{P}^1/(Z_2^D \times Z_2^{D'})$ and $\tau_S = T^1/8$.

[6] We use the notation $\tau_S = 4\pi S$.

Comparison of Eqs. (2.18) and (4.18) is even more suggestive. By identifying T with T^2 and U with T^3, the perturbative corrections on the heterotic side as a function of T and U are identical to the corrections on the type II side as a function of T^2 and T^3. These identifications allow us to promote the large-$\mathrm{Im}\, S$ heterotic corrections obtained so far to finite values of $\mathrm{Im}\, S$. The type II result (2.18) therefore provides the full, perturbative and non-perturbative, correction. The sub-leading logarithmic dilaton dependence and the infrared running are obtained from the expressions (2.18), (2.23) by substituting the type IIA string mass $M^{(\mathrm{II})}$ and cut-off $\mu^{(\mathrm{II})}$ with the duality-invariant Planck mass M_{Planck} and the physical cut-off μ. By using the relations

$$\frac{M^{(\mathrm{II})}}{\mu^{(\mathrm{II})}} = \frac{M^{(\mathrm{Het})}}{\mu^{(\mathrm{Het})}} = \frac{M_{\mathrm{Planck}}}{\mu} \ , \tag{4.20}$$

we can express the full dilaton dependence and infrared running of the effective coupling constant $16\,\pi^2/g^2_{\mathrm{grav}}(\mu^{(\mathrm{Het})})$, for the various models, as:

$$\begin{aligned} &-\frac{3N_V}{4}\log\left|\eta\left(\frac{16\tau_S}{N_V}\right)\right|^4 - \left(2-\frac{N_V}{4}\right)\log\left|\vartheta_4\left(\frac{16\tau_S}{N_V}\right)\right|^4 \\ &-\left(\frac{B_4-B_2}{3} - 2b_{\mathrm{grav}}\right)\log\,\mathrm{Im}\,\tau_S \ + \ \frac{B_4-B_2}{3}\log\frac{M_{\mathrm{Planck}}}{\mu}, \end{aligned} \tag{4.21}$$

where by $B_4 - B_2$ we actually mean the constant, massless contribution at a generic point in moduli space: the coefficients of the two terms in the second line are given in Eq. (2.24). By expanding the above expression as

$$4\pi\,\mathrm{Im}\,\tau_S - \left(2+\frac{N_V}{2}\right)\log\,\mathrm{Im}\,\tau_S + 2\,\mathrm{Re}\sum_k n_k\, e^{i2k\pi\tau_S}$$

and comparing it with the corresponding expression valid for the $N = 4$ heterotic string compactified on T^6 ($\sim 2\,\mathrm{Re}\log|\eta(\tau_S)|$) [7, 8], we can see that the instanton numbers k are restricted to integers which are respectively multiples of 2 ($N_V = 8$ and $N_V = 4$) and 4 ($N_V = 2$) [7].

The case of heterotic models based on Φ^V deserves several remarks. The one-loop amplitude of the modified gravitational operator Q'^2_{grav} reads:

$$\overline{P}'^2_{\mathrm{grav}}\, B_2\left(\Phi^V\right) = 2\sum_{(H^{\mathrm{f}},G^{\mathrm{f}})}{}' \,\Gamma^{\lambda=1}_{2,2}\begin{bmatrix}H^{\mathrm{f}}\\G^{\mathrm{f}}\end{bmatrix}\overline{\chi}\begin{bmatrix}H^{\mathrm{f}}\\G^{\mathrm{f}}\end{bmatrix}. \tag{4.22}$$

Expression (4.22) can still be integrated to give the thresholds (see [24] for details on such integrals). However, the heterotic result in this case does not match the perturbative gravitational thresholds of the type II models. In particular, we observe $N = 2$ singularities along lines where $\Delta b'_{\mathrm{grav}} \neq 0$, since $\Delta b(P_{2,2}) \neq -12\Delta b_{\mathrm{grav}}$ [8], while $b(P_{\mathrm{gauge}})$ remains zero everywhere in the (T, U) plane. This means that in the case at hand the relation between

[7] This feature was already observed for the $N_V = 8$ model, in [20].

[8] This is due to the appearance of hypermultiplets uncharged under the gauge group.

the type IIA and heterotic perturbative moduli is not simply $T^2 = T$, $T^3 = U$. The same conclusion was drawn when analysing the restoration of $N = 4$ supersymmetry with respect to the various decompactification limits.

The difference between the two classes of models constructed in Section 3 lies in the choice of the discrete Wilson lines. The models that are based on $\tilde{\Phi}^V$ are the duals of the type IIA constructions of Section 2, with the identifications $\tau_S = \frac{N_V}{16} T^1$, $T = T^2$, $U = T^3$. The models based on Φ^V correspond instead to a choice of discrete Wilson lines that do not correspond to the type II constructions presented in Section 2. In Ref. [20], assuming the heterotic/type II duality, the authors made a proposal for the full non-perturbative gravitational corrections in the $N_V = 8$ model, as a function of all the vector-multiplet moduli: $\Delta_{\rm grav}(\tau_S, T, U, Y_1, \ldots, Y_8)$. This proposal is based on the uniqueness properties of the automorphic forms of the Calabi–Yau threefold with $h^{1,1} = h^{2,1} = 11$. In order to test this through a comparison of heterotic and type IIA strings, special values of the Wilson lines $\vec{Y}$ must be chosen; this necessarily leads to the explicit constructions we have considered, which allow a test of the heterotic/type II duality conjecture.

5 Comments

In this work we explicitly constructed heterotic and type II dual pairs; we verified the duality conjecture not only for a model with $N_V = 8$, but also for models with $N_V = 4$ and 2. A relevant choice of Wilson lines $\vec{Y}$ on the heterotic side defines the $\tilde{\Phi}^V$ constructions. The heterotic/type II duality is not verified for the gravitational corrections, but for a modified gravitational and gauge combination associated to the operator $Q'^2_{\rm grav}$ introduced in Section 4. This operator has the property that it is regular, without singularities in the entire (T, U) moduli space. We found that *the type II corrections* $\Delta_{\rm II}^{N_V = N_H}(T^1, T^2, T^3)$ *provide the complete, perturbative and non-perturbative heterotic corrections.* This remarkable property is due to the universality of the $N = 2$ sector on heterotic side and of the corresponding $N = (2, 2)$ sectors on type II. Indeed, the heterotic $N = 2$ sector is universal and independent of the particular choice of Wilson lines $\vec{Y}$: it is the same for all the $N_V = N_H$ models with a separation of the $\Gamma^{\lambda}_{2,2}\left[{H^{\rm f} \atop G^{\rm f}}\right]$ lattice as in the $\tilde{\Phi}^V$ models.

The heterotic instanton corrections $n_k\, e^{ik\pi\tau_S}$ are due to the Euclidean five-brane wrapped on the six-dimensional internal space; they depend only on τ_S and not on the other moduli. The explicit expressions for these corrections are given in Eq. (4.21). The permitted integers k depend on N_V and the multiplicity coefficients n_k are fully determined from type IIA. The Olive–Montonen duality group is a subgroup of $SL(2, Z)_{\tau_S}$, which depends on N_V: it is $\Gamma(2)$ when $N_V = 8$, $\Gamma(8)$ when $N_V = 4$, and $\Gamma(16)$ when $N_V = 2$.

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