

# NERO: A CODE FOR EVALUATION OF NONLINEAR RESONANCES IN 4D SYMPLECTIC MAPPINGS

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## Abstract

A code to evaluate the stability, the position and the width of nonlinear resonances in four-dimensional symplectic mappings is described. NERO is based on the computation of the resonant perturbative series through the use of Lie transformation implemented in the code ARES, and on the analysis of the resonant orbits of the interpolating Hamiltonian. The code is aimed at studying the nonlinear motion of a charged particle moving in a circular accelerator under the influence of nonlinear forces.

## 1 INTRODUCTION

The single-particle dynamics in a magnetic lattice under the effect of nonlinear fields is a crucial issue for the construction of large circular accelerators such as the Large Hadron Collider [1]. These problems are described in terms of four-dimensional (4D) symplectic mappings representing of the transverse motion of a single particle over one turn of the machine. The comprehension of the relation between resonances, nonlinearities, tunes and the stability domain (dynamic aperture) in these models is a difficult task. The dynamic aperture is usually determined through numerical integration based on tracking [2, 3]. The perturbative theory based either on Hamiltonian flows [4–6], or on symplectic mappings [7–10], provides much analytical information on the detuning and on the resonance parameters.

In the case of unstable resonances, the dynamic aperture is usually determined by the hyperbolic resonant orbits (fixed lines) [11]. The stable resonances, on the other hand, feature families of islands that do not limit the stability domain, and therefore there is no direct relation with the dynamic aperture. However, several studies have shown that analytical indicators (Quality Factors, QFs) extracted through perturbative tools can be well-correlated with the dynamic aperture [6, 12, 13].

During the past years, arbitrary-order codes have been developed to compute both nonresonant and resonant perturbative series (normal forms) of a generic truncated one-turn map [7, 8, 10, 14]. We present the features of a new program NERO [15] that includes the previously developed code ARES that evaluates resonant normal forms [14]. NERO postprocesses ARES output in order to provide the following quantities:

- The automatic evaluation of quality factors defined in previous papers [13].

- The reconstruction of the phase space dynamics such as the network of resonances involved in the nonlinear motion, the position and the width of the islands.
- The automatic analysis of several lattices at the same time and produce correlation plots of the QFs with the dynamic aperture obtained through standard tracking. This feature is relevant, for instance, in the analysis of the effect of random errors [13], and in general for all the optimisation procedures.

## 2 THEORY

We consider betatronic one-turn maps that describe the transverse oscillations of charged particles in a magnetic lattice [7, 8, 10, 16]. The one-turn map can be expanded as a Taylor series truncated at order  $M$  in the physical coordinates and the conjugated momenta at a given section of the machine.

### 2.1 Normal forms

The perturbative theory of normal forms consists in transforming the one-turn map to a simpler, more symmetric map  $U$  (the normal form) that explicitly shows the motion invariants and the geometry of the orbits [7, 8, 10]. The normal form  $U$  is then written as the Lie series of an interpolating Hamiltonian  $h$ . The Hamiltonian is more easily expressed in terms of the amplitude-angle coordinates  $(\theta_1, \theta_2, \rho_1, \rho_2)$ , obtained in terms of the complex normal form coordinates as  $\zeta_1 = \sqrt{\rho_1}e^{i\theta_1}$  and  $\zeta_2 = \sqrt{\rho_2}e^{i\theta_2}$  ( $\rho_1$  and  $\rho_2$  are the generalisation of the emittances to the nonlinear case).

As a first step, one can build a nonresonant normal form: in this case the Hamiltonian only depends on the amplitudes  $\rho_1$  and  $\rho_2$ :

$$h(\rho_1, \rho_2) = \sum_{k_1, k_2} h_{k_1, k_2} \rho_1^{k_1} \rho_2^{k_2}. \quad (1)$$

The phase space described by this Hamiltonian is given by 2D KAM tori whose nonlinear frequencies are the partial derivatives of  $h$  with respect to  $\rho_1$  and  $\rho_2$ .

When the linear tunes are close to a resonance  $(q, p)$ , where  $q \in \mathbf{N}$  and  $p \in \mathbf{Z}$ , it may happen that the resonant condition on the nonlinear frequencies

$$q\nu_x + p\nu_y = m + \epsilon \quad \epsilon \ll 1 \quad m \in \mathbf{Z} \quad (2)$$

is exactly satisfied (i.e.,  $\epsilon = 0$ ) for some positive amplitudes. In this specific case, the general topology of the

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2D invariant KAM tori breaks down, and a family of islands arises. Such islands can be described by the single-resonance normal form theory, that consists of retaining the resonant combinations of the phases  $q\theta_1 + p\theta_2$  in the normal form  $U$  and in the interpolating Hamiltonian. Thus  $h$  reads

$$h(\rho_1, \rho_2, \theta_1, \theta_2) = \sum_{k_1, k_2, l} h_{k_1, k_2, l} \rho_1^{k_1 + lq/2} \rho_2^{k_2 + lp/2} \cos(l(q\theta_1 + p\theta_2) + \varphi_{k_1, k_2, l}).$$

## 2.2 Topology of single resonances

In order to analyse the dynamics of the above Hamiltonian, one can perform a canonical transformation such that in the new coordinates the Hamiltonian reads

$$h(\phi_1, r_1; r_2) = \sum_{k_1, k_2, l} h_{k_1, k_2, l} r_1^{k_1 + lq/2} \left( \frac{p}{q} r_1 + r_2 \right)^{k_2 + lp/2} \cos(lq\phi_1 + \varphi_{k_1, k_2, l}).$$

Since there is no dependence on  $\phi_2$ , the quantity  $r_2$  is invariant and the problem is reduced to an analysis of a 2D Hamiltonian with a parametric dependence on  $r_2$ . For each value of  $r_2$  one can compute the fixed points, the eigenvalues, and the position of the separatrices. Finally, one has to check that the obtained solutions  $r_1$  and  $r_2$  correspond to positive values of the original amplitudes  $\rho_1$  and  $\rho_2$  (see Ref. [15] for more details on the topology of a single resonance).

## 2.3 Quality factors

In previous studies [6, 12, 13] we have shown the importance of analytical quality factors to understand the nonlinear dynamics. NERO has the capability of computing the following quantities: norm of the map  $Q_1$ ; norm of the tunes  $Q_2$ ; norm of the resonance  $Q_3(q, p)$ ; hyper-volume of the resonance  $Q_4(q, p)$  (see Ref. [15] for more details).

## 3 ALGORITHMS

NERO incorporates the code ARES that evaluates the normal form series. The main part of the code NERO performs the analysis of the interpolating Hamiltonian to evaluate the existence, the position and the stability of the resonant orbits.

Consider a resonance  $(q, p)$  and the related interpolating Hamiltonian  $h$ . First of all, a scan over the tune values on the resonant line  $qv_x + pv_y = m$  is carried out and for each couple of tunes, the amplitudes  $\rho_1$  and  $\rho_2$  corresponding to that detuning are computed analytically. Then, a Newton method is applied to determine the values of the amplitudes using the nonresonant part of the Hamiltonian up to the truncation order (the first order solution analytically evaluated being the first guess). The obtained values of the amplitudes  $\rho_1^0$  and  $\rho_2^0$  correspond to the average amplitudes

of the resonance.

If the obtained amplitudes are positive, the second invariant is evaluated and the reduced Hamiltonian is used to find the position of the fixed points, by solving the system through a Newton method in the two variables  $r_1$  and  $\psi_1$

$$\begin{aligned} \frac{\partial h}{\partial r_1}(\psi_1, r_1; r_2) &= 0 \\ \frac{\partial h}{\partial \psi_1}(\psi_1, r_1; r_2) &= 0. \end{aligned} \quad (3)$$

The amplitude can be initialised using  $\rho_1^0$ , while the initial guess for the angle can be worked out by considering the first-order truncation of the second equation of the system (3): if there is only one sine term, the initial guess can be computed analytically. Otherwise, one can make a numerical scan over  $[0, 2\pi]$  in order to find out a good initial guess. In both cases two initial guesses are needed, corresponding to the elliptic and to the hyperbolic solution. The stability is evaluated by the computation of the Hessian of the Hamiltonian.

From the knowledge of the fixed points, one can determine the equation of the separatrix, through a first-order analytical guess for the position of the inner and outer separatrix for the angle  $\phi_{1e}$  corresponding to the elliptic fixed point. Then, each initial guess is used to start a Newton method that solves the separatrix equation to arbitrary order

$$h(\phi_{1e}, r_1; r_2) = E_h \equiv h(\phi_h, r_{1h}; r_2) \quad (4)$$

where  $(\phi_h, r_{1h})$  denote the position of the hyperbolic fixed point. The obtained values  $r_1^{min}$  and  $r_1^{max}$  are then transformed back to the original amplitudes to have the minimum and maximum amplitude of the separatrix  $(\rho_1^{min}, \rho_2^{min})$  and  $(\rho_1^{max}, \rho_2^{max})$ . Finally, the equation for the separatrix (4) is also solved for all the values of  $\phi_1$  between the elliptic and the hyperbolic fixed point thus obtaining the area of the islands.

## 4 NUMERICAL vs ANALYTICAL METHODS

The model used for our tests is the 4D Hénon mapping [10]. The linear tunes have been set to the values  $Q_x = 68.28, Q_y = 68.31$  corresponding to the working point of the LHC project [1]. As a starting point we have computed the resonance network. We consider a very dense scan in the plane of the initial conditions, and we tracked each orbit for 2048 turns. If the nonlinear frequencies of the orbit satisfy a resonant condition of order  $q + |p| \leq 15$ , with  $\epsilon < 0.0001$  [see Eq. (2)], we consider the orbit to be resonant and we plot the linear invariants  $(x^2, y^2)$  of the initial condition. From the resulting picture, shown in Fig. 1, one can see that some resonances are rather strong [for instance,  $(6, -2)$  and  $(3, -6)$ ], and that resonance  $(1, -4)$  splits the stability domain into two disconnected parts. The same kind of plot can be produced

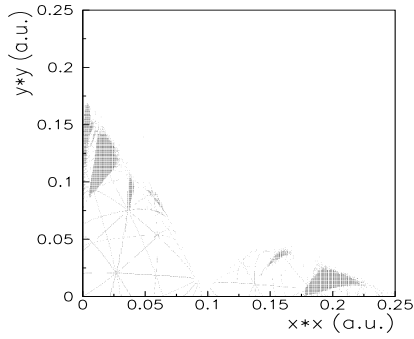


Figure 1: Resonance network of the 4D Hénon map at  $Q_x = 0.28, Q_y = 0.31$  via frequency analysis. Initial conditions satisfying a resonance condition up to order 15 are shown. Horizontal and vertical linear invariants are in the  $x$  and  $y$  axes, respectively.

by using resonant normal forms. In fact, by using NERO it is possible to compute analytically both the position of the resonances and their widths. In Fig. 2 a resonance network is shown, obtained by using normal forms: the resonance parameters are computed analytically, without tracking. In particular, resonances  $(3, -6)$  and  $(6, -2)$  are rather wide, in agreement with the previous figure obtained through numerical methods. The position of the resonance  $(1, -4)$ , marked as a thick line, is also in agreement with the previous figure. The slight mismatch between the two figures is due to the fact that in the first case the plot is shown in the plane of the linear invariants, while in the second one the nonlinear invariants obtained through normal forms are used. To compare frequency analysis and normal forms more precisely, the normal form guess for the resonance width of resonance  $(3, -6)$  (see Fig. 3, solid line) is compared with the numerical results projected in the plane of the nonlinear invariants: the quantitative agreement is good, even though the resonance starts at rather high amplitudes and extends to the dynamic aperture.

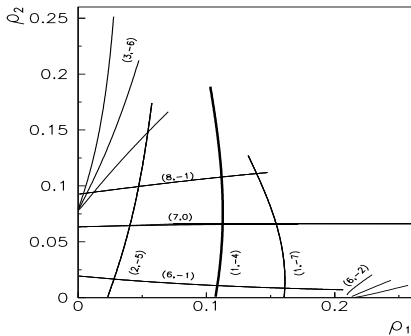


Figure 2: Resonance network of the 4D Hénon map at  $Q_x = 0.28, Q_y = 0.31$  via resonant normal forms. Resonances up to order 9 are evaluated. Horizontal and vertical nonlinear invariants are in the  $x$  and  $y$  axes, respectively.

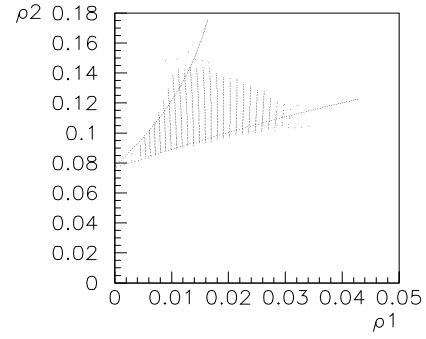


Figure 3: Initial conditions that are locked on the resonance  $(3, -6)$  for the 4D Hénon map at  $Q_x = 0.28, Q_y = 0.31$ . Numerical results (dots) are compared with the analytical estimate of the resonance width (solid lines) obtained using resonant normal forms at order 10. Horizontal and vertical invariants are plotted in the  $x$  and  $y$  axes, respectively.

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