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TUNE OPTIMIZATION FOR MAXIMUM DYNAMIC ACCEPTANCE, I: FORMULATION

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ABSTRACT

In order to combine the acceptance limitation due to a mechanical obstacle at radius $r_{\rm mech}$ with that due to magnetic imperfections present in the lattice, a quantity $\epsilon_{\rm da}$ to be called "dynamic acceptance" is introduced. Using lowest order theory (with transfer matrices and no Hamiltonian) perturbed linear betatron motion is calculated and used to derive the dependence $\epsilon_{\rm da}(r_{\rm mech})$. Being in analytic form, this *acceptance reduction* provides a figure of merit that can be used to optimize the lattice tunes (thereby refining the prescription "stay away from low order resonances"). Apart from its definition as an acceptance rather than an aperture, what distinguishes $\epsilon_{\rm da}(r_{\rm mech})$ from the commonly employed "dynamic aperture" is its dependence on $r_{\rm mech}$ and the importance of this distinction fades as $r_{\rm mech}$ becomes large. In this first part the method is formulated and, to demonstrate the method, optimal fractional tunes are found with only random errors present—the loss of acceptance is dominated by sextupole errors. But the intended application is for field errors that are systematic over sections of the lattice, but not necessarily over the whole lattice. Such field errors are unavoidable and are especially important in a high tune accelerator like the LHC.

1. Introduction

The performance of a high energy superconducting accelerator such as the LHC is limited at large particle amplitudes either by mechanical obstacles or by magnetic imperfections, with the latter having come to be expressed by the "dynamic aperture" of the lattice. Here it is argued that these limitations should be merged into a single quantity $\epsilon_{\rm da}(r_{\rm mech})$, to be called "dynamic acceptance",† which depends on the mechanical aperture $r_{\rm mech}$. In this paper, when the dynamic limitation is expressed at a particular point P it will be expressed as a radius $r_{\rm da}$ that is related to $\epsilon_{\rm da}$ by $r_{\rm da} = \sqrt{\beta^{\rm P}\epsilon_{\rm da}}$ where $\beta^{\rm P}$ is the appropriate beta function at P. Since most calculations will be referred to the point P at which the limiting mechanical aperture is located, formulas will mainly be expressed in terms of $r_{\rm da}$ applicable at that point and converted to $\epsilon_{\rm da}$ only at the end.

With the beam expected to be roughly Gaussian in transverse profile, it is customary to express apertures in terms of "beam sigmas" where $\sigma_x \approx \sigma_y \approx 1\,$mm is a typical value at a "typical" point where $\beta = \beta_{\rm typ}$. In order to make this introduction as informal as possible while retaining at least semi-quantitative accuracy let us accept this value so that the same unit can serve for both mechanical dimensions and beam sigma.

To make the discussion concrete let us define some characteristic radii and assign them plausible values. (Though they will be more reliable than their absolute values, their relative values are also not claimed to be precise.)

$r_{\rm min}$ $\overset{e.g.}{=}$ 6 mm, the smallest dynamic aperture for which measureable luminosity can be obtained assuming a perfectly centered closed orbit.

$r_{\rm mech,}$ the radius of the smallest mechanical aperture or scraper, assumed to be at the place in the lattice where $r_{\rm da}$ applies. In the LHC the minimum beam screen radius is 18 mm but practical beam cleaning scrapers will probably be at about 10 mm. In this paper $r_{\rm mech}$ is treated as variable.

$r_{\rm ref}$ = 17 mm, the "reference" radius at which multipole coefficients are expressed.

$r_{\rm nl}$ ≈ 20 mm radius at which particles are typically lost in a few turns according to element-by-element tracking with conservative field imperfections.

† It was Kjell Johnsen who, in coffee time conversation, objected to the term "dynamic aperture" and expressed the opinion that "dynamic acceptance" would be a more traditional and more useful measure. The recommendation was endorsed by the others present—Wolfgang Schnell and Albert Hoffman.

$r_{\rm conv} \approx 24\,$mm radius outside which the magnetic field is essentially unknown because of unknown convergence of the multipole series.

As amplitudes increase toward $r_{\rm nl}$ analytic calculation becomes impossible, but the other side of the coin is that as amplitudes decrease from $r_{\rm ref}$ toward $r_{\rm min}$ analytic theory becomes simpler and more accurate. This is the thesis on which this paper is based.

The theory just mentioned is *first order perturbation theory*, in which each nonlinear element is treated as independent of all others. Then the effect of all nonlinear elements acting in concert is obtained by simple superposition. Since this superposition is just like the superposition of waves in the diffraction theory of physical optics, it can be performed using phasor diagrams, and there is the possibility of constructive or destructive interference.† The number of phasor contributions is equal to the number of nonlinear elements, which in our case will usually be the number of half-cells $N_{1/2}$—with the tune Q being about 60 this is given approximately by $N_{1/2} \overset{e.g.}{=} 8Q \approx 500$. The maximum conceivable coherent sum can therefore be about 500 times greater than a typical individual term, but the sum will normally be much less constructive than this.

By performing this phasor summation we will obtain a "coherent" sum $\Delta r_{\rm coh}$ which is the maximum excursion away from a nominal, purely linear, betatron motion. The actual motion will therefore fill a band $\pm\Delta r_{\rm coh}$ and this will reduce the dynamic aperture to

$$\boxed{r_{\rm da}(r_{\rm mech}) = r_{\rm mech} - \Delta r_{\rm coh},} \tag{1.1}$$

and the dynamic acceptance to $\epsilon_{\rm da}(r_{\rm mech}) = r^2_{\rm da}(r_{\rm mech})/\beta^P$. The basis for this formula is that a particle with *linear* betatron amplitude $r_{\rm da}$ passes every possible obstacle with every possible phase of both its linear and nonlinear parts and hence will be lost if its amplitude exceeds the value given by the expression on the right hand side of (1.1). If there is no nonlinearity the formula reduces trivially to $r_{\rm da} = r_{\rm mech}$, and for sufficiently small values of $r_{\rm mech}$ (which can be varied arbitrarily in operations) Eq. (1.1) becomes arbitrarily more reliable. The thesis of this paper is that values of $r_{\rm mech}$ small enough to make Eq. (1.1) reliable in this sense will at the same time be large enough for practical operations. Though chaos makes it impossible to perform an accurate analytic calculation of the magnetic

† In the current context "destructive" interference is *good* and "constructive" interference is *bad* since the amplitudes being superimposed constitute undesirable distortion.

aperture in the absence of mechanical apertures, this limitation is somewhat academic since mechanical apertures are always present for operational reasons. We conjecture then, that amplitudes that are *practical* operationally are small enough for analytic formulation to be practical as well. It is very much in the spirit of the calculation to describe $\Delta r_{\rm coh}$ as the amount by which the mechanical aperture is "fuzzed out" by the nonlinear dynamics. The calculations of Lasheras and Jeanneret[1] are based on similar ideas.

If all nonlinear fields were known perfectly the coherent sum $\Delta r_{\rm coh}$ could be calculated exactly within the model, but for now we can only estimate the magnitudes of the field errors, concentrating on the nonlinearities associated with the main arc dipoles since they are expected to dominate, at least at injection. The formulas in this paper can be applied to random errors using a "random phase approximation" but they are more intended for systematic field errors, or rather on "somewhat systematic" field errors. Such errors are systematic over an appreciable fraction of the lattice, but not necessarily over the whole lattice. Because the arcs are themselves periodic structures, there is the likelihood of appreciable constructive interference over, say, one arc, even if the interference over the whole lattice is largely destructive. What is required therefore is a "somewhat random phase approximation". Though the enhancement factor from this source cannot approach the maximum possible value of 500 mentioned above, it can still be appreciably larger than the value $\sqrt{500}$ that might be expected if the elements contribute randomly, or the even smaller value that can be achieved if the errors are purely systematic and are intentionally arranged to be self-compensating.

Before proceeding to an accurate calculation of $\Delta r_{\rm coh}$, I make the following semi-quantitative estimate in order to provide guidance for the later formulation. As well as Q and $N_{1/2}$ defined previously we will use $2\pi R \overset{e.g.}{=} 2.7\times 10^4\,{\rm m}$, $\Delta\Theta = 2\pi/N_{1/2}$ which is the bend per half-cell, and $\beta_{\rm typ} = R/Q \overset{e.g.}{=} 72\,{\rm m}$ which is a typical value for the beta function. In particular we will use the combination

$$\frac{\beta_{\rm typ}\Delta\Theta}{r_{\rm ref}} \overset{e.g.}{=} \frac{0.938}{0.017} = 55. \tag{1.2}$$

When viewed at a particular point P in the lattice, the linear horizontal betatron displacement on turn t is given by[†]

$$x_t = a_x \cos \mu_x t; \tag{1.3}$$

the amplitude satisfies $a_x = \sqrt{\epsilon_x \beta_x^{\rm P}}$, where $\beta_x^{\rm P} \overset{e.g.}{=} \beta_{\rm typ}$. Assume that there is a nonlinear element at P with strength $\mathcal{M}_{n_x}$ that causes deflection

$$\Delta x_t' = \mathcal{M}_{n_x}\, x_t^{n_x}. \tag{1.4}$$

If this element is describable by conventional multipole coefficient b_{n_x,n_x+1} then

$$\mathcal{M}_{n_x}\, x_t^{n_x} = \Delta\Theta\, b_{n_x,n_x+1} \times 10^{-4} \left(\frac{x_t}{r_{\rm ref}}\right)^{n_x}, \tag{1.5}$$

where b_{n_x,n_x+1} is measured in standard "units" at $r_{\rm ref}$ and the two indices allow for American/European conventions. The effect of this nonlinear element at P is to "perturb" (1.3) so that, specializing to $n_x = 2$, the motion takes the form

$$x_t = a_x \left(\cos \mu_x t + \beta_{\rm typ}\, a_x^{2-1}\, \mathcal{M}_2\, \frac{\mathcal{N}_2}{\mathcal{D}_2} \cos \Omega t \right), \tag{1.6}$$

where Ω is a tune that in general is a sum of μ_x, μ_y, and μ_z with integer coefficients,[‡] that for $n_x = 2$ is given by $\Omega = 2\mu_x$. A typical value for the "numerator" factor is $\mathcal{N}_2 \approx 1/n_x$. The "denominator" factor is proportional to ΔQ, the "tune distance to the nearest resonance"; it is given by $\mathcal{D} = 2(\cos 2\mu_x - \cos \mu_x)$. "Avoiding the resonance" is done by adjusting the lattice parameters so that, say, $\mathcal{D} \approx 8\pi\Delta Q > 0.5$. Taking $a_x = r_{\rm ref}/2$, the fractional distortion is given by

$$\begin{aligned} \frac{\Delta a_x}{a_x} &= b_{2,3} \times 10^{-4} \frac{\Delta\Theta \beta_{\rm typ}}{r_{\rm ref}} \left(\frac{a_x}{r_{\rm ref}}\right)^{n_x - 1} \frac{\mathcal{N}_2}{\mathcal{D}_2} \\ &= b_{2,3} \times 10^{-4}\, 55\, \frac{1}{2}\, \frac{1/2}{0.5} \\ &\approx 0.003\, b_{2,3} \end{aligned} \tag{1.7}$$

By this estimate, the fractional distortion caused by 1 "unit" of sextupole component in the dipole in a single half-cell is approximately 0.3 percent.

† The main problem to be faced later is the coherent summation over phases, but for now we assume that the time origin has been chosen to make the phase vanish.

‡ All formulas in this paper can be generalized to full three dimensional motion, but for simplicity the discussion will mainly be restricted to two transverse dimensions

When two or more nonlinearities are superimposed it is necessary to take account of their relative phase shifts in Eq. (1.6) by the replacement $\Omega t \to \Omega t + \Phi_i$. It is the superposition of the resulting sinusoidal functions, appropriately performed using phasors, that gives the theory its diffraction-like character. The numerical factor quantifying this superposition will be called the "phasor factor". If every dipole has the same imperfection the estimate (1.7) has to be multiplied by a phasor factor that lies in the range from zero to 500 but which may typically be about $\sqrt{500} = 22$. The distortion caused by 1 unit of sextupole component would then be about $0.3 \times 22 = 7$ percent. If 14 percent were the largest tolerable acceptance reduction, this estimate suggests that the boundary between acceptable and unacceptable random sextupole imperfection would be about 1 unit.

In spite of the presence of nonlinearity, Eq. (1.6) and its generalizations appearing later in the paper, can be regarded as soundly based. What is problematical, because the actual magnet errors are unknown, is the phasor superposition by which $\Delta r_{\rm coh}$ is calculated. Also there is the minor nuisance of having to iterate Eq. (1.6) in order to find a self-consistent value of a_x.

Though I have emphasized the possible constructive interference of, say, the multipoles in one arc, it is probably inappropriate to visualize amplitude growth as being localized and occurring during any one passage through that particular arc. Note that Eq. (1.6) accounts for all nonlinear deflections from the distant past until the present. Also, the coherent superposition yielding $\Delta r_{\rm coh}$ can be performed at any point in the lattice, and not necessarily within the offending arc.

The essence of multipole nonlinearity (unlike the beam-beam force) is that particle orbits, though regular at "small amplitudes", say less than $r_{\rm min}$, "blow up" at large amplitudes such as $r_{\rm nl}$. Since this ratio of large to small is only a factor of two or three, it might be thought "unreasonable" to devote much effort to adjusting the lattice parameters in an attempt to recover a modest improvement, perhaps at most doubling the dynamic aperture. This is *wrong*, however. If the phase space densities of the beams are limited, then doubling the dynamic aperture in two transverse directions increases the potential limiting current of each beam by a factor of 2^4. This could result in a luminosity increase of $2^4 \times 2^4/2^2 = 64$.

Though one is accustomed to the mechanical aperture being "hard-edged" so that a particle can only miss it, and be entirely unaffected, or hit it, and be lost, the magnetic limitation is usually visualized as being more ephemeral. But, based on the discussion in the previous paragraph, my conjectured way of looking at the magnetic aperture gives it somewhat the same character as a mechanical edge. The "edge" region is "reasonably narrow", running from, say, 10 mm to say, 20 mm. From this, admittedly crude, point of view, formula (1.1) can be modified to become independent of the mechanical aperture, simply by taking $r_{\rm mech}$ to be the "edge radius" $r_{\rm edge} = 15\,{\rm mm}$. If the numbers have been chosen so that $r_{\rm edge} = r_{\rm mech}$, this change has no effect on the predicted value of $r_{\rm da}$. Of course the exact equality $r_{\rm edge} = r_{\rm mech}$ has been "put in by hand" and, as described, the procedure is inconsistent in that the first order formula is assumed to be valid out to amplitudes where it has previously been accepted to be invalid. Nevertheless, the fact that the mechanical aperture is comparable to the edge aperture can perhaps be regarded as natural and one expects the fraction error in $r_{\rm da}$ to be less than the fractional uncertainty in $r_{\rm edge}$. In any case, comparisons of dynamic apertures calculated in this paper with values calculated by tracking (with $r_{\rm mech} = \infty$) have to rely on the validity of this assumption.

The attitude just expressed may be a bit too optimistic but, even if it is, one hopes that compensation schemes and choices of tunes that yield optimal performance at intermediate amplitudes within the present model will yield near optimal performance in practice.

In practical accelerator operation the tunes are consciously chosen to avoid those resonances that are expected to be important by delicately balancing the distances to nearby resonances. (This will be called *level 0* application of the theory; in this report the procedure to accomplish it is illustrated in Fig. 9.1 to Fig. 9.4.) From a theoreticians's point of view this practice has the annoying effect of eliminating "the easy cases" in which the dynamics is dominated by a single resonance. A kind of *level 1* application of the theory can then be attempted in which all the resonances are combined by simple superposition. An immediate complication that arises however is that the nonlinear elements cause amplitude-dependent tune shifts. In lowest approximation it is only odd multipoles (U.S.) that do this and a *level 1a* can be defined in which these tune shifts are accounted for. Since these tune shifts almost surely disrupt the previously mentioned delicate balance, a useful estimate of dynamic aperture may result from calculating the amplitude at which the tunes have been shifted onto a particular nearby resonance. To this point lowest order theory is adequate. The theory can then be iterated to higher order. Probably the effect not yet considered at *level 1a* that is most likely to be important is the tune shift due to sextupoles. On the one hand they cause no tune shift in lowest order, but on the other hand they are invariably the strongest nonlinear elements in the ring because they are present for chromaticity compensation. It is relatively straightforward to complete by iteration a *level 2* of calculation that accounts for all nonlinear tune shifts that entered at *level 1.* But in this iteration many new resonances that were not present in *level 1* enter, making the calculation complex and causing the intuitive benefit of being able to concentrate on only one or two resonances to be lost. Nevertheless *level 2* can be completed by computer, and higher levels of iteration as well.

For labeling resonances and where they come from it is necessary to use numerous indices. In this report the letter n will be referred to as the *magnetic order.* It is the sum of the powers appearing in the formula for the deflection caused by a magnetic element. It is therefore also the multipole index (American convention), $n = 1$ for quadrupole, $n = 2$ for sextupole and so on. The letter m economically labels harmonic spectral lines that would be observed for example after Fourier analysing turn-by-turn beam position data. The observed signals are sinusoidal functions of sums and differences of tunes, $m_x\mu_x + m_y\mu_y$. (Mnemonic: m goes with μ.) The letter l labels resonance conditions in the form $l_xQ_x + l_yQ_y =$ integer. (Since these indices are the coefficients of *lines* in "resonance diagrams" there is a certain mnemonic value in this choice of l as index.) By convention l_x is positive, but l_y can have either sign. The quantity $l_x + |l_y|$ will be known as the *resonance order.*† Another index k also appears but only as an intermediate quantity. Though all these indices are related by simple formulas there are so many as to be rather confusing.

This is part I of a more extensive study of the effect of and correction of nonlinear resonances. It describes a theory having the same motivation and making the same general approximations as papers by Guignard.[2] The methodology is very different however since difference equations (obtained from transfer matrices) are used instead of Hamiltonian

† Usually the resonance order of a resonance caused by a pure multipole is equal to the European convention index for that multipole—for example the prominent resonances caused by sextupoles have resonance order 3. It is possible for a pure multipole to cause a resonance of lower resonance order than its European convention multipole index, however.

formalism. The most prominent effect of this is that the superposition over all time is performed before the superposition over all elements in the ring. Then the "variable of integration" (actually summation) for superimposing the effects of disjoint elements is the betatron phase angle, call it ϕ, ranging from 0 to $2\pi Q$. This contrasts with Guignard's treatment which has integrals (actually summations) over the range $0 < \theta < 2\pi$ where θ is the angle locating elements circumferentially in the ring. Integrals over θ are especially significant when Q is equal to a rational fraction Q_r, as is true for exact resonance, since they can be used to define "driving terms" for the corresponding resonance, and these terms dominate nonlinear distortion of particle motion when the "actual" tune Q is sufficiently close to Q_r. In this paper it is shown that this condition is usually *not satisfied* in practice, since one has choosen tunes to *make* it false by intentionally avoiding low order resonances. On the other hand, integrals over ϕ depend on the "actual" tune Q and hence are appropriate for superimposing the effects of all resonances as this report accomplishes (to lowest order.) This report greatly improves upon one of my ancient reports.[3]

In parts of the report not yet written the formulas in this part will be applied to LHC and to the Möbius-modified CESR accelerator. Though the same formulas apply to both cases the important issues are very different. While many multipoles are important for LHC only sextupoles are important for CESR. On the other hand the sextupole problem in CESR is made difficult not only by the absence of any superperiodicity (or even any periodicity) but also by the toggling between horizontal and vertical oscillations that makes it necessary to suppress all third integer resonances and not just $Q_x = 1/3$.

2. Difference equation description of perturbed betatron motion

Betatron motion in one dimension is described by a general, 2×2, Twiss parameterized, transfer matrix

$$\mathbf{T}\left(\beta_1,\alpha_1;\beta_2,\alpha_2;\varphi_{12}\right) = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\left(\cos\varphi_{12}+\alpha_1\sin\varphi_{12}\right) & \sqrt{\beta_1\beta_2}\sin\varphi_{12} \\ \frac{-\sin\varphi_{12}(1+\alpha_1\alpha_2)+\cos\varphi_{12}(\alpha_1-\alpha_2)}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}\left(\cos\varphi_{12}-\alpha_2\sin\varphi_{12}\right) \end{pmatrix}. \tag{2.1}$$

Operating on the vector $(x, x')^T$ with this matrix yields propagation from point P_1 to point P_2 with lattice functions (β_1, α_1) and (β_2, α_2) and betatron phase separation φ_{12}. Because

$\mathbf{T}$ is symplectic it satisfies

$$\mathbf{T}^{-1} = \overline{\mathbf{T}}, \tag{2.2}$$

where $\overline{\mathbf{T}}$ is the "symplectic conjugate" of $\mathbf{T}$ defined by

$$\overline{\mathbf{T}} = -\mathbf{S}\mathbf{T}^T\mathbf{S}, \quad \text{where} \quad \mathbf{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{or} \quad \mathbf{S} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.3}$$

in one or two dimensions. A 2×2 matrix $\mathbf{A}$ and its symplectic conjugate are related by

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{2.4}$$

which satisfy

$$\mathbf{A} + \overline{\mathbf{A}} = \mathrm{tr}\,\mathbf{A} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.5}$$

If $\mathbf{A}$ is given by a "once-around" transfer matrix $\mathbf{T}(\beta, \alpha; \beta, \alpha; \mu)$ then

$$\mathrm{tr}\,\mathbf{T} = a + d = 2\cos\mu. \tag{2.6}$$

Another combination that will be needed for $\mathbf{T}(\beta, \alpha; \beta, \alpha; \mu)$ is

$$\mathbf{T} - \overline{\mathbf{T}} = 2\sin\mu \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \tag{2.7}$$

Initially we will consider the effect of a single nonlinear element at some point P in the lattice and describe the turn-by-turn motion at that point. This perturbed betatron motion in two transverse dimensions is described by equations

$$\begin{pmatrix} \mathbf{X}_{t+1} - \Delta\mathbf{X}'_{t+1}/2 \\ \mathbf{Y}_{t+1} - \Delta\mathbf{Y}'_{t+1}/2 \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{X}_t + \Delta\mathbf{X}'_t/2 \\ \mathbf{Y}_t + \Delta\mathbf{Y}'_t/2 \end{pmatrix} \tag{2.8}$$

where

$$\mathbf{X}_t = \begin{pmatrix} x_t \\ x'_t \end{pmatrix}, \quad \mathbf{Y}_t = \begin{pmatrix} y_t \\ y'_t \end{pmatrix}, \tag{2.9}$$

give positions and slopes at a particular point in the lattice on "turn" t. Calling the once-around transfer matrix $\mathbf{M}$, if the accelerator is weakly coupled, as we will eventually (but not initially) assume, its off-diagonal block matrices $\mathbf{B}$ and $\mathbf{C}$ are small and the on-diagonal blocks are given approximately by

$$\mathbf{A} \approx \mathbf{T}\left(\beta_x, \alpha_x; \beta_x, \alpha_x; \mu_x\right), \quad \mathbf{D} \approx \mathbf{T}\left(\beta_y, \alpha_y; \beta_y, \alpha_y; \mu_y\right). \tag{2.10}$$

The extra terms $\mathbf{\Delta X}'_t$ and $\mathbf{\Delta Y}'_t$ in Eq. (2.8) represent the deflections occuring due to the perturbing element at the point P; they are assumed to have the form

$$\mathbf{\Delta X}'_t = \begin{pmatrix} 0 \\ \Delta x'_t\,(x_t, y_t) \end{pmatrix}, \quad \mathbf{\Delta Y}'_t = \begin{pmatrix} 0 \\ \Delta y'_t\,(x_t, y_t) \end{pmatrix}. \tag{2.11}$$

This form presupposes that the perturbing element has length short enough to be neglected (so the orbit is continuous) and causes a slope discontinuity or "kink" $(\Delta x'_t, \Delta y'_t)$ that depends only on the transverse position (x_t, y_t) and not on the slopes. The kink is treated as occuring half just before the point P, half just after. Since all linear terms, both erect and skew, can be included in $\mathbf{M}$ we can assume without loss of generality that the perturbing terms include only nonlinear parts—the (linear) effect of erect quadrupoles is to shift Λ_A and/or Λ_D, the effect of skew quadrupoles is included in the off-diagonal blocks of $\mathbf{M}$.

The matrix $\mathbf{M}$ and its symplectic conjugate are given by

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \quad \overline{\mathbf{M}} = \begin{pmatrix} \overline{\mathbf{A}} & \overline{\mathbf{C}} \\ \overline{\mathbf{B}} & \overline{\mathbf{D}} \end{pmatrix}, \tag{2.12}$$

and these satisfy

$$\mathbf{M} + \overline{\mathbf{M}} = \begin{pmatrix} (\operatorname{tr}\mathbf{A})\,\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\operatorname{tr}\mathbf{D})\,\mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \overline{\mathbf{E}} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}, \tag{2.13}$$

where

$$\mathbf{E} = \mathbf{C} + \overline{\mathbf{B}}, \quad \text{with determinant} \quad \mathscr{E} = \det|\mathbf{E}|. \tag{2.14}$$

Denoting the eigenvalues of $\mathbf{M} + \overline{\mathbf{M}}$ by Λ_A and Λ_D, they satisfy the simple equations

$$\Lambda_A + \Lambda_D = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{D}, \quad \Lambda_A\Lambda_D = \operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{D} - \mathscr{E}. \tag{2.15}$$

For stable lattices there are real angles $\mu_A = 2\pi Q_A$ and $\mu_D = 2\pi Q_D$ such that

$$\begin{aligned} \Lambda_A &\equiv \lambda_A + 1/\lambda_A = \exp(i\mu_A) + \exp(-i\mu_A) = 2\cos\mu_A \\ \Lambda_D &\equiv \lambda_D + 1/\lambda_D = \exp(i\mu_D) + \exp(-i\mu_D) = 2\cos\mu_D \end{aligned}, \tag{2.16}$$

where $\lambda_A, 1/\lambda_A, \lambda_D, 1/\lambda_D$ are the eigenvalues of $\mathbf{M}$ itself. Using these relations one can obtain an identity that will prove to be useful:

$$\left(\mathbf{M} + \overline{\mathbf{M}}\right)^2 = \begin{pmatrix} (\operatorname{tr}^2\mathbf{A} + \mathscr{E})\,\mathbf{1} & \mathbf{0} \\ \mathbf{0} & (\operatorname{tr}^2\mathbf{D} + \mathscr{E})\,\mathbf{1} \end{pmatrix} + (\operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{D}) \begin{pmatrix} \mathbf{0} & \overline{\mathbf{E}} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}. \tag{2.17}$$

Squaring $\mathbf{M} + \mathbf{M}^{-1}$ and subtracting the identity matrix multiplied by 2 yields

$$\mathbf{M}^2 + \mathbf{M}^{-2} = \begin{pmatrix} -2 + \operatorname{tr}^2\mathbf{A} + \mathscr{E} & \mathbf{0} \\ \mathbf{0} & -2 + \operatorname{tr}^2\mathbf{D} + \mathscr{E} \end{pmatrix} + (\Lambda_A + \Lambda_D) \begin{pmatrix} \mathbf{0} & \overline{\mathbf{E}} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}. \tag{2.18}$$

By eliminating the off-diagonal part of this equation using Eq. (2.13) and by liberal use of Eqs. (2.15) one obtains

$$\mathbf{M}^2 + \mathbf{M}^{-2} - (\Lambda_A + \Lambda_D)\left(\mathbf{M} + \mathbf{M}^{-1}\right) + (2 + \Lambda_A\Lambda_D)\,\mathbf{1} = 0. \tag{2.19}$$

This is a remarkable equation since it has to be satisfied by any 4×4 symplectic matrix describing a stable accelerator lattice. (This equation can also be obtained starting from the theorem that a matrix satisfies its own characteristic equation. This comment makes it also straightforward to obtain the analogous equation for the 6-dimensional matrix with longitudinal motion included.)

If the eigenfrequencies are known (as would be true if one were using Eq. (2.19) to analyse beam position data measured on an accelerator) then Eq. (2.19) is appropriate as it is, but if the elements of $\mathbf{M}$ are known it is more convenient to substitute from Eq. (2.15) to obtain

$$\boxed{\mathbf{M}^2 + \overline{\mathbf{M}}^2 - (\mathrm{tr}\,\mathbf{A} + \mathrm{tr}\,\mathbf{D})(\mathbf{M} + \overline{\mathbf{M}}) + (2 + \mathrm{tr}\,\mathbf{A}\,\mathrm{tr}\,\mathbf{D} - \mathscr{E})\mathbf{1} = \mathbf{0}.} \tag{2.20}$$

We wish next to manipulate Eq. (2.8) in such a way as to exploit this equation as far as the linear terms are concerned, while at the same time keeping track of the nonlinear perturbations. Because of Eq. (2.2), "backward propagation" can be described by

$$\begin{pmatrix} \mathbf{X}_{t-1} + \Delta\mathbf{X}'_{t-1}/2 \\ \mathbf{Y}_{t-1} + \Delta\mathbf{Y}'_{t-1}/2 \end{pmatrix} = \begin{pmatrix} \overline{\mathbf{A}} & \overline{\mathbf{C}} \\ \overline{\mathbf{B}} & \overline{\mathbf{D}} \end{pmatrix} \begin{pmatrix} \mathbf{X}_t - \Delta\mathbf{X}'_t/2 \\ \mathbf{Y}_t - \Delta\mathbf{Y}'_t/2 \end{pmatrix}. \tag{2.21}$$

By summing Eqs. (2.8) and (2.21) one obtains

$$\begin{aligned} \begin{pmatrix} \mathbf{X}_{t+1} + \mathbf{X}_{t-1} \\ \mathbf{Y}_{t+1} + \mathbf{Y}_{t-1} \end{pmatrix} &= \begin{pmatrix} \mathrm{tr}\,\mathbf{A} & \overline{\mathbf{E}} \\ \mathbf{E} & \mathrm{tr}\,\mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{X}_t \\ \mathbf{Y}_t \end{pmatrix} + \\ &+ \begin{pmatrix} \Delta\mathbf{X}'_{t+1}/2 - \Delta\mathbf{X}'_{t-1}/2 \\ \Delta\mathbf{Y}'_{t+1}/2 - \Delta\mathbf{Y}'_{t-1}/2 \end{pmatrix} + \begin{pmatrix} \mathbf{A} - \overline{\mathbf{A}} & \mathbf{B} - \overline{\mathbf{C}} \\ \mathbf{C} - \overline{\mathbf{B}} & \mathbf{D} - \overline{\mathbf{D}} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{X}'_t/2 \\ \Delta\mathbf{Y}'_t/2 \end{pmatrix} \end{aligned} \tag{2.22}$$

By translating indices the same equations can be used to describible several turns. For example,

$$\begin{aligned} \begin{pmatrix} \mathbf{X}_{t+2} - \Delta\mathbf{X}'_{t+2}/2 \\ \mathbf{Y}_{t+2} - \Delta\mathbf{Y}'_{t+2}/2 \end{pmatrix} &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{t+1} + \Delta\mathbf{X}'_{t+1}/2 \\ \mathbf{Y}_{t+1} + \Delta\mathbf{Y}'_{t+1}/2 \end{pmatrix} \\ \begin{pmatrix} \mathbf{X}_t + \Delta\mathbf{X}'_t/2 \\ \mathbf{Y}_t + \Delta\mathbf{Y}'_t/2 \end{pmatrix} &= \begin{pmatrix} \overline{\mathbf{A}} & \overline{\mathbf{C}} \\ \overline{\mathbf{B}} & \overline{\mathbf{D}} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{t+1} - \Delta\mathbf{X}'_{t+1}/2 \\ \mathbf{Y}_{t+1} - \Delta\mathbf{Y}'_{t+1}/2 \end{pmatrix} \\ \begin{pmatrix} \mathbf{X}_t - \Delta\mathbf{X}'_t/2 \\ \mathbf{Y}_t - \Delta\mathbf{Y}'_t/2 \end{pmatrix} &= \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{t-1} + \Delta\mathbf{X}'_{t-1}/2 \\ \mathbf{Y}_{t-1} + \Delta\mathbf{Y}'_{t-1}/2 \end{pmatrix} \\ \begin{pmatrix} \mathbf{X}_{t-2} + \Delta\mathbf{X}'_{t-2}/2 \\ \mathbf{Y}_{t-2} + \Delta\mathbf{Y}'_{t-2}/2 \end{pmatrix} &= \begin{pmatrix} \overline{\mathbf{A}} & \overline{\mathbf{C}} \\ \overline{\mathbf{B}} & \overline{\mathbf{D}} \end{pmatrix} \begin{pmatrix} \mathbf{X}_{t-1} - \Delta\mathbf{X}'_{t-1}/2 \\ \mathbf{Y}_{t-1} - \Delta\mathbf{Y}'_{t-1}/2 \end{pmatrix}. \end{aligned} \tag{2.23}$$

Adding these equations, the result is

$$\begin{aligned}\begin{pmatrix}\mathbf{X}_{t+2}+2\mathbf{X}_t+\mathbf{X}_{t-2}\\ \mathbf{Y}_{t+2}+2\mathbf{Y}_t+\mathbf{Y}_{t-2}\end{pmatrix} &= \begin{pmatrix}\operatorname{tr}\mathbf{A} & \overline{\mathbf{E}}\\ \mathbf{E} & \operatorname{tr}\mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{X}_{t+1}+\mathbf{X}_{t-1}\\ \mathbf{Y}_{t+1}+\mathbf{Y}_{t-1}\end{pmatrix}+\\ &+\begin{pmatrix}\Delta\mathbf{X}'_{t+2}/2-\Delta\mathbf{X}'_{t-2}/2\\ \Delta\mathbf{Y}'_{t+2}/2-\Delta\mathbf{Y}'_{t-2}/2\end{pmatrix}+\begin{pmatrix}\mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}}\\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_{t+1}/2+\Delta\mathbf{X}'_{t-1}/2\\ \Delta\mathbf{Y}'_{t+1}/2+\Delta\mathbf{Y}'_{t-1}/2\end{pmatrix};\end{aligned} \tag{2.24}$$

note that the explicitly linear terms are all in the first line. Using Eqs. (2.22) and (2.17) the first term on the right hand side can be eliminated;

$$\begin{aligned}&\begin{pmatrix}\mathbf{X}_{t+2}+2\mathbf{X}_t+\mathbf{X}_{t-2}\\ \mathbf{Y}_{t+2}+2\mathbf{Y}_t+\mathbf{Y}_{t-2}\end{pmatrix}\\ &=\left(\begin{pmatrix}\operatorname{tr}^2\mathbf{A}+\mathscr{E} & \mathbf{0}\\ \mathbf{0} & \operatorname{tr}^2\mathbf{D}+\mathscr{E}\end{pmatrix}+(\operatorname{tr}\mathbf{A}+\operatorname{tr}\mathbf{D})\begin{pmatrix}\mathbf{0} & \overline{\mathbf{E}}\\ \mathbf{E} & \mathbf{0}\end{pmatrix}\right)\begin{pmatrix}\mathbf{X}_t\\ \mathbf{Y}_t\end{pmatrix}+\\ &+\begin{pmatrix}\operatorname{tr}\mathbf{A} & \overline{\mathbf{E}}\\ \mathbf{E} & \operatorname{tr}\mathbf{D}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_{t+1}/2-\Delta\mathbf{X}'_{t-1}/2\\ \Delta\mathbf{Y}'_{t+1}/2-\Delta\mathbf{Y}'_{t-1}/2\end{pmatrix}+\\ &+\begin{pmatrix}\operatorname{tr}\mathbf{A} & \overline{\mathbf{E}}\\ \mathbf{E} & \operatorname{tr}\mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}}\\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_t/2\\ \Delta\mathbf{Y}'_t/2\end{pmatrix}\\ &+\begin{pmatrix}\Delta\mathbf{X}'_{t+2}/2-\Delta\mathbf{X}'_{t-2}/2\\ \Delta\mathbf{Y}'_{t+2}/2-\Delta\mathbf{Y}'_{t-2}/2\end{pmatrix}+\begin{pmatrix}\mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}}\\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_{t+1}/2+\Delta\mathbf{X}'_{t-1}/2\\ \Delta\mathbf{Y}'_{t+1}/2+\Delta\mathbf{Y}'_{t-1}/2\end{pmatrix}\end{aligned}$$

After this substitution one notes that off-diagonal blocks that "couple" $\mathbf{X}_t$ and $\mathbf{Y}_t$ still remain. But, guided by Eq. (2.20), one notes that terms corresponding to $(\operatorname{tr}\mathbf{A}+\operatorname{tr}\mathbf{D})(\mathbf{M}+\overline{\mathbf{M}})$ as given by Eq. (2.22) must still be subtracted; this yields

$$\boxed{\begin{pmatrix}\mathbf{X}_{t+2}+\mathbf{X}_{t-2}\\ \\ \mathbf{Y}_{t+2}+\mathbf{Y}_{t-2}\end{pmatrix}-(\Lambda_A+\Lambda_D)\begin{pmatrix}\mathbf{X}_{t+1}+\mathbf{X}_{t-1}\\ \\ \mathbf{Y}_{t+1}+\mathbf{Y}_{t-1}\end{pmatrix}+(2+\Lambda_A\Lambda_D)\begin{pmatrix}\mathbf{X}_t\\ \\ \mathbf{Y}_t\end{pmatrix}=\mathbf{\Delta}'\mathrm{s},} \tag{2.25}$$

where

$$\begin{aligned}\mathbf{\Delta}'\mathrm{s}= &\begin{pmatrix}\Delta\mathbf{X}'_{t+2}/2-\Delta\mathbf{X}'_{t-2}/2\\ \Delta\mathbf{Y}'_{t+2}/2-\Delta\mathbf{Y}'_{t-2}/2\end{pmatrix}+\begin{pmatrix}\mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}}\\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_{t+1}/2+\Delta\mathbf{X}'_{t-1}/2\\ \Delta\mathbf{Y}'_{t+1}/2+\Delta\mathbf{Y}'_{t-1}/2\end{pmatrix}\\ &+\begin{pmatrix}-\operatorname{tr}\mathbf{D} & \overline{\mathbf{E}}\\ \mathbf{E} & -\operatorname{tr}\mathbf{A}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_{t+1}/2-\Delta\mathbf{X}'_{t-1}/2\\ \Delta\mathbf{Y}'_{t+1}/2-\Delta\mathbf{Y}'_{t-1}/2\end{pmatrix}\\ &+\begin{pmatrix}-\operatorname{tr}\mathbf{D} & \overline{\mathbf{E}}\\ \mathbf{E} & -\operatorname{tr}\mathbf{A}\end{pmatrix}\begin{pmatrix}\mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}}\\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}}\end{pmatrix}\begin{pmatrix}\Delta\mathbf{X}'_t/2\\ \Delta\mathbf{Y}'_t/2\end{pmatrix}\end{aligned} \tag{2.25}$$

This is the master equation on which everything else is based. In spite of the nonlinear deflections it is exact, but this is mainly academic since the deflections themselves depend on the displacements. Being nonlinear this equation is subject to the well-known phenomena of chaos and dynamic aperture limitation. It will be greatly simplified in the remaining sections of this paper.

3. Lowest order approximation of Δ's

When viewed at point P in an arbitrarily coupled lattice, choosing $t = 0$ appropriately, the small amplitude turn-by-turn positions of one of the two pure normal mode oscillations is given by

$$\begin{aligned}
x_t &= A_x^{\mathrm{P}} \cos\left(\mu_A t + \Phi_A^{\mathrm{P},x}\right) \\
x'_t &= -\frac{A_x^{\mathrm{P}}}{\beta_x}\left[\sin\left(\mu_A t + \Phi_A^{\mathrm{P},x}\right) + \alpha_x \cos\left(\mu_A t + \Phi_A^{\mathrm{P},x}\right)\right] \\
y_t &= A_y^{\mathrm{P}} \cos\left(\mu_A t + \Phi_A^{\mathrm{P},y}\right) \\
y'_t &= -\frac{A_y^{\mathrm{P}}}{\beta_y}\left[\sin\left(\mu_A t + \Phi_A^{\mathrm{P},y}\right) + \alpha_y \cos\left(\mu_A t + \Phi_A^{\mathrm{P},y}\right)\right]
\end{aligned} \tag{3.1}$$

For the other mode: $A_x \to D_x$, $A_y \to D_y$, $\mu_A \to \mu_D$, $\Phi_A^{\mathrm{P},x} \to \Phi_D^{\mathrm{P},x}$, $\Phi_A^{\mathrm{P},y} \to \Phi_D^{\mathrm{P},y}$. We introduce the abbreviations $\Phi_i^{\mathrm{P},x} = \Phi^{\mathrm{P},x} + i\mu$, $\Phi_i^{\mathrm{P},y} = \Phi^{\mathrm{P},y} + i\mu$, applicable to either mode, (meaning the dependence on mode will no longer be indicated explicitly.) Also let a_x, a_y stand for either A_x, A_y or D_x, D_y. Then Eqs. (3.1) become

$$\begin{aligned}
x_{t+i}^{\mathrm{P}} &= a_x^{\mathrm{P}} \cos\left(\mu t + \Phi_i^{\mathrm{P},x}\right) \equiv \mathcal{X}_i^{\mathrm{P}}, \\
{x'}_{t+i}^{\mathrm{P}} &= -\frac{a_x^{\mathrm{P}}}{\beta_x}\left[\sin\left(\mu t + \Phi_i^{\mathrm{P},x}\right) + \alpha_x \cos\left(\mu t + \Phi_i^{\mathrm{P},x}\right)\right] \equiv {\mathcal{X}'}_i^{\mathrm{P}}, \\
y_{t+i}^{\mathrm{P}} &= a_y^{\mathrm{P}} \cos\left(\mu t + \Phi_i^{\mathrm{P},y}\right) \equiv \mathcal{Y}_i^{\mathrm{P}}, \\
{y'}_{t+i}^{\mathrm{P}} &= -\frac{a_y^{\mathrm{P}}}{\beta_y}\left[\sin\left(\mu t + \Phi_i^{\mathrm{P},y}\right) + \alpha_y \cos\left(\mu t + \Phi_i^{\mathrm{P},y}\right)\right] \equiv {\mathcal{Y}'}_i^{\mathrm{P}}.
\end{aligned} \tag{3.2}$$

Because these are normal modes the frequencies are the same in both planes, but in general the phases are different for x and y and for the two modes and they depend on position P and turn index i as well. (The strategy guiding the notation is to refer the turn index to t, to have nothing but cosines appear in x_t and y_t, and to have μt with unshifted t in all arguments.) Other than the *small amplitude* assumption these formulas are completely general—that is, when they are used to evaluate the $\boldsymbol{\Delta}'$s in Eqs. (2.25) the results are valid for arbitrarily coupled lattices provided the coupled-lattice Twiss functions are used.

If the lattice is approximately uncoupled the normal modes oscillations can be distinguished as nominally horizontal and vertical, satisfying

$$\begin{aligned} \mu_A \approx \mu_x, \quad 0 < a_y^{\mathrm{P}} << a_x^{\mathrm{P}} \approx n_{\sigma x}\sqrt{\beta_x^{\mathrm{P}}\epsilon_x}\ , \\ \mu_D \approx \mu_y, \quad 0 < a_x^{\mathrm{P}} << a_y^{\mathrm{P}} \approx n_{\sigma x}\sqrt{\beta_y^{\mathrm{P}}\epsilon_y}. \end{aligned} \tag{3.3}$$

The deflection terms Δ's in Eq. (2.25) can then be written

$$\begin{aligned} \boldsymbol{\Delta}'\mathrm{s} = {} & \frac{1}{2}\begin{pmatrix} 0 \\ \Delta x'\left(\mathcal{X}_2^{\mathrm{P}},\mathcal{Y}_2^{\mathrm{P}}\right) - \Delta x'\left(\mathcal{X}_{-2}^{\mathrm{P}},\mathcal{Y}_{-2}^{\mathrm{P}}\right) \\ 0 \\ \Delta y'\left(\mathcal{X}_2^{\mathrm{P}},\mathcal{Y}_2^{\mathrm{P}}\right) - \Delta y'\left(\mathcal{X}_{-2}^{\mathrm{P}},\mathcal{Y}_{-2}^{\mathrm{P}}\right) \end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix} \mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}} \\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}} \end{pmatrix}\begin{pmatrix} 0 \\ \Delta x'\left(\mathcal{X}_1^{\mathrm{P}},\mathcal{Y}_1^{\mathrm{P}}\right) + \Delta x'\left(\mathcal{X}_{-1}^{\mathrm{P}},\mathcal{Y}_{-1}^{\mathrm{P}}\right) \\ 0 \\ \Delta y'\left(\mathcal{X}_1^{\mathrm{P}},\mathcal{Y}_1^{\mathrm{P}}\right) + \Delta y'\left(\mathcal{X}_{-1}^{\mathrm{P}},\mathcal{Y}_{-1}^{\mathrm{P}}\right) \end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix} -\operatorname{tr}\mathbf{D} & \overline{\mathbf{E}} \\ \mathbf{E} & -\operatorname{tr}\mathbf{A} \end{pmatrix}\begin{pmatrix} 0 \\ \Delta x'\left(\mathcal{X}_1^{\mathrm{P}},\mathcal{Y}_1^{\mathrm{P}}\right) - \Delta x'\left(\mathcal{X}_{-1}^{\mathrm{P}},\mathcal{Y}_{-1}^{\mathrm{P}}\right) \\ 0 \\ \Delta y'\left(\mathcal{X}_1^{\mathrm{P}},\mathcal{Y}_1^{\mathrm{P}}\right) - \Delta y'\left(\mathcal{X}_{-1}^{\mathrm{P}},\mathcal{Y}_{-1}^{\mathrm{P}}\right) \end{pmatrix} \\ & + \frac{1}{2}\begin{pmatrix} -\operatorname{tr}\mathbf{D} & \overline{\mathbf{E}} \\ \mathbf{E} & -\operatorname{tr}\mathbf{A} \end{pmatrix}\begin{pmatrix} \mathbf{A}-\overline{\mathbf{A}} & \mathbf{B}-\overline{\mathbf{C}} \\ \mathbf{C}-\overline{\mathbf{B}} & \mathbf{D}-\overline{\mathbf{D}} \end{pmatrix}\begin{pmatrix} 0 \\ \Delta x'\left(\mathcal{X}_0^{\mathrm{P}},\mathcal{Y}_0^{\mathrm{P}}\right) \\ 0 \\ \Delta y'\left(\mathcal{X}_0^{\mathrm{P}},\mathcal{Y}_0^{\mathrm{P}}\right) \end{pmatrix}, \end{aligned} \tag{3.4}$$

where the motion has been approximated by Eq. (3.2) .

When the off-diagonal terms of this equation are neglected under the no-coupling assumption it is curious that the resulting horizontal equation seems to depend on $\operatorname{tr}\mathbf{D}$. Equating to zero the coefficients of $\operatorname{tr}\mathbf{D}$ in the nominally horizontal (*i.e.* upper) terms of Eq. (2.25) yields

$$\begin{aligned} & \begin{pmatrix} x_{t+1} - 2\cos\mu_x x_t + x_{t-1} \\ x'_{t+1} - 2\cos\mu_x x'_t + x'_{t-1} \end{pmatrix} \\ & = \begin{pmatrix} 0 \\ \Delta x'\left(\mathcal{X}_1^{\mathrm{P}},\mathcal{Y}_1^{\mathrm{P}}\right)/2 - \Delta x'\left(\mathcal{X}_{-1}^{\mathrm{P}},\mathcal{Y}_{-1}^{\mathrm{P}}\right)/2 \end{pmatrix} + \sin\mu_x \begin{pmatrix} \beta_x \\ -\alpha_x \end{pmatrix} \Delta x'\left(\mathcal{X}_0^{\mathrm{P}},\mathcal{Y}_0^{\mathrm{P}}\right) \end{aligned} \tag{3.5}$$

This is the same difference equation one would have derived in the first place for uncoupled x motion; it can be obtained immediately from Eq. (2.22). Hence the coefficient of $\operatorname{tr}\mathbf{D}$ in the fourth order difference equation vanishes identically in this uncoupled case. Our problem has therefore been reduced to solving two second order difference equations, (3.5) and the corresponding equation for vertical oscillations. The upper equations are

$$\boxed{\begin{aligned} x_{t+1} - 2\cos\mu_x x_t + x_{t-1} &= \beta_x \sin\mu_x \Delta x'(\mathcal{X}_0^{\rm P}, \mathcal{Y}_0^{\rm P}) \\ y_{t+1} - 2\cos\mu_x y_t + y_{t-1} &= \beta_y \sin\mu_y \Delta y'(\mathcal{X}_0^{\rm P}, \mathcal{Y}_0^{\rm P}) \end{aligned}.} \tag{3.6}$$

It is often sufficient to solve only these since the slopes can be obtained from

$$x'_t + \frac{\Delta x'_t}{2} = \frac{x_{t+1} - (\cos\mu_x + \alpha_x \sin\mu_x)\, x_t}{\beta_x \sin\mu_x}. \tag{3.7}$$

This expressions gives x'_{t+}, the slope just after the nonlinear element.

Since the deflections are nonlinear the Courant-Snyder invariant calculated just after passage through the nonlinear element,

$$\epsilon_t^{(\rm nl)} = \frac{1}{\beta}\left({x_t}^2 + \left(\alpha x_t + \beta x'_{t+}\right)^2\right) = \frac{1}{\beta}\left({x_t}^2 + \left(\frac{x_{t+1} - \cos\mu\, x_t}{\sin\mu}\right)^2\right), \tag{3.8}$$

is not conserved indefinitely, but it *is* conserved until the next nonlinear element is encountered. Note that its value is independent of α, meaning that the impulsive discontinuity in Courant-Snyder invariant caused by a nonlinear element depends only on β.

If the time variation of the deflection varies proportional to $\cos\Omega t$ it induces a response $\hat{x}\cos\Omega t$. (It will be shown in the next section). The corresponding variation of $\epsilon_t^{(\rm nl)}$ is

$$\epsilon_t^{(\rm nl)} = \frac{\hat{x}^2}{\beta}\left(\cos^2\Omega t + \left(\frac{\cos\Omega\,(t+1) - \cos\mu\,\cos\Omega t}{\sin\mu}\right)^2\right) \tag{3.9}$$

The quantity

$$\begin{aligned} \alpha x_t + \beta x_{t+} &= \hat{x}\left(\frac{\cos\Omega\,(t+1) - \cos\mu\,\cos\Omega t}{\sin\mu}\right) \\ &= \hat{x}\,\frac{-\sin\Omega\sin\Omega t + (\cos\Omega - \cos\mu)\cos\Omega t}{\sin\mu} \end{aligned} \tag{3.10}$$

is called "the slope component" in "normalized phase space". In practice it will turn out that "resonance" occurs only for $\cos\Omega \approx \cos\mu$, in which case the second term becomes negligible, and the first becomes approximately $\pm\hat{x}\sin\Omega t$. Eq. (3.10) therefore shows that the phase shift relative to drive of the response in normalized phase space is small. Furthermore, to the extent there is a phase shift, it depends only on Ω and μ. This greatly simplifies the superimposition of the effects of nonlinear elements at different locations in the lattice, since phases simply add in normalized phase space and, to the extent there is phase shift, it is common to all elements.

Later we will need to obtain the maximum value $\epsilon_{\max}^{(\mathrm{nl})}$ from Eq. (3.9). In general this is complicated, but if we assume $\sin\mu \approx \pm 1$ and $\cos\mu \approx 0$ we get

$$\epsilon_{\max}^{(\mathrm{nl})} = \frac{\hat{x}^2}{\beta}\mathcal{R}_{\Omega,\mu}, \quad \text{where} \quad \mathcal{R}_{\Omega,\mu} = 1 + \left(\frac{\cos\Omega - \cos\mu}{\sin\mu}\right)^2. \tag{3.11}$$

In practice the second term will be fractionally important only for non-resonant terms that are themselves small; in other words the "correction factor" $\mathcal{R}_{\Omega,\mu}$ is approximately 1 for resonant terms so simply setting $\mathcal{R}_{\Omega,\mu} = 1$ constitutes a reasonably consistent approximation. This approximation will be made from here on in this report.

4. Lowest order solution of the perturbed betatron equations

Because the left hand side of Eq. (2.25) is completely uncoupled, in the absence of perturbation all components of $\mathbf{X}_t$ satisfy the same equation. Setting the right hand side to zero the uppermost equation is

$$x_{t+2} + x_{t-2} - (\Lambda_A + \Lambda_D)(x_{t+2} + x_{t-2}) + (2 + \Lambda_A\Lambda_D)\, x_t = 0. \tag{4.1}$$

Seeking a "homogeneous" solution of Eqs. (2.25) of the form

$$x_t = \cos\mu t, \quad \text{or} \quad x_t = \sin\mu t, \tag{4.2}$$

to be known as the "zero'th order motion", the equation becomes

$$2\left(\cos 2\mu - 2\left(\cos\mu_A + \cos\mu_D\right)\cos\mu + 1 + 2\cos\mu_A\cos\mu_D\right)(\cos\mu t \text{ or } \sin\mu t) = 0, \tag{4.3}$$

and these give the same condition for μ;

$$\cos 2\mu - 2\left(\cos\mu_A + \cos\mu_D\right)\cos\mu + 1 + 2\cos\mu_A\cos\mu_D = 0. \tag{4.4}$$

The solutions to this equation can be seen to be $\cos\mu = \cos\mu_A$ and $\cos\mu = \cos\mu_D$ which is re-assuring.

Consider next the "inhomogeneous" response in Eq. (2.25) to a sinusoidal "drive term" of the form

$$\Delta'_x = c_\Omega \cos(\Omega t + \phi_\Omega). \tag{4.5}$$

The perturbed betatron equation is

$$x_{t+2}+x_{t-2}-2\left(\cos\mu_A+\cos\mu_D\right)\left(x_{t+2}+x_{t-2}\right)+\left(2+2\cos\mu_A\cos\mu_D\right)x_t=c_\Omega\cos\left(\Omega t+\phi_\Omega\right). \tag{4.6}$$

Seeking a solution in the form

$$x_t=a_\Omega\cos\left(\Omega t+\phi_\Omega\right) \tag{4.7}$$

one finds

$$\begin{aligned} a_\Omega &= \frac{c_\Omega/2}{\cos 2\Omega-2\left(\cos\mu_A+\cos\mu_D\right)\cos\Omega+1+2\cos\mu_A os\mu_D} \\ &= \frac{c_\Omega/4}{\left(\cos\Omega-\cos\mu_A\right)\left(\cos\Omega-\cos\mu_D\right)} \\ &= \frac{c_\Omega/4}{\cos\mu_A-\cos\mu_D}\left(\frac{1}{\cos\Omega-\cos\mu_A}-\frac{1}{\cos\Omega-\cos\mu_D}\right) \end{aligned} \tag{4.8}$$

where the last step used partial fraction expansion. The conditions under which a denominator factor can vanish is made more transparent by re-expressing this in the form

$$\boxed{a_\Omega=\frac{c_\Omega/16}{\sin\frac{\mu_A+\mu_D}{2}\sin\frac{\mu_A-\mu_D}{2}}\left(\frac{1}{\sin\frac{\Omega+\mu_A}{2}\sin\frac{\Omega-\mu_A}{2}}-\frac{1}{\sin\frac{\Omega+\mu_D}{2}\sin\frac{\Omega-\mu_D}{2}}\right)} \tag{4.9}$$

One notes in passing that if we include longitudinal motion then Eq. (4.8) generalizes to

$$a_\Omega=\frac{c_\Omega/8}{\left(\cos\Omega-\cos\mu_A\right)\left(\cos\Omega-\cos\mu_D\right)\left(\cos\Omega-\cos\mu_L\right)}, \tag{4.10}$$

where $\mu_L/(2\pi)$ is the synchrotron tune. As in Eq. (4.8) this expression can be expanded into three separate terms by partial fraction expansion and much the same inferences could be drawn concerning the resonances caused by vanishing denominator factors.

When the lattice is approximately uncoupled only the *second order* difference equations (3.6) have to be solved;

$$x_{t+1}-2\cos\mu_x x_t+x_{t-1}=c_\Omega\cos\left(\Omega t+\Phi_\Omega\right). \tag{4.11}$$

The inhomogeneous solution of this equation is

$$x_t=\frac{c_\Omega/2}{\cos\Omega-\cos\mu_x}\cos\left(\Omega t+\Phi_\Omega\right)=\frac{-c_\Omega/4}{\sin\frac{\Omega+\mu_x}{2}\sin\frac{\Omega-\mu_x}{2}}\cos\left(\Omega t+\Phi_\Omega\right). \tag{4.12}$$

This and the corresponding equation for y lead to the same resonances we had already come to expect.

In the context of this paper deflections like (4.5) arise in the iterative solution of nonlinear equations. To show this consider a deflection due to a sextupole of "strength" $\mathcal{M}$;

$$\Delta x'_t = \mathcal{M}\, x_t^2 = \mathcal{M}\, \left(a_x \cos\mu_x t\right)^2 = \mathcal{M}\, \frac{a_x^2}{2}\left(1+\cos 2\mu_x t\right), \tag{4.13}$$

so that (dropping the constant term) $\Omega = 2\mu_x$ and Eq. (3.5) is

$$\begin{pmatrix} x_{t+1} - 2\cos\mu_x x_t + x_{t-1} \\ x'_{t+1} - 2\cos\mu_x x'_t + x'_{t-1} \end{pmatrix} = \frac{\mathcal{M}\, a_x^2}{4}\begin{pmatrix} 0 \\ \cos\left(2\mu_x t + 2\mu_x\right) - \cos\left(2\mu_x t - 2\mu_x\right)\end{pmatrix} + \frac{\mathcal{M}\, a_x^2 \sin\mu_x}{2}\begin{pmatrix}\beta_x \\ -\alpha_x\end{pmatrix}\cos 2\mu_x t, \tag{4.14}$$

with solution

$$\begin{pmatrix} x_t \\ x'_t \end{pmatrix} = \mathcal{M}\, a_x^2 \frac{\sin\mu_x/2}{2\left(\cos 2\mu_x - \cos\mu_x\right)}\begin{pmatrix} \beta_x \cos 2\mu_x t \\ -2\cos\mu_x\, \sin 2\mu_x t - \alpha_x \cos 2\mu_x t\end{pmatrix}. \tag{4.15}$$

Including the zero'th order motion the upper component takes the general form

$$x_t = a_x\left(\cos\mu_x t + \mathcal{P}'\,\frac{\mathcal{N}}{\mathcal{D}}\mathcal{C}\left(t\right)\right), \tag{4.16}$$

where symbols have been introduced that will be used for variable-form, standard-role expressions through the paper: $n_{\sigma x}$ is the amplitude in units of beam sigmas, $\mathcal{N} = \sin\mu_x/2$, $\mathcal{D} = 2(\cos 2\mu_x - \cos\mu_x)$, $\mathcal{P}' = n_{\sigma x}\sqrt{\epsilon_x}\,\mathcal{P}$, $\mathcal{P} = \beta_x^{3/2}\mathcal{M}$, and $\mathcal{C}(t) = \cos 2\mu_x t$. The factor a_x has been replaced by the factor $n_{\sigma x}\sqrt{\beta_x\epsilon_x}$ to take advantage of the constancy of ϵ_x over the lattice as in Eq. (3.3). The factors $\mathcal{N}$, $\mathcal{D}$, $\mathcal{P}$ or $\mathcal{P}'$, and $\mathcal{C}(t)$ will be known respectively as "numerator", "denominator", "phasor", and "time-varying" factors of the response. (When there are more than one nonlinear elements in the ring, $\mathcal{P}$ will have to be replaced by a summation over them, but the other factors will not change.)

In greater generality, if $\Delta x' = x_t^{n_x}$ where $n_x = 2, 4, 6, \ldots$ is an even power of x

$$\Delta x'_t = a_x^{n_x}\cos^{n_x}\left(\mu_x t + \Phi\right) = \left(\frac{a_x}{2}\right)^{n_x}\left\{\sum_{k_x=0}^{n_x/2-1} 2\binom{n_x}{k_x}\cos\left(\left(n_x - 2k_x\right)\left(\mu_x t + \Phi\right)\right) + \binom{n_x}{n_x/2}\right\},$$

$$\frac{\Delta x'_{t+1} - \Delta x'_{t-1}}{2} = -2\left(\frac{a_x}{2}\right)^{n_x}\left\{\sum_{k_x=0}^{n_x/2-1}\binom{n_x}{k_x}\sin\left(\left(n_x - 2k_x\right)\mu_x\right)\, \sin\left(\left(n_x - 2k_x\right)\left(\mu_x t + \Phi\right)\right)\right\} \tag{4.17}$$

where the lower series is only required for the lower of Eqs. (3.5). There is a similar expansions for odd n_x; both are given in Table 4.1.

Table 4.1: Expansion coefficients $\mathcal{N}_{n,k}$ in $\cos^n\phi = \sum_k \mathcal{N}_{n,k}\cos(n-2k)\phi$. The first entry is appropriate if $n-2k$ ranges over non-negative possibilities. The second entry is appropriate if $n-2k$ ranges over non-positive possibilities. For the latter case some elements are truncated.

n		0	1	2	3	4	5	6	7	8	9
k	common	1	1	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
0		1	1 -	1 -	1 -	1 -	1 -	1 -	1 -	$2\binom{8}{0}$	$\binom{9}{0}$
1			- 1	1 1	3 -	4 -	5 -	6 -	7 -	$2\binom{8}{1}$	$\binom{9}{1}$
2				- 1	- 3	3 3	10 -	15 -	21 -	$2\binom{8}{2}$	$\binom{9}{2}$
3					- 1	- 4	- 10	10 10	35 -	$2\binom{8}{3}$	$\binom{9}{3}$
4						- 1	- 5	- 15	- 35	$2\binom{8}{4}$	$\binom{9}{4}$

For reasons that will only become clear when we discuss two dimensional motion this table has been made "unnecessarily" complicated by allowing for two possible ranges for the k index. The reason for this freedom is that the cosine is an even function. For one dimensional motion one should simply use the first entry so that $n-2k$ remains non-negative. (For the other choice $n-2k$ remains non-positive.) For the one-dimensional case presently under discussion, for temporary simplicity, we assume the first choice.†

Motivated by formulas in section 6 giving the deflections caused by pure magnetic multipoles, and introducing a "strength coefficient" $\mathcal{M}^{x\to x}_{n_x}$, the first of Eqs. (4.17) can be written

$$\Delta x'_t = \mathcal{M}^{x\to x}_{n_x} x_t^{n_x} = \mathcal{M}^{x\to x}_{n_x} a_x^{n_x}\cos^{n_x}\left(\mu_x t + \Phi^{\mathrm{P},x}\right) = \mathcal{M}^{x\to x}_{n_x} a_x^{n_x}\sum_{k_x}\mathcal{N}_{n_x,k_x}\mathcal{C}^{\mathrm{P},x}_{m_x}(t)\,,$$
$$\text{where}\quad \mathcal{C}^{\mathrm{P},x}_{m_x}(t) = \cos\left(m_x\left(\mu_x t + \Phi^{\mathrm{P},x}\right)\right),\quad m_x = n_x - 2k_x. \tag{4.18}$$

This same notation also serves for $n_x = 1,3,5,\ldots$, though Eq. (4.17) is not valid in this case and the summation runs up to $k_x = (n_x-1)/2$ in Eq. (4.18). From now on the

† To aid in making rough estimates, one may note that the coefficient of $\cos(n_x(\mu_x t + \Phi))$ (the top row in the table) is $1/2^{n_x-1}$ which is "atypically" small since the coefficients are positive and have to add up to one, so "typical" values of the remaining coefficients are roughly the inverse of the number of terms. The only reason this is mentioned is that the resonance caused by nonlinearity x^n that is likely to spring to mind first come from the replacement $\cos^n\phi \to \cos(n\phi)$ and this is the one that is atypically small.

upper limit will be left off with the understanding that the sum terminates before $n_x - 2k_x$ changes sign.

For the n_x, k_x term in this expansion the "drive frequency" is $\Omega = m_x\mu_x$ and the resonant denominator factor is

$$\frac{1}{\mathcal{D}^{x\to x}_{m_x}} = \frac{1/2}{\cos(m_x\mu_x) - \cos\mu_x} = \frac{-1/4}{\sin((m_x+1)\,\pi Q_x)\ \sin((m_x-1)\,\pi Q_x)}. \tag{4.19}$$

This factor contributes a potentially large factor to a solution much like Eq. (4.15). Before writing this down one notes that the same Fourier term coming from the same, nominally horizontal, x-mode oscillation may also cause a vertical oscillation with strength $\mathcal{M}^{x\to y}_{n_x}$. The left hand side of the relevant equation is the same as Eq. (4.14) but with $x_t \to y_t$ and the corresponding resonant factor is

$$\frac{1}{\mathcal{D}^{x\to y}_{m_x}} = \frac{-1/4}{\sin((m_xQ_x + Q_y)\,\pi)\ \sin((m_xQ_x - Q_y)\,\pi)}. \tag{4.20}$$

Just as the x-mode motion drives both x and y response, there are corresponding responses to y-mode motion. All drive terms discussed so far are included in

$$\begin{aligned}\Delta x'_t &= \mathcal{M}^{x\to x}_{n_x} x_t^{n_x} + \mathcal{M}^{y\to x}_{n_y} y_t^{n_y},\\ \Delta y'_t &= \mathcal{M}^{x\to y}_{n_x} x_t^{n_x} + \mathcal{M}^{y\to y}_{n_y} y_t^{n_y}.\end{aligned} \tag{4.21}$$

The "free oscillations" that include the effects of these nonlinear deflections are then†

$$\boxed{\begin{aligned} x_t &= a_x(\cos\mu_x t + \textstyle\sum_{k_x=0} \beta_x \mathcal{M}^{x\to x}_{n_x} a_x^{n_x-1} \frac{\mathcal{N}_{n_x,k_x}}{\mathcal{D}^{x\to x}_{m_x}}\ \mathcal{C}^{\mathrm{P},x}_{m_x}(t))\\ y_t &= \textstyle\sum_{k_x=0} \beta_y \mathcal{M}^{x\to y}_{n_x} a_x^{n_x} \frac{\mathcal{N}_{n_x,k_x}}{\mathcal{D}^{x\to y}_{m_x}}\ \mathcal{C}^{\mathrm{P},x}_{m_x}(t)\\ x_t &= \textstyle\sum_{k_y=0} \beta_x \mathcal{M}^{y\to x}_{n_y} a_y^{n_y} \frac{\mathcal{N}_{n_y,k_y}}{\mathcal{D}^{y\to x}_{m_y}}\ \mathcal{C}^{\mathrm{P},y}_{m_y}(t)\\ y_t &= a_y(\cos\mu_y t + \textstyle\sum_{k_y=0} \beta_y \mathcal{M}^{y\to y}_{n_y} a_y^{n_y-1} \frac{\mathcal{N}_{n_y,k_y}}{\mathcal{D}^{y\to y}_{m_y}}\ \mathcal{C}^{\mathrm{P},y}_{m_y}(t))\end{aligned}} \tag{4.22}$$

These formulas account (to terms of the same order $n = n_x$ or $n = n_y$) for the nonlinear element situated at point P. It would not be difficult to iterate this solution to get an approximation to higher order, but there would be no point in doing that before first

† For consistency with later formulas it may be necessary to accept the "other" choice in Table 4.1 for k_y and hence $n_y - 2k_y$.

including the effects of all other nonlinear elements in the lattice. One substitution that will be appropriate for this is to substitute $a_x \to \sqrt{\epsilon_x \beta_x^{\rm P}}$ and $a_y \to \sqrt{\epsilon_y \beta_y^{\rm P}}$.

We have now covered all the possibilities for pure-but-perturbed normal mode oscillations. But a typical particle is oscillating in both transverse normal modes simultaneously (and longitudinally as well, but we skip consideration of this possibility.) It may happen, especially in electron storage rings, that one or the other of the modes is negligibly small relative to the other, say $a_y << a_x$, and in that case Eq. (4.22) (the upper pair) is all that is required for a consistent calculation to order $n = n_x$. But if a_x and a_y are comparable in size it is necessary to account for deflections proportional to $x^{n_x} y^{n_y}$ where $n = n_x + n_y$. Let us therefore consider the effect of deflections of the form

$$\begin{aligned} \Delta x'_t &= \mathcal{M}^{xy\to x}_{n_x,n_y} x_t^{n_x} y_t^{n_y}, \quad n_x + n_y = n, \\ \Delta y'_t &= \mathcal{M}^{xy\to y}_{n_x,n_y} x_t^{n_x} y_t^{n_y}, \quad n_x + n_y = n. \end{aligned} \tag{4.23}$$

When the unperturbed normal mode motions are substituted into these expressions, the same formula (4.18) as before can be used to Fourier expand the individual $x_t^{n_x}$ and $y_t^{n_y}$ factors, and then the product can be expanded using Eq. (4.13). (This just brings in factors of 1/2.)

$$\boxed{\begin{aligned} x_t &= a_x(\cos\mu_x t + \textstyle\sum_{k_x}\sum_{k_y} \beta_x \mathcal{M}^{xy\to x}_{n_x,n_y} a_x^{n_x-1} a_y^{n_y} \dfrac{\mathcal{N}_{n_x,k_x}\mathcal{N}_{n_y,k_y}/2}{\mathcal{D}^{xy\to x}_{m_x,m_y}}\, \mathcal{C}^{\rm P}_{m_x,m_y}(t)) \\ y_t &= a_y(\cos\mu_y t + \textstyle\sum_{k_x}\sum_{k_y} \beta_y \mathcal{M}^{xy\to y}_{n_x,n_y} a_x^{n_x} a_y^{n_y-1} \dfrac{\mathcal{N}_{n_x,k_x}\mathcal{N}_{n_y,k_y}/2}{\mathcal{D}^{xy\to y}_{m_x,m_y}}\, \mathcal{C}^{\rm P}_{m_x,m_y}(t)) \\ &\text{where}\quad \mathcal{C}^{\rm P}_{m_x,m_y}(t) = \cos(m_x(\mu_x t + \Phi^{{\rm P},x}) + m_y(\mu_y t + \Phi^{{\rm P},y})). \end{aligned}} \tag{4.24}$$

The summation over k_y is extended to include terms with m_y both positive and negative. A sample denominator factor is

$$\frac{1}{\mathcal{D}^{xy\to x}_{m_x,m_y}} = \frac{-1/4}{\sin\left((l_{x1}Q_x + l_{y1}Q_y)\,\pi\right)\ \sin\left((l_{x2}Q_x + l_{y2}Q_y)\,\pi\right)}. \tag{4.25}$$

where

$$l_{x1} = m_x + 1, \quad l_{y1} = m_y, \quad l_{x2} = m_x - 1, \quad l_{y2} = m_y. \tag{4.26}$$

These coupled motion formulas will have to be replaced by more compact formulas later on in order to correctly combine the terms coming from pure multipoles.

We have now exhausted all possibilities for the $n = n_x + n_y$ order of perturbation. We anticipate that the perturbation terms that have been calculated will be unimportant unless one of the denominator factors $\mathcal{D}$, is small. After having made the conditions for this to happen more explicit it will be necessary to superimpose the effects of all lattice elements contributing to any particular "resonance".†

In Eqs. (4.22) and (4.24) individual terms within the summations "distort" the linear betatron motion. Since their frequencies are incommensurate (except for cases to be discussed shortly) the sum of the absolute values of the coefficients of the $\mathcal{C}(t)$ factors can be interpreted as $\Delta a_x/a_x$, $\Delta a_y/a_x$, $\Delta a_x/a_y$ or $\Delta a_y/a_y$, as appropriate.

5. Resonance conditions

It is the possible vanishing of one of the denominators that is the source of resonance. This is the only possible source however, since the $\mathbf{\Delta}'$ factors are finite and appear only in numerators. In an iterative procedure the right hand side of Eq. (2.25) will be approximated by substituting a previously-determined approximate formula for x_t. When this (necessarily periodic or multi-periodic) function is Fourier analysed it consists of a sum of "harmonics" of the form $c_\Omega \cos(\Omega t + \Phi_\Omega)$ where Ω is formed from some or all of the possible integer combinations of μ_A and μ_D. The response to each of these "drive terms" is given by Eq. (4.9).‡ But to the order of accuracy of the current calculation (after one iteration, that is) it is only the resonances corresponding to Eqs. (4.22) and (4.24) that enter.

It is possible for a tune combination not to cause a resonance in spite of the fact that it causes a denominator to vanish. This would be because the corresponding numerator term coming from the right hand side of Eq. (2.25) vanishes. This occurs either naturally,

† To lowest order, which is to say Eqs. (4.22) and (4.24), there is no possibility of an exactly vanishing denominator, and hence, strictly speaking, no resonance. However, when higher orders are included the tunes can shift in such a way as to shift the tunes and cause true resonance.

‡ In the Fourier expansion of $\mathbf{\Delta}$'s, as well as the various nonlinear harmonics, "fundamental terms" $\cos(\mu_A t + \Phi_\Omega)$ or $\cos(\mu_D t + \Phi_\Omega)$ will in general appear. These terms reflect the fact that the nonlinear forces can shift the tunes and/or cause linear coupling. Since terms like this correspond to linear motion they must be cancelled by subtracting appropriate terms from both sides of the equation. This amounts to "renormalizing" the coefficients on the left hand side of the equation. In the approximately-uncoupled approximation the linear terms that may enter have the form $\cos(\mu_x t + \Phi_\Omega)$ or $\cos(\mu_y t + \Phi_\Omega)$ and these have the unfortunate ability to couple the linear motion. This forces one to revert from the simpler second order difference equations (3.6) to the fourth order equations (2.25).

because of some symmetry, or because some nonlinear compensation scheme has arranged for it to happen. It occurs, for example, in the uncoupled case when $\mathcal{N}_{n_x,k_x} = 0$.

The tunes Q_A and Q_D can be split into integer and fractional parts and both parts influence the resonant behavior of the accelerator. But it can be seen from the structure of Eq. (4.10) that with only one nonlinear element in the ring we need only be concerned with the fractional parts. For the time being, this will be taken for granted.

Superficially it might appear that all factors in the denominator of Eq. (4.9) are capable of vanishing and hence will result in similar behavior and be subject to similar analysis. This is wrong however. First consider the situations near "linear sum resonances" where $Q_A + Q_D$ is close to an integer, or near "linear difference resonances" where $Q_A - Q_D$ is close to an integer. For linear sum resonances, since the tunes "attract", exact resonance is possible and accelerator operation is never attempted in that vicinity. The possibility of linear sum resonances will therefore be ignored from now on in this report. On the other hand, difference resonances "repel" making it impossible for the difference resonance condition to be satisfied exactly. Hence, though the factor $\cos\mu_A - \cos\mu_D$ is small near the difference resonance, and hence amplifies other response terms, it cannot vanish. The only possibilities of resonance in lowest order therefore are $\cos\Omega \approx \cos\mu_A$ or $\cos\Omega \approx \cos\mu_D$.

One thing that should be remembered is that in the present formalism the eigenvalues Λ_A and Λ_D are exact and include all linear coupling effects. When operating close to a difference resonance these eigenvalues may differ markedly from the nominal, uncoupled values, $\cos\mu_x$ and $\cos\mu_y$ (which might even be equal, for example.) Even more so, the eigenmotions may be far from the nominal, uncoupled, pure horizontal or pure vertical, motions that one tends to visualize. The eigenmotions might, for example, be at angles comparable to $\pm 45°$. If this happened to be the case, then the erect sextupoles in the ring would be capable of causing resonances (for example $Q_D = 1/3$) that would not have been expected for uncoupled motion.

6. Deflections caused by pure multipole fields

The deflection terms $\mathbf{\Delta}'$ have to be calculated for the particular magnet being traversed, whose field is expressed as a multipole series. In order to coalesce common factors, dimensionless scaled multipole coefficients $\tilde{a}_{n,n+1}$ and $\tilde{b}_{n,n+1}$ are defined by

$$\tilde{a}_{n,n+1} = \Delta\Theta\, 10^{-4}\, a_{n,n+1}, \quad \tilde{b}_{n,n+1} = \Delta\Theta\, 10^{-4}\, b_{n,n+1}, \quad \text{where} \quad \Delta\Theta = \frac{LB_0}{p_0/e} \tag{6.1}$$

is the bend angle in radians of a particle of momentum p_0 passing through a dipole with field B_0 and arc length L. (If $B_0 = 0$ it is necessary to replace it by, say, $\partial B/\partial x|_0$ if it is nominally a quadrupole.) The two indices correspond to American/European conventions. In terms of multipole coefficients, the (dimensionless) magnetic field components are

$$\begin{aligned}
\tilde{B}_y &= \frac{B_y L}{p_0/e} = \Re \sum_{n=0,1}^{M} \frac{\tilde{b}_{n,n+1} + i\tilde{a}_{n,n+1}}{r_{\rm ref}^n} \Big((x - \underline{\Delta x}) + i\,(y - \underline{\Delta y}) \Big)^n, \\
\tilde{B}_x &= \frac{B_x L}{p_0/e} = \Im \sum_{n=0,1}^{M} \frac{\tilde{b}_{n,n+1} + i\tilde{a}_{n,n+1}}{r_{\rm ref}^n} \Big((x - \underline{\Delta x}) + i\,(y - \underline{\Delta y}) \Big)^n.
\end{aligned} \tag{6.2}$$

These definitions differ from earlier formulations by the inclusion of the factor $r_{\rm ref}^n$ which has the effect that the $b_{n,n+1}$ and $a_{n,n+1}$ are dimensionless and are to be interpreted as fractional field deviations at $r_{\rm ref}$ (traditionally in parts per 10^4). The coefficients in the multipole series are related to other conventional parameters as shown in Table 6.1. Formulas relating transverse momentum deviations caused by magnetic fields are

$$\Delta x' = -\tilde{B}_y, \quad \Delta y' = \tilde{B}_x.$$

Suppressing the offsets $(\underline{\Delta x}, \underline{\Delta y})$, the multipole series and the deflections it causes are then given by

$$-\Delta x' + i\Delta y' = \sum \frac{\tilde{b}_{n,n+1} + i\tilde{a}_{n,n+1}}{r_{\rm ref}^n} (x + iy)^n. \tag{6.3}$$

Real and imaginary coefficients R_n and I_n are defined by

$$(x + iy)^n = R_n + iI_n. \tag{6.4}$$

	n	R_n	I_n	$\tilde{b}_{n,n+1}$	$\tilde{a}_{n,n+1}$	$\Delta x' = -\tilde{B}_y$	$\Delta y' = \tilde{B}_x$
Horizontal bend	0	1	0	$\Delta\theta_x$	0	$-\Delta\theta_x$	0
Vertical bend				0	$\Delta\theta_y$	0	$\Delta\theta_y$
Erect quadrupole	1	x	y	$q = 1/f$	0	$-qx$	qy
Skew quadrupole				0	$q_s = 1/f_s$	$q_s y$	$q_s x$
Erect sextupole	2	$x^2 - y^2$	$2xy$	$S/2$	0	$-\frac{S}{2}(x^2 - y^2)$	$\frac{S}{2}2xy$
Skew sextupole				0	$S_s/2$	$\frac{S_s}{2}2xy$	$\frac{S_s}{2}(x^2 - y^2)$
Erect octupole	3	$x^3 - 3xy^2$	$3x^2y - y^3$	$O/6$	0	$-\frac{O}{6}(x^3 - 3xy^2)$	$\frac{O}{6}(3x^2y - y^3)$
Skew octupole				0	$O_s/6$	$\frac{O_s}{6}(3x^2y - y^3)$	$\frac{O_s}{6}(x^3 - 3xy^2)$
Erect decapole	4	$x^4 - 6x^2y^2$	$4xy(x^2 - y^2)$	$D/24$	0	$-\frac{D}{24}(x^4 - 6x^2y^2 + y^4)$	$\frac{D}{24}4xy(x^2 - y^2)$
Skew decapole		$+y^4$		0	$D_s/24$	$\frac{D_s}{24}4xy(x^2 - y^2)$	$\frac{D_s}{24}(x^4 - 6x^2y^2 + y^4)$

Table 6.1: Deflections caused by standard magnets and notations for their strengths

The factors 1!, 2!, 3! entering the definitions of quad strength q, sextupole strength S, octupole strength O, *etc.* are conventional. Formulas giving transverse momentum deviations caused by the magnetic field components of pure multipole are

$$\begin{aligned} \Delta x'_n &= -\tilde{B}_y|_n = -\frac{\tilde{b}_{n,n+1}}{r_{\text{ref}}^n} R_n + \frac{\tilde{a}_{n,n+1}}{r_{\text{ref}}^n} I_n, \\ \Delta y'_n &= \tilde{B}_x|_n = \frac{\tilde{b}_{n,n+1}}{r_{\text{ref}}^n} I_n + \frac{\tilde{a}_{n,n+1}}{r_{\text{ref}}^n} R_n. \end{aligned} \tag{6.5}$$

7. Fourier harmonics caused by low order multipole fields

When deflection terms are drawn from Table 6.1 and evaluated using first approximations

$$x_t = a_x \cos(\mu_x t + \Phi_\Omega), \quad y_t = a_y \cos(\mu_y t + \Phi_\Omega)$$

the following notations and formulas (which employ the abbreviations $m_x = n_x - 2k_x$ $m_y = n_y - 2k_y$) can be used to produce Fourier expansions:

$$\begin{aligned}
\mathcal{C}_{m_x,m_y} \equiv \cos\left((m_x Q_x + m_y Q_y)\,2\pi t\right), \quad \mathcal{C}_{m_x,\pm m_y} &\equiv \mathcal{C}_{m_x,m_y} + \mathcal{C}_{m_x,-m_y}, \quad 1 = \mathcal{C}_{0,0} \\
x &= a_x \cos(2\pi Q_x t) = a_x \mathcal{C}_{1,0} \\
y &= a_y \cos(2\pi Q_y t) = a_y \mathcal{C}_{0,1} \\
x^2 - y^2 &= \frac{a_x^2}{2}\mathcal{C}_{2,0} - \frac{a_y^2}{2}\mathcal{C}_{0,2} + \frac{a_x^2 - a_y^2}{2}\mathcal{C}_{0,0} \\
2xy &= a_x a_y \mathcal{C}_{1,\pm 1} \\
x^3 - 3xy^2 &= \frac{a_x^3}{4}\mathcal{C}_{3,0} - \frac{3a_x a_y^2}{4}\mathcal{C}_{1,\pm 2} + \frac{3a_x^3 - 6a_x a_y^2}{4}\mathcal{C}_{1,0} \\
3x^2 y - y^3 &= -\frac{a_y^3}{4}\mathcal{C}_{0,3} + \frac{3a_x^2 a_y}{4}\mathcal{C}_{2,\pm 1} - \frac{3a_y^3 - 6a_x^2 a_y}{4}\mathcal{C}_{0,1} \\
x^4 - 6x^2y^2 + y^4 &= \frac{a_x^4}{8}\mathcal{C}_{4,0} - \frac{6a_x^2 a_y^2}{8}\mathcal{C}_{2,\pm 2} + \frac{a_y^4}{8}\mathcal{C}_{0,4} \\
&\quad + \frac{4a_x^4 - 12a_x^2 a_y^2}{8}\mathcal{C}_{2,0} + \frac{4a_y^4 - 12a_x^2 a_y^2}{8}\mathcal{C}_{0,2} + \frac{3a_x^4 - 12a_x^2 a_y^2 + 3a_y^4}{8}\mathcal{C}_{0,0} \\
4x^3 y - 4xy^3 &= \frac{4a_x^3 a_y}{8}\mathcal{C}_{3,\pm 1} - \frac{4a_x a_y^3}{8}\mathcal{C}_{1,\pm 3} + \frac{12a_x^3 a_y - 12a_x a_y^3}{8}\mathcal{C}_{1,\pm 1}
\end{aligned} \tag{7.1}$$

Because everything has been expressed in terms of cosines, which are even functions of their argument, it is valid, without loss of generality, to assume $m_x \geq 0$.

When viewing the spectrum obtained by Fourier analysing turn-by-turn beam position data taken in the presence of nonlinearity the observed lines can be labelled with the same indices as on $\mathcal{C}_{m_x,m_y}$. Since the factors a_x and a_y are presumably, in some sense, "small", the dominant lines tend to be those having minimal powers of these factors. Furthermore, if only one normal mode is excited, for example because the beam deflection is purely horizontal or purely vertical and the uncoupled approximation of Eq. (4.11) is adequate, one or the other of a_x and a_y is non-vanishing, so terms containing their product cannot contribute.

m_x	0	1	0	2	1	0	3	2	1	0	4	3	2	1	0
m_y	0	0	1	0	±1	2	0	±1	±2	3	0	±1	±2	±3	4
$b_{0,1}$	1														
$a_{0,1}$															
$b_{1,2}$		a_x													
$a_{1,2}$			a_y												
$b_{2,3}$	a_x^2, a_y^2			a_x^2		a_y^2									
$a_{2,3}$					$a_x a_y$										
$b_{3,4}$		$a_x^3, a_x a_y^2$					a_x^3		$a_x a_y^2$						
$a_{3,4}$			$a_x^2 a_y, a_y^3$					$a_x^2 a_y$		a_y^3					
$b_{4,5}$	$a_x^4, a_x^2 a_y^2, a_y^4$			$a_x^4, a_x^2 a_y^2$		$a_x^2 a_y^2, a_y^4$					a_x^4		a_x^2, a_y^2		a_y^4
$a_{4,5}$					$a_x^3 a_y, a_x a_y^3$							$a_x^3 a_y$		$a_x a_y^3$	

Table 7.1: Spectral lines in X-spectrum (horizontal) caused by particular multipoles. a_x and a_y are "fundamental" amplitudes. There are also numerical factors, of order one, not shown.

m_x	0	1	0	2	1	0	3	2	1	0	4	3	2	1	0
m_y	0	0	1	0	±1	2	0	±1	±2	3	0	±1	±2	±3	4
$b_{0,1}$															
$a_{0,1}$	1														
$b_{1,2}$			a_y												
$a_{1,2}$		a_x													
$b_{2,3}$					$a_x a_y$										
$a_{2,3}$	a_x^2, a_y^2			a_x^2		a_y^2									
$b_{3,4}$			$a_x^2 a_y, a_y^3$					$a_x^2 a_y$		a_y^3					
$a_{3,4}$		$a_x^3, a_x a_y^2$					a_x^3		$a_x a_y^2$						
$b_{4,5}$					$a_x^3 a_y, a_x a_y^3$							$a_x^3 a_y$		$a_x a_y^3$	
$a_{4,5}$	$a_x^4, a_x^2 a_y^2, a_y^4$			$a_x^4, a_x^2 a_y^2$		$a_x^2 a_y^2, a_y^4$					a_x^4		a_x^2, a_y^2		a_y^4

Table 7.2: Spectral lines in Y-spectrum (vertical) caused by particular multipoles. a_x and a_y are "fundamental" amplitudes. There are also numerical factors, of order one, not shown.

8. Simultaneous x and y motion

Because of the large number of factors to be kept track of, it is advisable to formulate the solution as compactly as possible, and because the multipole coefficients are originally defined as coefficients in a complex power series, it is appropriate to re-formulate the calculation using complex algebra. Let h_t be the value on turn t of a real or imaginary harmonic response (that will later be identified either with x_t or y_t.) It satisfies an equation

$$h_{t+1} - \cos\mu_h \; h_t + h_{t-1} = \sum_0^{n_{\max}} c_n \beta_{x,y} \sin\mu_{x,y} \left(x + iy\right)^n , \tag{8.1}$$

where, coming from Eqs. (3.6) and (6.3),

$$c_n = \left(\frac{\tilde{a}_{n,n+1}}{r_{\rm ref}^n} + i \frac{\tilde{b}_{n,n+1}}{r_{\rm ref}^n} \right) ; \tag{8.2}$$

$\mu_{x,y}$ and $\beta_{x,y}$ will eventually be identified as μ_x and β_x or as μ_y and β_y. With $n = n_x + n_y$, $x = a_x \cos\mu_x t$ and $y = a_y \cos\mu_y t$, a_x and a_y both real, $(x+iy)^n$ can be expanded by the binomial formula and each of the factors further binomial expanded to yield

$$\begin{aligned} &\frac{1}{2^n} \left(a_x \left(e^{i\mu_x t} + e^{-i\mu_x t} \right) + i a_y \left(e^{i\mu_y t} + e^{-i\mu_y t} \right) \right)^n = \\ &= \frac{1}{2^n} \sum_{n_y} \binom{n}{n_y} \; a_x^{n_x} \; \left(e^{i\mu_x t} + e^{-i\mu_x t} \right)^{n_x} \; (i a_y)^{n_y} \; \left(e^{i\mu_y t} + e^{-i\mu_y t} \right)^{n_y} \\ &= \frac{1}{2^n} \sum_{n_y} \binom{n}{n_y} \; a_x^{n_x} \; (i a_y)^{n_y} \sum_{m_x} \sum_{m_y} \binom{n_x}{(n_x - m_x)/2} \binom{n_y}{(n_y - m_y)/2} e^{i\Omega(m_x, m_y) t}. \end{aligned} \tag{8.3}$$

where

$$\Omega\left(m_x, m_y\right) = m_x \mu_x + m_y \mu_y . \tag{8.4}$$

The index n is determined by the particular multipole being analysed and the index n_x fixes the amplitude dependent factor $a_x^{n_x} \left(i a_y\right)^{n_y}$. While calculating the other factors these are held fixed as we vary the m_x and m_y indices.† Substituting into Eq. (8.1) the right hand side becomes

$$\sin\mu_{x,y} \sum_{m_x} \sum_{m_y} \left(\beta_{x,y} \sum_{n=2}^{n_{\max}} \sum_{n_y} \frac{c_n}{2^n} a_x^{n_x} \; (i a_y)^{n_y} \; \binom{n}{n_y} \binom{n_x}{(n_x - m_x)/2} \binom{n_y}{(n_y - m_y)/2} \; e^{i\Omega(m_x, m_y)t} \right) . \tag{8.5}$$

† It is useful to remember that $\binom{n}{0} = \binom{n}{n} = 1$. When evaluating low order resonances most terms have $k_x = 0$, which yields $\binom{n_x}{k_x} = 1$, and/or $k_y = 0$, which yields $\binom{n_y}{k_y} = 1$.

The summations have been re-ordered since all terms with the same values of m_x and m_y are coherent and must be summed before taking absolute values. Any one term in this summation yields inhomogeneous response

$$\frac{c_n\beta_{x,y}\sin\mu_{x,y}\,\frac{1}{2^n}\binom{n}{n_y}\,a_x^{n_x}\,(ia_y)^{n_y}\binom{n_x}{k_x}\binom{n_y}{k_y}\,e^{i\Omega(m_x,m_y)t}}{2\cos\Omega\left(m_x,m_y\right)-2\cos\mu_h}$$
$$=\frac{-c_n\beta_{x,y}\sin\mu_{x,y}\binom{n}{n_y}\left(\frac{a_x}{2}\right)^{n_x}\binom{n_x}{k_x}\,e^{im_x\mu_x t}\left(\frac{ia_y}{2}\right)^{n_y}\binom{n_y}{k_y}\,e^{im_y\mu_y t}}{4\sin\left(\Omega\left(m_x,m_y\right)+\mu_h\right)\,\sin\left(\Omega\left(m_x,m_y\right)-\mu_h\right)} \quad (8.6)$$

Notice that the coefficient of the exponential is unchanged if the signs of both m_x and m_y are reversed. This makes it possible to pair-wise sum the term with indices m_x, m_y to the term with $-m_x, -m_y$. There is only one aberrant case, $m_x = m_y = 0$, for which this switch yields the same term. This case only occurs when n_x and n_y are both even and is also the only case when the total number of terms in the double summation over k_x and k_y in (8.3) is odd. Planning to correct its double counting straightaway we simply ignore the $m_x = m_y = 0$ possibility and from now on limit m_x to be non-negative for purposes of keeping track of the summed pairs of terms. Note that in this grouping the pairs $(m_x = 0, m_y)$ and $(m_x = 0, -m_y)$ have also been summed so from now on if $m_x = 0$ then $m_y \geq 0$; otherwise m_y can be either positive or negative. The typical drive term and its response then contain the factor

$$e^{i(m_x\mu_x+m_y\mu_y)}+e^{i(-m_x\mu_x-m_y\mu_y)}=2\cos\left(m_x\mu_x+m_y\mu_y\right),$$

and the response is $\Delta a_h(n, n_x; m_x, m_y; \mu_h)\cos(\Omega(m_x, m_y)t)$ where

$$\boxed{\Delta a_h=\frac{2\binom{n}{n_y}\left(\frac{a_x}{2}\right)^{n_x}\binom{n_x}{k_x}\left(\frac{a_y}{2}\right)^{n_y}\binom{n_y}{k_y}\left(1-\delta^0_{m_x}\delta^0_{m_y}/2\right)}{2\cos\Omega(m_x,m_y)-2\cos\mu_h}\beta_{x,y}\sin\mu_{x,y}\,i^{n_y}c_n} \quad (8.7)$$

The only complex numbers are now contained in the final factor: one factor i enters depending on skew ($c_n \sim a_n$) or erect ($c_n \sim ib_n$) multipolarity; the factor i^{n_y} depends on n_y (as n_y advances $0, 1, 2, 3, \ldots$ the factor advances $i, -1, -i, 1, i, \ldots$); and one factor i depends on whether the deflection being calculated is $\Delta x'$ or $\Delta y'$; see Eq. (6.3).

Depending on whether $\mu_h = \mu_x$ or $\mu_h = \mu_y$, the conditions for resonance can be represented by equations whose coefficients are integers l_x, l_y;

$$\begin{aligned} (m_x \pm 1)\,Q_x + m_yQ_y = l_xQ_x + l_yQ_y = 0, \pm1, \pm2, \ldots \quad &\text{for } \mu_h = \mu_x \\ m_xQ_x + (m_y \pm 1)\,Q_y = l_xQ_x + l_yQ_y = 0, \pm1, \pm2, \ldots \quad &\text{for } \mu_h = \mu_y. \end{aligned} \quad (8.8)$$

For given Q_x and Q_y numerical "accidents" determine particular pairs m_x, m_y that cause one of the rightmost of (8.8) to be approximately satisfied. One then finds corresponding values for l_x, l_y by equating coefficients of the leftmost equations in (8.8). This yields

$$\boxed{\begin{array}{lll} l_y = m_y, & l_x = m_x \pm 1 & \text{for } \mu_h = \mu_x \\ l_x = m_x, & l_y = m_y \pm 1 & \text{for } \mu_h = \mu_y \end{array}} \tag{8.9}$$

The content of Eqs. (8.7) and (4.24) is essentially the same but Eqs. (8.7) more explicitly includes the summation over n_y of all the terms coming from a single multipole. The reason the new indices l_x, l_y had to be introduced is that there is yet another summation required which is over the elements in the lattice and this summation is directly a function of l_x and l_y rather than m_x and m_y. We have arranged for n_x and n_y to be non-negative and can insist that m_x also be non-negative but it is possible for m_x and m_y to have opposite signs, which forces m_y to be negative, in turn forcing l_y to be non-positive.

9. (Frustrated) attempt to identify dominant resonance(s)

Using formulas given earlier for the deflections caused by pure low order multipole elements, the conditions under which a denominator factor can vanish are exhibited graphically in the following pages, one each for sextupoles and octupoles, two for decapoles. Most, but not all, nonlinear terms are shown. The straight lines are contours on which one of the sine function factors of one of the $\mathcal{D}$ factors vanishes. In these figures the axes are unlabelled and have no scales, but in every case the scales are $0 \leq Q_x < 1$ and $0 \leq Q_y < 1$. The purpose of the circles is to give a rough visual representation of the "distance to nearest resonance" of a sample tune combination $Q_x = 0.28$, $Q_y = 0.31$. The lines, having equations of the form

$$l_x Q_x + l_y Q_y = \text{ integer}, \tag{9.1}$$

are contours on which $\sin(l_x\mu_x + l_y\mu_y)$ vanishes. The radii $0.05/(|l_x| + |l_y|)$ of "circles of influence" have been chosen to be inversely proportional to the number of bands in the plot. Roughly speaking then, a possible resonance can be discarded unless its circle intersects a line.

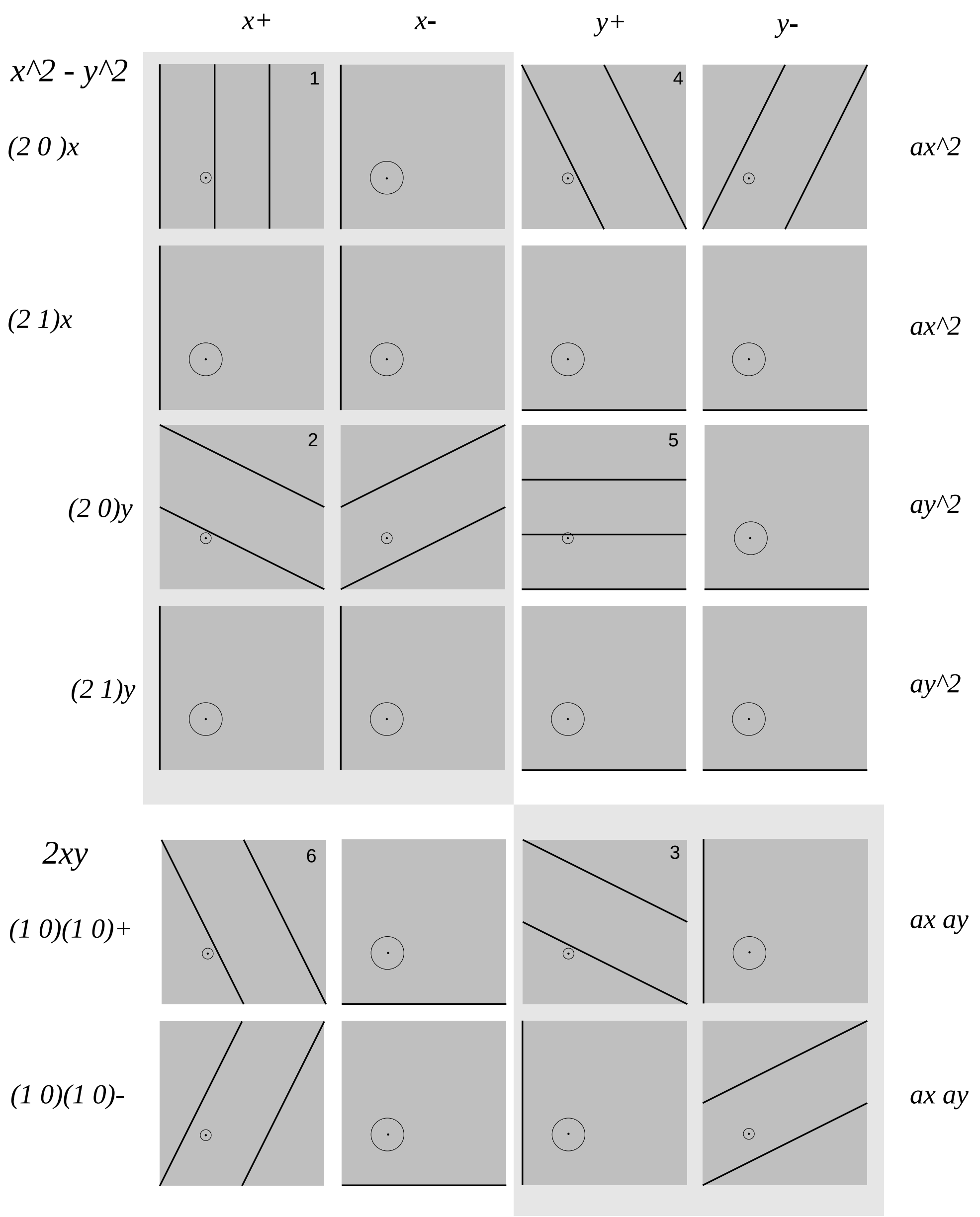

Figure 9.1: Resonance lines caused by *sextupoles*. Horizontal responses are in the two left columns, vertical responses are in the two right colums. Erect sextupoles cause the lines in the upper left and lower right (very lightly shaded.) Skew sextupoles are on the opposite diagonal. A possible choice of fractional tunes, $Q_x = 0.28$, $Q_y = 0.31$ is plotted, centered in "circles of inluence". Notations on the left are (n_x, k_x) or (n_y, k_y) or $(n_x, k_x)(n_y, k_y)\pm$ as appropriate.

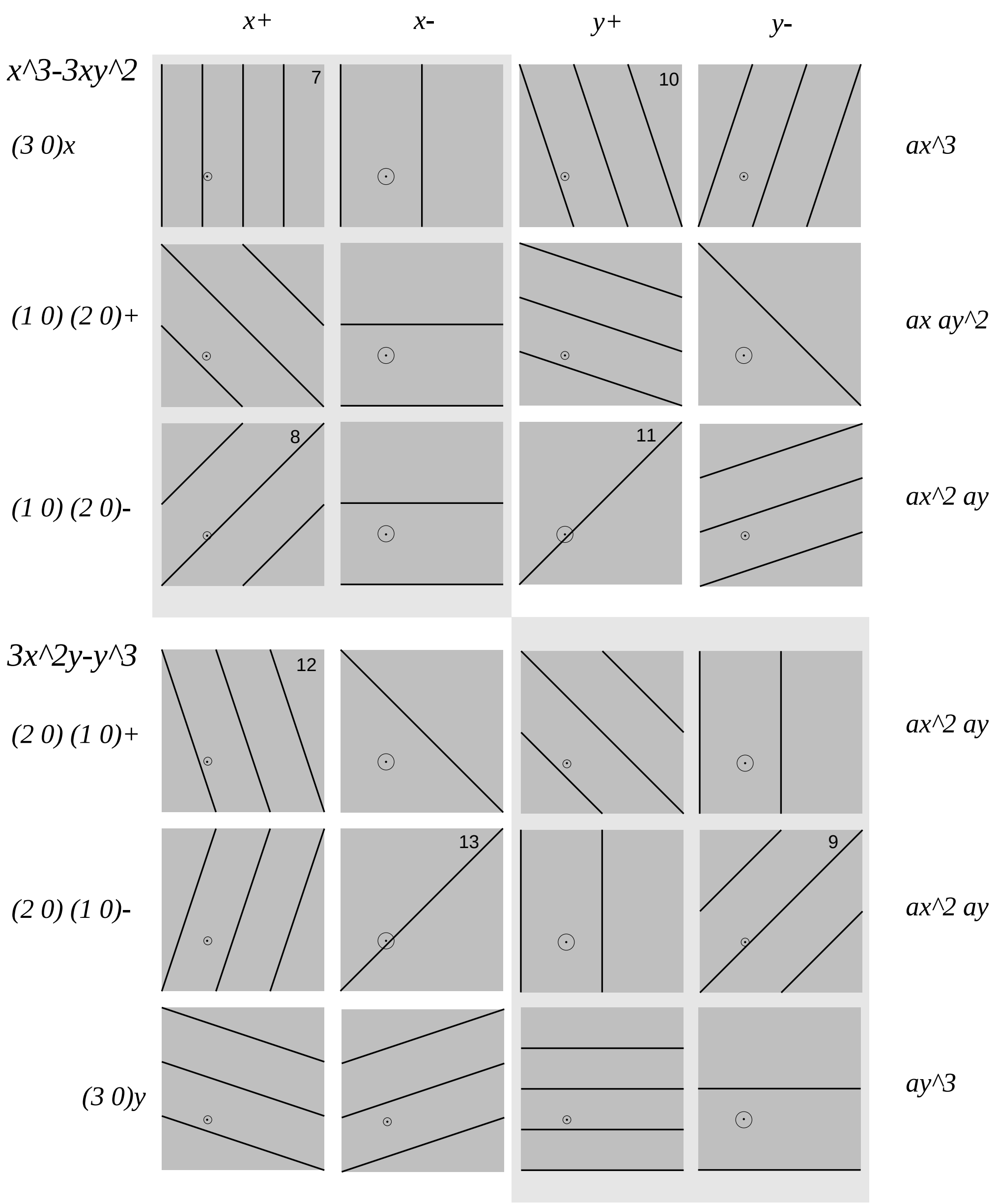

Figure 9.2: Resonance lines caused by *octupoles*. Horizontal responses are in the two left columns, vertical responses are in the two right colums. Erect octupoles cause the lines in the upper left and lower right (very lightly shaded.) Skew octupoles are on the opposite diagonal. A possible choice of fractional tunes, $Q_x = 0.28$, $Q_y = 0.31$ is plotted, centered in "circles of inluence". Notations on the left are (n_x, k_x) or (n_y, k_y) or $(n_x, k_x)(n_y, k_y)\pm$ as appropriate. Terms such as $(n_x, k_x) = (3, 1)$ or $(n_x, k_x)(n_y, k_y)\pm = (2, 1)(1, 0)\pm$ that renormalize linear motion have been dropped. A row that would have been labeled $(31)y$ was overlooked in making the figure.

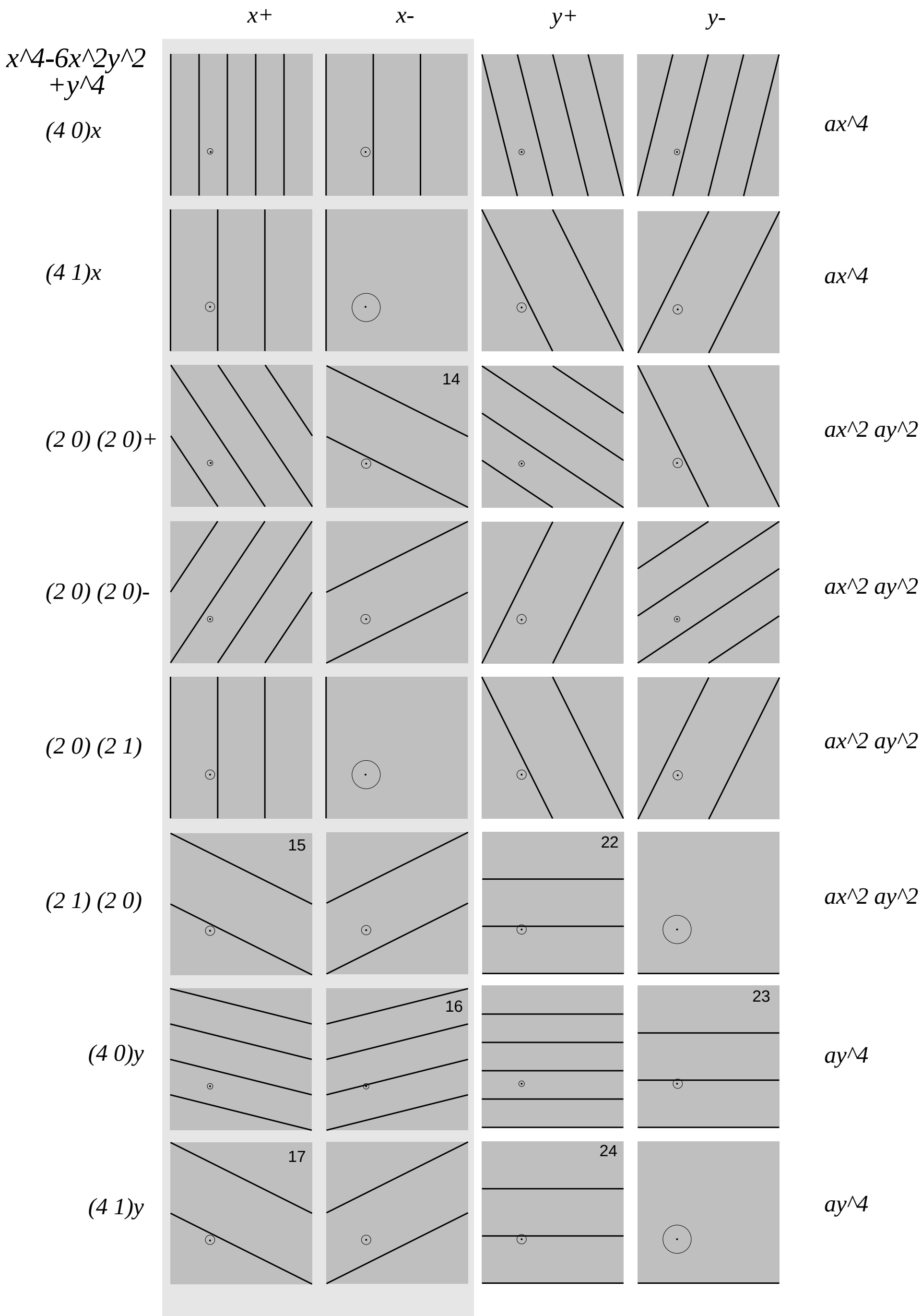

Figure 9.3: Resonance lines caused by *decapoles* (upper part of figure). Horizontal/vertical responses are in the two left/right columns. Erect decapoles cause the lines on the left (faintly shaded) of this figure and the right of the next. A possible choice of fractional tunes, $Q_x = 0.28$, $Q_y = 0.31$ is plotted, centered in "circles of inluence". Some integer resonance cases have been dropped.

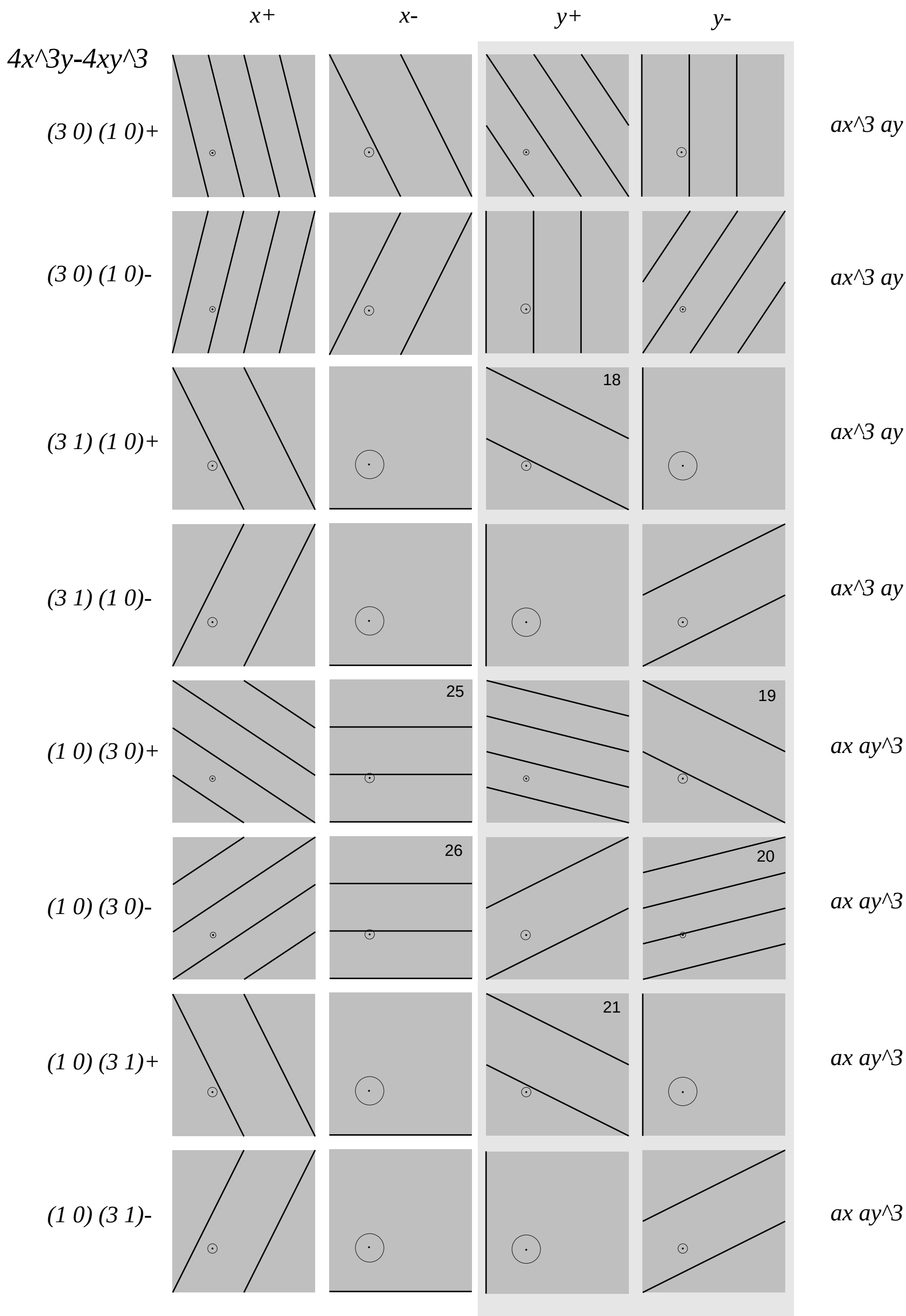

Figure 9.4: Resonance lines caused by *decapoles* (continued). Horizontal/vertical responses are in the two left/right columns. Erect decapoles cause the lines on the right of this figure (faintly shaded) and the left of the previous figure.

Accepting the circle radii as drawn in these figures as a guide, Table 9.1 shows cases where the circles intersect (or almost intersect) a line for the various low order *erect* multipoles assuming $Q_x = 0.28$, $Q_y = 0.31$. Table 9.2 shows the same thing for *skew* multipoles. These are the cases that deserve most accurate treatment in predicting the dynamic acceptance reductions they cause. The entry Δ indicates "distance to nearest resonance" more quantitatively. No case is extremely close to resonance since, for example, $1/(\sin 0.1\pi) = 3.2$, which is not much greater than 1.

In making the resonance plots there is one row for each term of the form $x^{n_x} y^{n-n_x}$ in the formula for the deflection caused by the particular multipole with index n. This can result in "double counting" as for example with entries 25 and 26. Since all indices match for these two terms they must not be added. On the other hand terms with matching values for m_x and m_y, but different values of other indices are "coherent" and their contributions must be summed. This is the one exception, mentioned earlier, where it is necessary to keep track of the signs. See entries 22, 23, and 24. In the table the factors in question are listed in the column with heading i^{n_y}. But since terms differing by the factor i are never summed it is necessary only to keep track of the sign and not the i. These factors have been included in the column with heading "factor" which combines all other integer factors as well.

The final columns in Table 9.1 and Table 9.2 contain values calculated using Eq. (11.1). For the reasons just mentioned these numbers cannot simply be added. But their order of magnitude can be assessed by comparing with the roughly comparable factor $\frac{\mathcal{N}_2}{\mathcal{D}_2}\frac{a_x}{r_{\rm ref}} \approx 0.5$ appearing in Eq. (1.7).

Though the resonances included in Table 9.1 and Table 9.2 were intended to be the "strong ones" it cannot be said that any particular one is dominant and there is no obvious way to keep only one or two and reject the rest. There is a certain inevitableness to this circumstance in that the nominal tunes were initially chosen to "stay away from low order resonances". It is therefore not surprising that at least a few have comparable strengths. We conclude therefore that the only consistent procedure is to keep *all* terms that appear in lowest order.

Table 9.1: Potentially important resonances due to *erect* multipoles, based on intersection or near intersection of circles and lines on resonance diagrams, for $Q_x = 0.28$,$Q_y = 0.31$. $\Delta =$ nearest integer $- 0.28l_x - 0.31l_y$. Factors $\binom{n}{k}$ are not shown because they are 1 in most cases. In cases with $k = 1$ (marked by *) extra factors $\binom{n}{1} = n$ have to be included.

num.	m-pole	$\binom{n}{n_x}$	i^{n_y}	$(\frac{a_x}{2})^{n_x}$	k_x	m_x	$(\frac{a_y}{2})^{n_y}$	k_y	m_y	l_x	l_y	Δ	factor	coeff.
1	sext.	1	1	$a_x^2/4$	0	2	$a_y^0/1$	0	0	3	0	0.16	1/4	-0.3368
2	$-\Delta x'$	1	-1	$a_x^0/1$	0	0	$a_y^2/4$	0	2	1	2	0.10	-1/4	0.4616
3	$i\Delta y'$	2	i	$a_x^1/2$	0	1	$a_y^1/2$	0	1	1	2	0.10	1/2	-1.050
7	oct.	1	1	$a_x^3/8$	0	3	$a_y^0/1$	0	0	4	0	-0.12	1/8	0.1728
8	$-\Delta x'$	3	-1	$a_x^1/2$	0	1	$a_y^2/4$	2	-2	2	-2	0.06	-3/8	1.0762
9	$i\Delta y'$	3	i	$a_x^2/4$	0	2	$a_y^1/2$	1	-1	2	-2	0.06	3/8	1.019
14	dec.	6	-1	$a_x^2/4$	0	2	$a_y^2/4$	0	2	1	2	0.10	-3/8	-0.6116
15	$-\Delta x'$	6	-1	$a_x^2/4$	1*	0	$a_y^2/4$	0	2	1	2	0.10	-3/4	1.3848
16		1	1	$a_x^0/1$	0	0	$a_y^4/16$	4	-4	1	-4	-0.04	1/16	0.2498
17		1	1	$a_x^0/1$	0	0	$a_y^4/16$	1*	2	1	2	0.10	1/4	-0.4616
18	$i\Delta y'$	4	i	$a_x^3/8$	1*	1	$a_y^1/2$	0	1	1	2	0.10	3/4	-1.5750
19		4	-i	$a_x^1/2$	0	1	$a_y^3/8$	0	3	1	2	0.10	-1/4	-0.4052
20		4	-i	$a_x^1/2$	0	1	$a_y^3/8$	3	-3	1	-4	-0.04	-1/4	1.1382
21		4	-i	$a_x^1/2$	0	1	$a_y^3/8$	1*	1	1	2	0.10	-3/4	1.5750

Table 9.2: Potentially important resonances due to *skew* multipoles, based on intersection or near intersection of circles and lines on resonance diagrams, for $Q_x = 0.28$,$Q_y = 0.31$. $\Delta =$ nearest integer $- 0.28 l_x - 0.31 l_y$. When l_y is negative the "other" choice is made for k_y. Factors $\binom{n_x}{k_x}$ are not shown since in most cases they are 1. In cases with $k = 1$ (marked by *) extra factors $\binom{n}{1} = n$ have to be included.

num.	m-pole	$\binom{n}{n_x}$	i^{n_y}	$(\frac{a_x}{2})^{n_x}$	k_x	m_x	$(\frac{a_y}{2})^{n_y}$	k_y	m_y	l_x	l_y	Δ	factor	coeff.
4	sext.	1	1	$a_x^2/4$	0	2	$a_y^0/1$	0	0	2	1	0.13	1/4	-0.4452
5	$i\Delta y'$	1	-1	$a_x^0/1$	0	0	$a_y^2/4$	0	2	0	3	0.07	-1/4	0.6922
6	$-\Delta x'$	2	i	$a_x^1/2$	0	1	$a_y^1/2$	0	1	2	1	0.13	1/2	-0.7610
10	oct.	1	1	$a_x^3/8$	0	3	$a_y^0/1$	0	0	3	1	-0.15	1/8	0.1382
11	$i\Delta y'$	3	-1	$a_x^1/2$	0	1	$a_y^2/4$	2	-2	1	-1	0.03	-3/8	2.2362
12	$-\Delta x'$	3	i	$a_x^2/4$	0	2	$a_y^1/2$	0	1	3	1	-0.15	3/8	0.4300
13		3	i	$a_x^2/4$	0	2	$a_y^1/2$	1	-1	1	-1	0.03	3/8	2.0020
22	dec.	6	-1	$a_x^2/4$	1*	0	$a_y^2/4$	0	2	0	3	0.07	-3/4	2.0784
23	$i\Delta y'$	1	1	$a_x^0/1$	0	0	$a_y^4/16$	0	4	0	3	0.07	1/16	0.1450
24		1	1	$a_x^0/1$	0	0	$a_y^4/16$	1*	2	0	3	0.07	1/4	-0.6922
25	$-\Delta x'$	4	-i	$a_x^1/2$	0	1	$a_y^3/8$	0	3	0	3	0.07	-1/4	-0.5734
26		4	-i	$a_x^1/2$	0	1	$a_y^3/8$	0	3	0	3	0.07	-1/4	-0.5734

10. Coherent superposition of amplitudes

It is worth noting that the presence of a nonlinear element in a lattice causes a local kink, whether or not all elements have been arranged to cancel the resonance drive globally. This kink is fractionally unimportant only close to resonance. For example, referring to Eq. (4.14), if one wishes to propagate solution (4.14) away from the point where it has been evaluated it is necessary first to add the small deflection $\Delta x'_t/2$. If this term is comparable with the other drive terms (because the resonance is not close) then it is not really consistent to ignore $\Delta x'_t/2$.

In this sense then, the condition for a resonance to be a potential strong contributor to nonlinear distortion is $\mathcal{D} << 1$. This is the condition in which the discussion following Eq. (3.10) applies and terms for which it does not apply, to have their effect included correctly, require the inclusion of the correction factor $\mathcal{R}_{\Omega,\mu}$. Even when the condition $\mathcal{D} << 1$ is not met the phasor construction to be described is applicable however.

To understand how to superimpose perturbed responses from disjoint sources it is important to understand the "phase" $\mu_x t + \Phi_x^{\rm P}$ in the unperturbed expression $x_t = a_x \cos(\mu_x t + \Phi_x^{\rm P})$. Because t is a turn index rather than time, it has the same value for every element on any one turn and advances discontinuously by 1 unit each time a reference particle passes the origin. Hence the phase *increases* either as t *increases* or the longitudinal coordinate *increases.*

Before beginning to combine the effects of more than one nonlinear element, one can contemplate the source of the possible divergence exhibited with just one sextupole, say in Eq. (4.15). The denominator factor $\cos 2\mu_x - \cos\mu_x$ vanishes if either $2\mu_x - \mu_x$ or $2\mu_x + \mu_x$ is an integer multiple of 2π. If either of these conditions is met then subsequent passages through the element induce responses that "interfere" constructively and cause divergence. The first of the resonance conditions can be understood as follows: the zero'th order time dependence at P is $\cos\mu_x t$ which the sextupole converts to $\cos 2\mu_x t$ (and a constant term that is being dropped.) This drive causes synchronous (though not necessarily in-phase) perturbed response at the same frequency. Setting aside the out-of-phase deviation because it is common, this response can be compared with the one-turn-later response after it has been propagated back by one turn to become $\cos(2\mu_x(t+1) - \mu_x)$. The condition for these to be in-phase is that μ_x be an integral multiple of 2π which, as claimed, is the first condition. The second condition relies on the fact that the deflection depends on the position x_t but not on the slope x'_t. This means that an "aliased" motion (through angle $(2\pi - \mu_x)$/turn in phase space) given by $x_t = \cos{(2\pi - \mu_x)t}$ induces the same $\cos 2\mu_x t$ response. Evaluating this at $t+1$ and propagating its response backward leads to $\cos(4\pi t - 2\mu_x t - 2\mu_x - \mu_x)$; this yields the condition that $3\mu_x$ be an integer multiple of 2π. Another way of performing the same process (which will simplify later calculations) is to take advantage of the fact that, since the cosine function is an even function of its argument, one can reverse the sign of the μ_x correction term instead of the sign of the $2\mu_x t$ drive term. This amounts to pretending the particle is rotating counter-clockwise in phase space by the angle μ_x/turn; it doesn't matter that the slope is wrong almost everywhere. Recapitulating the single sextupole case, the harmonic frequency is $\Omega = 2\mu_x$ and the resonance conditions are that $\Omega \pm \mu_x$ be multiples of 2π which conforms directly with the result (4.9) obtained using trigonometric identities.

If there are multiple nonlinear elements their contributions can be compared only if they are referred to a common origin. Whether they interfere constructively or destructively depends on their lattice (betatron phase) locations and on the particular resonance.

Let us generalize solution (4.16), referring to Eq. (4.17) if necessary to keep track of the phase factors. Taking the phase locations of two sextupoles to be $\Phi_x^{(1)}$ and $\Phi_x^{(2)}$, let us perform the superposition at a location for which $\Phi_x = 0$. The perturbed responses, referred back to this origin, are $\cos(2\mu_x t + 2\Phi_x^{(1)} - \Phi_x^{(1)})$ and $\cos(2\mu_x t + 2\Phi_x^{(2)} - \Phi_x^{(2)})$. When these sinusoids are represented by phasors, the angle between them is $\Phi_x^{(2)} - \Phi_x^{(1)}$. If we employ the "counter-clockwise" correction the two responses vary like $\cos(2\mu_x t + 2\Phi_x^{(1)} + \Phi_x^{(1)})$ and $\cos(2\mu_x t + 2\Phi_x^{(2)} + \Phi_x^{(2)})$ and the angle between phasors is $3(\Phi_x^{(2)} - \Phi_x^{(1)})$.

A phasor diagram appropriate for "near third integer" horizontal motion with two erect sextupoles is plotted in Fig. 10.1. For a particular resonance applicable to the values n_x and k_x, the phase factors are multiplied by the factor $n_x - 2k_x = m_x$ in Eq. (4.17) by which $\Delta x'_t$ is calculated, and the response is shifted in phase by the same amount. Since it is the horizontal deflection being evaluated it is further necessary to shift one phase by $\pm\Phi^{(1,x)}$ and the other by $\pm\Phi^{(2,x)}$ to refer them both to the agreed upon origin. Altogether then, each response has to be shifted in phase by $l_x \Phi^{(i,x)}$ where $l_x = m_x + 1$ or $l_x = m_x - 1$ before performing the coherent summation. For Eq. (4.16), $n_x = 2$ and $k_x = 0$, and hence $m_x = 2$, and the phase shifts are 3Φ or Φ. In the more general indexing scheme these possibilities correspond to $(l_x, l_y) = (3, 0)$ or $(1, 0)$. This extra generality is required to cover the possibility that the vertical deflection will later be evaluated and then the corrections to common origin will be $\pm\Phi^{(i,y)}$ and the overall phase shift will be $l_x\Phi^{(i,x)} + l_y\Phi^{(i,y)}$ where $(l_x, l_y) = (2, 1)$ or $(2, -1)$. The four examples given in this paragraph correspond to the top row of Fig. 9.1, reading in order from left to right.

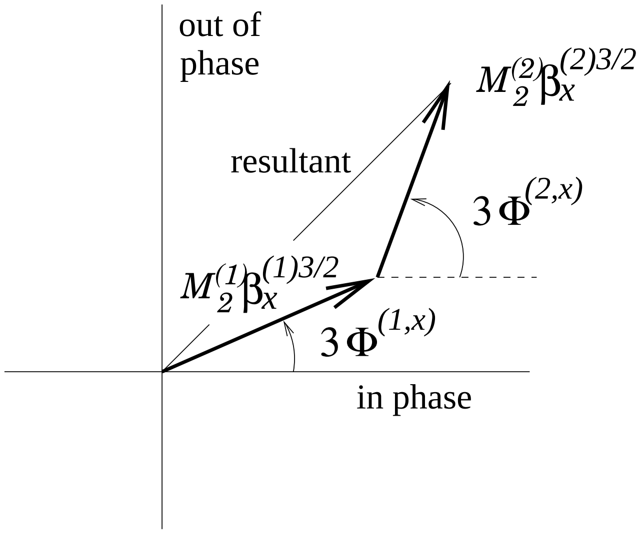

Figure 10.1: Phasor diagram appropriate for superimposing the contributions of two erect sextupoles to the resonance $3Q_x =$ integer for a deflecting term $l_x = 3$ coming from $n_x = 2$, $k_x = 0$ to obtain phasor $\mathcal{P}$. For the general resonance the phasor angle is $l_x\Phi^x + l_y\Phi^y$ and the phasor length is $M_n\beta_x^{(1+n_x)/2}\beta_y^{n_y/2}$ for x-resonance and $M_n\beta_x^{n_x/2}\beta_y^{(1+n_y)/2}$ for y-resonance.

In general the multipole strengths also depend on the position P and so also do the zero'th order amplitudes a_x and a_y. Assume they are given by

$$a_x = n_{\sigma x}\sqrt{\epsilon_x\beta_x^{\rm P}}, \quad a_y = n_{\sigma y}\sqrt{\epsilon_y\beta_y^{\rm P}}, \tag{10.1}$$

where $n_{\sigma x}$ and $n_{\sigma y}$ are beam sizes in sigma-units. Except for obtaining absolute values the factors $n_{\sigma x}\sqrt{\epsilon_x}$ and $n_{\sigma y}\sqrt{\epsilon_y}$ can be dropped since they do not depend on P. When substituting into Eqs. (4.22) or (4.24) for x_t, terms proportional to $a_x^{n_x}a_y^{n_y}$ acquire factors $\beta_x^{(n_x-1)/2}\beta_y^{n_y/2}$ to go with the explicit factor β_x; this results in an overall factor

$\beta_x^{(1+n_x)/2}\beta_y^{n_y/2}$. Similarly the y_t response acquires a factor $\beta_x^{n_x/2}\beta_y^{(1+n_y)/2}$. For our simple, two sextupole example, the lengths of the two phasors are $\beta_x^{(1)^{3/2}}\mathcal{M}_2^{(1)}$ and $\beta_x^{(2)^{3/2}}\mathcal{M}_2^{(2)}$.

For general one dimensional motion the variation $\cos m_x\mu_x t$ at point P, after correction to a common origin, becomes $\cos(m_x\mu_x t + l_x\Phi_x^{(P)})$ with $l_x = m_x \pm 1$ or $\cos(m_x\mu_x t + l_y\Phi_y^{(P)})$ with $l_y = \pm 1$. The second row of Fig. 9.1 illustrates these possibilities with $k_x = 1$. Before combining the effects of disjoint elements the phasor angle correction $l_x\Phi_x^{(P)} + l_y\Phi_y^{(P)}$ must therefore be applied and the multiplicative factor is $\beta_x^{(n_x+1)/2}$.

For two dimensional motion, the most general possible deflection can be expressed $\cos(m_x\mu_x t + m_y\mu_y t)$ with $m_x \geq 0$ but m_y allowed to be either positive or negative. The response to this deflection, after correction to a common origin is $\cos(m_x\mu_x t + m_y\mu_y t + l_x\Phi_x^{(P)} + l_y\Phi_y^{(P)})$ with $l_x = m_x \pm 1, l_y = m_y$ or $l_x = m_x, l_y = m_y \pm 1$. The condition for resonance can then be expressed as

$$l_x Q_x + l_y Q_y = 0, \pm 1, \pm 2, \ldots. \tag{10.2}$$

The coefficients l_x and l_y of these equations are the same coefficients that multiply the betatron phase differences when constructing the phasor diagram. It is possible to infer the indices l_x and l_y from figures like Fig. 9.1 through Fig. 9.4 since the equations of the straight lines (from left to right) in those figures are

$$\begin{aligned} (m_x \pm 1)\, Q_x + m_y Q_y &= 0, \pm 1, \pm 2, \ldots, \\ m_x Q_x + (m_y \pm 1)\, Q_y &= 0, \pm 1, \pm 2, \ldots, \end{aligned} \tag{10.3}$$

with $m_x \geq 0$ but m_y allowed to be either positive or negative.†

Note that, since Q_x and Q_y are just the fractional parts of the tunes, they necessarily lie in the range from 0 to 1, and at most a few of the integers on the right hand side of Eq. (10.3) are actually required to include all the lines shown. Of course the equations as written remain valid if the integer parts are included but there are then an infinite number of equations without bringing in anything new.

The phasor diagram for $(n_x, n_y, l_x, l_y) = (2, 0, 3, 0)$ has been exhibited in Fig. 10.1. The indices (n_x, n_y), along with the strengths c_n determine the lengths of the phasors and the

† The coefficients l_x and l_y can be read off the figures just by counting *intervals between lines* along the Q_x and Q_y axes.

indices (l_x, l_y) determine their angles. For the general phasor calculation it is convenient to factor out the amplitude dependence and to define x and y-specific phasor factors

$$\mathcal{P}'_x\left(n_{\sigma x}, n_{\sigma x}; n, n_y; l_x, l_y\right) = n_{\sigma x}^{(n_x-1)} n_{\sigma y}^{n_y} \left(\frac{\epsilon_x}{r_{\rm ref}^2}\right)^{\frac{n_x-1}{2}} \left(\frac{\epsilon_y}{r_{\rm ref}^2}\right)^{\frac{n_y}{2}} r_{\rm ref}^{\frac{n-1}{2}} \mathcal{P}_x\left(n, n_y; l_x, l_y\right),$$

$$\mathcal{P}'_y\left(n_{\sigma x}, n_{\sigma x}; n, n_y; l_x, l_y\right) = n_{\sigma x}^{n_x} n_{\sigma y}^{(n_y-1)} \left(\frac{\epsilon_x}{r_{\rm ref}^2}\right)^{\frac{n_x}{2}} \left(\frac{\epsilon_y}{r_{\rm ref}^2}\right)^{\frac{n_y-1}{2}} r_{\rm ref}^{\frac{n-1}{2}} \mathcal{P}_y\left(n, n_y; l_x, l_y\right), \tag{10.4}$$

Construction of $\mathcal{P}_x$ and $\mathcal{P}_y$ has been described previously; they depend on the (normally complicated) distribution of element strengths and phases over the lattice. The factor $\mathcal{P}'_x$ (respectively $\mathcal{P}'_y$) has been defined with one power of a_x (respectively a_y) divided out so that it yields a fractional (and hence β independent) quantity; via the factors $\mathcal{P}_x$ and $\mathcal{P}_y$ it includes the factor β_x or β_y from Eq. (8.7) and also the factor c_n (which has dimension $[\text{length}]^{-n}$). There are the correct powers of β_x and β_y in $\mathcal{P}_x$ to go with the powers of ϵ_x and ϵ_y to render $\mathcal{P}'_x$ and $\mathcal{P}'_y$ dimensionless. (The $\mathcal{P}$ factors vary like $\beta^{(n+1)/2} c^n \sim [\text{length}]^{-(n-1)/2}$ which cancels the dimensions of $\epsilon^{(n-1)}$.) Powers of $r_{\rm ref}$ have been introduced to make all factors separately dimensionless. A final dimensional rationalization will occur when c_n is replaced by $\widetilde{b}_{n,n+1}/r_{\rm ref}^n$ or $10^{-4} b_{n,n+1}/r_{\rm ref}^n \Delta\Theta$ in accordance with Eqs. (8.2) and (6.1). Then the phasor lengths will vary like $(\beta/r_{\rm ref})^{(n+1)/2} b_{n,n+1}$ with both factors dimensionless. Finally, to obtain $\frac{\Delta x}{a_x}$ due to all elements in the lattice, it is necessary to multiply $\mathcal{P}'_x$ by the remaining factors in Eq. (8.7).

The indices (m_x, m_y) have no direct influence on the phasor diagram but, along with (n_x, n_y), they determine the numerator factor $\mathcal{N}$ that multiplies the overall resonance strength. Also they control another complication that remains to be faced. Their values determine the frequencies of turn-by-turn sinusoidal motion observed at a fixed point in the lattice and it is possible for response at particular values of (m_x, m_y) to come from more than one source. Since these responses combine coherently it is necessary to add them before taking the absolute value to obtain the maximum amplitude excursion due to nonlinear distortion. One can say therefore that (l_x, l_y) (in conjunction with betatron phase advances around the ring) control the coherency of different elements while (m_x, m_y) (in conjunction with the lattice tunes) control the turn-by-turn coherency.

Under the special assumptions that $a_x = a_y = a$, $\beta_x = \beta_y = \beta_{\rm typ}$, $n_{\sigma_x} = n_{\sigma_y} = n_\sigma$, with $\mathcal{N}_{1/2}$ random phasor contributions from erect multipole $b_{n,n+1}$, all phasor factors are

$$r_{\rm ref}^{\frac{n-1}{2}} \mathcal{P}\,(n) = \sqrt{\mathcal{N}_{1/2}}\,\left(\frac{\beta_{\rm typ}}{r_{\rm ref}}\right)^{\frac{n+1}{2}} 10^{-4} b_{n,n+1}\,\Delta\Theta, \tag{10.5}$$

independent of n_y, l_x, and l_y.

11. Figure of merit

The nonlinear distortion that has been calculated can be used as a figure of merit whose minimization leads to “optimal” lattice parameters. Though the distortion is likely to be dominated by a few resonances it is convenient to be able to sum over all resonances rather than selecting just the particular large contributors. As well as being small, the contributions from non-resonant terms should be relatively insensitive to the lattice tunes and should hence have little affect on the location of the minimum of a figure of merit. To obtain the “worst case” it is appropriate to sum the absolute values of the responses of individual resonances. But it is not legitimate to sum the absolute values of the terms in Eq. (8.7) because some of them contribute to the same resonance coherently and with opposite signs.

We wish to calculate the coherent sum $\Delta x(m_x, m_y; a_x, a_y)$ of all terms that contribute to the response for a particular pair of values of $m_x \geq 0$, m_y. The combinations $(m_x, m_y) = (1, 0), (0, 1), (0, -1)$ will be excluded however since they correspond to linear motion. (By keeping track of these terms the amplitude dependent detuning and coupling can be obtained.) Initially we consider only the terms of Eq. (8.7), for which n_y is *even* (*i.e.* for erect multipoles). Still, more than one multipole order can contribute coherently to the same Fourier term. Since it is x-response, all contributing terms have $l_y = m_y$. The formula can then be embellished to include skew multipoles by introducing a quantity (E/S) which is the integer 0/1 for erect/skew multipoles. Also symmetry is exploited to restrict the summation over multipole orders by introducing (e/o) which is 0/1 if $|m_x| + |m_y|$ is even/odd.

$$
\begin{aligned}
\frac{\Delta x^{(\mathrm{E/S})}(m_x,m_y;a_x,a_y)}{a_x} &= \frac{\sin\mu_x\,\mathcal{R}_{\Omega(m_x,m_y),\mu_x}}{2\cos\Omega(m_x,m_y)-2\cos\mu_x}\Big(1-\tfrac{1}{2}\delta^0_{m_x}\delta^0_{m_y}\Big)\times \\
&\sum_{n'=(2+(e/o)),2}^{n'\le n_{\max}}\ \sum_{n'_y=(E/S),2}^{n'_y\le n'}\ \sum_{k'_y=0}^{n'_y}\ \sum_{k'_x=0}^{k'_x\le(n'-n'_y)/2} n_{\sigma_x}^{(n'_x-1)} n_{\sigma_y}^{n'_y}\Big(\frac{\epsilon_x}{r^2_{\mathrm{ref}}}\Big)^{\frac{n'_x-1}{2}}\Big(\frac{\epsilon_y}{r^2_{\mathrm{ref}}}\Big)^{\frac{n'_y}{2}}\times \\
&(-1)^{\mathrm{int}(\frac{n'_y}{2})}\frac{1}{2^{n'-1}}\binom{n'}{n'_y}\binom{n'_y}{k'_y}\binom{n'-n'_y}{k'_x}\,\delta^0_{n'_y-2k'_y-m_y}\,\delta^0_{n'-n'_y-2k'_x-m_x}\times \\
&\Big(r_{\mathrm{ref}}^{\frac{n'-1}{2}}\mathcal{P}_x(n_{\sigma_x},n_{\sigma_y};n',n'_y;m_x+1,m_y)+r_{\mathrm{ref}}^{\frac{n'-1}{2}}\mathcal{P}_x(n_{\sigma_x},n_{\sigma_y};n',n'_y;m_x-1,m_y)\Big)
\end{aligned}
\tag{11.1}
$$

The ranges of m_x and m_y have been defined previously (below Eq. (8.6)). Formally the summations run over unphysical combinations but the Kronecker delta factors filter out the correct contributions. Note that the two phasor factors $\mathcal{P}_x$ shown explicitly in this formula (as well as all the others implied by the summations) contribute coherently to the same Fourier motion in spite of the fact that their rules for superimposing multiple disjoint elements are different. This complicates the calculation seriously because it prevents factorizing the equation into a lattice dependent part and a combinatorial part and prevents completing one multipole at a time. In the end one will take absolute values, but it is important that the appropriate coherent sums be evaluated before this is done. The coefficients contained in Eq. (11.1) can be matched with those given with the same power of a_x and a_y in Eq. (7.1) (setting all factors in the second and fourth lines to 1).

The factor $\mathcal{R}_{\Omega,\mu_x}$, defined in Eq. (3.11), will usually be approximated by 1. Its purpose is to correct the maximum values attained by non-resonant amplitudes—notice the cancellation of its second term with a denominator factor.† The importance of this correction can be investigated by comparing the results with and without the approximation $\mathcal{R}_{\Omega,\mu_x}=1$. As argued previously, the inclusion of the $\mathcal{R}_{\Omega,\mu_x}$ correction is unlikely to effect the location of the minimum value of the figure of merit to be introduced shortly, but it may alter its absolute value noticeably because of the large number of non-resonant terms.

When reconciling Eq. (11.1) with the resonance diagrams Fig. 9.1 through Fig. 9.4, it is important to appreciate that pairs of resonance plots have the same denominator factor $2\cos\Omega(m_x,m_y)-2\cos\mu_x$ in Eq. (11.1); they have $l_x=m_x-1$ or $l_x=m_x+1$ and the same values for l_y. This pairing occurs because two phasor superpositions (note the sum of two $\mathcal{P}'$ phasor factors in Eq. (11.1)) have the same "quadratic" denominator which is

† As $\mathcal{R}_{\Omega,\mu_x}$ deviates from 1 the phase of the response relative to the zero'th order betatron motion varies. The shift is common for all elements of the same type. The correction also assumes the phase shift is the same for any amplitudes that "interfere" coherently.

the product of two "linear" factors, $\sin((\Omega(m_x, m_y) \pm \mu_x)/2)$. The product of these linear factors is unaffected by which of the two is small and it is not consistent to keep one term and drop the other. For x response the members of the pair differ by whether μ_x is added or subtracted in the formula for the location of the "pole"; for y response whether μ_y is added or subtracted. Hence the paired values have one of l_x or l_y the same and the other different by two units.† For this reason the prescription determining what entries were made in Table 9.1 and Table 9.2 was not really consistent, since the amplitude of the unshown case is comparable with that shown in some cases. The factors other than the denominator are not necessarily equal either in magnitude or in sign in these paired cases and the phasor constructions for summing them over lattice elements are different. They are nevertheless "coherent" and they must be added before absolute values are taken.

If Eq. (11.1) is simplified according to Eq. (10.5), as is appropriate for random multipoles, the result is

$$\frac{\Delta x^{(\mathrm{E/S})}\left(m_x, m_y; a, a\right)}{a} = 10^{-4}\sqrt{\mathcal{N}_{1/2}}\,\frac{\beta_{\mathrm{typ}}\Delta\Theta}{r_{\mathrm{ref}}}\,\mathcal{F}_{1x}\left(m_x, m_y; a\right)\;\mathcal{F}_{2x}^{(E/S)}\left(m_x, m_y; a\right) \quad \text{where}$$

$$\mathcal{F}_{1x}\left(m_x, m_y\right) = \frac{\sqrt{2}\,\sin\mu_x\,\mathcal{R}_{\Omega(m_x,m_y),\mu_x}}{2\cos\Omega\left(m_x, m_y\right) - 2\cos\mu_x}$$

$$\mathcal{F}_{2x}^{(E/S)}\left(m_x, m_y; a\right) = \left(1 - \frac{\delta^0_{m_x}\delta^0_{m_y}}{2}\right) \sum_{n'=(2+(e/o)),2}^{n'\le n_{\mathrm{max}}} \left(\frac{a}{r_{\mathrm{ref}}}\right)^{n'-1} (b/a)_{n',n'+1}$$

$$\sum_{n'_y=(E/S),2}^{ny'\le n'} \;\sum_{k'_y=0}^{n'_y} \;\sum_{k'_x=0}^{k'_x\le\frac{n'-n'_y}{2}} (-1)^{\mathrm{int}\left(\frac{n'_y}{2}\right)} \frac{1}{2^{n'-1}} \binom{n'}{n'_y}\binom{n'_y}{k'_y}\binom{n'-n'_y}{k'_x}\;\delta^0_{n'_y-2k'_y-m_y}\delta^0_{n'-n'_y-2k'_x-m_x} \tag{11.2}$$

The two factors $\mathcal{F}_{1x}$ and $\mathcal{F}_{2x}^{(E/S)}$ have been introduced so that values of $\mathcal{F}_{2x}^{(E/S)}$ can be compared directly with the coefficients in Eqs. (7.1) (with $(b/a)_{n',n'+1} = 1$). The fact $\sqrt{2}$ in $\mathcal{F}_{1x}$ comes from combining quadratically the two phasor terms in Eq. (11.1), as is appropriate for the assumed random distribution of strengths. In spite of our approximations

† A likely source of confusion concerning the pairing of resonances can be illustrated by noting (in the plot numbered 2) that its resonance $(l_x, l_y) = (1, 2)$ is necessarily paired with the case immediately on its right with $(l_x, l_y) = (1, -2)$. The latter label would have been $(l_x, l_y) = (-1, 2)$ except for our convention of keeping l_x positive. Except for this, paired poles would always be consistent with the rule that their l_x values differ by 2 and their l_y values are the same. (Unless they refer to y response, of course, in which case their l_y values differ by 2 and their l_x values are the same.)

the multipole strengths $(b/a)_{n,n+1}$ are still entangled inside the summation and it is not possible to proceed without knowing how the strengths depend on n.

Returning to the exact formulation, from the quantities $\frac{\Delta x}{a_x}$ given by Eq. (11.1) we wish to obtain maximum amplitude excursions $\frac{\Delta a_x}{a_x}$ so that the quantity $r_{\rm da}\frac{\Delta a_x}{a_x}$ is interpretable as contributing additively to $\Delta r_{\rm coh}$ in Eq. (1.1). From Eq. (11.1) "x-erect and x-skew figures of merit" are defined by

$$\begin{aligned}\mathrm{FOM}_x^{(E/S)}(a_x,a_y;Q_x,Q_y) &= \sum_{m_y=(E/S),2}^{m_y\le n_{\max}} \left|\frac{\Delta x^{(E/S)}(0,m_y;a_x,a_y)}{a_x}\right| + \\ &+ \sum_{m_x=1}^{n_{\max}} \sum_{m_y=(E/S),2}^{m_y\le n_{\max}-m_x} \Big(1-\frac{\delta^0_{m_y}}{2}\Big)\left(\left|\frac{\Delta x^{(E/S)}(m_x,m_y;a_x,a_y)}{a_x}\right| + \left|\frac{\Delta x^{(E/S)}(m_x,-m_y;a_x,a_y)}{a_x}\right|\right). \end{aligned} \tag{11.3}$$

All terms that interfere coherently are within the absolute value signs.† Figures of merit $\mathrm{FOM}_y^{(E/S)}$ applicable to y response are defined similarly but starting from Eq. (11.4).

The phasor factors $\mathcal{P}$ account for the phasor summation over all elements of one multipole type in the lattice. This is the only factor in the theory that depends on the detailed lattice design or on whatever statistical assumptions are made concerning the distribution of errors. Also, any influence the integer parts of the tunes have on the figure of merit is through this factor. The theory is completely deterministic only when the strength of every nonlinear element is known so that $\mathcal{P}$ can be evaluated precisely.

For reference the formula for y-deflections is also given;

$$\begin{aligned}\frac{\Delta y^{(\mathrm{E/S})}(m_x,m_y;a_x,a_y)}{a_y} &= \frac{\sin\mu_y\,\mathcal{R}_{\Omega(m_x,m_y),\mu_y}}{2\cos\Omega(m_x,m_y)-2\cos\mu_y}\Big(1-\frac{1}{2}\delta^0_{m_x}\delta^0_{m_y}\Big)\times \\ &\sum_{n'=(2+(e/o)),2}^{n'\le n_{\max}} \sum_{n'_y=(S/E),2}^{n'_y\le n'} \sum_{k'_y=0}^{n'_y} \sum_{k'_x=0}^{k'_x\le(n'-n'_y)/2} n_{\sigma x}^{n'_x} n_{\sigma y}^{n'_y-1} \Big(\frac{\epsilon_x}{r^2_{\rm ref}}\Big)^{\frac{n'_x}{2}} \Big(\frac{\epsilon_y}{r^2_{\rm ref}}\Big)^{\frac{n'_y-1}{2}} \times \\ &(-1)^{\mathrm{int}(\frac{n'_y}{2})} \frac{1}{2^{n'-1}} \binom{n'}{n'_y}\binom{n'_y}{k'_y}\binom{n'-n'_y}{k'_x}\,\delta^0_{n'_y-2k'_y-m_y}\,\delta^0_{n'-n'_y-2k'_x-m_x}\times \\ &\Big(r_{\rm ref}^{\frac{n'-1}{2}}\mathcal{P}_y(n_{\sigma x},n_{\sigma y};n',n'_y;m_x,m_y+1) + r_{\rm ref}^{\frac{n'-1}{2}}\mathcal{P}_y(n_{\sigma x},n_{\sigma y};n',n'_y;m_x,m_y-1)\Big)\end{aligned} \tag{11.4}$$

† The order of evaluation of absolute values has treated skew and erect multipoles as uncorrelated. This assumption is not necessarily valid. In fact it seems entirely likely that there are correlations among the measured multipole strengths. Such correlations are routinely destroyed even in "reliable" tracking determinations of dynamic aperture when Monte Carlo randomized errors are assigned. In the numerical example below, since we keep only sextupole, octupole and decapole, the only possibility of interference (other than that shown explicitly in the last factor of Eq. (11.1)) is between sextupole and decapole. If one or the other is negligibly small the coherence effect will be unimportant and if they are comparable the statistical factor will not have been included quite correctly by the simple $\sqrt{\mathcal{N}_{1/2}}$ factors. But the calculation is shown in detail primarily for pedagogical purposes. All these complications will be avoided when every nonlinear element is treated individually.

In the clumsy notation of this formula (S/E) is the integer 1/0 for erect/skew multipoles, and the factor (e/o) has the same meaning as in Eq. (11.1). The y figure of merit is given by

$$\mathrm{FOM}_y^{(E/S)}(a_x, a_y; Q_x, Q_y) = \sum_{m_y=(S/E),2}^{m_y \le n_{\max}} \left| \frac{\Delta y^{(S/E)}(0,m_y;a_x,a_y)}{a_x} \right| + \\ + \sum_{m_x=1}^{n_{\max}} \sum_{m_y=(E/S),2}^{m_y \le n_{\max}-m_x} (1 - \frac{\delta^0_{m_y}}{2}) \left(\left| \frac{\Delta y^{(E/S)}(m_x,m_y;a_x,a_y)}{a_x} \right| + \left| \frac{\Delta y^{(E/S)}(m_x,-m_y;a_x,a_y)}{a_x} \right| \right). \tag{11.5}$$

The figures of merit still depend on the amplitudes a_x and a_y. Since the relative importance of different multipole orders depends on the magnitudes of a_x and a_y, these amplitudes have to be chosen "appropriately", perhaps iterating to obtain self-consistency. Assuming that typical magnitudes of a_x and a_y are comparable one can define more nearly isotropic figures of merit by

$$\mathrm{FOM}^{(E/S)} = \mathrm{FOM}_x^{(E/S)} + \mathrm{FOM}_y^{(E/S)}. \tag{11.6}$$

Finally an overall figure of merit can be defined by

$$\mathrm{FOM} = \left(\mathrm{FOM}^{(E)} + \mathrm{FOM}^{(S)} \right). \tag{11.7}$$

Then the "optimum" tunes Q_x^{opt}, Q_y^{opt} are those that minimize FOM.

Some sort of tentatively optimal fractional tunes can be determined by minimizing FOM under the hypothesis that the phasor factors $\mathcal{P}$ are independent of l_x and l_y. A numerical example is given in the next section. Naturally the tunes found this way depend on the assumed multipole strengths, and they do not in general remain optimal when more realistic assumptions are made about the distribution of multipole strengths.

For the special case $a_x = a_y = a$ the effect of the nonlinearities is to shift $a_x = a \to a_x = a(1 + \mathrm{FOM}^{(x)})$ and $a_y = a \to a_y = a(1 + \mathrm{FOM}^{(y)})$ and $a_x^2 + a_y^2 \to 2a^2(1 + (\mathrm{FOM}^{(x)} + \mathrm{FOM}^{(y)})))$. Letting $r_{\mathrm{da}} = \sqrt{a_x^2 + a_y^2} = \sqrt{2}a$, this corresponds to

$$\frac{\Delta r_{\mathrm{da}}}{r_{\mathrm{da}}} = \frac{\mathrm{FOM}}{2}. \tag{11.8}$$

Substituting into Eq. (1.1) yields $r_{\mathrm{da}} \approx r_{\mathrm{mech}}/(1 + \mathrm{FOM}/2)$ or

$$\boxed{\epsilon_{\mathrm{da}} \approx \frac{\epsilon_{\mathrm{mech}}}{1+\mathrm{FOM}}.} \tag{11.9}$$

This equation has been manipulated starting with the assumption that the fractional distortion is $\mathrm{FOM} \times r_{\mathrm{da}}/2$ which makes it self-consistent to lowest order, but in principle it is necessary to solve iteratively since FOM itself depends on amplitude. The superficial proportionality of $\epsilon_{\rm da}$ to $\epsilon_{\rm mech}$ is illusory since increasing $r_{\rm mech}$ also increases FOM. If the lowest order formulation were valid to all amplitudes (which it is not) the value $r_{\rm da}$ given by Eq. (11.9) would approach the mechanical-aperture-independent "dynamic aperture", as it is customarily defined, for example in element-by-element tracking calculations.

12. Numerical example, purely random errors

With $(Q_x, Q_y) = (0.28, 0.31)$, $a_x = a_y = r_{\rm ref}$, values of $\mathcal{F}_{1x,y}(m_x, m_y)\mathcal{F}^{(E/S)}_{2x,y}(m_x, m_y)$ calculated using (11.2) for sextupoles, octupoles and decapoles are shown in Fig. 12.1.

SEXT X — ERECT

mx \ my	-2	0	2	4
0			.9069	
1				
2		-.6616		
3				
4				

SEXT X — SKEW

	-3	-1	1	3
0				
1		.8398	-1.4952	
2				
3				
4				

SEXT Y — SKEW

mx \ my	-2	0	2	4
0			1.2883	
1				
2		-.8277		
3				
4				

SEXT Y — ERECT

	-3	-1	1	3
0				
1		.6885	-1.9525	
2				
3				
4				

OCT X

	-2	0	2	4
0				
1	2.1143		-.7394	
2				
3		.3396		
4				

	-3	-1	1	3
0				-.2248
1				
2		3.9316	.8449	
3				
4				

OCT Y

	-2	0	2	4
0				
1	4.1582		-.5924	
2				
3		.2571		
4				

	-3	-1	1	3
0				-.1826
1				
2		1.8943	.6624	
3				
4				

DEC X

	-2	0	2	4
0		-.6205	1.8137	.4908
1				
2	-.6595	1.3231	-1.2015	
3				
4		.1340		

	-3	-1	1	3
0				
1	1.2266			-1.1263
2				
3		-.6179	.6336	
4				

DEC Y

	-2	0	2	4
0		-.5097	2.5767	.2697
1				
2	-.5373	1.6554	-.8784	
3		.2571		
4		.1059		

	-3	-1	1	3
0				
1	2.1164			-.7537
2				
3		-.7569	.4863	
4				

TOT X

	-2	0	2	4
0		-.6205	2.7206	.4908
1	2.1143		-.7394	
2	-.6595	.6615	-1.2015	
3		.3396		
4		.1340		

	-3	-1	1	3
0				-.2248
1	1.2266	.8398	-1.4952	-1.1263
2		3.9316	.8449	
3		-.6179	.6336	
4				

TOT Y

	-2	0	2	4
0		-.5097	3.865	.2697
1	4.1582		-.5924	
2	-.5373	.8277	-.8784	
3				
4		.1059		

	-3	-1	1	3
0				-.1826
1	2.1164	.6885	-1.9525	-.7357
2		1.8943	.6624	
3		-.7569	.4863	
4				

Figure 12.1: Values of $\sqrt{2}\,\mathcal{F}_{1x,y}(m_x, m_y)\mathcal{F}^{(E/S)}_{2x,y}(m_x, m_y)$ calculated from Eqs. (11.2) and its y-analog. The bottom row of tables contains sums of the upper three rows. Entries affected by coherence are underlined.

In each case $(b/a)_{n',n'+1} = 1$ for the particular multipole and all others are set to zero. In the lowest row of tables the result of setting all $(b/a)_{n',n'+1} = 1$ and forming the coherent sum is shown. Cases where more than one multipole order contributes to the same coherent sum are underlined. These sums have been formed by simple addition which corresponds to their phasors having been assumed to be parallel. For purely random errors their orientations would be random and it would be appropriate to take the quadratic sum instead. It can be seen that our scrupulous attention to forming these sums before taking absolute values has been largely academic in that there are coherent contributions to the same sum in only four cases, some of them exhibiting constructive interference, some destructive. Nevertheless the effort has undoubtedly been justified because it has illustrated the nature of the coherence.

As well as determining optimal tunes, FOM provides the effective reduction of acceptance below that implied by the mechanical aperture. According to Eq. (11.9), the *fractional acceptance* reduction is given by FOM, but we will quote results as 100 FOM which is the *acceptance* reduction as a *percentage.* Incorporating also the remaining factors in Eq. (11.2), the FOM values in tables must still be multiplied by the factors

$$100 \times 10^{-4}\,(a/b)_{n,n+1}\left(\frac{a}{r_{\rm ref}}\right)^{n-1}\sqrt{\mathcal{N}_{1/2}}\,\frac{\Delta\Theta\,\beta}{r_{\rm ref}} \approx 12.6\ (a/b)_{n,n+1}\left(\frac{a}{r_{\rm ref}}\right)^{n-1}$$

where $(a/b)_{n,n+1}$ is expressed in "units" at $r_{\rm ref}$. Of course the factor $\sqrt{\mathcal{N}_{1/2}}$ is only appropriate for purely random phases. For the LHC the anticipated r.m.s. dipole errors during injection, in "units" at 17 mm are†

$$\begin{aligned}\sigma_{b2,3} &= 0.607 \times 1.7^2/\sqrt{3}, \quad \sigma_{b3,4} = 0.113 \times 1.7^3/\sqrt{3}, \quad \sigma_{b4,5} = 0.112 \times 1.7^4/\sqrt{3},\\ \sigma_{a2,3} &= 0.283 \times 1.7^2/\sqrt{3}, \quad \sigma_{a3,4} = 0.167 \times 1.7^3/\sqrt{3}, \quad \sigma_{a4,5} = 0.046 \times 1.7^4/\sqrt{3},\end{aligned} \tag{12.1}$$

The factor 1.7^n corrects the values from the "old" reference radius of 1 cm to the "new" value of 1.7 cm. The factor $1/\sqrt{3}$ corrects for the fact that there are three dipoles per half cell. With these values of $(b/a)_{n',n'+1}$ values of 100 FOM calculated from Eqs. (11.3) through (11.7) are in the table shown next. The particle amplitude has been taken to be 7 mm which is about 10σ. Since the numbers are small compared to 100 percent, it makes little difference whether they are referred to $r_{\rm da}$ or $r_{\rm mech}$. The table is calculated for a grid

† Private communication from Jean-Pierre Koutchouk.

of Q_x and Q_y values.[†] Locating the minimum in this table yields the best figure of merit and hence the best operating point (under the assumptions). According to these numbers the loss of acceptance at the nominal tunes is 10 percent, and the acceptance could be improved by only about 1.5 percent by more advantageous choice of fractional tunes since the minimum is broad.

100*FOM = percentage acceptance reduction at 10 sigma due to randoms

qy = qx=	0.270	0.278	0.286	0.294	0.302	0.310	0.318	0.326	0.334	0.342	0.350
0.260	11.5	9.5	8.8	8.5	8.6	8.9	9.7	11.1	34.4	12.4	13.6
0.264	14.3	10.1	9.0	8.7	8.7	9.0	10.0	11.3	34.7	12.8	14.5
0.268	28.9	11.3	9.5	8.9	8.9	9.2	10.6	11.5	35.0	13.3	15.6
0.272	28.9	14.3	10.2	9.3	9.1	9.4	999.9	11.8	35.4	14.0	17.0
0.276	14.3	29.0	11.5	9.8	9.4	9.6	11.1	12.1	35.9	14.8	18.9
0.280	11.4	29.0	14.5	10.6	9.9	10.0	11.0	12.5	36.4	15.7	21.6
0.284	10.2	14.4	29.3	12.0	10.5	10.4	11.2	13.0	37.1	17.0	25.5
0.288	9.6	11.6	29.4	15.1	11.3	10.9	11.6	13.6	37.9	18.5	32.1
0.292	9.2	10.5	14.9	29.9	12.8	11.6	12.1	14.3	38.8	20.6	45.0
0.296	9.1	10.0	12.2	30.1	16.0	12.6	12.8	15.3	40.0	23.5	83.3
0.300	9.0	9.7	11.1	15.7	30.9	14.2	13.8	16.7	41.5	27.7	999.9

When the multipole coefficients are turned on one at a time, the contributions to FOM(.28,.31) in the order $b_{2,3}, b_{3,4}, b_{4,5}, a_{2,3}, a_{3,4}, a_{4,5}$ are 5.1, 0.6, 0.4, 2.5, 1.3, and 0.1. The fact that these numbers add to roughly the same value as when all multipoles are on at once, implies that interference effects are unimportant, (except as regards the two terms in Eq. (11.1).) They also indicate that the loss of dynamic acceptance is due primarily to sextupoles.

It provides a handy rule of thumb, and is perhaps not entirely a coincidence, that the percent reductions in acceptance are roughly proportional to the multipole strengths (in "units" at 1 cm). If a particular resonance were dominant such a rule of thumb could not work, but the *optimal* tunes presumably avoid such operating points. Setting a limit on the simple sum of the (absolute values of) the multipole coefficients is not so different from designating a "good field region" (in which the fractional field error does not exceed some tolerance) as was the "old fashioned" practice in accelerator magnet design. (Of course this only makes sense if r_{ref} is comparable with the good field region.) The suggestion then is that, in the absence of dominant resonance, it is not silly to characterize an accelerator

[†] The 999.9 entry near the center of the table actually stands for infinity; it corresponds exactly to the resonance $1 \times 0.272 - 4 \times 0.318 = -1.0$. The "narrowness" of this resonance and the granularity of the table is such that the resonance does not show up as a line.

magnet by its "good field region". One can even conceive of this being *more* reliable than the enumeration of all the multipole coefficients in that the good field region properly accounts for correlations that are commonly lost in the field representation by multipoles.†

Paper II of this report will contain calculations of FOM(Q_x, Q_y) for the LHC under "realistic" assumptions concerning systematic field errors, with the aim of finding the optimal operating point with integer tunes permitted to deviate by as much as several units from their nominal values. From the results obtained so far one anticipates that the 8 entries in the upper row of Fig. 12.1 will continue to be the most important contributors, and that the optimal fractional tunes will always be situated more or less equidistant from nearby low order resonances.‡

The calculations in this paper and the paper to follow were suggested originally by Jacques Gareyte and Jean-Pierre Koutchouk, though none of us correctly estimated the complexity of the task.

References

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† Losing its correlations among multipole coefficients is likely to make the quality of a magnet appear worse than it actually is because a deviant term r^n with positive coefficient can, over a limited range of radius, be compensated by a term r^{n+1} with negative coefficient.

‡ Continuing to analyse only regular arc dipoles, it will be a good approximation to assume $\beta_x = \beta_y \approx \beta_{\text{typ}}$ and it simplifies the calculation markedly to assume $a_x = a_y = a$. Under these conditions the dependence on integer tunes is primarily through the phasor factors $\mathcal{P}$ appearing in the last line of Eq. (11.1) and these factors, though dependent on n, do not depend explicitly on n_y. See Eqs. (7.1).