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Combinatorial Formulae for Vassiliev Invariants from Chern-Simons Gauge Theory

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Abstract

We analyse the perturbative series expansion of the vacuum expectation value of a Wilson loop in Chern-Simons gauge theory in the temporal gauge. From the analysis emerges the notion of the *kernel* of a Vassiliev invariant. The kernel of a Vassiliev invariant of order n is not a knot invariant, since it depends on the regular knot projection chosen, but it differs from a Vassiliev invariant by terms that vanish on knots with n singular crossings. We conjecture that Vassiliev invariants can be reconstructed from their kernels. We present the general form of the kernel of a Vassiliev invariant and we describe the reconstruction of the full primitive Vassiliev invariants at orders two, three and four. At orders two and three we recover known combinatorial expressions for these invariants. At order four we present new combinatorial expressions for the two primitive Vassiliev invariants present at this order.

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1 Introduction

Topological quantum field theories have provided important connections between different types of topological invariants. These connections are obtained by exploiting the multiple approaches inherent in quantum field theory. Chern-Simons gauge theory constitutes a very successful case in this respect. Its analysis, from both the perturbative and the non-perturbative points of view, has provided numerous important insights in the theory of knot and link invariants. Non-perturbative methods [1, 2, 3, 4, 5, 6] have established the connection of Chern-Simons gauge theory with polynomial invariants as the Jones polynomial [7] and its generalizations [8, 9, 10]. Perturbative methods [11, 12, 13, 14, 15, 16, 17, 18] have provided representations of Vassiliev invariants.

Gauge theories can be analysed by performing different gauge fixings. Vacuum expectation values of gauge-invariant operators are gauge-independent and they can therefore be computed in different gauges. Covariant gauges are simple to treat and its analysis in the case of perturbative Chern-Simons gauge theory has shown to lead to covariant combinatorial formulae for Vassiliev invariants [11, 12, 14, 15, 16]. These formulae involve multidimensional space and path integrals which, in general, are rather involved to carry out explicit computations of Vassiliev invariants. Non-covariant gauges seem to lead to simpler formulae. However, the subtleties inherent in non-covariant gauges [19] plague their analysis with difficulties. The two non-covariant gauges that have been more widely studied are the light-cone gauge and the temporal gauge [20, 21, 22]. Both belong to the general category of axial gauges. In the light-cone gauge the resulting expressions for the Vassiliev invariants turn out to be the ones involving Kontsevich integrals [23]. These integrals, although simpler than the ones appearing in covariant gauges, are still too complicated to carry out explicit computations of Vassiliev invariants. Simpler expressions in which no integrals are involved, *i.e.*combinatorial ones, are desirable. The aim of this paper is to reach this goal by studying the theory in the temporal gauge.

From the analysis in the light-cone gauge we have learned an important lesson: a naive treatment of the perturbative series expansion in a non-covariant gauge leads to non-invariant quantities. One needs to introduce correction terms to render the perturbative series invariant. At present it is not known what the 'physical' reasons are for being forced to introduce a correction term, but it certainly must be inherent in the subtleties involved in the use of non-covariant gauges. It is very likely that the understanding of this problem in Chern-Simons gauge theory will shed some light on the general solution to these problems.

In this paper we concentrate our attention on the analysis of the perturbative series expansion of the vacuum expectation value of a Wilson loop in the temporal gauge. Some aspects of this gauge have been studied in [13, 20]. In our analysis we encounter all the problems, inherent in non-covariant gauges, which were present in the light-cone gauge. A key ingredient in the analysis of the perturbative series expansion is the gauge propagator. The computation of the gauge propagator in non-covariant gauges is plagued with ambiguities, which are solved by demanding some properties for the correlation functions of the theory. These properties are usually based on physical grounds. In our case we must demand invariance of the vacuum expectation values of Wilson loops. As we encounter these ambiguities in our analysis we are forced to work with a rather general propagator in which some of the terms are not known explicitly. Fortunately, the complete explicit form of the propagator is not needed to compute vacuum expectation values of Wilson loops. Consistency, however, forces the introduction of a correction term similar to the one needed in the light-cone gauge. Since in our analysis some hypotheses are introduced to cope with the ambiguities, we must check that our final expressions are indeed knot invariants. We will prove this to be the case for the terms of the perturbative series expansion under consideration.

The propagator possesses two terms, one whose explicit form is known and that depends on the signatures at the crossings, and one whose complete explicit form is not known but is independent of the signatures at the crossings. Taking into account only the first term, we construct what we call the kernels of the Vassiliev invariants. These are quantities that are not knot invariants but depend on the regular projection chosen. These kernels have the property that they differ from an invariant by terms that vanish on singular knots with a high enough number of singular crossings. More precisely, if one considers an order-m kernel, it differs from an order-m Vassiliev invariant by terms that vanish after performing the m subtractions needed to get the invariant for a singular knot with m singular crossings.

The three main goals of this paper are:

– to provide the general formulae for the kernels of the Vassiliev invariants,

– to conjecture that the information contained in the kernels is sufficient to reconstruct all the Vassiliev invariants at a given order,

– to sustain this conjecture by showing how the reconstruction procedure is implemented at orders two, three and four.

Our results agree with the known combinatorial expression for Vassiliev invariants at orders two and three. The combinatorial formulae obtained for the two primitive Vassiliev invariants of order four are new. At present we lack an all-order reconstruction theorem but, as it will become clear from our analysis, the reconstruction procedure can be generalized. The key ingredients of our analysis are the structure of Chern-Simons gauge theory and the factorization theorem proved in [24]. Thus our approach is valid only for Vassiliev invariants based on Lie algebras.

The paper is organized as follows. In sect. 2 we formulate the perturbative series expansion of the vacuum expectation value of a Wilson loop in the temporal gauge. In sect. 3 we present the kernels of Vassiliev invariants and we analyse their properties. In sect. 4 we carry out the reconstruction procedure at order two, three and four. In sect. 5 we prove that the quantities obtained at order four are invariant under Reidemeister moves. Finally, in sect. 6 we state our conclusions. An appendix contains tables where the output of our combinatorial expressions for the primitive Vassiliev invariants of orders two to four for prime knots up to nine crossings is compiled.

2 Chern-Simons perturbation theory in the temporal gauge

In this section we formulate Chern-Simons gauge theory in the temporal gauge. Let us consider a semi-simple gauge group G and a G-connection on a three-space M. The action of the theory is the integral of the Chern-Simons form:

$$S_{\rm CS}(A) = \frac{k}{4\pi}\int_M {\rm Tr}\Big(A\wedge {\rm d}A + \frac{2}{3}A\wedge A\wedge A\Big), \tag{2.1}$$

where Tr denotes the trace over the fundamental representation of G, and k is a real parameter. The exponential $\exp(iS_{\rm CS})$ of this action is invariant under the gauge transformation

$$A_\mu \to h^{-1}A_\mu h + h^{-1}\partial_\mu h, \tag{2.2}$$

where h is a map from M to G, when the parameter k is an integer. Of special interest in Chern-Simons gauge theory are the Wilson loops. These are gauge-invariant operators labelled by a loop C embedded in M and a representation R of the gauge group G. They are defined by the holonomy along the loop C of the gauge connection A:

$$\mathcal{W}_R(C,G) = {\rm Tr}\left[{\rm P}_R \exp g\oint A\right], \tag{2.3}$$

where ${\rm P}_R$ denotes that the integral is path-ordered and that A must be considered in the representation R of G. As shown in [1], the vacuum expectation values of products of these operators lead to invariants associated to links corresponding to sets of non-intersecting loops.

Gauge-invariant theories need to undergo a gauge-fixing procedure to make their associated functional integrals well defined. Different choices of gauge fixing lead to different representations of the same quantities. For vacuum expectation values of products of Wilson lines one obtains different expressions for knot and link invariants. The aim of this paper is to study the perturbative series expansion corresponding to these quantities when one chooses the temporal gauge. In the temporal gauge the condition imposed on the gauge connection A is

$$n^\mu A_\mu = 0, \tag{2.4}$$

where n is the vector $n^\mu = (1, 0, 0)$. This gauge is a particular case of a more general class of non-covariant gauges called axial gauges in which just (2.4) is imposed, n being a constant vector satisfying some condition. The light-cone gauge studied in [17] is another particular case of this type of gauges. We showed in [17] that in the light-cone gauge the perturbative series expansion of the vacuum expectation value of a Wilson line leads to the Kontsevich integral [23] representation for Vassiliev invariants.

Condition (2.4) is imposed in the functional integral, adding the following gauge-fixing term to the action:

$$S_{\text{gf}} = \int_M \mathrm{d}^3x \mathrm{Tr}(dn^\mu A_\mu + bn^\mu D_\mu c + \alpha d^2), \tag{2.5}$$

where d is an auxiliary field, c and b are ghost fields, and α is an arbitrary constant. In defining perturbative series expansions, it is convenient to rescale the fields by $A \to gA$, where $g = \sqrt{\frac{4\pi}{k}}$, and to integrate out d. The quantum action becomes:

$$S = -\frac{1}{2} \int_M \mathrm{d}^3x \Big[\epsilon^{\mu\nu\rho}\Big(A^a_\mu \partial_\nu A^a_\rho - \frac{g}{3} f_{abc} A^a_\mu A^b_\nu A^c_\rho\Big) - \frac{1}{\alpha}(n^\mu A^a_\mu)^2 + b^a n^\mu D^{ab}_\mu c^b\Big]. \tag{2.6}$$

We will study the theory in the gauge $\alpha \to 0$. In this case we can impose the condition (2.4) for the terms in the action and it turns out that all terms but the quadratic ones vanish. Thus, the corresponding Feynman rules do not have vertices and all the information is contained in the form of the propagator. This observation might not hold for some types of three-manifolds M since there can be zero modes that cannot be gauged away and some interaction terms could remain. For the case of $M = \mathbf{R}^3$, the one on which we concentrate our attention, this does not occur. However, they may play an important role in other situations [13]. We are therefore left with the quadratic part of (2.6). The ghost contribution is trivial and for the gauge fields one obtains the following propagator in momentum space:

$$\Delta_{\mu\nu}(p) = \frac{\alpha}{(np)^2}(p_\mu p_\nu - \frac{i}{\alpha}(np)\epsilon_{\mu\rho\nu} n^\rho), \tag{2.7}$$

which, in the limit $\alpha \to 0$, becomes:

$$\Delta_{\mu\nu}(p) \to -i\epsilon_{\mu\rho\nu} \frac{n^\rho}{np}. \tag{2.8}$$

This propagator presents a pole at $np = 0$ and a prescription to regulate it is needed. This type of problems is standard in non-covariant gauges and several prescriptions have been proposed to avoid the pole (see [19] for a review on the subject). To construct the perturbative series expansion of the vacuum expectation value of a Wilson loop, we need the Fourier transform of (2.8) and therefore the problem related to the presence of the pole is unavoidable. In the temporal gauge, the momentum-space integral that has to be carried out has the form:

$$\Delta(x_0, x_1, x_2) = \int_M \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{e}^{i(p^0 x_0 + p^1 x_1 + p^2 x_2)}}{p^0}. \tag{2.9}$$

This integral is ill-defined due to the pole at $p^0 = 0$. To make sense of it a prescription has to be given to circumvent the pole. But, before studying possible prescriptions, let us first analyse the dependence of $\Delta(x_0, x_1, x_2)$ in (2.9) on x_0. The pole in p^0 is avoided if, instead of (2.9), one analyses the derivative of $\Delta(x_0, x_1, x_2)$ with respect to x_0. Considering $\Delta(x_0, x_1, x_2)$ as a distribution one obtains:

$$\frac{\partial \Delta}{\partial x_0} = i\delta(x_0)\delta(x_1)\delta(x_2). \tag{2.10}$$

Integrating this expression with respect to x_0, one finds that any prescription would lead to a result of the following form:

$$\Delta(x_0, x_1, x_2) = \frac{i}{2}\mathrm{sign}(x_0)\delta(x_1)\delta(x_2) + f(x_1, x_2), \tag{2.11}$$

where $f(x_1, x_2)$ is a prescription-dependent function. The important consequence of the result (2.11) is that the dependence of $\Delta(x_0, x_1, x_2)$ on x_0 has to be in the form $\mathrm{sign}(x_0)\delta(x_1)\delta(x_2)$. This observation will be crucial in our analysis. We will actually work with the rather general formula (2.11) for $\Delta(x_0, x_1, x_2)$. This form of the propagator will allow us to introduce the notion of kernel of a Vassiliev invariant and to design a procedure to compute combinatorial expressions for these invariants.

Although we will not use an explicit prescription to compute the propagator (2.9) let us analyse one of them to check that indeed it has the form (2.11). We will choose a Mandelstam-like prescription [19] to show that the propagator in the right-hand side of (2.8) has the form advocated in [20]. Let us consider:

$$\Delta_\epsilon(x_0,x_1,x_2) = \int_M \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{\mathrm{e}^{i(p^0x_0+p^1x_1+p^2x_2)}}{p^0 - i\epsilon \mathrm{sign}(p^2)}. \qquad (2.12)$$

The integral in the left-hand side of (2.12) has poles at $p^0 = i\epsilon\mathrm{sign}(p^2)$. To carry out the p^0-integration, we close for $x_0 > 0$ (for $x_0 < 0$) the contour integration in the upper (lower) half plane:

$$\begin{aligned} \Delta_\epsilon &= i\Theta(x_0)\delta(x_1)\int_{p^2>0}\frac{\mathrm{d}p^2}{2\pi}\,\mathrm{e}^{ip^2(x_2+i\epsilon x_0)} - i\Theta(-x_0)\delta(x_1)\int_{p^2<0}\frac{\mathrm{d}p^2}{2\pi}\,\mathrm{e}^{ip^2(x_2+i\epsilon x_0)} \\ &= -\frac{\delta(x_1)}{2\pi}\frac{1}{x_2+i\epsilon x_0}. \end{aligned} \qquad (2.13)$$

Using the relation:

$$\frac{1}{x_2+i\epsilon x_0} = \mathrm{P}\Big(\frac{1}{x_2}\Big) - i\pi\mathrm{sign}(x_0)\delta(x_2), \qquad (2.14)$$

one finally obtains:

$$\Delta_\epsilon = \frac{i}{2}\mathrm{sign}(x_0)\delta(x_1)\delta(x_2) - \frac{1}{2\pi}\mathrm{P}\Big(\frac{1}{x_2}\Big)\delta(x_1), \qquad (2.15)$$

which has the general form (2.11). This propagator is the one used in the analysis performed in [20]. Notice that the prescription that we have used breaks the symmetry under rotations in the x_1, x_2 plane, which is present in the temporal gauge. A more symmetric prescription in which this symmetry is kept would be preferable. Although such a prescription could be constructed easily, we will not do it here. As stated above, we will not need in our analysis an explicit form of the distribution f in (2.11).

Taking the expression (2.11) for $\Delta(x_0,x_1,x_2)$ and (2.8) we can easily obtain the form of all the components of the propagator:

$$\begin{aligned} \langle A_0^a(x)A_\mu^b(x')\rangle &= 0, \\ \langle A_m^a(x)A_n^b(x')\rangle &= \frac{i}{2}\delta^{ab}\varepsilon_{mn}\mathrm{sign}(x_0-x_0')\delta(x_1-x_1')\delta(x_2-x_2') \\ &\quad +f(x_1-x_1',x_2-x_2'), \end{aligned} \qquad (2.16)$$

where $m,n=1,2$ and ε_{mn} is antisymmetric with $\varepsilon_{12}=1$. This propagator contains the basic information of the theory and constitutes the essential ingredient in the construction of the perturbative series expansion of the vacuum expectation value of a Wilson loop.

3 Kernels of Vassiliev invariants

Wilson loops are the gauge-invariant operators (2.3) whose vacuum expectation value leads to knot invariants. Our aim is to compute the normalized vacuum expectation value:

$$\langle \mathcal{W}_R(C,G) \rangle = \frac{1}{Z_k} \int [DA] \mathcal{W}_R(C,G)\, \mathrm{e}^{iS_{CS}(A)}, \tag{3.1}$$

where Z_k is the partition function of the theory:

$$Z_k = \int [DA]\, \mathrm{e}^{iS_{CS}(A)}. \tag{3.2}$$

This quantity leads to knot invariants and possesses a perturbative series expansion in the coupling constant g. This series can be constructed diagrammatically from the Feynman rules of the theory. One assigns an external circle to the loop C carrying a representation R, and internal lines to the propagator (2.16). These internal lines are attached to the external circle by the vertex dictated by the form of the Wilson loop (2.3):

$$V_i^{j\,\mu a}(x) = g(T^a_{(R)})_i^j \int \mathrm{d}x^\mu. \tag{3.3}$$

The perturbative series is constructed by expanding the Wilson loop operator (2.3) and contracting the gauge fields with the propagator (2.16). It has the following general form [14]:

$$\langle \mathcal{W}_R(C,G) \rangle = \dim R \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(C)\, r_{ij}(R)\, x^i, \tag{3.4}$$

where $x = ig^2/2$ is the expansion parameter. The quantities $\alpha_{ij}(C)$, or geometrical factors, are combinations of path integrals along the loop C, and the r_{ij} are traces of products of generators of the Lie algebra associated to the gauge group G. The index i corresponds to the order in perturbation theory, and j labels independent contributions at a given order, d_i being the number of these at order i. In (3.4) $\dim R$ denotes the dimension of the representation R. Notice the convention $\alpha_{01}(C) = 1$. For a given order in perturbation theory, $\{r_{ij}\}_{\{j=1\ldots d_i\}}$ represents a basis of independent group factors.

The quantities $\alpha_{ij}(C)$ in (3.4) are Vassiliev invariants of order i [25]. Our goal is to compute them in the temporal gauge. Their form in one of the more studied covariant gauges, the Landau gauge, involves path and multidimensional space integrals. In the light-cone gauge its form is simpler, but one is forced to multiply the resulting perturbative series expansion by a factor to obtain true invariants. We will discover that there must be, in the temporal gauge, also a multiplicative factor to render the terms of the perturbative series expansion invariant. We will obtain some conditions that this factor must satisfy and we will present an ansatz for it possessing the same structure as the one present in the light-cone gauge. With this problem around and the arbitrary distribution f present in the propagator (2.16), one would not expect that it is possible to obtain concrete combinatorial expressions for the Vassiliev invariants. However, it turns out that the structure of the perturbative series expansion is so much constrainted that making a natural hypotheses on the form of the multiplicative factor we are able to achieve our goal.

The structure of the gauge propagator (2.16) indicates that in the temporal gauge we must consider loops in three-space, which do not make the argument of the delta functions vanish along a finite segment. This fact restricts us to knot configurations which possess a regular projection into the plane x_1, x_2. This does not imply any loss of generality since any knot can be continuously deformed to one of that type.

Given a regular projection $\mathcal{K}$ of a knot K onto the x_1, x_2 plane we will construct first the perturbative series expansion of the vacuum expectation value of the corresponding Wilson loop using only the first term of the propagator (2.16). At order i the terms in the series involve i propagators and so, in considering only the first term of (2.16), we are missing terms with all powers of $f(x_1, x_2)$ from 1 to i. The term with f^i does not depend on x_0 and therefore it only contains information on the shadow of the knot projection $\mathcal{K}$ on the plane x_1, x_2, *i.e.* it does not contain information on the signs of the crossings. Terms with lower powers of f contain some information on the crossings, but they vanish after considering the signed sums involved in the invariants associated to singular knots. Vassiliev invariants for singular knots are defined as an iterated sum of the differences when a singular point is resolved by an overcrossing and an undercrossing. Terms with f^j vanish for singular knots with $i-j+1$ singularities. Thus what we are missing by considering at order i only the first term in the propagator (2.16) is something that vanishes on knots with i singularities. Of course, being an invariant of finite type, the whole term vanishes for knots with more than i singularities. The main goal of this paper is to conjecture that the full invariant can be reconstructed from the quantities that result from a consideration of only the first term in (2.16). We will call those quantities *kernels* of Vassiliev invariants. We will show explicitly the reconstruction from the kernels at orders two, three and four.

In order to understand the role played by the first term of (2.16) in the perturbative series expansion, let us analyse in detail the second-order contribution. The quantity to be computed is of the form:

$$\int_{v<w} dvdw\dot{x}^m(v)\dot{x}^n(w)\varepsilon_{mn}\mathrm{sign}(x_0(v)-x_0(w))\delta(x_1(v)-x_1(w))\delta(x_2(v)-x_2(w)). \tag{3.5}$$

Two types of contributions can be encountered. There are other contributions when v and w get close to each other. These contributions are related to framing and they can be analysed as in [17], giving the standard framing factor. Since they are simple to control, we will not consider them here. There are contributions when $x_1(v) = x_1(w)$ and $x_2(v) = x_2(w)$, but $v \neq w$. These situations correspond precisely to the crossings. Let us suppose that the knot projection $\mathcal{K}$ has n crossings labelled by j, $j = 1, \ldots, n$. At a crossing j the parameters v and w take values $v = s_j$ and $w = t_j$, with $s_j \neq t_j$. The delta functions present in (3.5) can be evaluated very easily. Equation (3.5) becomes:

$$\sum_{j=1}^{n} \int_{v<w} dvdw \frac{\dot{x}^m(v)\dot{x}^n(w)\varepsilon_{mn}}{|\dot{x}^m(v)\dot{x}^n(w)\varepsilon_{mn}|}\mathrm{sign}(x_0(v)-x_0(w))\delta(v-s_j)\delta(w-t_j), \tag{3.6}$$

which is precisely a sum of the crossing signs ϵ_j at the crossings $j = 1, \ldots, n$:

$$\sum_{j=1}^{n} \epsilon_j. \tag{3.7}$$

The structure of the computation of (3.5) clearly generalizes. Whenever a term containing the first part of the propagator (2.16) appears in the perturbative series expansion, it can be traded by crossing signs. In general, one obtains powers of crossing signs multiplying quantities which depend only on the shadow of the regular projection. This proves that, as stated above, an

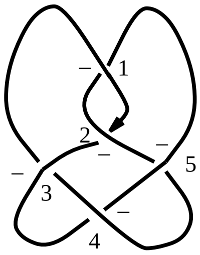


Figure 1: Example of a knot projection.

order-i contribution with j powers of f (and therefore $i-j$ powers of crossing signs) vanishes for singular knots with $i-j+1$ singularities.

The argument leading to (3.7) generalizes and allows us to write the general form of the perturbative series expansion when only the first term in (2.16) is taken into account. Let K be a knot whose regular projection $\mathcal{K}$ presents n crossings with crossing signs ϵ_j, $j=1,\ldots,n$. The resulting perturbative series expansion turns out to be:

$$
\begin{aligned}
\mathcal{N}(\mathcal{K}) \;=\; \sum_{k=0}^{\infty} \Bigg(& \sum_{i_1<\cdots<i_k} \epsilon_{i_1}\cdots\epsilon_{i_k}\mathcal{T}(i_1,\ldots,i_k) \\
& + \frac{1}{(2!)^2} \sum_{\substack{\sigma\in P_2 \\ j\neq i_1,\ldots,i_{k-2} \\ i_1<\cdots<i_{k-2}}} \epsilon_j^2\epsilon_{i_1}\cdots\epsilon_{i_{k-2}}\mathcal{T}(j,\sigma;i_1,\ldots,i_{k-2}) \\
& + \frac{1}{(3!)^2} \sum_{\substack{\sigma\in P_3 \\ j\neq i_1,\ldots,i_{k-3} \\ i_1<\cdots<i_{k-3}}} \epsilon_j^3\epsilon_{i_1}\cdots\epsilon_{i_{k-3}}\mathcal{T}(j,\sigma;i_1,\ldots,i_{k-3}) \\
& \qquad \cdots \\
& + \frac{1}{(m!)^2} \sum_{\substack{\sigma\in P_m \\ j\neq i_1,\ldots,i_{k-m} \\ i_1<\cdots<i_{k-m}}} \epsilon_j^m\epsilon_{i_1}\cdots\epsilon_{i_{k-m}}\mathcal{T}(j,\sigma;i_1,\ldots,i_{k-m}) \\
& \qquad \cdots
\end{aligned}
$$

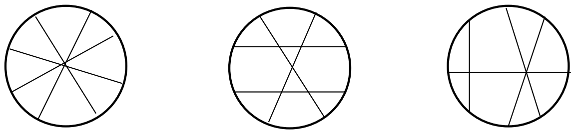


Figure 2: Group factors.

$$+\frac{1}{(k!)^2}\sum_{\substack{\sigma\in P_k\\ j}} \epsilon_j^k \mathcal{T}(j,\sigma)\Bigg) \tag{3.8}$$

Several comments are in order relative to this expression. The first term comes from the contribution in which all the propagators are attached to different crossings. The second when two propagators are attached to the same crossing and the rest to different crossings, and so on. Since we are dealing with an ordered integral, a correction must be included when there are more than one propagator at a crossing. This correction accounts for the fact that we do not have full integrations over the delta functions and corresponds to the factor $1/(m!)^2$. In (3.8), P_m denotes the permutation group. When several propagators coincide at a crossing one must include all the permutations in which they can be arranged. The factors $\mathcal{T}(j,\sigma;i_1,\dots,i_{k-m})$ are group factors, and they are computed in the following way: given a set of crossings $i_1, i_2, \dots, i_{i-m}$ and a permutation $\sigma\in P_m$, the corresponding group factor $\mathcal{T}(j,\sigma;i_1,\dots,i_{k-m})$ is the result of taking a trace over the product of group generators which is obtained after assigning a group generator to each of the crossings $i_1, i_2, \dots, i_{i-m}$, and m generators to crossing j, and travelling along the knot starting from a base point. The first time one encounters j a product of m group generators is introdiced; the second time the product is similar, but with the indices rearranged according to the permutation σ.

In order to clarify the content of (3.8) we will work out an example. Let us consider the knot shown in fig. 1 and let us concentrate on the fourth order term ($k=4$) containing some permutation $\sigma\in P_2$:

$$\frac{1}{(2!)^2}\sum_{\substack{j\neq i_1,i_2\\ i_1<i_2}} \epsilon_j^2\epsilon_{i_1}\epsilon_{i_2}\mathcal{T}(j,\sigma;i_1,i_2). \tag{3.9}$$

Examples of the group factors entering this expression are:

$$\mathcal{T}(1,\sigma;2,3) \;=\; \mathrm{Tr}(T^{b_1}T^{b_2}T^{a_1}T^{a_2}T^{\sigma(b_1)}T^{\sigma(b_2)}T^{a_1}T^{a_2}),$$

$$
\begin{aligned}
\mathcal{T}(1,\sigma;3,5) &= \mathrm{Tr}(T^{b_1}T^{b_2}T^{a_1}T^{a_2}T^{\sigma(b_1)}T^{\sigma(b_2)}T^{a_2}T^{a_1}), \\
\mathcal{T}(3,\sigma;1,4) &= \mathrm{Tr}(T^{b_1}T^{b_2}T^{a_1}T^{a_2}T^{a_1}T^{\sigma(b_1)}T^{\sigma(b_2)}T^{a_2}), \qquad (3.10)
\end{aligned}
$$

where we have used the labels specified in fig. 1. Group factors can be represented by chord diagrams. For example if one chooses $\sigma = (12)$ the three chord diagrams corresponding to the group factors in (3.10) are the ones pictured in fig. 2.

The terms of the series (3.8) are not knot invariants. Besides the knot K, they clearly depend on the knot projection chosen. However, at order k they are knot invariants modulo terms that vanish when an order-k signed sum is considered. We will call the terms appearing in (3.8) at each order in perturbation theory *kernels* of the Vassiliev invariants of order k. We conjecture that these kernels contain enough information to allow reconstruction of the full Vassiliev invariants at each order. In the next section we will present the reconstruction procedure for all primitive Vassiliev invariants up to order four.

4 Reconstruction of Vassiliev invariants up to order four

4.1 Outline of the calculation procedure

In this section we will obtain combinatorial formulae for the primitive Vassiliev invariants up to order four using the kernels (3.8) presented in the previous section. In order to be able to reconstruct the invariant we need to exploit as much as possible all the information contained in the perturbative series expansion of a Wilson loop in Chern-Simons theory. The crucial ingredient is, as we observed above, that all the dependence on the crossings comes from the first term of the propagator (2.16). Contributions not involving that part are crossing independent, *i.e.* they would not distinguish between over- or undercrossings, and consequently neither between the diagram of a knot projection $\mathcal{K}$ and its standard ascending diagram $\alpha(\mathcal{K})$. Recall that the ascending diagram of a knot projection is defined as the diagram obtained by switching, when travelling along the knot from the base point, all the undercrossings to overcrossings. There is a straightforward consequence of this fact that will help us in symplifying our calculations. Under the action of an inversion of space, a Vassiliev invariant of even order does not vary, while one of odd order change sign. This means that all the signature contributions should be of even order in the former and odd order in the later. As we claim that all these contributions come from the first term of the propagator (2.16), it follows that integrals with an odd number of powers of the function f in (2.16) will not contribute to the invariant.

Many of the ingredients entering the reconstruction procedure of the full invariants rely on the use of the factorization theorem in Chern-Simons theory proved in [24]. According to this theorem, once a canonical basis is chosen for the group factors, any non-primitive Vassiliev invariant of a given order can be written in terms of invariants of lower orders. Using this theorem we will obtain a series of relations involving unknown integrals, which will allow their solution to be such that a combinatorial formula for the Vassiliev invariants will be obtained.

As stated above, we will have to deal also with a Kontsevich-type global factor. We will assume that this factor can be written as the invariant of the unknot raised to some exponent that depends on some features of the knot projection under consideration. This will modify the perturbative series at every even order. We will obtain a series of consistency relations for the exponent, which admit simple solutions.

As crucial as the calculation itself is finding a convenient way to deal with the integrals appearing in the perturbative series expansion. On the one hand, we would like to write them as explicitly as possible so that, when we know how they behave, relations appearing at a given order can be used in higher orders; on the other hand, we would like to describe them in a compact way so that the notation does not become too clumsy. We will introduce a notation that we think satisfies these two conditions.

We will basically denote an integral made out of the f-dependent part of the propagator (2.16) by a capital D and a subindex which will actually be the Feynman diagram it comes from. Our calculations, though, require a more subtle labelling. Given a Feynman diagram, each chord in it usually represents the propagator of the theory. Our propagator (2.16) contains two pieces: the explicit one, which leads to the signatures of the crossings, and the f-dependent one. A Feynman integral will be a sum over all the possible ways of identifying the chords with each of them. So for a given Feynman diagram we will end up with different types of D integrals, depending on how many f-terms they contain. When all the propagators in the Feynman integral are of this kind, we will simply denote it by $D_{\bigcirc}$, where the subindex represents the corresponding Feynman diagram (the void circle stands for any diagram). If only one chord stands for the signature-dependent part, its evaluation will simply result in a crossing sign, ϵ_i, plus a restriction of the original integration domain. We will say that the chord standing for this factor is attached to the i-th crossing, meaning that the ordered integration domain of the other chords of the D integral is now limited by the position of that crossing. We will write down the resulting integral as:

$$\epsilon_i D^i_{\bigcirc} \,, \tag{4.1}$$

with the superindex of D denoting that one of the chords in the diagram is attached to the i-th crossing. This time D is in fact a sum over all the possible choices of placing the signature-dependent part of the propagator (2.16) in the i-th crossing (and of course a sum over the permutations of the given diagram is understood everywhere). Some examples are shown in fig. 3. There the integrals are represented directly by their Feynman diagrams,

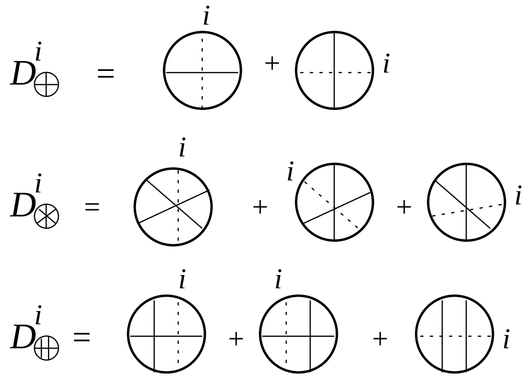


Figure 3: Examples of D^i integrals.

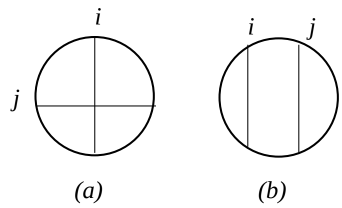


Figure 4: Possible configurations of two crossings.

with a dashed line standing for a signature-dependent term of the propagator, attached to a crossing i, and a continuous line representing the f-dependent term.

A more involved case arises when the integrand contains two signature-dependent terms of the propagator (2.16). In this case we will distinguish three subcases:

- when both are attached to the same crossing the integral will then be written as:

$$\frac{1}{4}\epsilon_i^2 D^{ii}_{\bigcirc}\,, \tag{4.2}$$

- when the crossings are different and, while travelling along the knot path, their labels follow the pattern in fig. 4(a); then the integral will be denoted as:

$$\epsilon_i \epsilon_j D^{ij,a}_{\bigcirc}\,, \tag{4.3}$$

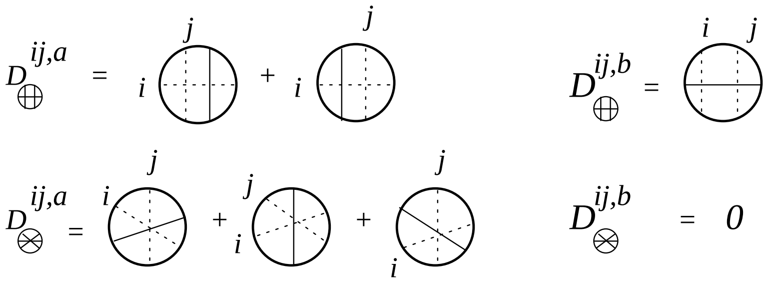


Figure 5: Examples of D^{ij} integrals.

- and when they are as in figure 4(b):

$$\epsilon_i \epsilon_j D^{ij,b}_{\bigcirc} . \tag{4.4}$$

We will denote by $\mathcal{C}_a$ the set of all pairs of crossings like those in 4(a), and by $\mathcal{C}_b$ the pairs like in fig. 4(b). Examples of these cases are drawn in fig. 5. As we are dealing with invariants up to order four, we do not need to handle the case where three or more signature-dependent terms of the propagator (but not all) are fixed to crossings. When the contribution does not contain f-dependent terms, the Feynman integral may be read from the kernels (3.8). We will denote by $E_{\bigcirc}$ the sum of terms in (3.8) corresponding to the diagram specified in its subindex, encoded in that formula in the form of the group factor $\mathcal{T}$.

In order to organize the perturbative series expansion, we have to make a choice of basis for the group factors entering the Feynman diagrams. Once this is done, the coefficients of the basis elements will be built out of a sum of Feynman integrals, with the apropppriate factors. We will denote these sums by $S^E_{\bigcirc}$ when the terms involve only the signature-dependent part of the propagator, and by $S^D_{\bigcirc}$ when they involve the f-dependent part. Indexes in the capital D will have the same meaning as above. This time, the diagram will stand for the independent group factor as obtained following the group Feynman rules given below, in fig. 7.

Additional notation is needed to write explicit combinatorial formulae for the invariants. These involve the so-called *crossing numbers* [26], which build up the signature contributions in every $E_{\bigcirc}$ integral. We will use the notation introduced in [26] for some of these functions, as well as new ones. The key

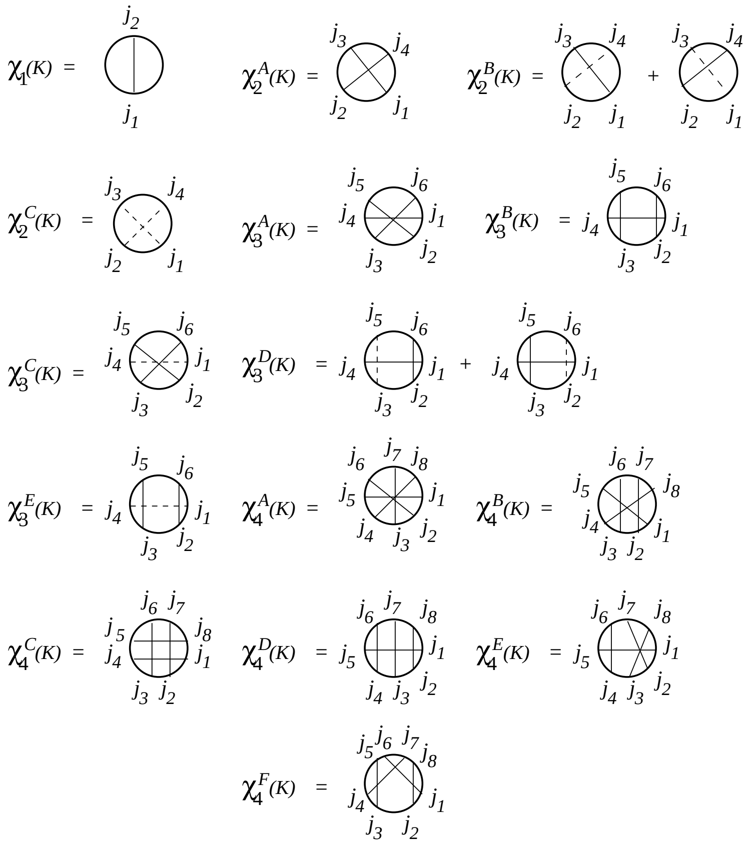


Figure 6: Diagrammatic expression for crossing numbers.

ingredient of that notation (see [26] or [27] for a more detailed explanation) is the following definition of the signature function:

Let $\pi : S^1 \longrightarrow \mathbf{R}^2$ be the projection, into the x_1, x_2 plane of a knot diagram $\mathcal{K}$. Let $s_i \in S^1$, $i \in \mathcal{I}$, be the pre-images of the n crossings in $\mathcal{K}$, with $\mathcal{I} = \{1, \ldots, 2n\}$ the index set of the labellings of the crossings. Then, following [26], we define:

$$\epsilon(i,j) = \begin{cases} \epsilon(\pi(s_i)) & \text{if } \pi(s_i) = \pi(s_j) \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

This function is such that whenever two labellings happen to label the same crossing, it gives its signature, and if they do not, it returns zero.

With the help of this function one defines quantities involving powers of the signatures. The following is a list of all the definitions of these quantities required in our computations (notice that our notation for the order-two functions is slightly different from that in [26]). The sums are taken over all possible ways of choosing the labellings in the index set $\mathcal{I}$ within the given ordering. In fig. 6 we present a diagrammatic notation for these definitions. There, the solid lines stand for the signature function, and the dashed ones for its square. The diagram tells us in a straightforward way the ordering that the labels must follow when we travel along the knot, and thus which collection of crossings would contribute to a given crossing number. Only one representative is chosen from those entering each sum. The others can be obtained very simply from the representative performing a rotation of the diagram while keeping the labels fixed. The list of functions that will be needed below is the following:

$$
\begin{aligned}
\chi_1(\mathcal{K}) &= \sum_{j_1>j_2} \epsilon(j_1,j_2) \qquad (4.6)\\
\chi_2^A(\mathcal{K}) &= \sum_{j_1>j_2>j_3>j_4} \epsilon(j_1,j_3)\epsilon(j_2,j_4)\\
\chi_2^B(\mathcal{K}) &= \sum_{j_1>j_2>j_3>j_4} [\epsilon(j_1,j_3)^2\epsilon(j_2,j_4)+\epsilon(j_1,j_3)\epsilon(j_2,j_4)^2]\\
\chi_2^C(\mathcal{K}) &= \sum_{j_1>j_2>j_3>j_4} \epsilon(j_1,j_3)^2\epsilon(j_2,j_4)^2\\
\chi_3^A(\mathcal{K}) &= \sum_{j_1>\cdots>j_6} \epsilon(j_1,j_4)\epsilon(j_2,j_5)\epsilon(j_3,j_6)\\
\chi_3^B(\mathcal{K}) &= \sum_{j_1>\cdots>j_6} [\epsilon(j_1,j_3)\epsilon(j_2,j_5)\epsilon(j_4,j_6)+\epsilon(j_1,j_4)\epsilon(j_2,j_6)\epsilon(j_3,j_5)\\
&+ \epsilon(j_1,j_5)\epsilon(j_2,j_4)\epsilon(j_3,j_6)]\\
\chi_3^C(\mathcal{K}) &= \sum_{j_1>\cdots>j_6} [\epsilon(j_1,j_4)^2\epsilon(j_2,j_5)\epsilon(j_3,j_6)+\epsilon(j_1,j_4)\epsilon(j_2,j_5)^2\epsilon(j_3,j_6)\\
&+ \epsilon(j_1,j_4)\epsilon(j_2,j_5)\epsilon(j_3,j_6)^2]\\
\chi_3^D(\mathcal{K}) &= \sum_{j_1>\cdots>j_6} [\epsilon(j_1,j_3)^2\epsilon(j_2,j_5)\epsilon(j_4,j_6)+\epsilon(j_1,j_3)\epsilon(j_2,j_5)\epsilon(j_4,j_6)^2\\
&+ \epsilon(j_1,j_4)\epsilon(j_2,j_6)^2\epsilon(j_3,j_5)+\epsilon(j_1,j_4)\epsilon(j_2,j_6)\epsilon(j_3,j_5)^2\\
&+ \epsilon(j_1,j_5)^2\epsilon(j_2,j_4)\epsilon(j_3,j_6)+\epsilon(j_1,j_5)\epsilon(j_2,j_4)^2\epsilon(j_3,j_6)]\\
\chi_3^E(\mathcal{K}) &= \sum_{j_1>\cdots>j_6} [\epsilon(j_1,j_3)\epsilon(j_2,j_5)^2\epsilon(j_4,j_6)+\epsilon(j_1,j_4)^2\epsilon(j_2,j_6)\epsilon(j_3,j_5)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \epsilon(j_1,j_5)\epsilon(j_2,j_4)\epsilon(j_3,j_6)^2] \\
\chi_4^A(\mathcal{K}) &= \sum_{j_1>\cdots>j_8} \epsilon(j_1,j_5)\epsilon(j_2,j_6)\epsilon(j_3,j_7)\epsilon(j_4,j_8) \\
\chi_4^B(\mathcal{K}) &= \sum_{j_1>\cdots>j_8} [\epsilon(j_1,j_5)\epsilon(j_2,j_7)\epsilon(j_3,j_6)\epsilon(j_4,j_8) \\
&+ \epsilon(j_1,j_6)\epsilon(j_2,j_5)\epsilon(j_3,j_7)\epsilon(j_4,j_8) + \epsilon(j_1,j_4)\epsilon(j_2,j_6)\epsilon(j_3,j_7)\epsilon(j_5,j_8) \\
&+ \epsilon(j_1,j_5)\epsilon(j_2,j_6)\epsilon(j_3,j_8)\epsilon(j_4,j_7)]
\end{aligned}
$$

$$
\begin{aligned}
\chi_4^C(\mathcal{K}) &= \sum_{j_1>\cdots>j_8} [\epsilon(j_1,j_6)\epsilon(j_2,j_5)\epsilon(j_3,j_8)\epsilon(j_4,j_7) \\
&+ \epsilon(j_1,j_4)\epsilon(j_2,j_7)\epsilon(j_3,j_6)\epsilon(j_5,j_8)]
\end{aligned}
$$

$$
\begin{aligned}
\chi_4^D(\mathcal{K}) &= \sum_{j_1>\cdots>j_8} [\epsilon(j_1,j_7)\epsilon(j_2,j_6)\epsilon(j_3,j_5)\epsilon(j_4,j_8) \\
&+ \epsilon(j_1,j_5)\epsilon(j_2,j_4)\epsilon(j_3,j_7)\epsilon(j_6,j_8) + \epsilon(j_1,j_3)\epsilon(j_2,j_6)\epsilon(j_4,j_8)\epsilon(j_5,j_7) \\
&+ \epsilon(j_1,j_5)\epsilon(j_2,j_8)\epsilon(j_3,j_7)\epsilon(j_4,j_6)]
\end{aligned}
$$

$$
\begin{aligned}
\chi_4^E(\mathcal{K}) &= \sum_{j_1>\cdots>j_8} [\epsilon(j_1,j_6)\epsilon(j_2,j_7)\epsilon(j_3,j_5)\epsilon(j_4,j_8) \\
&+ \epsilon(j_1,j_6)\epsilon(j_2,j_4)\epsilon(j_3,j_7)\epsilon(j_5,j_8) + \epsilon(j_1,j_3)\epsilon(j_2,j_6)\epsilon(j_4,j_7)\epsilon(j_5,j_8) \\
&+ \epsilon(j_1,j_5)\epsilon(j_2,j_8)\epsilon(j_3,j_6)\epsilon(j_4,j_7) + \epsilon(j_1,j_7)\epsilon(j_2,j_5)\epsilon(j_3,j_6)\epsilon(j_4,j_8) \\
&+ \epsilon(j_1,j_4)\epsilon(j_2,j_5)\epsilon(j_3,j_7)\epsilon(j_6,j_8) + \epsilon(j_1,j_4)\epsilon(j_2,j_6)\epsilon(j_3,j_8)\epsilon(j_5,j_7) \\
&+ \epsilon(j_1,j_5)\epsilon(j_2,j_7)\epsilon(j_3,j_8)\epsilon(j_4,j_6)]
\end{aligned}
$$

$$
\begin{aligned}
\chi_4^F(\mathcal{K}) &= \sum_{j_1>\cdots>j_8} [\epsilon(j_1,j_4)\epsilon(j_2,j_8)\epsilon(j_3,j_6)\epsilon(j_5,j_7) \\
&+ \epsilon(j_1,j_7)\epsilon(j_2,j_5)\epsilon(j_3,j_8)\epsilon(j_4,j_6) + \epsilon(j_1,j_4)\epsilon(j_2,j_7)\epsilon(j_3,j_5)\epsilon(j_6,j_8) \\
&+ \epsilon(j_1,j_6)\epsilon(j_2,j_4)\epsilon(j_3,j_8)\epsilon(j_5,j_7) + \epsilon(j_1,j_3)\epsilon(j_2,j_7)\epsilon(j_4,j_6)\epsilon(j_5,j_8) \\
&+ \epsilon(j_1,j_6)\epsilon(j_2,j_8)\epsilon(j_3,j_5)\epsilon(j_4,j_7) + \epsilon(j_1,j_7)\epsilon(j_2,j_4)\epsilon(j_3,j_6)\epsilon(j_5,j_8) \\
&+ \epsilon(j_1,j_3)\epsilon(j_2,j_5)\epsilon(j_4,j_7)\epsilon(j_6,j_8)]
\end{aligned}
$$

4.2 Vassiliev invariants of order two and three

In this subsection we will present the reconstruction procedure to obtain a combinatorial expression for each of the primitive Vassiliev invariants at orders two and three. We will obtain the same combinatorial expressions as the ones computed in [26, 27], working in a covariant gauge.

As we argued above we will assume that the perturbative series expansion emerging in the temporal gauge must be accompanied by a global factor which involves the topological invariant for the unknot to some power. The topological invariant vacuum expectation value of the Wilson line corresponding to the knot K has the form

$$\langle W(K,G)\rangle = \langle W(\mathcal{K},G)\rangle_{\rm temp} \times \langle W(U,G)\rangle^{b(\mathcal{K})}, \tag{4.7}$$

where, as before, $\mathcal{K}$ denotes the regular projection into the x_1, x_2 plane chosen, and $b(\mathcal{K})$ is an unknown function. As in the case of the light-cone, gauge we will assume that this function depends only on the shadow corresponding to the projection $\mathcal{K}$ of the knot K. In other words, the quantity $b(\mathcal{K})$ is insensitive to crossing changes in $\mathcal{K}$.

We will denote the perturbative series expansion of $W(K,G)$ by:

$$\frac{1}{d}\langle W(K,G)\rangle = 1 + \sum_{i=1}^{\infty} v_i(K) x^i, \tag{4.8}$$

where $v_i(K)$ stands for the combination of Vassiliev invariants appearing at order i, while that of $W(\mathcal{K},G)_{\rm temp}$ denoted by:

$$\frac{1}{d}\langle W(\mathcal{K},G)\rangle_{\rm temp} = 1 + \sum_{i=1}^{\infty} \hat{v}_i(\mathcal{K}) x^i. \tag{4.9}$$

The quantities $\hat{v}_i(\mathcal{K})$ do not need to be topological invariants. Actually, as explicitly shown in its labelling, they depend on the projection $\mathcal{K}$ of the knot K. In (4.8) and (4.9), $d = \dim R$, the dimension of the representation carried by the Wilson loop.

As explained in detail in [14], and summarized in eq. (3.4), to obtain universal Vassiliev invariants (just depending on the knot class, and not on the chosen gauge group) we first express the contribution from a given diagram in the perturbative series as a sum of products of two factors, geometrical and group factors; then we choose a basis for the independent group factors.

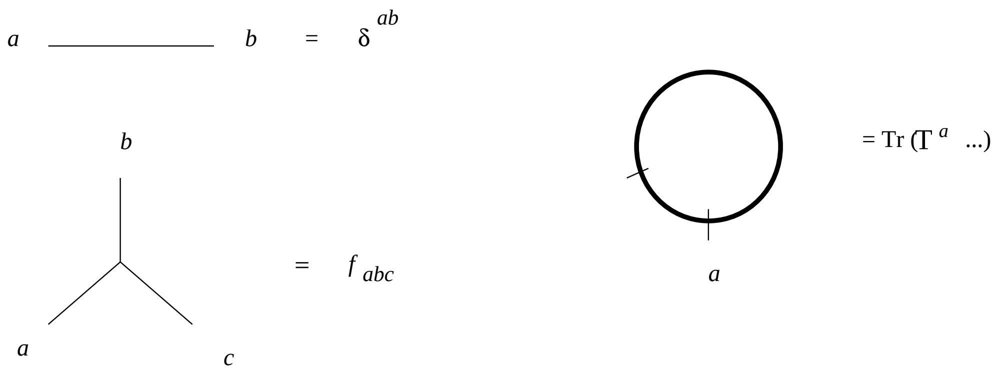


Figure 7: Group factor Feynman rules.

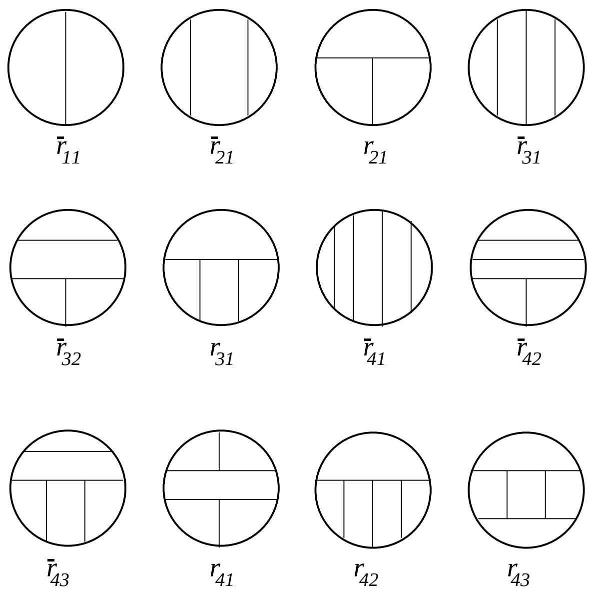


Figure 8: Choice of canonical basis up to order four.

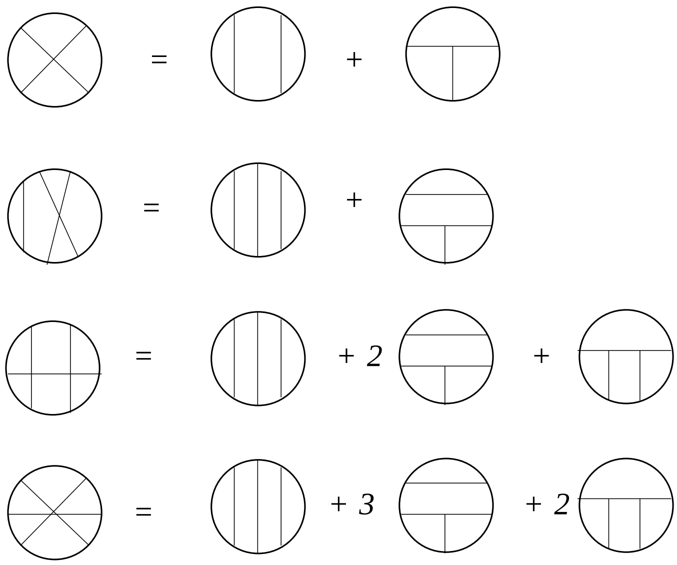


Figure 9: Expansion of chord diagrams in the canonical basis: orders two and three.

The coefficients of the basis elements will turn out to be Vassiliev invariants. In order to obtain the primitive invariants, and also the relations holding for the non-primitive ones, there is a preferred family of bases called canonical [24]. Our choice of basis will be the same as in [14], but here we will refer to it using diagrams. In fig. 7 we have drawn the Feynman rules needed to build up a group factor out of its diagram. Our choice of canonical basis is depicted in fig. 8. Notice that we are including diagrams with isolated chords or collapsible propagators. The reason for this is that their inclusion provides useful information when working in the non-trivial vertical framing. Instead of factorizing them out as in [14], we will keep them in our analysis. This implies that the number of elements in the basis at a given order will increase with respect to ref. [14]. The expressions of all the chord diagrams in terms of the elements of the canonical basis have been collected in figs. 9 and 10.

The perturbative series expansions entering (4.7) get some modifications

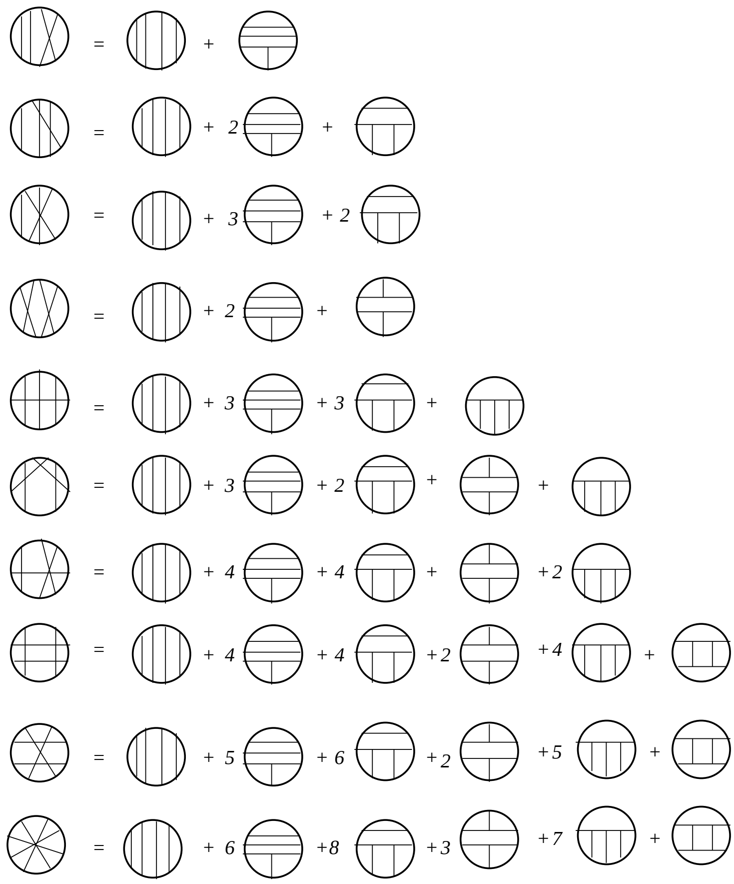


Figure 10: Expansion of chord diagrams in the canonical basis: order four.

relative to their form in (3.4). We will write them in the form:

$$
\begin{aligned}
\frac{1}{d}\langle W(K,G)\rangle &= 1+\sum_{i=1}^{\infty}\sum_{j=1}^{d_i}\alpha_{ij}(K)r_{ij}(G)x^i+\sum_{i=1}^{\infty}\sum_{j=1}^{\tilde{d}_i}\gamma_{ij}(K)\tilde{r}_{ij}(G)x^i,\\
\frac{1}{d}\langle W(\mathcal{K},G)\rangle_{\rm temp} &= 1+\sum_{i=1}^{\infty}\sum_{j=1}^{d_i}\hat{\alpha}_{ij}(\mathcal{K})r_{ij}(G)x^i+\sum_{i=1}^{\infty}\sum_{j=1}^{\tilde{d}_i}\hat{\gamma}_{ij}(\mathcal{K})\tilde{r}_{ij}(G)x^i.
\end{aligned}
\tag{4.10}
$$

Notice that we have split the perturbative series into two sums. In the first sum the group factors, and their corresponding coefficients, are exactly those appearing in (3.4), while in the second sum they are all the non-primitive elements coming from diagrams with collapsible propagators. The quantities $r_{ij}(G)$ and $\tilde{r}_{ij}(G)$ denote the respective group factors, while d_i and $\tilde{d}_i$ are the dimension of their basis at order i. As for the geometrical factors, $\alpha_{ij}(K)$ and $\gamma_{ij}(K)$ denote the Vassiliev invariants, primitive or not, we are looking for, while $\hat{\alpha}_{ij}(\mathcal{K})$ and $\hat{\gamma}_{ij}(\mathcal{K})$ are just the geometrical coefficients in the canonical basis of the perturbative Chern-Simons theory in the temporal gauge.

Our strategy is the following. First we will analyse the behaviour of the unknown integrals entering $\hat{\alpha}_{ij}(\mathcal{K})$ and $\hat{\gamma}_{ij}(\mathcal{K})$; then we will build the whole invariant, taking into account the corresponding global term as dictated by (4.7). Since, as shown in [24], the perturbative series expansion of the vacuum expectation value of the Wilson loop exponentiates in terms of the primitive basis elements, we have the following simple relation among primitives:

$$
\alpha_{ij}(K)=\hat{\alpha}_{ij}(\mathcal{K})+b(\mathcal{K})\,\alpha_{ij}(U). \tag{4.11}
$$

Let us begin with the analysis of $\hat{v}_i(\mathcal{K})$ in (4.9). At first order we have no correction term (recall we are using the vertical framing), and the temporal gauge series provides the full regular invariant:

$$
v_1(K)=\hat{v}_1(\mathcal{K})=\Big(\, E_{\bigcirc\!\!\!\!\!\mid}\;+\;D_{\bigcirc\!\!\!\!\!\mid}\,\Big)\times\bigcirc\!\!\!\!\!\mid\,. \tag{4.12}
$$

From the expression (3.8) for the kernels we easily find, extracting the $k=1$ contribution:

$$
E_{\bigcirc\!\!\!\!\!\mid}=\sum_{i=1}^{n}\epsilon_i, \tag{4.13}
$$

where n is the number of crossings in $\mathcal{K}$. This corresponds to the linking number in the vertical framing, which is known to be the correct answer for $v_1(K)$. Thus we must have

$$D_{\bigcirc\!\!\!\!\!|} = 0, \tag{4.14}$$

which agrees with our general arguments, showing that contributions with an odd number of f-dependent terms vanish.

At order two, the series expansion of (4.9) can be expressed as:

$$\hat{v}_2(\mathcal{K}) = \Big(E_{\bigcirc\!\!\!\!\!\parallel} + D_{\bigcirc\!\!\!\!\!\parallel} \Big) \times \bigcirc\!\!\!\!\!\!\parallel + \Big(E_{\bigoplus} + D_{\bigoplus} \Big) \times \bigoplus. \tag{4.15}$$

Notice that we have not included terms of the form $\sum\limits_{i=1}^{n} \epsilon_i D^i_{\bigcirc}$, since they have an odd number of f-dependent terms, and should not contribute. In terms of the group factors of the chosen canonical basis, the last expression takes the form:

$$\begin{aligned} \hat{v}_2(\mathcal{K}) &= \hat{\gamma}_{21}(\mathcal{K}) \times \bigcirc\!\!\!\!\!\!\parallel + \hat{\alpha}_{21}(\mathcal{K}) \times \bigcirc\!\!\!\!\!\!\top \\ &= \Big(E_{\bigcirc\!\!\!\!\!\parallel} + E_{\bigoplus} + D_{\bigcirc\!\!\!\!\!\parallel} + D_{\bigoplus} \Big) \times \bigcirc\!\!\!\!\!\!\parallel \\ &+ \Big(E_{\bigoplus} + D_{\bigoplus} \Big) \times \bigcirc\!\!\!\!\!\!\top. \end{aligned} \tag{4.16}$$

We can easily compute from the expression (3.8) for the kernels, the two signature-dependent terms entering this expression. One finds:

$$\begin{aligned} E_{\bigcirc\!\!\!\!\!\parallel} &= \frac{1}{4} n(\mathcal{K}) + \sum_{j_1 > j_2 > j_3 > j_4} \big(\epsilon(j_1, j_2)\epsilon(j_3, j_4) + \epsilon(j_1, j_4)\epsilon(j_2, j_3)\big), \\ E_{\bigoplus} &= \frac{1}{4} n(\mathcal{K}) + \sum_{j_1 > j_2 > j_3 > j_4} \epsilon(j_1, j_3)\epsilon(j_2, j_4), \end{aligned} \tag{4.17}$$

where $n(\mathcal{K})$ is the total number of crossings of the knot projection $\mathcal{K}$. These give the following contribution to the sum entering the first group factor in (4.15):

$$S^E_{\bigcirc\!\!\!\!\!\parallel} \equiv E_{\bigcirc\!\!\!\!\!\parallel} + E_{\bigoplus} = \frac{1}{2}\Big(\sum_{i=1}^{n} \epsilon_i\Big)^2. \tag{4.18}$$

According to the factorization theorem [24] this is the whole non-primitive regular invariant of order two, $\gamma_{21} = \frac{1}{2}(\sum \epsilon_i)^2$. Thus, we conclude that the order-two D integrals must satisfy:

$$S^D_{\bigcirc\!\!\!\!\!\parallel} \equiv D_{\bigcirc\!\!\!\!\!\parallel} + D_{\bigoplus} = 0. \tag{4.19}$$

The second equation in (4.17) gives us the crossing-dependent part of the primitive element $\hat{\alpha}_{21}(\mathcal{K})$, which can alternatively be written, using the crossing numbers in (4.6), as:

$$E_{\bigoplus} = \frac{1}{4} n(\mathcal{K}) + \chi_2^A(\mathcal{K}). \tag{4.20}$$

Adding the corresponding global factor term from (4.11) we end with the following expression for the primitive invariant at order two:

$$\alpha_{21}(K) = \frac{1}{4} n(\mathcal{K}) + \chi_2^A(\mathcal{K}) + D_{\bigoplus}(\mathcal{K}) + b(\mathcal{K})\, \alpha_{21}(U), \tag{4.21}$$

where $\alpha_{21}(U)$ stands for the value of this invariant for the unknot. The function $b(\mathcal{K})$ is the unknown exponent in the global factor in (4.7). Using the fact that $D_{\bigcirc}$ and $b(\mathcal{K})$ are equal in $\mathcal{K}$ and $\alpha(\mathcal{K})$, and that the latter is equivalent under ambient isotopy to the unknot, we find:

$$D_{\bigoplus}(\mathcal{K}) = \alpha_{21}(U)\,[1 - b(\mathcal{K})] - \chi_2^A(\alpha(\mathcal{K})) - \frac{1}{4} n(\mathcal{K}). \tag{4.22}$$

The final expression for the invariant is:

$$\alpha_{21}(K) = \alpha_{21}(U) + \chi_2^A(\mathcal{K}) - \chi_2^A(\alpha(\mathcal{K})), \tag{4.23}$$

which agrees with the formulae given in [26] and [28]. Notice that its dependence on $b(\mathcal{K})$ has disappeared, so up to this order we do not get any condition on this function. It might be identically zero. It is important to remark that the derivation of (4.23) that we have presented is much simpler than the one in the covariant gauge obtained in [26]. This simplicity is rooted in the special features of the temporal gauge that permit to have the compact expression (3.8) for the kernels, which are the essential building blocks of the combinatorial expressions for Vassiliev invariants. These features will become more prominent in the third-order analysis to which we now turn.

At order three, the expression of the perturbative series of (4.9) takes the form:

$$\begin{aligned}
\hat{v}_3(\mathcal{K}) &= \hat{\gamma}_{31}(\mathcal{K}) \times \textcircled{|||} + \hat{\gamma}_{32}(\mathcal{K}) \times \textcircled{\top} + \hat{\alpha}_{31}(\mathcal{K}) \times \textcircled{\sqcup} \\
&= \Big(S^E_{\textcircled{|||}} + \sum_i \epsilon_i\, S^{Di}_{\textcircled{|||}} \Big) \times \textcircled{|||} + \Big(S^E_{\textcircled{\top}} + \sum_i \epsilon_i\, S^{D^i}_{\textcircled{\top}} \Big) \times \textcircled{\top} \\
&+ \Big(S^E_{\textcircled{\sqcup}} + \sum_i \epsilon_i\, S^{Di}_{\textcircled{\sqcup}} \Big) \times \textcircled{\sqcup},
\end{aligned} \tag{4.24}$$

where we have made the choice of canonical basis shown in fig. 8. In order to write down the sums $S^E_{\bigcirc}$ and $S^D_{\bigcirc}$ in terms of their Feynman integrals, one has to take into account the change of basis described in fig. 9. Given a basis element, its sum will be built up with all the chord diagrams whose expansion in the canonical basis contains that element, each multiplied by the corresponding coefficient.

At this order there are three independent group factors, but only one is primitive. The factorization theorem provides a sufficiently large number of relations between the D integrals, so that we will be able to solve for the primitive invariant. The Feynman integrals proportional to ϵ^0 or ϵ^2 times some D integral are not written in (4.24) since, as we argued before, they do not contribute. Recall that the integrals D^i are built out of two f-dependent terms of the propagator (2.16). The third factor, which is a signature-dependent one, leads to the sign ϵ_i and a restriction in the integration domain. Thus, we may expect them to be related to the order-two independent integral $D_{\oplus}$. As we show below, this is indeed the case. With the help of the factorization theorem we will find relations for the non-primitive diagrams. Our task is to use these relations to find expressions for the unknown integrals in the primitive factor $\hat{\alpha}_{31}$ in terms of the order-two integrals evaluated in the whole knot or in some closed piece of it.

Similarly to the case of lower order, the computation of the signature contributions is easily obtained from the kernels (3.8). For the case $k = 3$ in (3.8) and the first group factor in (4.24) one finds:

$$\begin{aligned} S^E_{\bigcirc\!\!\!\!\!\!\!\;|||} &\equiv E_{\bigcirc\!\!\!\!\!\!\!\;|||} + E_{\otimes\!\!\!\!\ominus} + E_{\oplus\!\!\!\!\ominus} + E_{\otimes\!\!\!\!\ominus\!\!\!\!\oplus} \\ &= \frac{1}{3}\Big(\sum_i \epsilon_i\Big)^3 = \gamma_{31}(K), \end{aligned} \tag{4.25}$$

where we have used the factorization theorem [24]. The other term associated to the group factor under consideration must therefore vanish:

$$S^{Di}_{\bigcirc\!\!\!\!\!\!\!\;|||} \equiv D^i_{\bigcirc\!\!\!\!\!\!\!\;|||} + D^i_{\otimes\!\!\!\!\ominus} + D^i_{\oplus\!\!\!\!\ominus} + D^i_{\otimes\!\!\!\!\ominus\!\!\!\!\oplus} = 0 \quad \forall\, i. \tag{4.26}$$

We thus end with a non-trivial relation for the $D_{\bigcirc}$ integrals of order three: they sum up to zero. Notice that it is the same kind of relation that we found in (4.19) at order two.

To the second group factor in (4.24) is associated the other non-primitive factor $\hat{\gamma}_{32}(\mathcal{K})$:

$$\hat{\gamma}_{32}(\mathcal{K}) = \Big(S^{E}_{\ominus} + \sum_i \epsilon_i \; S^{D^i}_{\ominus} \Big), \qquad (4.27)$$

whose relation with the corresponding regular invariant, following (4.7), is:

$$\gamma_{32}(K) = \hat{\gamma}_{32}(\mathcal{K}) + b(\mathcal{K})\, \alpha_{21}(U) \, \sum_i \epsilon_i. \qquad (4.28)$$

Due to the Chern-Simons factorization theorem [24], the invariant must fulfil the relation:

$$\gamma_{32}(K) = \alpha_{21}(K) \, \sum_i \epsilon_i. \qquad (4.29)$$

These last two equations trivially imply that the following relation must hold:

$$\hat{\gamma}_{32}(\mathcal{K}) + b(\mathcal{K})\, \alpha_{21}(U) \, \sum_i \epsilon_i = \alpha_{21}(K) \, \sum_i \epsilon_i. \qquad (4.30)$$

This equation will provide important relations to solve the unknown quantities in (4.27). The strategy to obtain them is the following. First we extract the signature-dependent part of (4.27), using the kernels (3.8); then one substitutes (4.21) and (4.28) into (4.30). The signature contributions in (4.27) turn out to be, using the definition of the signature function given in (4.5) and the crossing numbers (4.6):

$$\begin{aligned} S^{E}_{\ominus} \;&\equiv\; E_{\circledast} + 2 \; E_{\oplus} + 3 \; E_{\otimes} \qquad (4.31) \\ &= \Big(\sum_{j_1 > \cdots > j_6} \epsilon(j_1, j_2)\epsilon(j_3, j_5)\epsilon(j_4, j_6) + \text{ c. p.} \Big) + 2\, \chi_3^B + 3\, \chi_3^A, \end{aligned}$$

where c. p. stands for the five inequivalent cyclic permutations of the indices. The right-hand side of (4.31) factorizes as :

$$S^{E}_{\ominus} = E_{\oplus} \, \sum_i \epsilon_i \qquad (4.32)$$

where $E_{\oplus}$ is given in (4.17). Substituting this result into (4.30), we find a relation which involves three of the four D^i integrals present at this order:

$$S^{D^i}_{\ominus} \equiv D^i_{\circledast} + 2 \; D^i_{\oplus} + 3 \; D^i_{\otimes} = D_{\oplus} \,. \qquad (4.33)$$

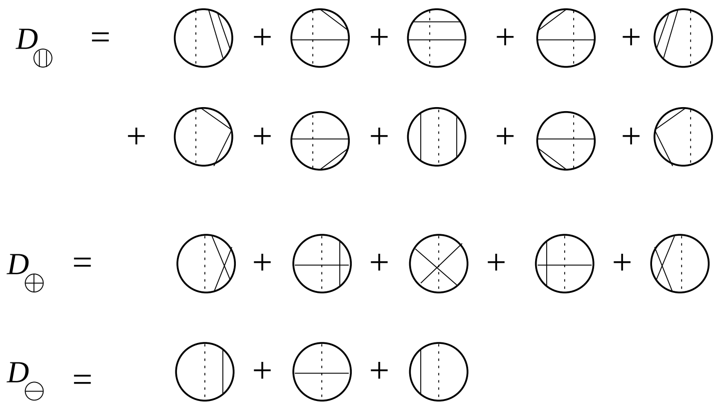


Figure 11: Splitting of D integrals.

Notice that the left-hand side of (4.33) depends on the crossing i, while the right-hand side does not, *i.e.* the precise combination of D integrals in the left, whose domain of integration in principle depends on the crossing i, is in fact equal to an order-two D integral evaluated in the whole knot.

Equations (4.26) and (4.33) can also be proved without making use of the factorization theorem. It is worthwhile to describe how this is so, since it provides some insight that will be useful later. The left-hand side of both equations is a sum over Feynman integrals, with one propagator fixed at a crossing and the other two running over the two regions in which that crossing divides the path of the knot in a way consistent with the corresponding diagram. To better understand the argument, let us first think that we want to compute the integral $D_{\oplus}$, or $D_{(||)}$, splitting the knot path into two regions from a selected crossing i, so one runs over the parametric interval (s_{i_-}, s_{i_+}), and the other over (s_{i_+}, s_{i_-}). The integration region can be decomposed in a sum of partial contributions as depicted in fig. 11, where the dashed line represents the crossing i. The linearity of the integral guarantees that the sum of all these partial contributions leads to the full $D_{\oplus}$, or $D_{(||)}$. The order-two integrals D^i appearing in (4.26) can be organized in a similar way, leading to:

$$D^i_{(||)} + D^i_{\otimes} + D^i_{\oplus} + D^i_{\otimes} = D_{(||)} + D_{\oplus}, \tag{4.34}$$

so (4.26) is a consequence of (4.19). For (4.33) the splitting procedure is a bit more elaborated. The D integrals entering in its left-hand side can be read from fig. 11: given some $D^i_{\bigcirc}$, all the diagrams in fig. 11 whose three chords build the Feynman diagram in the subscript (no matter whether they are dashed or not) contribute to it. We can see that some cancellations occur between them. For instance, when the first three integrals in the second column of fig. 11 (the first two entering $D^i_{\otimes}$ and the third $D^i_{\oplus}$) are summed up, they factorize to the first integral times the second in the last equality of fig. 11. This factorization can be written as:

$$D_{\bigcirc}(\mathcal{K}_{i_+}) \times D_{\bigcirc}(\mathcal{K}_{i_-}, \mathcal{K}_{i_+}), \tag{4.35}$$

where the notation used is the following: $\mathcal{K}_{i_+}$ and $\mathcal{K}_{i_-}$ are the two components obtained from the original knot when the i crossing is removed; $D_{\bigcirc}(\mathcal{K}_{i_+})$ stands for the integral of the f-dependent part of the propagator over the component $\mathcal{K}_{i_+}$, while $D_{\bigcirc}(\mathcal{K}_{i_-}, \mathcal{K}_{i_+})$ represents the integral of the f-part of the propagator with one of its end-points running over $\mathcal{K}_{i_-}$ and the other over $\mathcal{K}_{i_+}$. Recall that the diagram in the subscript denotes the way that the propagators are attached to the knot. The first factor in (4.35) is zero because of (4.14), as it is the evaluation of an odd number of f-dependent parts of the propagator (2.16) over the knot $\mathcal{K}_{i_+}$. Similar arguments hold for the first three integrals in the fourth column of fig. 11, which again sum up to zero. Also, the sum of the second and third integrals in the third column of fig. 11 factorize to:

$$\frac{1}{2}\left(D_{\bigcirc}(\mathcal{K}_{i_-}, \mathcal{K}_{i_+})\right)^2. \tag{4.36}$$

But this integral can be seen to be zero by using the decomposition of the first-order integral in the last equality of fig. 11, and (4.14). One can now see that the remaining integrals in (4.33) build the decomposition of $D_{\oplus}$ shown in fig. 11, so (4.33) is proved.

The splitting argument will again lead to a relation for the unknown integrals entering the primitive diagram of eq. (4.24). These are:

$$S^{Di}_{\ominus} \equiv D^i_{\oplus} + 2\, D^i_{\otimes}\,. \tag{4.37}$$

All the integrals entering (4.37) can again be read from fig. 11: just add up those diagrams whose chords build the subscripts in (4.37) with the apropppriate factors. A factorization of the type explained in (4.36) occurs, and the remaining terms build up the decomposition of $D_{\oplus}$, except for the first and last terms in the second equality of fig. 11, leading to the following result:

$$S^{Di}_{\ominus} = D_{\oplus}(\mathcal{K}) - D_{\oplus}(\mathcal{K}_{i_+}) - D_{\oplus}(\mathcal{K}_{i_-}). \tag{4.38}$$

Thus, we have achieved our goal: we have expressed the unknown integrals in the primitive factor at order three in terms of an known integrals of order two. Given a crossing i, the D integrals of order three in the left-hand side of (4.38) (made out of two f-dependent parts of propagator (2.16)) can be expressed as a combination of the order-two integral $D_{\oplus}$, evaluated in $\mathcal{K}$ and in the two components $\mathcal{K}_{i_+}$ and $\mathcal{K}_{i_-}$ in which the original knot projection is divided when the i crossing is removed. Notice that now both sides of (4.38) depend on the crossing i.

Equation (4.38) can also be obtained by using the factorization theorem and the relations (4.26) and (4.33). Taking into account that

$$S^{Di}_{\ominus} - S^{Di}_{\ominus} = D^{i}_{\otimes} + D^{i}_{\oplus} + D^{i}_{\otimes} = - D^{i}_{\textcircled{|||}}, \tag{4.39}$$

where, in the last equality, we have made use of (4.26), and using (4.33) and (4.37), one easily finds:

$$S^{Di}_{\ominus} = D_{\oplus} + D^{i}_{\textcircled{|||}}. \tag{4.40}$$

The last term of this equation can be written as:

$$\begin{aligned} D^{i}_{\textcircled{|||}}(\mathcal{K}) &= D_{\textcircled{|||}}(\mathcal{K}_{i_+}) + D_{\textcircled{|||}}(\mathcal{K}_{i_-}) \\ &+ D_{\textcircled{|}}(\mathcal{K}_{i_+}) \times D_{\textcircled{|}}(\mathcal{K}_{i_-}). \end{aligned} \tag{4.41}$$

Taking into account that the third and fourth terms of this equation vanishes because of (4.14) and that, using (4.19), the first two can be substituted by the corresponding order-two D-terms, one finds that (4.38) holds.

From the procedure that we have developed to obtain (4.38), we learn that the factorization theorem provides enough relations to express all the unknown parts of the primitive invariants in terms of the lower-order integral $D_{\oplus}$. Alternatively to the use of the factorization theorem, we also possess a splitting procedure which leads to the same results and that sheds some light on the origin of the relations involved.

In order to write down the order-three primitive invariant explicitly, we need to compute in detail the signature contributions. These are easily obtained by using the general formula for the kernels in (3.8). The resulting contribution can be written in a compact form, using the crossing functions (4.6):

$$\begin{aligned} S^E_{\ominus} &\equiv E_{\oplus} + 2\, E_{\otimes} \\ &= \frac{1}{9}\chi_1(\mathcal{K}) + \frac{3}{4}\chi_2^B(\mathcal{K}) + \chi_3^B(\mathcal{K}) + 2\chi_3^A(\mathcal{K}). \end{aligned} \tag{4.42}$$

Using (4.22), (4.38), (4.42), and the following two relations,

$$\chi_2^B(\mathcal{K}) = \sum_i \epsilon_i \, n(\mathcal{K}_{i_+}, \mathcal{K}_{i_-}), \tag{4.43}$$

$$n(\mathcal{K}_{i_+}, \mathcal{K}_{i_-}) + 1 = n(\mathcal{K}) - n(\mathcal{K}_{i_+}) - n(\mathcal{K}_{i_-}), \tag{4.44}$$

where $n(\mathcal{K}_{i_+}, \mathcal{K}_{i_-})$ stands for the number of crossings between the two components $\mathcal{K}_{i_+}$ and $\mathcal{K}_{i_-}$, we end with the formula (recall that, according to (4.7), there is no contribution from the global factor at odd orders):

$$\begin{aligned} \alpha_{31}(K) &= \Big[\frac{1}{9} - \frac{1}{4} - \alpha_{21}(U)\Big]\,\chi_1(\mathcal{K}) + \frac{1}{2}\chi_2^B(\mathcal{K}) + \chi_3^B(\mathcal{K}) + 2\chi_3^A(\mathcal{K}) \\ &- \sum_i \epsilon_i \Big[\chi_2^A(\alpha(\mathcal{K})) - \chi_2^A(\alpha(\mathcal{K}_{i_+})) - \chi_2^A(\alpha(\mathcal{K}_{i_-}))\Big] \\ &- \alpha_{21}(U) \sum_i \epsilon_i \, [b(\mathcal{K}) - b(\mathcal{K}_{i_+}) - b(\mathcal{K}_{i_-})]. \end{aligned} \tag{4.45}$$

Contrary to the order-two result (4.23) we now find an expression depending explicitly on the unknown function b. Invariance of $\alpha_{31}(K)$ provides a relation for the terms involving b in (4.45), which, in turn, leads to a b-independent expression. The simplest way to achieve this is to consider a knot K with two projections which differ by a Reidemeister move of type I. It is easy to find that the value of χ_1 varies in one unit while all the other terms, except the last one involving the function b in (4.45), remain invariant (this will be shown in full detail in the next section). Thus, the contribution from the last term in (4.45) must cancel the one from the first term. This implies that the unknown function b must satisfy:

$$b(\mathcal{K}) - b(\mathcal{K}_{i_+}) - b(\mathcal{K}_{i_-}) = x, \tag{4.46}$$

where x is a constant such that:

$$\frac{1}{9} - \frac{1}{4} - \alpha_{21}(U) - \alpha_{21}(U)\, x = 0. \tag{4.47}$$

Throughout this paper we will use the following normalization for the unknot:

$$\alpha_{21}(U) = -\frac{1}{6}, \tag{4.48}$$

which implies:

$$x = -\frac{1}{6}. \tag{4.49}$$

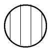
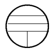
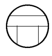
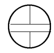
Taking into account (4.46) and (4.49), expression (4.45) for the order-three primitive Vassiliev invariant turns out to be:

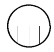
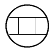
$$\begin{aligned} \alpha_{31}(K) \;&=\; \frac{1}{2}\chi_2^B(\mathcal{K}) + \chi_3^B(\mathcal{K}) + 2\chi_3^A(\mathcal{K}) \\ &-\; \sum_i \epsilon_i \Big[\chi_2^A(\alpha(\mathcal{K})) - \chi_2^A(\alpha(\mathcal{K}_{i_+})) - \chi_2^A(\alpha(\mathcal{K}_{i_-}))\Big]. \end{aligned} \tag{4.50}$$

This combinatorial formula is the same as the one obtained in [27], using Chern-Simons gauge theory in a covariant gauge and in [28] using other methods. As compared to the calculation in the covariant gauge, our computation is much simpler. It is very unlikely that with the covariant-gauge methods utilized in [27] one could obtain combinatorial expressions for higher-order invariants. It turns out that our procedure goes beyond and can be implemented at higher order. We will show in the next section how this is achieved at order four, obtaining combinatorial formulae for the two primitive Vassiliev invariants present at that order.

4.3 Vassiliev invariants of order four

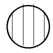
In this section we will apply our reconstruction procedure to compute the two combinatorial expressions for the two primitive invariants at order four. Using our diagrammatic notation for the group factors and the choice of basis shown in fig. 8, the perturbative series expansion in the temporal gauge at this order takes the form:

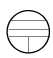
$$\hat{v}_4(\mathcal{K}) = \hat{\gamma}_{41}(\mathcal{K}) \times$$  $$+ \hat{\gamma}_{42}(\mathcal{K}) \times$$  $$+ \hat{\gamma}_{43}(\mathcal{K}) \times$$  $$+ \hat{\alpha}_{41}(\mathcal{K}) \times$$ 

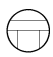
$$+ \hat{\alpha}_{42}(\mathcal{K}) \times$$  $$+ \hat{\alpha}_{43}(\mathcal{K}) \times$$  , (4.51)

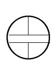
which, after writing down the geometrical contributions more explicitly, making use of the notation introduced above to separate the D integrals from the E integrals, becomes:

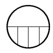
$$\hat{v}_4(\mathcal{K}) =$$

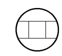
$$\Big(S^{E} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij} + S^{D} \Big) \times$$  $+$

$$\Big(S^{E} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij} + S^{D} \Big) \times$$  $+$

$$\Big(S^{E} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij} + S^{D} \Big) \times$$  $+$

$$\Big(S^{E} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij} + S^{D} \Big) \times$$  $+$

$$\Big(S^{E} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij} + S^{D} \Big) \times$$  $+$

$$\Big(S^{E} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij} + S^{D} \Big) \times$$  . (4.52)

Notice that in this expression we have not included the Feynman integrals proportional to ϵ and ϵ^3, as they do not contribute. Also it is worthwhile to point out that $D^{ii}_{\bigcirc}$ denotes an integral where two chords for the signature-dependent part of (2.16) are attached to the same crossing i, while $D^{ij}_{\bigcirc}$ corresponds to one in which the two chords are attached to two different crossings. In the latter, there are in fact two different sums: one for $i, j \in \mathcal{C}_a$, and another for $i, j \in \mathcal{C}_b$, where $\mathcal{C}_a$ and $\mathcal{C}_b$ are the sets which entered in (4.3) and (4.4). All these integrals are built out of products of two f-terms, while the ones of $D_{\bigcirc}$ contain four f-terms.

At order four there are six independent group factors, but only two are primitive. The factorization theorem will allow us to obtain ways of relating all the D-integrals in terms of the second-order one $D_{\oplus}$. This will lead to an expression for the ones associated to the primitive group factors, $S^{Dij}_{\textcircled{\scriptsize $\boxplus$}}$ and $S^{Dij}_{\textcircled{\scriptsize $\boxminus$}}$, similar to that obtained at third order in (4.38).

As in previous orders, one easily finds, with the aid of the kernels (3.8) and of the factorization theorem, that the sum over all the signature contributions coming from the propagator (2.16), which are contained in $S^{E}_{\textcircled{\scriptsize |||}}$, builds up the whole regular invariant:

$$\begin{aligned}
S^{E}_{\textcircled{\scriptsize |||}} &\equiv E_{\textcircled{\scriptsize |||}} + E_{\otimes} + E_{\theta} + E_{\otimes} + E_{\otimes} + E_{\oplus} \\
&+ E_{\oplus} + E_{\otimes} + E_{\oplus} + E_{\otimes} + E_{\otimes} \\
&= \frac{1}{4}(\sum_i \epsilon_i)^4 = \gamma_{41}(K).
\end{aligned} \tag{4.53}$$

This implies that the rest of the coefficients associated to that group factor vanish:

$$\begin{aligned}
S^{Dij}_{\textcircled{\scriptsize |||}} &\equiv D^{ij}_{\textcircled{\scriptsize |||}} + D^{ij}_{\otimes} + D^{ij}_{\theta} + D^{ij}_{\otimes} + D^{ij}_{\otimes} \\
&+ D^{ij}_{\oplus} + D^{ij}_{\oplus} + D^{ij}_{\otimes} + D^{ij}_{\oplus} + D^{ij}_{\otimes} \\
&+ D^{ij}_{\otimes} = 0 \ \ \forall i, j\,, \\
S^{D}_{\textcircled{\scriptsize |||}} &\equiv D_{\textcircled{\scriptsize |||}} + D_{\otimes} + D_{\theta} + D_{\otimes} + D_{\otimes} + D_{\oplus} \\
&+ D_{\oplus} + D_{\otimes} + D_{\oplus} + D_{\otimes} + D_{\otimes} = 0.
\end{aligned} \tag{4.54}$$

The next non-primitive factor, $\hat{\gamma}_{42}(\mathcal{K})$, has the form:

$$\hat{\gamma}_{42}(\mathcal{K}) = S^{E}_{\ominus} + \frac{1}{4}\sum_i \epsilon_i^2\, S^{Dii}_{\ominus} + \sum_{i>j} \epsilon_i \epsilon_j\, S^{Dij}_{\ominus} + S^{D}_{\ominus}\,, \tag{4.55}$$

and, as follows from (4.7), its relation with the corresponding invariant is:

$$\gamma_{42}(K) = \hat{\gamma}_{42}(\mathcal{K}) + \frac{1}{2} b(\mathcal{K})\, \alpha_{21}(U)\, (\sum_i \epsilon_i)^2. \tag{4.56}$$

From the factorization theorem, it follows that this invariant factorizes as:

$$\gamma_{42}(K) = \frac{1}{2}(\sum_i \epsilon_i)^2\, \alpha_{21}(K). \tag{4.57}$$

Following the procedure used at order three, *i.e.* computing the signature-dependent part of $\hat{\gamma}_{42}(\mathcal{K})$ with the aid of the kernels in (3.8), and comparing these last two equations, we find that the signature contributions match,

$$S^{E}_{\ominus} = \frac{1}{2}(\sum_i \epsilon_i)^2 \; E_{\oplus} \,, \tag{4.58}$$

so the D integrals have to fulfil the following relations:

$$\begin{aligned} S^{Dii}_{\ominus} \;&\equiv\; D^{ii}_{\otimes} + 2\, D^{ii}_{\oslash} + 3\, D^{ii}_{\otimes} + 2\, D^{ii}_{\otimes} + 3\, D^{ii}_{\boxplus} \\ &+\; 3\, D^{ii}_{\oplus} + 4\, D^{ii}_{\otimes} + 4\, D^{ii}_{\boxplus} + 5\, D^{ii}_{\otimes} + 6\, D^{ii}_{\otimes} \\ &=\; 2\, D_{\oplus} \qquad \forall i \,, \end{aligned} \tag{4.59}$$

$$\begin{aligned} S^{Dij}_{\ominus} \;&\equiv\; D^{ij}_{\otimes} + 2\, D^{ij}_{\oslash} + 3\, D^{ij}_{\otimes} + 2\, D^{ij}_{\otimes} + 3\, D^{ij}_{\boxplus} \\ &+\; 3\, D^{ij}_{\oplus} + 4\, D^{ij}_{\otimes} + 4\, D^{ij}_{\boxplus} + 5\, D^{ij}_{\otimes} + 6\, D^{ij}_{\otimes} \\ &=\; D_{\oplus} \qquad \forall i \neq j \,, \end{aligned} \tag{4.60}$$

$$\begin{aligned} S^{D}_{\ominus} \;&\equiv\; D_{\otimes} + 2\, D_{\oslash} + 3\, D_{\otimes} + 2\, D_{\otimes} + 3\, D_{\boxplus} \\ &+\; 3\, D_{\oplus} + 4\, D_{\otimes} + 4\, D_{\boxplus} + 5\, D_{\otimes} + 6\, D_{\otimes} \\ &=\; 0. \end{aligned} \tag{4.61}$$

Recall that the coefficients multiplying the D-integrals come from the choice of basis that we have made. They can be computed with the aid of fig. 10. Notice that in principle the left-hand side of (4.59) and (4.60) could depend upon the pair of crossings chosen. The factorization theorem, however, implies that this is not the case. Actually, these relations are even more remarkable. In (4.59) and (4.60), we are dealing with D integrals where two of the propagators are placed in the same crossing (D^{ii}), in two different crossings belonging to $\mathcal{C}_a$ ($D^{ij,a}$), or in another two in $\mathcal{C}_b$ ($D^{ij,b}$). A given pair of chords in a given Feynman diagram will fulfil only one of the last two conditions, as is easily seen from their picture. So in fact eq. (4.60) is not one but two different relations. It is also worthwhile to point out that factorization provides also a check for the kernels (3.8): the computation of the signature contributions encoded in the symbol $S^{E}_{\ominus}$, done with the aid of (3.8), has to match that coming from the factorized expression of the invariant given in (4.57). Equation (4.58) shows that this is indeed the case.

For the other non-primitive factors one proceeds similarly. For $\hat{\gamma}_{43}(\mathcal{K})$ we have:

$$\hat{\gamma}_{43}(\mathcal{K}) = S^{E}_{\ominus} + \frac{1}{4}\sum_i \epsilon_i^2\, S^{Dii}_{\ominus} + \sum_{i>j} \epsilon_i\epsilon_j\, S^{Dij}_{\ominus} + S^{D}_{\ominus}, \tag{4.62}$$

$$\gamma_{43}(K) = \hat{\gamma}_{43}(K), \tag{4.63}$$

$$\gamma_{43}(K) = \sum_i \epsilon_i\, \alpha_{31}(K). \tag{4.64}$$

Comparing (4.63) with (4.64), and making use of (3.8) to check that the signature contributions in $\gamma_{43}(K)$ match those of the right-hand side of (4.64), we find the following relations for the D integrals:

$$\begin{aligned} S^{Dii}_{\ominus} &\equiv D^{ii}_{\oslash} + 2\, D^{ii}_{\otimes} + +2\, D^{ii}_{\oplus} + 3\, D^{ii}_{\oplus} + 4\, D^{ii}_{\otimes} \\ &+ 4\, D^{ii}_{\oplus} + 6\, D^{ii}_{\otimes} + 8\, D^{ii}_{\otimes} \\ &= 4\, S^{Di}_{\ominus} = 4[\, D_{\oplus}(\mathcal{K}) - D_{\oplus}(\mathcal{K}_{i_+}) - D_{\oplus}(\mathcal{K}_{i_-})], \end{aligned} \tag{4.65}$$

$$\begin{aligned} S^{Dij}_{\ominus} &\equiv D^{ij}_{\oslash} + 2\, D^{ij}_{\otimes} + +2\, D^{ij}_{\oplus} + 3\, D^{ij}_{\oplus} + 4\, D^{ij}_{\otimes} \\ &+ 4\, D^{ij}_{\oplus} + 6\, D^{ij}_{\otimes} + 8\, D^{ij}_{\otimes} \\ &= S^{Di}_{\ominus} = D_{\oplus}(\mathcal{K}) - D_{\oplus}(\mathcal{K}_{i_+}) - D_{\oplus}(\mathcal{K}_{i_-}), \end{aligned} \tag{4.66}$$

$$S^{D}_{\ominus} \equiv D_{\oslash} + 2\, D_{\otimes} + 2\, D_{\oplus} + 3\, D_{\oplus} + 4\, D_{\otimes} + 4\, D_{\oplus} + 6\, D_{\otimes} + 8\, D_{\otimes} = 0, \tag{4.67}$$

where, for the last step in eqs. (4.65) and (4.66), we have made use of (4.38).

For the last non-primitive factor $\hat{\alpha}_{41}(\mathcal{K})$, we find:

$$\hat{\alpha}_{41}(\mathcal{K}) = S^{E}_{\oplus} + \frac{1}{4}\sum_i \epsilon_i^2\, S^{Dii}_{\oplus} + \sum_{i>j} \epsilon_i \epsilon_j\, S^{Dij}_{\oplus} + S^{D}_{\oplus}, \tag{4.68}$$

$$\alpha_{41}(K) = \hat{\alpha}_{41}(\mathcal{K}) + b(\mathcal{K})\, \frac{1}{2}(\alpha_{21}(U))^2, \tag{4.69}$$

$$\alpha_{41}(K) = \frac{1}{2}(\alpha_{21}(K))^2, \tag{4.70}$$

and the relations obtained for the D-integrals turn out to be:

$$S^{Dii}_{\oplus} \equiv D^{ii}{}_{\otimes} + D^{ii}{}_{\oplus} + D^{ii}{}_{\otimes} + 2\, D^{ii}{}_{\oplus} + 2\, D^{ii}{}_{\otimes} + 3\, D^{ii}{}_{\otimes} = D_{\oplus} \quad \forall i\,, \tag{4.71}$$

$$S^{Dij,a}_{\oplus} \equiv D^{ij,a}{}_{\otimes} + D^{ij,a}{}_{\oplus} + D^{ij,a}{}_{\otimes} + 2\, D^{ij,a}{}_{\oplus} + 2\, D^{ij,a}{}_{\otimes} + 3\, D^{ij,a}{}_{\otimes} = D_{\oplus} \quad \forall i,j \in \mathcal{C}_a\,, \tag{4.72}$$

$$S^{Dij,b}_{\oplus} \equiv D^{ij,b}{}_{\otimes} + D^{ij,b}{}_{\oplus} + D^{ij,b}{}_{\otimes} + 2\, D^{ij,b}{}_{\oplus} + 2\, D^{ij,b}{}_{\otimes} + 3\, D^{ij,b}{}_{\otimes} = 0 \quad \forall i,j \in \mathcal{C}_b\,, \tag{4.73}$$

$$S^{D}_{\oplus} \equiv D_{\otimes} + D_{\oplus} + D_{\otimes} + 2\, D_{\oplus} + 2\, D_{\otimes} + 3\, D_{\otimes} = \frac{1}{2}\Big(D_{\oplus} \Big)^2. \tag{4.74}$$

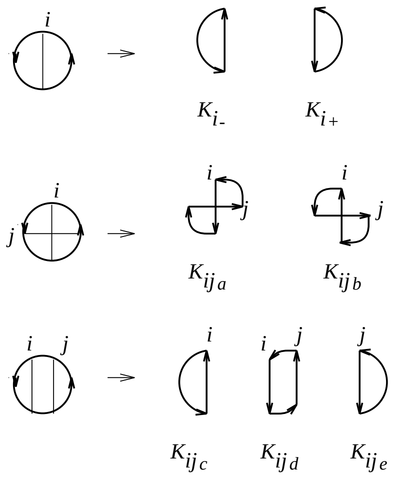


Figure 12: Dividing the knot in other knots.

Notice that in this case we have now found different relations for the $D^{ij}_{\bigcirc}$ integrals depending on the relative position between the crossing labels.

We have obtained a series of relations which will be used in the determination of the unknown terms in the primitive factors. As in order three, a fundamental step to carry out the computation is the expression of the integral $D_{\oplus}$ in terms of all the integrals appearing when we split its integration domain in the pieces defined by two selected crossings i and j. Now there are two ways of doing this, depending on which are the relative positions of the crossings labels. The notation we will use when a closed path is split after removing two crossings is shown in fig. 12. Notice that, when the crossings are alternating, orientation has to be reversed in two of the four segments the knot is divided into, so as to have actual *closed* paths.

Using all the previous relations for the D integrals at this order, and applying a splitting procedure analogous to the one at order three, one finds the following expressions for the unknown integrals present in the two primitive factors:

$$
\begin{aligned}
S^{Dii}_{\ominus} \;\equiv\;& D^{ii}_{\oplus} + D^{ii}_{\oplus} + 2\, D^{ii}_{\otimes} + 4\, D^{ii}_{\oplus} \\
&+ 5\, D^{ii}_{\otimes} + 7\, D^{ii}_{\otimes}
\end{aligned}
$$

$$= 3\Big[D_{\bigoplus}(\mathcal{K}) - D_{\bigoplus}(\mathcal{K}_{i_+}) - D_{\bigoplus}(\mathcal{K}_{i_-})\Big], \quad (4.75)$$

$$\begin{aligned} S^{Dij,a}_{\ominus} \equiv\ & D^{ij,a}_{\bigoplus} + D^{ij,a}_{\bigoplus} + 2\, D^{ij,a}_{\otimes} + 4\, D^{ij,a}_{\bigoplus} \\ +\ & 5\, D^{ij,a}_{\otimes} + 7\, D^{ij,a}_{\otimes} \\ =\ & 3\, D_{\bigoplus}(\mathcal{K}) \\ -\ & 2\Big[D_{\bigoplus}(\mathcal{K}_{i_+}) + D_{\bigoplus}(\mathcal{K}_{i_-}) + D_{\bigoplus}(\mathcal{K}_{j_+}) + D_{\bigoplus}(\mathcal{K}_{j_-})\Big] \\ +\ & D_{\bigoplus}(\mathcal{K}_{ij_a}) + D_{\bigoplus}(\mathcal{K}_{ij_b}), \end{aligned} \quad (4.76)$$

$$\begin{aligned} S^{Dij,b}_{\ominus} \equiv\ & D^{ij,b}_{\bigoplus} + D^{ij,b}_{\bigoplus} + 2\, D^{ij,b}_{\otimes} + 4\, D^{ij,b}_{\bigoplus} \\ +\ & 5\, D^{ij,b}_{\otimes} + 7\, D^{ij,b}_{\otimes} \\ =\ & D_{\bigoplus}(\mathcal{K}) \\ -\ & \Big[D_{\bigoplus}(\mathcal{K}_{i_+}) + D_{\bigoplus}(\mathcal{K}_{i_-}) + D_{\bigoplus}(\mathcal{K}_{j_+}) + D_{\bigoplus}(\mathcal{K}_{j_-})\Big] \\ +\ & D_{\bigoplus}(\mathcal{K}_{ij_c}) + D_{\bigoplus}(\mathcal{K}_{ij_d}) + D_{\bigoplus}(\mathcal{K}_{ij_e}), \end{aligned} \quad (4.77)$$

$$S^{Dii}_{\ominus} \equiv D^{ii}_{\bigoplus} + D^{ii}_{\otimes} + D^{ii}_{\otimes} = 0, \quad (4.78)$$

$$S^{Dij,b}_{\ominus} \equiv D^{ij,b}_{\bigoplus} + D^{ij,b}_{\otimes} + D^{ij,b}_{\otimes} = 0, \quad (4.79)$$

$$S^{Dij,a}_{\ominus} \equiv D^{ij,a}_{\bigoplus} + D^{ij,a}_{\otimes} + D^{ij,a}_{\otimes}$$

$$
\begin{aligned}
&= D_{\bigoplus}(\mathcal{K}) \\
&- \Big[D_{\bigoplus}(\mathcal{K}_{i_+}) + D_{\bigoplus}(\mathcal{K}_{i_-}) + D_{\bigoplus}(\mathcal{K}_{j_+}) + D_{\bigoplus}(\mathcal{K}_{j_-}) \Big] \\
&+ D_{\bigoplus}(\mathcal{K}_{ij_a}) + D_{\bigoplus}(\mathcal{K}_{ij_b}).
\end{aligned}
\tag{4.80}
$$

As before, the relations we have found depend on how the two crossings are related. Notice that all the possible ways of dividing the knot into other closed knots when one or two crossings are removed appear. The alternating case is a bit subtle because, to form closed paths, orientation in some segments has to be reversed, and so the integrals in the left-hand side of (4.76) and (4.80) have to appear with the aproppriate sign.

Our aim is now to calculate $\hat{\alpha}_{42}(\mathcal{K})$ and $\hat{\alpha}_{43}(\mathcal{K})$, the primitive factors at this order. It follows from (4.52) that they have the form:

$$
\hat{\alpha}_{42}(\mathcal{K}) = S^{E}_{\ominus} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii}_{\ominus} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij}_{\ominus} + S^{D}_{\ominus}, \tag{4.81}
$$

$$
\hat{\alpha}_{43}(\mathcal{K}) = S^{E}_{\boxminus} + \frac{1}{4}\sum_i \epsilon_i^2 \, S^{Dii}_{\boxminus} + \sum_{i>j} \epsilon_i \epsilon_j \, S^{Dij}_{\boxminus} + S^{D}_{\boxminus}. \tag{4.82}
$$

The signature contributions appearing in them can be computed from the kernels (3.8). Using the crossing numbers notation introduced in eq. (4.6), we find:

$$
\begin{aligned}
S^{E}_{\ominus} &\equiv E_{\bigoplus} + E_{\bigoplus} + 2\, E_{\bigotimes} + 4\, E_{\bigoplus} \\
&+ 5\, E_{\bigotimes} + 7\, E_{\bigotimes} \\
&= \frac{38}{(4!)^2} n(\mathcal{K}) + \frac{44}{(3!)^2}\chi_2^A(\mathcal{K}) + \frac{21}{16}\chi_2^C(\mathcal{K}) + 3\chi_3^C(\mathcal{K}) \\
&+ \frac{3}{4}\chi_3^E(\mathcal{K}) + \frac{9}{4}\chi_3^D(\mathcal{K}) + 7\chi_4^A(\mathcal{K}) + 5\chi_4^B(\mathcal{K}) + 4\chi_4^C(\mathcal{K}) \\
&+ \chi_4^D(\mathcal{K}) + 2\chi_4^E(\mathcal{K}) + \chi_4^F(\mathcal{K}),
\end{aligned}
\tag{4.83}
$$

$$
\begin{aligned}
S^{E}_{\ominus} &\equiv E_{\oplus} + E_{\otimes} + E_{\otimes} \\
&= \frac{5}{(4!)^2} n(\mathcal{K}) + \frac{1}{3!}\chi_2^A(\mathcal{K}) + \frac{1}{4}\chi_2^C(\mathcal{K}) + \frac{1}{2}\chi_3^C(\mathcal{K}) \\
&+ \frac{1}{2}\chi_3^E(\mathcal{K}) + \chi_4^A(\mathcal{K}) + \chi_4^B(\mathcal{K}) + \chi_4^C(\mathcal{K}).
\end{aligned}
\tag{4.84}
$$

Making use of all these formulae and adding the corresponding global terms coming from (4.7) we obtain the following expressions for the primitive invariants at order four:

$$
\begin{aligned}
\alpha_{42}(K) &= \frac{38}{(4!)^2} n(\mathcal{K}) + \frac{44}{(3!)^2}\chi_2^A(\mathcal{K}) + \frac{21}{16}\chi_2^C(\mathcal{K}) + 3\chi_3^C(\mathcal{K}) \\
&+ \frac{3}{4}\chi_3^D(\mathcal{K}) + \frac{9}{4}\chi_3^E(\mathcal{K}) + 7\chi_4^A(\mathcal{K}) + 5\chi_4^B(\mathcal{K}) + 4\chi_4^C(\mathcal{K}) \\
&+ \chi_4^D(\mathcal{K}) + 2\chi_4^E(\mathcal{K}) + \chi_4^F(\mathcal{K}) \\
&+ \frac{3}{4}\sum_i \epsilon_i^2 \Big[D_{\oplus}(\mathcal{K}) - D_{\oplus}(\mathcal{K}_{i_+}) - D_{\oplus}(\mathcal{K}_{i_-}) \Big] \\
&+ \sum_{i,j\in\mathcal{C}_a} \epsilon_i\epsilon_j \Big\{ 3\, D_{\oplus}(\mathcal{K}) - 2\Big[D_{\oplus}(\mathcal{K}_{i_+}) + D_{\oplus}(\mathcal{K}_{i_-}) \\
&+ D_{\oplus}(\mathcal{K}_{j_+}) + D_{\oplus}(\mathcal{K}_{j_-}) \Big] + D_{\oplus}(\mathcal{K}_{ij_a}) + D_{\oplus}(\mathcal{K}_{ij_b}) \Big\} \\
&+ \sum_{i,j\in\mathcal{C}_b} \epsilon_i\epsilon_j \Big\{ D_{\oplus}(\mathcal{K}) - \Big[D_{\oplus}(\mathcal{K}_{i_+}) + D_{\oplus}(\mathcal{K}_{i_-}) + D_{\oplus}(\mathcal{K}_{j_+}) \\
&+ D_{\oplus}(\mathcal{K}_{j_-}) \Big] + D_{\oplus}(\mathcal{K}_{ij_c}) + D_{\oplus}(\mathcal{K}_{ij_d}) + D_{\oplus}(\mathcal{K}_{ij_e}) \Big\} \\
&+ D_{\boxplus}(\mathcal{K}) + b(\mathcal{K})\, \alpha_{42}(U),
\end{aligned}
\tag{4.85}
$$

$$
\alpha_{43}(K) = \frac{5}{(4!)^2} n(\mathcal{K}) + \frac{1}{3!}\chi_2^A(\mathcal{K}) + \frac{1}{4}\chi_2^C(\mathcal{K}) + \frac{1}{2}\chi_3^C(\mathcal{K})
$$

$$
\begin{aligned}
+ \quad & \frac{1}{2}\chi_3^E(\mathcal{K}) + \chi_4^A(\mathcal{K}) + \chi_4^B(\mathcal{K}) + \chi_4^C(\mathcal{K}) \\
+ \quad & \sum_{i,j\in\mathcal{C}_a} \epsilon_i\epsilon_j \Big\{ D_{\bigoplus}(\mathcal{K}) - \big[D_{\bigoplus}(\mathcal{K}_{i_+}) + D_{\bigoplus}(\mathcal{K}_{i_-}) + D_{\bigoplus}(\mathcal{K}_{j_+}) \\
+ \quad & D_{\bigoplus}(\mathcal{K}_{j_-}) \big] + D_{\bigoplus}(\mathcal{K}_{ij_a}) + D_{\bigoplus}(\mathcal{K}_{ij_b}) \Big\} \\
+ \quad & D_{\boxplus}(\mathcal{K}) + b(\mathcal{K})\,\alpha_{43}(U).
\end{aligned} \tag{4.86}
$$

To get rid of the D-integrals appearing in this expression one has to first replace $D_{\bigoplus}$ by (4.22), and then take into account the fact that all D integrals, as well as the unknown function $b(\mathcal{K})$, only depend on the shadow of the knot. In order to simplify the equations, we will use in (4.85) and (4.86) the following set of relations:

$$
\begin{aligned}
\chi_3^E(\mathcal{K}) &= \sum_{i>j\in\mathcal{C}_b} \epsilon_i\epsilon_j(\mathcal{K}) \Big[n(\mathcal{K}) - n(\mathcal{K}_{i_+}) - n(\mathcal{K}_{i_-}) \\
-n(\mathcal{K}_{j_+}) &- n(\mathcal{K}_{j_-}) + n(\mathcal{K}_{ij_c}) + n(\mathcal{K}_{ij_d}) + n(\mathcal{K}_{ij_e}) \Big],
\end{aligned}
$$

$$
\begin{aligned}
6\,\chi_2^A(\mathcal{K}) + 4\,\chi_3^C(\mathcal{K}) + \chi_3^D(\mathcal{K}) &= \sum_{i>j\in\mathcal{C}_a} \epsilon_i\epsilon_j(\mathcal{K}) \Big[3\,n(\mathcal{K}) - 2\,n(\mathcal{K}_{i_+}) - 2\,n(\mathcal{K}_{i_-}) \\
-2\,n(\mathcal{K}_{j_+}) &- 2\,n(\mathcal{K}_{j_-}) + n(\mathcal{K}_{ij_a}) + n(\mathcal{K}_{ij_b}) \Big], \\
2\,\chi_2^A(\mathcal{K}) + 2\,\chi_3^C(\mathcal{K}) &= \sum_{i>j\in\mathcal{C}_a} \epsilon_i\epsilon_j(\mathcal{K}) \Big[n(\mathcal{K}) - n(\mathcal{K}_{i_+}) - n(\mathcal{K}_{i_-}) \\
-n(\mathcal{K}_{j_+}) &- n(\mathcal{K}_{j_-}) + n(\mathcal{K}_{ij_a}) + n(\mathcal{K}_{ij_b}) \Big].
\end{aligned} \tag{4.87}
$$

These relations are analogous to (4.43) and their use will make (4.85) and (4.86) independent of the function $n(\mathcal{K})$. Evaluating (4.85) and (4.86) for the ascending diagram $\alpha(\mathcal{K})$, a projection of the unknot, one obtains expressions for the order-four integrals $D_{\boxminus}$ and $D_{\boxplus}$. Substituting them back into (4.85) and (4.86), using the normalization for the unknot invariant at order two given in (4.48) and the relations (4.87), one obtains the following expressions:

$$
\alpha_{42}(K) = \alpha_{42}(U) + \frac{2}{9}\left[\chi_2^A(\mathcal{K}) - \chi_2^A(\alpha(\mathcal{K}))\right] + 2\left[\chi_3^C(\mathcal{K}) - \chi_3^C(\alpha(\mathcal{K}))\right]
$$

$$+\ \frac{1}{2}\left[\chi_3^D(\mathcal{K}) - \chi_3^D(\alpha(\mathcal{K}))\right] + 2\left[\chi_3^E(\mathcal{K}) - \chi_3^E(\alpha(\mathcal{K}))\right] + 7\left[\chi_4^A(\mathcal{K}) - \chi_4^A(\alpha(\mathcal{K}))\right]$$

$$+\ 5\left[\chi_4^B(\mathcal{K}) - \chi_4^B(\alpha(\mathcal{K}))\right] + 4\left[\chi_4^C(\mathcal{K}) - \chi_4^C(\alpha(\mathcal{K}))\right] + \chi_4^D(\mathcal{K}) - \chi_4^D(\alpha(\mathcal{K}))$$

$$+\ 2\left[\chi_4^E(\mathcal{K}) - \chi_4^E(\alpha(\mathcal{K}))\right] + \chi_4^F(\mathcal{K}) - \chi_4^F(\alpha(\mathcal{K}))$$

$$-\ \sum_{i>j\in\mathcal{C}_a}\left[\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{\ 3\chi_2^A(\alpha(\mathcal{K})) -\ 2\Big[\chi_2^A(\alpha(\mathcal{K}_{i_+})) + \chi_2^A(\alpha(\mathcal{K}_{i_-}))$$

$$+\ \chi_2^A(\alpha(\mathcal{K}_{j_+})) + \chi_2^A(\alpha(\mathcal{K}_{j_-}))\Big] + \chi_2^A(\alpha(\mathcal{K}_{ij_a})) + \chi_2^A(\alpha(\mathcal{K}_{ij_b}))\Big\}$$

$$-\ \sum_{i>j\in\mathcal{C}_b}\left[\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{\ \chi_2^A(\alpha(\mathcal{K})) - \chi_2^A(\alpha(\mathcal{K}_{i_+})) - \chi_2^A(\alpha(\mathcal{K}_{i_-}))$$

$$-\ \chi_2^A(\alpha(\mathcal{K}_{j_+})) - \chi_2^A(\alpha(\mathcal{K}_{j_-})) + \chi_2^A(\alpha(\mathcal{K}_{ij_c})) + \chi_2^A(\alpha(\mathcal{K}_{ij_d})) + \chi_2^A(\alpha(\mathcal{K}_{ij_e}))\Big\}$$

$$+\ \frac{1}{6}\sum_{i>j\in\mathcal{C}_a}\left[\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{3b(\mathcal{K}) - 2b(\mathcal{K}_{i_+}) - 2b(\mathcal{K}_{i_-})$$
$$-\ 2b(\mathcal{K}_{j_+}) - 2b(\mathcal{K}_{j_-}) + b(\mathcal{K}_{ij_a}) + b(\mathcal{K}_{ij_b})\Big\}$$
$$+\ \frac{1}{6}\sum_{i>j\in\mathcal{C}_b}\left[\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{b(\mathcal{K}) - b(\mathcal{K}_{i_+}) - b(\mathcal{K}_{i_-})$$
$$-\ b(\mathcal{K}_{j_+}) - b(\mathcal{K}_{j_-}) + b(\mathcal{K}_{ij_c}) + b(\mathcal{K}_{ij_d}) + b(\mathcal{K}_{ij_e})\Big\}, \qquad (4.88)$$

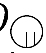
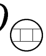
$$\alpha_{43}(K) = \alpha_{43}(U) - \frac{1}{6}\left[\chi_2^A(\mathcal{K}) - \chi_2^A(\alpha(\mathcal{K}))\right] + \frac{1}{2}\left[\chi_3^E(\mathcal{K}) - \chi_3^E(\alpha(\mathcal{K}))\right]$$

$$+\ \chi_4^A(\mathcal{K}) - \chi_4^A(\alpha(\mathcal{K})) + \chi_4^B(\mathcal{K}) - \chi_4^B(\alpha(\mathcal{K})) + \chi_4^C(\mathcal{K}) - \chi_4^C(\alpha(\mathcal{K}))$$

$$-\ \sum_{i>j\in\mathcal{C}_a}\left[\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{\ \chi_2^A(\alpha(\mathcal{K})) - \chi_2^A(\alpha(\mathcal{K}_{i_+})) - \chi_2^A(\alpha(\mathcal{K}_{i_-}))$$

$$-\ \chi_2^A(\alpha(\mathcal{K}_{j_+})) - \chi_2^A(\alpha(\mathcal{K}_{j_-})) + \chi_2^A(\alpha(\mathcal{K}_{ij_a)})) + \chi_2^A(\alpha(\mathcal{K}_{ij_b}))\Big\}$$

$$+\ \frac{1}{6}\sum_{i>j\in\mathcal{C}_a}\left[\epsilon_i\epsilon_j(\mathcal{K})-\epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{b(\mathcal{K})-b(\mathcal{K}_{i_+})-b(\mathcal{K}_{i_-})$$
$$-\ b(\mathcal{K}_{j_+})-b(\mathcal{K}_{j_-})+b(\mathcal{K}_{ij_a})+b(\mathcal{K}_{ij_b})\Big\}. \qquad (4.89)$$

Notice that some coefficients of the crossing functions have changed because there are new contributions coming from the use of (4.22) and (4.87). Also, the terms depending only on $\alpha(\mathcal{K})$ (like $n(\mathcal{K})$ or χ_2^C) disappear after substituting back D and D .

The expressions (4.88) and (4.89) contain sums involving the function $b(\mathcal{K})$ evaluated in different closed paths. In analogy with order three, we will require the factors of these sums to be constants. Actually, as we argue below, this is the only possibility for (4.88) and (4.89) to be invariants. Making use of the constraint imposed at order three (see (4.46)) we define the following:

$$\begin{aligned}
2x &- [b(\mathcal{K})-b(\mathcal{K}_{ij_c})-b(\mathcal{K}_{ij_d})-b(\mathcal{K}_{ij_e})]=y,\\
2x &- [b(\mathcal{K})-b(\mathcal{K}_{ij_a})-b(\mathcal{K}_{ij_b})]=z,\\
4x &- [b(\mathcal{K})-b(\mathcal{K}_{ij_a})-b(\mathcal{K}_{ij_b})]=t,
\end{aligned} \qquad (4.90)$$

where y, z and t are the constants and $x=-\frac{1}{6}$ follows from (4.49). Recall that the labels of $\mathcal{K}$ refer to the closed paths in which the original knot is split when two of its crossings are removed (see fig. 12). Consistency between the last two equations leads to $t=2x+z$. A solution for the other two can be obtained by using the values of α_{42} and α_{43} for some non-trivial knot (for example, for the trefoil knot T: $\alpha_{42}(T)=62/3+1/360$ and $\alpha_{43}(T)=10/3-1/360$). They turn out to be:

$$y=z=0. \qquad (4.91)$$

The final combinatorial expressions for the two order-four primitive Vassiliev invariants are:

$$\alpha_{42}(K)=\alpha_{42}(U)+\frac{1}{6}\left[\chi_2^A(\mathcal{K})-\chi_2^A(\alpha(\mathcal{K}))\right]+2\left[\chi_3^C(\mathcal{K})-\chi_3^C(\alpha(\mathcal{K}))\right]$$
$$+\ \frac{1}{2}\left[\chi_3^D(\mathcal{K})-\chi_3^D(\alpha(\mathcal{K}))\right]+2\left[\chi_3^E(\mathcal{K})-\chi_3^E(\alpha(\mathcal{K}))\right]+7\left[\chi_4^A(\mathcal{K})-\chi_4^A(\alpha(\mathcal{K}))\right]$$
$$+\ 5\left[\chi_4^B(\mathcal{K})-\chi_4^B(\alpha(\mathcal{K}))\right]+4\left[\chi_4^C(\mathcal{K})-\chi_4^C(\alpha(\mathcal{K}))\right]+\chi_4^D(\mathcal{K})-\chi_4^D(\alpha(\mathcal{K}))$$

$$+\ 2\left[\chi_4^E(\mathcal{K})-\chi_4^E(\alpha(\mathcal{K}))\right]+\chi_4^F(\mathcal{K})-\chi_4^F(\alpha(\mathcal{K}))$$

$$-\sum_{i>j\in\mathcal{C}_a}\left[\epsilon_i\epsilon_j(\mathcal{K})-\epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{\,3\chi_2^A(\alpha(\mathcal{K}))-\ 2\Big[\chi_2^A(\alpha(\mathcal{K}_{i_+}))+\chi_2^A(\alpha(\mathcal{K}_{i_-}))$$

$$+\ \chi_2^A(\alpha(\mathcal{K}_{j_+}))+\chi_2^A(\alpha(\mathcal{K}_{j_-}))\Big]+\chi_2^A(\alpha(\mathcal{K}_{ij_a}))+\chi_2^A(\alpha(\mathcal{K}_{ij_b}))\Big\}$$

$$-\sum_{i>j\in\mathcal{C}_b}\left[\epsilon_i\epsilon_j(\mathcal{K})-\epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{\,\chi_2^A(\alpha(\mathcal{K}))-\chi_2^A(\alpha(\mathcal{K}_{i_+}))-\chi_2^A(\alpha(\mathcal{K}_{i_-}))$$

$$-\ \chi_2^A(\alpha(\mathcal{K}_{j_+}))-\chi_2^A(\alpha(\mathcal{K}_{j_-}))+\chi_2^A(\alpha(\mathcal{K}_{ij_c}))+\chi_2^A(\alpha(\mathcal{K}_{ij_d}))+\chi_2^A(\alpha(\mathcal{K}_{ij_e}))\Big\} \tag{4.92}$$

$$\alpha_{43}(K)=\alpha_{43}(U)-\frac{1}{6}\left[\chi_2^A(\mathcal{K})-\chi_2^A(\alpha(\mathcal{K}))\right]+\frac{1}{2}\left[\chi_3^E(\mathcal{K})-\chi_3^E(\alpha(\mathcal{K}))\right]$$

$$+\ \chi_4^A(\mathcal{K})-\chi_4^A(\alpha(\mathcal{K}))+\chi_4^B(\mathcal{K})-\chi_4^B(\alpha(\mathcal{K}))+\chi_4^C(\mathcal{K})-\chi_4^C(\alpha(\mathcal{K}))$$

$$-\sum_{i>j\in\mathcal{C}_a}\left[\epsilon_i\epsilon_j(\mathcal{K})-\epsilon_i\epsilon_j(\alpha(\mathcal{K}))\right]\Big\{\,\chi_2^A(\alpha(\mathcal{K}))-\chi_2^A(\alpha(\mathcal{K}_{i_+}))-\chi_2^A(\alpha(\mathcal{K}_{i_-}))$$

$$-\ \chi_2^A(\alpha(\mathcal{K}_{j_+}))-\chi_2^A(\alpha(\mathcal{K}_{j_-}))+\chi_2^A(\alpha(\mathcal{K}_{ij_a})))+\chi_2^A(\alpha(\mathcal{K}_{ij_b}))\Big\}. \tag{4.93}$$

Again, the coefficient of χ_2^A in (4.92) has changed because of the contribution coming from the last equation in (4.90). Note that both formulae have the same structure: a sum over some crossing numbers evaluated in $\mathcal{K}$ minus the same sum evaluated in the ascending diagram $\alpha(\mathcal{K})$. In addition, there are residual sums involving some combination of the functions χ_2^A evaluated in the different pieces the knot is divided into when two crossings are selected. In $\alpha_{42}(K)$ we have two of these sums: one for all the pairs of crossings belonging to $\mathcal{C}_a$, and another for those in $\mathcal{C}_b$. In $\alpha_{43}(K)$, however, only the former set contributes. This, together with the fact that there is a larger number of order-four crossing numbers appearing in (4.92), makes the expression for $\alpha_{42}(K)$ more complicated.

There is another important comment to be made: the term in both invariants proportional to $\chi_2^A(\mathcal{K}) - \chi_2^A(\alpha(\mathcal{K}))$ is in fact a Vassiliev invariant by itself, that of order two with the unknot normalized to zero. So the rest of the sum also has to be a topological invariant (as we prove in the next section). Then, the value of the coefficient of $\chi_2^A(\mathcal{K}) - \chi_2^A(\alpha(\mathcal{K}))$ does not affect the topological invariance of our formulae. The last two constraints for the function $b(\mathcal{K})$ in (4.90) only affect that term, so that any other values for t and z, as long as they are constants, would not spoil the topological invariance of our formulae. With our present knowledge, the only way to fix t and z is to compare our expression for $\alpha_{42}(K)$ and $\alpha_{43}(K)$ to a known one for some non-trivial knot, as we did to get (4.91). Of course, this would not be necessary if we had an independent argument to obtain the function $b(\mathcal{K})$. To fix the constant y, however, there is no need to make explicit comparisons: it follows from invariance, as was the case of x at order three. Indeed, under the first Reidemeister move, the variation of the sum that multiplies y,

$$\sum_{i>j\in\mathcal{C}_b} \epsilon_i\epsilon_j(\mathcal{K}), \tag{4.94}$$

is proportional to the writhe, $\chi_1(\mathcal{K})$, while the rest of the terms remain invariant (this will be explicitly shown in the next section). Thus $y = 0$ is the only solution.

We have implemented the combinatorial expressions (4.92) and (4.93), as well as the known ones, (4.23) and (4.50) into a Mathematica algorithm. In the tables 1 and 2 of the appendix we present a list of the values of the four primitive invariants α_{21}, α_{31}, α_{42} and α_{43} for all prime knots up to nine crossings. Actually, the values presented in those tables are α_{21}, α_{31}, α_{42} and α_{43} once their value for the unknot has been substracted. The values for the new combinatorial expressions for α_{42} and α_{43} agree for all knots for which those quantities are known [14, 15, 29].

The constraints that we have obtained for the function $b(\mathcal{K})$ can be summarized in the following equations:

$$\begin{aligned} b(\mathcal{K}) - b(\mathcal{K}_{i_+}) - b(\mathcal{K}_{i_-}) &= -\frac{1}{6}, \\ b(\mathcal{K}) - b(\mathcal{K}_{ij_a}) - b(\mathcal{K}_{ij_b}) &= -\frac{1}{3}, \\ b(\mathcal{K}) - b(\mathcal{K}_{ij_c}) - b(\mathcal{K}_{ij_d}) - b(\mathcal{K}_{ij_e}) &= -\frac{1}{3}. \end{aligned} \tag{4.95}$$

These constraints on b have a very simple solution. Let us consider a representative of $\mathcal{K}$ which is a Morse function in both the x and the y directions. Certainly this can always be done without lost of generality for any projection $\mathcal{K}$. This representative has well defined numbers of critical points in both the x and the y directions. Let us denote these numbers by n_x and n_y respectively. A solution of the equations (4.95) is:

$$b(\mathcal{K}) = \frac{1}{12}(n_x + n_y). \tag{4.96}$$

To prove that this is indeed a solution, let us consider the three possible splittings of $\mathcal{K}$ contained in eq. (4.95), which are represented in fig. 12. Under the first splitting we find that $n_x + n_y \to n_x + n_y + 2$, *i.e.* that the number of extrema is increased by 1 in each of the resulting components. Under the second splitting, the number of extrema is increased by 2 in each of the resulting components, and therefore $n_x + n_y \to n_x + n_y + 4$. Finally, in the third splitting $n_x + n_y \to n_x + n_y + 4$, since one component increases by 2 while in the other two it increases by 1. Thus (4.96) satisfies the relations (4.95). Notice that the ansatz (4.96) is symmetric under the interchange of x and y. This is consistent with the rotational invariance on the plane normal to the time direction present in the temporal gauge. We conjecture that (4.96) is the correct form of the function $b(\mathcal{K})$ when a representative of $\mathcal{K}$, which is a Morse function in both the x and the y directions, is chosen.

R-I

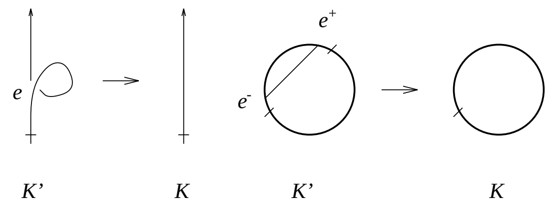


Figure 13: Reidemeister I .

5 Invariance under Reidemeister moves

In this section we will prove that the combinatorial expressions for the two primitive invariants of order four, (4.92) and (4.93), are actually topological invariants, showing that they are invariant under the three Reidemeister moves. To do so, we have to know how the crossing functions behave under these moves. For some of them this has been done in [26, 27]. For the others, we will work out the form of their variations proceeding in an analogous way.

The Reidemeister moves are depicted in figs. 13, 14, 15. For concreteness, we have made a choice of base points and orientations, as well as a choice of joining the three curves in R-III, but it should be clear that these choices do not affect our proof. They have also been taken such that the signature of the crossings involved in the moves does not vary when we change from $\mathcal{K}$ to $\alpha(\mathcal{K})$ or from $\mathcal{K}'$ to $\alpha(\mathcal{K}')$. As the number of different functions is quite large, we are going to provide details only for one case, χ_4^A, and give a list of the variations for the rest of the crossing functions.

It is easier to understand the procedure to obtain the variations of the crossing functions using diagrams. Recall that in fig. 6 we gave a diagrammatic definition of the crossing functions in which a circle represented the ordered set of crossing labels on $\mathcal{K}$, and a series of chords joining the labels i and j represented the signature function $\epsilon(i,j)$. From those diagrams one can immediately figure out that, given a selected group of crossings, only those that follow the pattern given by the diagram contribute to the corresponding function. In figs. 13, 14, 15 we have drawn similar diagrams to represent the three Reidemeister moves, which are labelled R-I, R-IIA and R-IIB, and R-III. In these diagrams, the circle representing $\mathcal{K}$ is divided into sections,

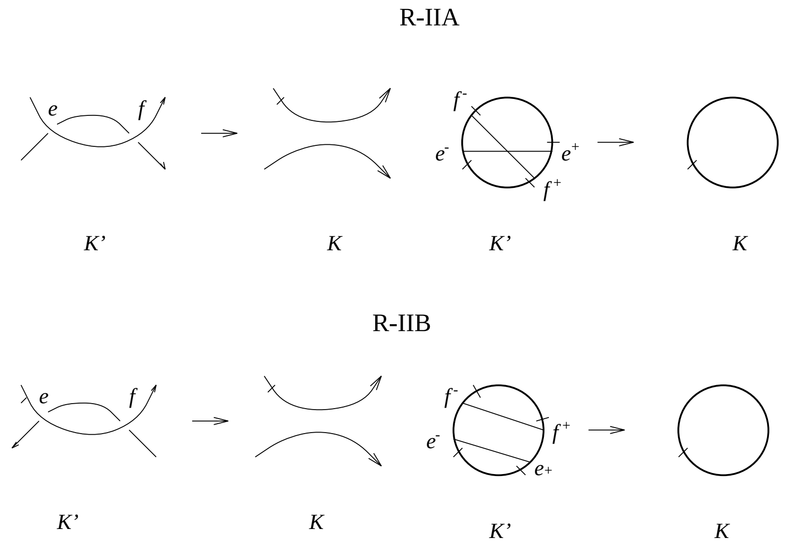


Figure 14: Reidemeister II.

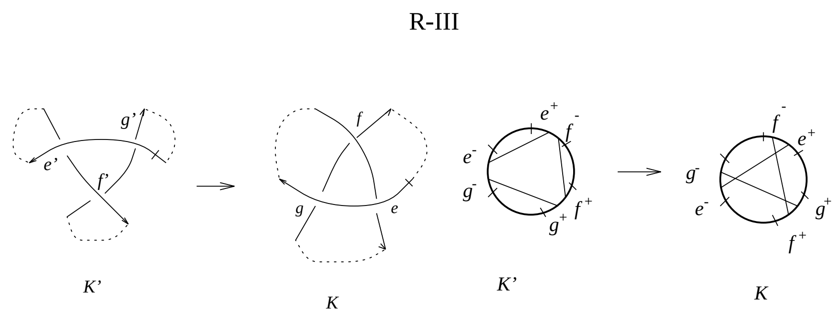


Figure 15: Reidemeister III.

some of them affected by the move and some not. Only the labels affected by the move are depicted and a chord is pictured as joining them. All other crossings do not change under the moves.

Under R-I, the function χ_4^A does not vary:

$$\chi_4^A(\mathcal{K}') - \chi_4^A(\mathcal{K}) = 0. \tag{5.1}$$

The e crossing does not contribute to any term in χ_4^A, as can be seen from the symbolic expression of the move in fig. 13: there is no way to choose e and other three crossings so that they reproduce the diagrammatic expression of this function, because all the others lay in the region outside (e^-, e^+). The same argument holds for every other crossing function appearing in eqs. (4.92) and (4.93) for $\alpha_{42}(K)$ and $\alpha_{43}(K)$, because none has a diagrammatic expression with isolated chords.

Under a move of type R-IIA we find:

$$\chi_4^A(\mathcal{K}') - \chi_4^A(\mathcal{K}) = \epsilon_e\,\epsilon_f \sum_{\substack{i_1,i_2\in(f^-,e^+)\\ i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4), \tag{5.2}$$

where the crossings e and f are such that $\epsilon_e = -\epsilon_f$. In this expression and throughout this section, the crossing labels $i_1,\dots,i_n$ fulfil the natural order: $i_1 < i_2 < \cdots < i_n$. In the variation (5.2) there is a potential term proportional to $\epsilon_e + \epsilon_f$ but it does not contribute because its coefficient turns out to be zero. The sum on the right-hand side of (5.2) can be expressed in diagrammatic form. The regions to which the labels are attached are the regions not affected by the move. The signature functions fix how to draw the chords. These two chords, together with the ones corresponding to e and f, build the diagram associated to χ_4^A.

Under R-IIB we find,

$$\chi_4^A(\mathcal{K}') - \chi_4^A(\mathcal{K}) = 0. \tag{5.3}$$

As in the previous case the crossings e and f are such that $\epsilon_e = -\epsilon_f$, and the terms linear in e and f cancel. This time no quadratic contribution is left, because there is no way to choose e, f and two other crossings so as to end up with the chord diagram corresponding to χ_4^A.

Under R-III the three crossings involved in the move are such that $\epsilon_f = \epsilon_g = -\epsilon_e$. Their signature values do not vary from $\mathcal{K}'$ to $\mathcal{K}$. The variation turns out to be:

$$\chi_4^A(\mathcal{K}') - \chi_4^A(\mathcal{K}) = -\,\epsilon_f\,\epsilon_g \sum_{\substack{i_1,i_2\in(f^+,e^-)\\ i_3,i_4\in(e^+,g^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)$$

$$-\quad \epsilon_e\,\epsilon_g \sum_{\substack{i_1,i_2\in(f^+,e^-)\\ i_3,i_4\in(g^-,f^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) - \epsilon_f\,\epsilon_e \sum_{\substack{i_1,i_2\in(g^-,f^-)\\ i_3,i_4\in(e^+,g^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4). \quad (5.4)$$

In order to make the resulting expressions simpler, from now on we will assume that $\epsilon_f = 1$ in any of the moves. Certainly this does not imply a loss of generality. We now present the lists of variations of the crossing functions. Under R-IIA one finds:

$$\chi_2^A(\mathcal{K}') - \chi_2^A(\mathcal{K}) = -1$$

$$\chi_3^C(\mathcal{K}') - \chi_3^C(\mathcal{K}) = -\sum_{\substack{i_1\in(f^+,e^-)\\ i_2\in(f^-,e^+)}} \epsilon^2(i_1,i_2) + 2\sum_{\substack{i_1,i_2\in(f^+,e^-)\\ i_3,i_4\in(f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_3^D(\mathcal{K}') - \chi_3^D(\mathcal{K}) = 2\sum_{\substack{i_1,i_2,i_3\in(f^+,e^-)\\ i_4\in(f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) + 2\sum_{\substack{i_1\in(f^+,e^-)\\ i_2,i_3,i_4\in(f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_3^E(\mathcal{K}') - \chi_3^E(\mathcal{K}) = 2\sum_{\substack{i_1,i_2\in(f^+,e^-)\\ i_3,i_4\in(f^-,e^+)}} \epsilon(i_2,i_3)\epsilon(i_1,i_4),$$

$$\chi_4^B(\mathcal{K}') - \chi_4^B(\mathcal{K}) = -\sum_{\substack{i_1,i_2\in(f^+,e^-)\\ i_3,i_4\in(f^-,e^+)}} \epsilon(i_2,i_3)\epsilon(i_1,i_4),$$

$$\chi_4^C(\mathcal{K}') - \chi_4^C(\mathcal{K}) = 0,$$

$$\chi_4^D(\mathcal{K}') - \chi_4^D(\mathcal{K}) = 0,$$

$$\chi_4^E(\mathcal{K}') - \chi_4^E(\mathcal{K}) = -\sum_{\substack{i_1,i_2,i_3\in(f^+,e^-)\\ i_4\in(f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) - \sum_{\substack{i_1\in(f^+,e^-)\\ i_2,i_3,i_4\in(f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_4^F(\mathcal{K}') - \chi_4^F(\mathcal{K}) = 0. \quad (5.5)$$

Under R-IIB the variations turn out to be:

$$\chi_2^A(\mathcal{K}') - \chi_2^A(\mathcal{K}) = 0,$$

$$\chi_3^C(\mathcal{K}') - \chi_3^C(\mathcal{K}) = 2 \sum_{\substack{i_1,i_2 \in (f^+,e^-) \\ i_3,i_4 \in (f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_3^D(\mathcal{K}') - \chi_3^D(\mathcal{K}) = 2 \sum_{\substack{i_1,i_2,i_3 \in (f^+,e^-) \\ i_4 \in (f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) \; + 2 \sum_{\substack{i_1 \in (f^+,e^-) \\ i_2,i_3,i_4 \in (f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_3^E(\mathcal{K}') - \chi_3^E(\mathcal{K}) = - \sum_{\substack{i_1 \in (f^+,e^-) \\ i_2 \in (f^-,e^+)}} \epsilon^2(i_1,i_2) \; + 2 \sum_{\substack{i_1,i_2 \in (f^+,e^-) \\ i_3,i_4 \in (f^-,e^+)}} \epsilon(i_2,i_3)\epsilon(i_1,i_4),$$

$$\chi_4^B(\mathcal{K}') - \chi_4^B(\mathcal{K}) = - \sum_{\substack{i_1,i_2 \in (f^+,e^-) \\ i_3,i_4 \in (f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_4^C(\mathcal{K}') - \chi_4^C(\mathcal{K}) = - \sum_{\substack{i_1,i_2 \in (f^+,e^-) \\ i_3,i_4 \in (f^-,e^+)}} \epsilon(i_2,i_3)\epsilon(i_1,i_4),$$

$$\chi_4^D(\mathcal{K}') - \chi_4^D(\mathcal{K}) = - \sum_{\substack{i_1,i_2,i_3 \in (f^+,e^-) \\ i_4 \in (f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) \; - \sum_{\substack{i_1 \in (f^+,e^-) \\ i_2,i_3,i_4 \in (f^-,e^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),$$

$$\chi_4^E(\mathcal{K}') - \chi_4^E(\mathcal{K}) = 0,$$

$$\chi_4^F(\mathcal{K}') - \chi_4^F(\mathcal{K}) = 0. \tag{5.6}$$

Under R-III moves we find the variations of the crossing numbers collected below. For simplicity, we have not written explicitly the terms where the signature function is squared (these terms appear in χ_3^C, χ_3^D and χ_3^E). The reason is that, whatever they might be, they are trivially cancelled when we compute $\alpha_{42}(K') - \alpha_{42}(K)$, or $\alpha_{43}(K') - \alpha_{43}(K)$, because they always appear in terms of the form $\chi(\mathcal{K}) - \chi(\alpha(\mathcal{K}))$ and $\epsilon(i,j)^2$ has the same value in both $\mathcal{K}$ and $\alpha(\mathcal{K})$ for any i, j. We will generally denote these contributions by $F(\epsilon^2)$. The variations under R-III moves are:

$$\chi_2^A(\mathcal{K}') - \chi_2^A(\mathcal{K}) \;\; = \;\; 1$$

$$\chi_3^C(\mathcal{K}') - \chi_3^C(\mathcal{K}) = 1 - 2 \sum_{\substack{i_1 \in (e^+, g^+) \\ i_2 \in (f^+, e^-)}} \epsilon(i_1, i_2) + F(\epsilon^2),$$

$$\chi_3^D(\mathcal{K}') - \chi_3^D(\mathcal{K}) = 0 + F(\epsilon^2),$$

$$\chi_3^E(\mathcal{K}') - \chi_3^E(\mathcal{K}) = 2 \sum_{\substack{i_1 \in (e^+, g^+) \\ i_2 \in (f^+, e^-)}} \epsilon(i_1, i_2) - 2 \sum_{\substack{i_1 \in (g^-, f^-) \\ i_2 \in (f^+, e^-)}} \epsilon(i_1, i_2) - 2 \sum_{\substack{i_1 \in (g^-, f^-) \\ i_2 \in (e^+, g^+)}} \epsilon(i_1, i_2),$$

$$\begin{aligned}
\chi_4^B(\mathcal{K}') - \chi_4^B(\mathcal{K}) = & - \sum_{\substack{i_1, i_2 \in (e^+, g^+) \\ i_3, i_4 \in (f^+, e^-)}} [\epsilon(i_1, i_4)\epsilon(i_2, i_3) - \epsilon(i_1, i_3)\epsilon(i_2, i_4) - \epsilon(i_1, i_3)] \\
& + \sum_{\substack{i_1, i_2 \in (g^-, f^-) \\ i_3, i_4 \in (e^+, g^+)}} [\epsilon(i_1, i_4)\epsilon(i_2, i_3) - \epsilon(i_1, i_3)\epsilon(i_2, i_4) + \epsilon(i_1, i_3)] \\
& + \sum_{\substack{i_1, i_2 \in (g^-, f^-) \\ i_3, i_4 \in (e^+, g^+)}} [\epsilon(i_1, i_4)\epsilon(i_2, i_3) - \epsilon(i_1, i_3)\epsilon(i_2, i_4) + \epsilon(i_1, i_3)] \\
& + 2 \sum_{\substack{i_1, i_2 \in (g^-, f^-) \\ i_3 \in (e^+, g^+), i_4 \in (f^+, e^-)}} \epsilon(i_1, i_3)\epsilon(i_2, i_4)
\end{aligned}$$

$$\begin{aligned}
\chi_4^C(\mathcal{K}') - \chi_4^C(\mathcal{K}) = & - \sum_{\substack{i_1, i_2 \in (g^-, f^-) \\ i_3 \in (e^+, g^+), i_4 \in (f^+, e^-)}} \epsilon(i_1, i_3)\epsilon(i_2, i_4) + \sum_{\substack{i_2, i_3 \in (e^+, g^+) \\ i_1 \in (g^-, f^-), i_4 \in (f^+, e^-)}} \epsilon(i_1, i_3)\epsilon(i_2, i_4) \\
& + \sum_{\substack{i_3, i_4 \in (f^+, e^-) \\ i_1 \in (g^-, f^-), i_2 \in (e^+, g^+)}} \epsilon(i_1, i_3)\epsilon(i_2, i_4) - \sum_{\substack{i_1, i_2 \in (g^-, f^-) \\ i_3, i_4 \in (e^+, g^+)}} \epsilon(i_1, i_4)\epsilon(i_2, i_3) \\
& + \sum_{\substack{i_1, i_2 \in (e^+, g^+) \\ i_3, i_4 \in (f^+, e^-)}} \epsilon(i_1, i_4)\epsilon(i_2, i_3) - \sum_{\substack{i_1, i_2 \in (g^-, f^-) \\ i_3, i_4 \in (f^+, e^-)}} \epsilon(i_1, i_4)\epsilon(i_2, i_3)
\end{aligned}$$

$$\chi_4^D(\mathcal{K}') - \chi_4^D(\mathcal{K}) = 2 \sum_{\substack{i_1, i_2 \in (e^+, g^+) \\ i_3, i_4 \in (f^+, e^-)}} \epsilon(i_1, i_4)\epsilon(i_2, i_3) + \sum_{\substack{i_1, i_2, i_3 \in (e^+, g^+) \\ i_4 \in (f^+, e^-)}} \epsilon(i_1, i_3)\epsilon(i_2, i_4)$$

$$
\begin{aligned}
&+ \sum_{\substack{i_1\in(e^+,g^+)\\ i_2,i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) - \sum_{\substack{i_1,i_2,i_3\in(g^-,f^-)\\ i_4\in(e^+,g^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)\\
&- \sum_{\substack{i_1\in(g^-,f^-)\\ i_2,i_3,i_4\in(e^+,g^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) - \sum_{\substack{i_1,i_2,i_3\in(g^-,f^-)\\ i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)\\
&- \sum_{\substack{i_1\in(g^-,f^-)\\ i_2,i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),
\end{aligned}
$$

$$
\begin{aligned}
\chi_4^E(\mathcal{K}') - \chi_4^E(\mathcal{K}) &= -2 \sum_{\substack{i_1,i_2\in(g^-,f^-)\\ i_3\in(e^+,g^+),i_4\in(f^+,e^-)}} [\epsilon(i_1,i_3)\epsilon(i_2,i_4) - \epsilon(i_1,i_4)\epsilon(i_2,i_3)]\\
&+ 2 \sum_{\substack{i_1,i_2\in(e^+,g^+)\\ i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) - \sum_{\substack{i_1,i_2,i_3\in(e^+,g^+)\\ i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)\\
&- \sum_{\substack{i_1\in(e^+,g^+)\\ i_2,i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) + \sum_{\substack{i_1,i_2,i_3\in(g^-,f^-)\\ i_4\in(e^+,g^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)\\
&+ \sum_{\substack{i_1\in(g^-,f^-)\\ i_2,i_3,i_4\in(e^+,g^+)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) + \sum_{\substack{i_1,i_2,i_3\in(g^-,f^-)\\ i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)\\
&+ \sum_{\substack{i_1\in(g^-,f^-)\\ i_2,i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4),
\end{aligned}
$$

$$
\begin{aligned}
\chi_4^F(\mathcal{K}') - \chi_4^F(\mathcal{K}) &= \sum_{\substack{i_1,i_2\in(g^-,f^-)\\ i_3\in(e^+,g^+),i_4\in(f^+,e^-)}} [\epsilon(i_1,i_3)\epsilon(i_2,i_4) - 3\,\epsilon(i_1,i_4)\epsilon(i_2,i_3)]\\
&+ \sum_{\substack{i_2,i_3\in(e^+,g^+)\\ i_1\in(g^-,f^-),i_4\in(f^+,e^-)}} [\epsilon(i_1,i_4)\epsilon(i_2,i_3) - \epsilon(i_1,i_3)\epsilon(i_2,i_4)]
\end{aligned}
$$

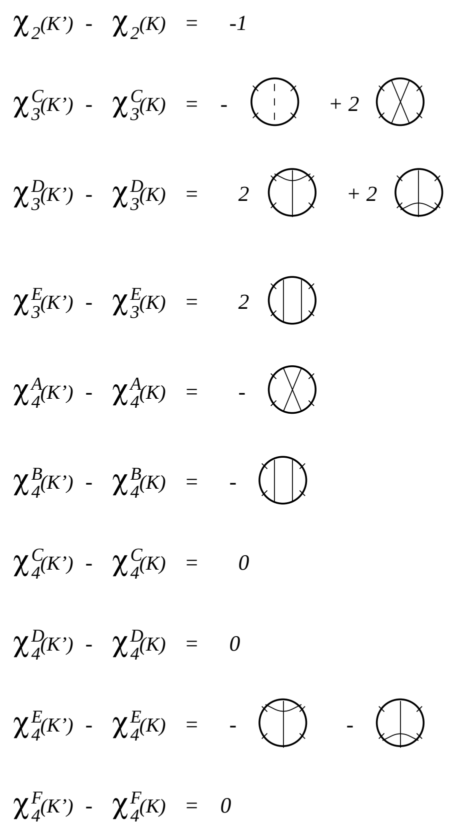


Figure 16: Behaviour of crossing functions under Reidemeister IIA.

$$
\begin{aligned}
&+ \sum_{\substack{i_3,i_4\in(f^+,e^-)\\ i_1\in(g^-,f^-),i_2\in(e^+,g^+)}} [\epsilon(i_1,i_4)\epsilon(i_2,i_3)-\epsilon(i_1,i_3)\epsilon(i_2,i_4)] \\
&+ \ 2\sum_{\substack{i_1,i_2,i_3\in(e^+,g^+)\\ i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4) + \ 2\sum_{\substack{i_1\in(e^+,g^+)\\ i_2,i_3,i_4\in(f^+,e^-)}} \epsilon(i_1,i_3)\epsilon(i_2,i_4)
\end{aligned}
\tag{5.7}
$$

All the variations in the previous equations possess simple diagrammatic expressions. They have been depicted in figs. 16, 17 and 18.

Applying the formulae for the variations to the expressions (4.92) and (4.93) for α_{42} and α_{43}, we obtain the transformation under the moves of all the terms containing only crossing numbers. The behaviour of the terms containing sums over $\mathcal{C}_a$ and $\mathcal{C}_b$ has to be studied separately. One has to

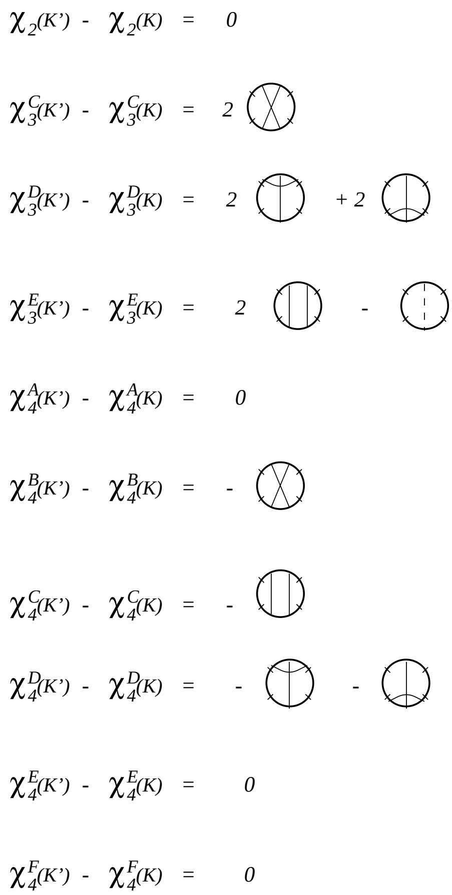


Figure 17: Behaviour of crossing functions under Reidemeister IIB.

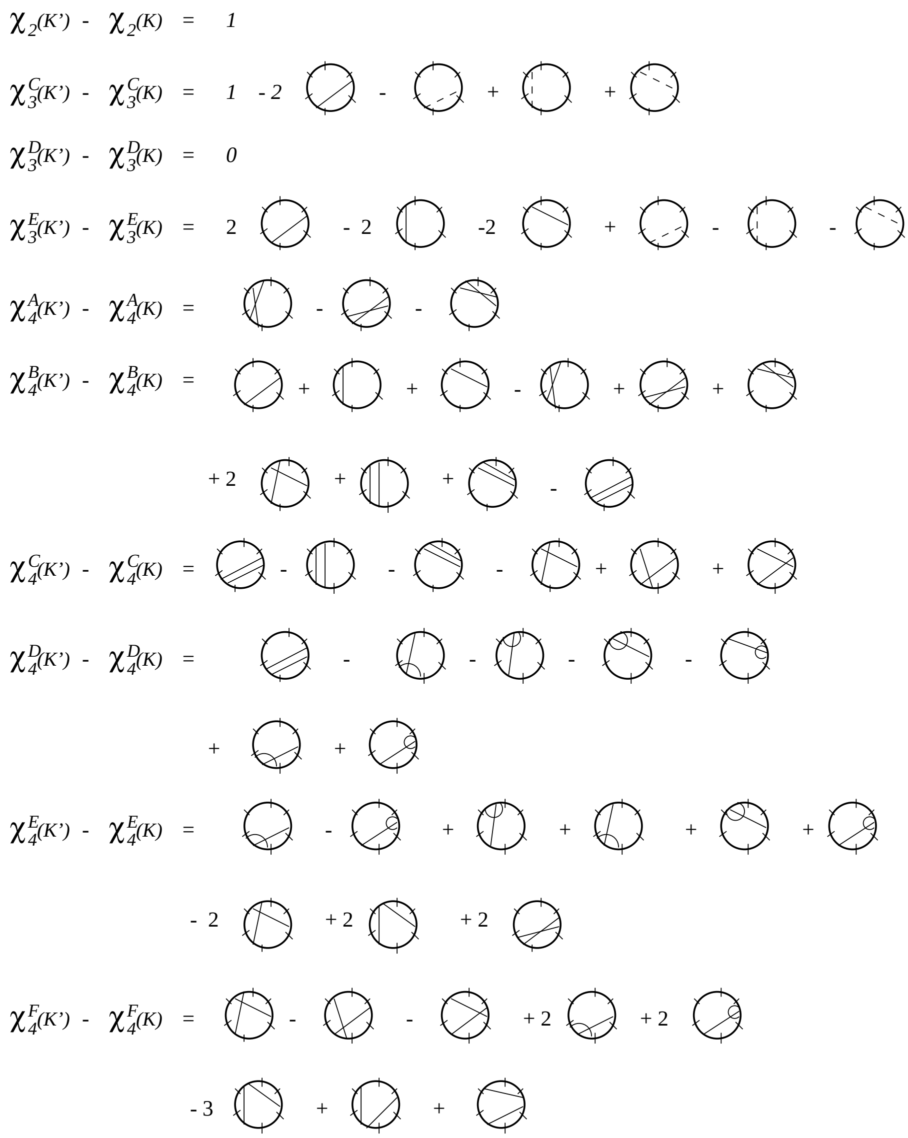


Figure 18: Behaviour of crossing functions under Reidemeister III.

analyse the three different cases: both crossings i and j belong to the set affected by the moves, only one of them, or none. Recall that these sums are essentially made out of the crossing number χ_2^A evaluated in the ascending diagram of different closed pieces of the knot. In order to clarify the analysis, let us reproduce those terms here:

$$
\begin{aligned}
I_1(\mathcal{K}) \;&=\; \sum_{i>j\in\mathcal{C}_a} [\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))] \Big\{ \chi_2^A(\alpha(\mathcal{K})) \\
&-\; \chi_2^A(\alpha(\mathcal{K}_{i_+})) - \chi_2^A(\alpha(\mathcal{K}_{i_-})) - \chi_2^A(\alpha(\mathcal{K}_{j_+})) - \chi_2^A(\alpha(\mathcal{K}_{j_-})) \\
&+\; \chi_2^A(\alpha(\mathcal{K}_{ij_a})) + \chi_2^A(\alpha(\mathcal{K}_{ij_b}))\Big\}, \qquad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
I_2(\mathcal{K}) \;&=\; \sum_{i>j\in\mathcal{C}_a} [\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))] \Big\{ 3\,\chi_2^A(\alpha(\mathcal{K})) \\
&-\; 2\chi_2^A(\alpha(\mathcal{K}_{i_+})) - 2\chi_2^A(\alpha(\mathcal{K}_{i_-})) - 2\chi_2^A(\alpha(\mathcal{K}_{j_+})) - 2\chi_2^A(\alpha(\mathcal{K}_{j_-})) \\
&+\; \chi_2^A(\alpha(\mathcal{K}_{ij_a})) + \chi_2^A(\alpha(\mathcal{K}_{ij_b})\Big\}, \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
I_3(\mathcal{K}) \;&=\; \sum_{i>j\in\mathcal{C}_b} [\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))] \Big\{ \chi_2^A(\alpha(\mathcal{K})) \\
&-\; \chi_2^A(\alpha(\mathcal{K}_{i_+})) - \chi_2^A(\alpha(\mathcal{K}_{i_-})) - \chi_2^A(\alpha(\mathcal{K}_{j_+})) - \chi_2^A(\alpha(\mathcal{K}_{j_-})) \\
&+\; \chi_2^A(\alpha(\mathcal{K}_{ij_c})) + \chi_2^A(\alpha(\mathcal{K}_{ij_d})) + \chi_2^A(\alpha(\mathcal{K}_{ij_e}))\Big\}. \qquad (5.10)
\end{aligned}
$$

These three expressions possess the same structure, so we will refer to them in the following compact way, whenever we do not need to take into account particular details:

$$
I_k(\mathcal{K}) = \sum_{i>j\in c(k)} [\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))]\, F_{ij}^k(\mathcal{K}), \qquad (5.11)
$$

with $k = 1, 2, 3$, F_{ij}^k standing for the combination of functions entering a given sum. The superindex in F_{ij}^k denotes that this combination depends on the sum, and the subindexes that it also depends on the pair of crossings. The set over which the sum is taken is specified by $c(k)$, where $c(k) = \mathcal{C}_a$ for $k = 1, 2$ and $c(k) = \mathcal{C}_b$ for $k = 3$.

Under R-I the variation of all these sums is trivially zero for the same reasons as stated above: as the crossing e involved in this move is isolated, there is no other crossing that could give a contribution to any of the χ_2^A functions. As we have already seen, the variation of all the other terms in $\alpha_{42}(K)$ and $\alpha_{43}(K)$ also vanishes; then it follows trivially that our formulae (4.92) and (4.93) are invariant under this move.

Under R-IIA or R-IIB the general behaviour of the sums can be written as:

$$
\begin{aligned}
I_k(\mathcal{K}') - I_k(\mathcal{K}) \;&=\; \sum_{\substack{i>j\neq\{e,f\}\\ i,j\in c(k)}} [\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))]\,(F^k_{ij}(\mathcal{K}') - F^k_{ij}(\mathcal{K})) \\
&+ \sum_{\substack{i\neq\{e,f\}\\ i,e\in c(k)}} \epsilon_e[\epsilon_i(\mathcal{K}) - \epsilon_i(\alpha(\mathcal{K}))]\,F^k_{ie}(\mathcal{K}') \\
&+ \sum_{\substack{i\neq\{e,f\}\\ i,f\in c(k)}} \epsilon_f[\epsilon_i(\mathcal{K}) - \epsilon_i(\alpha(\mathcal{K}))]\,F^k_{if}(\mathcal{K}').
\end{aligned}
\tag{5.12}
$$

Notice that in this equation there are no additional terms proportional to $\epsilon_e\epsilon_f$ because the diagram in fig. 15 has been chosen so that

$$
\epsilon_e\epsilon_f(\mathcal{K}) - \epsilon_e\epsilon_f(\alpha(\mathcal{K})) = 0. \tag{5.13}
$$

Recall that all the crossings $i, j \neq \{e, f\}$ do not change when going from $\mathcal{K}'$ to $\mathcal{K}$. In the last two terms, there is no subtraction of the function $F^k(\mathcal{K})$ because the crossings e and f are not present in $\mathcal{K}$.

After these general comments on (5.11) we will evaluate (5.12) and then we will find out the behaviour of (5.8), (5.9) and (5.10) under the second Reidemeister move. We need to specify the two labellings on each crossing, as in the case of the crossing numbers, to distinguish on which section of $\mathcal{K}$ they lay. We will write the signature function as given in (4.5) and the labels will fulfil the ordering $i_1 < i_2 < i_3 < i_4$. Under R-IIA we find:

$$
I_1(\mathcal{K}') - I_1(\mathcal{K}) \;=\; -\sum_{\substack{i_1,i_2\in(f^-,e^+)\\ i_3,i_4\in(f^+,e^-)}} \Big[\epsilon(i_1,i_3)\epsilon(i_2,i_4)(\mathcal{K}) - \epsilon(i_1,i_3)\epsilon(i_2,i_4)(\alpha(\mathcal{K}))\Big],
$$

$$
\begin{aligned}
I_2(\mathcal{K}') - I_2(\mathcal{K}) &= \Big\{ -3 \sum_{\substack{i_1,i_2\in(f^-,e^+) \\ i_3,i_4\in(f^+,e^-)}} - \sum_{\substack{i_1,i_2,i_3\in(f^-,e^+) \\ i_4\in(f^+,e^-)}} - \sum_{\substack{i_1\in(f^-,e^+) \\ i_2,i_3,i_4\in(f^+,e^-)}} \Big\} \\
&\times \Big[\epsilon(i_1,i_3)\epsilon(i_2,i_4)(\mathcal{K}) - \epsilon(i_1,i_3)\epsilon(i_2,i_4)(\alpha(\mathcal{K}))\Big], \\
I_3(\mathcal{K}') - I_3(\mathcal{K}) &= -\sum_{\substack{i_1,i_2\in(f^-,e^+) \\ i_3,i_4\in(f^+,e^-)}} \Big[\epsilon(i_1,i_4)\epsilon(i_2,i_3)(\mathcal{K}) - \epsilon(i_1,i_4)\epsilon(i_2,i_3)(\alpha(\mathcal{K}))\Big],
\end{aligned} \tag{5.14}
$$

and, under R-IIB:

$$
\begin{aligned}
I_1(\mathcal{K}') - I_1(\mathcal{K}) &= -\sum_{\substack{i_1,i_2\in(f^-,f^+) \\ i_3,i_4\in(e^+,e^-)}} \Big[\epsilon(i_1,i_3)\epsilon(i_2,i_4)(\mathcal{K}) - \epsilon(i_1,i_3)\epsilon(i_2,i_4)(\alpha(\mathcal{K}))\Big], \\
I_2(\mathcal{K}') - I_2(\mathcal{K}) &= -\sum_{\substack{i_1,i_2\in(f^-,f^+) \\ i_3,i_4\in(e^+,e^-)}} \Big[\epsilon(i_1,i_3)\epsilon(i_2,i_4)(\mathcal{K}) - \epsilon(i_1,i_3)\epsilon(i_2,i_4)(\alpha(\mathcal{K}))\Big], \\
I_3(\mathcal{K}') - I_3(\mathcal{K}) &= 0.
\end{aligned} \tag{5.15}
$$

The invariance under the second Reidemeister move of $\alpha_{42}(K)$ in (4.92) and $\alpha_{43}(K)$ in (4.93) then follows, after summing up the contributions coming from the crossing numbers in (5.5) and (5.6), and the expressions (5.14) and (5.15), respectively, and finding out that they cancel.

To prove the invariance under R-III we will study first the behaviour of the I_k sums (5.8–5.10) under R-III. As in the previous case, we start by writing down the general structure of their variation:

$$
\begin{aligned}
I_k(\mathcal{K}') - I_k(\mathcal{K}) &= \sum_{\substack{i>j\neq\{e,f,g\} \\ i,j\in c(k)}} [\epsilon_i\epsilon_j(\mathcal{K}) - \epsilon_i\epsilon_j(\alpha(\mathcal{K}))]\,(F^k_{ij}(\mathcal{K}') - F^k_{ij}(\mathcal{K})) \\
&+ \sum_{\substack{i\neq\{e,f,g\} \\ i,e\in c(k)}} \epsilon_e[\epsilon_i(\mathcal{K}) - \epsilon_i(\alpha(\mathcal{K}))]\,(F^k_{ie}(\mathcal{K}') - F^k_{ie}(\mathcal{K})) \\
&+ \sum_{\substack{i\neq\{e,f,g\} \\ i,f\in c(k)}} \epsilon_f[\epsilon_i(\mathcal{K}) - \epsilon_i(\alpha(\mathcal{K}))]\,(F^k_{if}(\mathcal{K}') - F^k_{if}(\mathcal{K})) \\
&+ \sum_{\substack{i\neq\{e,f,g\} \\ i,g\in c(k)}} \epsilon_g[\epsilon_i(\mathcal{K}) - \epsilon_i(\alpha(\mathcal{K}))]\,(F^k_{ig}(\mathcal{K}') - F^k_{ig}(\mathcal{K})).
\end{aligned} \tag{5.16}
$$

Again, the terms proportional to $\epsilon_e\epsilon_f$, $\epsilon_e\epsilon_g$ or $\epsilon_g\epsilon_f$ do not contribute because their signature values do not vary when going from $\mathcal{K}$ to $\alpha(\mathcal{K})$. The computation of the variation of the different F^k functions appearing in (5.16) is more complicated than before. Instead of changes in the crossings involved in the move, we are now dealing with a change in the configuration of the three crossings affected by the move. This implies that many of the crossings contributing to the functions χ_2^A change. For example, the value of the subtraction $\chi_2^A(\alpha(\mathcal{K}'_{ij_a})) - \chi_2^A(\alpha(\mathcal{K}_{ij_a}))$ (or any other of the functions evaluated in the splitt knot) depends on the sections of the knot in which the crossings i and j lay in between. Let us work out some examples. In these examples, we will specify both labels of the crossings: $i_1 < i_2$ for one and $j_1 < j_2$ for the other.

If a crossing happens to have all the labels in the region (e^-, e^+) we find:

$$\chi_2^A(\alpha(\mathcal{K}'_{ij_a})) - \chi_2^A(\alpha(\mathcal{K}_{ij_a})) = -\epsilon_e\epsilon_f - \epsilon_e\epsilon_g - \epsilon_g\epsilon_f = 1, \tag{5.17}$$

while if the situation is $i_1 \in (e^-, e^+)$ and $i_2, j_1, j_2 \in (f^-, f^+)$:

$$\chi_2^A(\alpha(\mathcal{K}'_{ij_a})) - \chi_2^A(\alpha(\mathcal{K}_{ij_a})) = 0. \tag{5.18}$$

In some other cases there are apparently non-trivial contributions linear in the signature of the crossings e, f or g. An example is $\chi_2^A(\alpha(\mathcal{K}'_{ie_a})) - \chi_2^A(\alpha(\mathcal{K}_{ie_a}))$ for some crossing i such that $i, e \in \mathcal{C}_a$ and whose two labels, $i_1 < i_2$, lay in the following knot regions: $i_1 \in (e^-, e^+)$ and $i_2 \in (g^+, g^-)$. We then find that:

$$\chi_2^A(\alpha(\mathcal{K}'_{ie_a})) - \chi_2^A(\alpha(\mathcal{K}_{ie_a})) = \epsilon_f \Big[-\sum_{\substack{j_1 \in (e^-, i_1) \\ j_2 \in (f^-, f^+)}} \epsilon(j_1, j_2)(\alpha(\mathcal{K})) + \sum_{\substack{j_1 \in (f^-, f^+) \\ j_2 \in (g^+, i_2)}} \epsilon(j_1, j_2)(\alpha(\mathcal{K}))\Big] - \epsilon_g \Big[\sum_{\substack{j_1 \in (e^-, i_1) \\ j_2 \in (f^-, f^+)}} \epsilon(j_1, j_2)(\alpha(\mathcal{K})) - \sum_{\substack{j_1 \in (f^-, f^+) \\ j_2 \in (g^+, i_2)}} \epsilon(j_1, j_2)(\alpha(\mathcal{K}))\Big]. \tag{5.19}$$

The minus sign in front of some of the terms inside the brackets is due to the fact that in order to close the knot $\mathcal{K}_{ie_a}$, we had to reverse the orientation in some piece of the original knot; this implies a change of sign in the signature functions affected by this reversing. In this example we are reversing the orientation of the region (e^-, i_1). The key point is to notice that to the contributions inside the brackets sum up to the linking number between some specific knots and that this linking number is always zero:

$$\begin{aligned}\chi_2^A(\alpha(\mathcal{K}'_{ie_a})) - \chi_2^A(\alpha(\mathcal{K}_{ie_a})) &= \epsilon_f \cdot \mathcal{L}(\alpha(\mathcal{K}^{f_+}_{ie_a}), \alpha(\mathcal{K}^{f_-}_{ie_a})) \\ &- \epsilon_g \cdot \mathcal{L}(\alpha(\mathcal{K}^{g_+}_{ie_a}), \alpha(\mathcal{K}^{g_-}_{ie_a})) = 0, \quad (5.20)\end{aligned}$$

where $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$ stands for the linking number between $\mathcal{K}_1$ and $\mathcal{K}_2$. In the first term the knots are the two pieces into which the knot $\alpha(\mathcal{K}_{ie_a})$ is divided when splitting the f-crossing, and in the second those obtained after the splitting of the g-crossing. As the knot $\alpha(\mathcal{K}_{ie_a})$ is just an ascending diagram, these two pieces lay one on top of the other, and so their linking number is zero. This kind of argument can be applied to other contributions of the same type (*i.e.* for other choices of crossings in the sums I_k and other χ_2^A functions appearing in them). The computation of all the contributions to (5.16) of the sums (5.8 − 5.10) can now be done without difficulty, leading to the following formulae for their behaviour under R-III (where again $i_1 < i_2 < i_3 < i_4$):

$$I_1(\mathcal{K}') - I_1(\mathcal{K}) = \Big\{ \sum_{\substack{i_3,i_4 \in (g^+,g^-) \\ i_1 \in (e^-,e^+), i_2 \in (f^-,f^+)}} + \sum_{\substack{i_1 \in (e^-,e^+), i_4 \in (g^+,g^-) \\ i_2,i_3 \in (f^-,f^+)}} + \sum_{\substack{i_1,i_2 \in (e^-,e^+) \\ i_3 \in (f^-,f^+), i_4 \in (g^+,g^-)}} \Big\}$$

$$\times\ \epsilon(i_1,i_3)\epsilon(i_2,i_4) \ + 2 \sum_{\substack{i_2 \in (g^+,g^-) \\ i_1 \in (f^-,f^+)}} \epsilon(i_1,i_2),$$

$$I_2(\mathcal{K}') - I_2(\mathcal{K}) = 3 \Big\{ \sum_{\substack{i_3,i_4 \in (g^+,g^-) \\ i_1 \in (e^-,e^+), i_2 \in (f^-,f^+)}} + \sum_{\substack{i_1 \in (e^-,e^+), i_4 \in (g^+,g^-) \\ i_2,i_3 \in (f^-,f^+)}} + \sum_{\substack{i_1,i_2 \in (e^-,e^+) \\ i_3 \in (f^-,f^+), i_4 \in (g^+,g^-)}} \Big\}$$

$$\times\ \epsilon(i_1,i_3)\epsilon(i_2,i_4) + \Big\{ \sum_{\substack{i_1 \in (e^-,e^+) \\ i_2,i_3,i_4 \in (f^-,f^+)}} + \sum_{\substack{i_2,i_3,i_4 \in (g^+,g^-) \\ i_1 \in (f^-,f^+)}} + \sum_{\substack{i_1 \in (e^-,e^+) \\ i_2,i_3,i_4 \in (g^+,g^-)}} \Big\} \epsilon(i_1,i_3)\epsilon(i_2,i_4)$$

$$+ \Big\{ \sum_{\substack{i_4 \in (f^-,f^+) \\ i_1,i_2,i_3 \in (e^-,e^+)}} + \sum_{\substack{i_1,i_2,i_3 \in (f^-,f^+) \\ i_4 \in (g^+,g^-)}} + \sum_{\substack{i_4 \in (g^+,g^-) \\ i_1,i_2,i_3 \in (e^-,e^+)}} \Big\} \epsilon(i_1,i_3)\epsilon(i_2,i_4) + \ 6 \sum_{\substack{i_2 \in (g^+,g^-) \\ i_1 \in (f^-,f^+)}} \epsilon(i_1,i_2),$$

$$I_3(\mathcal{K}') - I_3(\mathcal{K}) = \Big\{ \sum_{\substack{i_3,i_4 \in (g^+,g^-) \\ i_1 \in (e^-,e^+), i_2 \in (f^-,f^+)}} + \sum_{\substack{i_1,i_2 \in (e^-,e^+) \\ i_3 \in (f^-,f^+), i_4 \in (g^+,g^-)}} \Big\} \epsilon(i_1,i_4)\epsilon(i_2,i_3)$$

$$+ \sum_{\substack{i_1 \in (e^-, e^+), i_4 \in (g^+, g^-) \\ i_2, i_3 \in (f^-, f^+)}} \epsilon(i_1, i_2)\epsilon(i_3, i_4) \tag{5.21}$$

$$+ \Big\{ \sum_{\substack{i_1, i_2 \in (e^-, e^+) \\ i_3, i_4 \in (f^-, f^+)}} + \sum_{\substack{i_3, i_4 \in (g^+, g^-) \\ i_1, i_2 \in (f^-, f^+)}} + \sum_{\substack{i_1, i_2 \in (e^-, e^+) \\ i_3, i_4 \in (g^+, g^-)}} \Big\} \epsilon(i_1, i_4)\epsilon(i_2, i_3)$$

Taking into account (5.21) and the behaviour under R-III of the crossing numbers given in (5.7), one can see that all the terms appearing in computing the variation of (4.92) and (4.93) under R-III cancel, and thus the topological invariance of the combinatorial expressions (4.92) and (4.93) for $\alpha_{42}(K)$ and $\alpha_{43}(K)$ is established.

6 Conclusions

In this paper we have analysed Chern-Simons gauge theory in the temporal gauge. The main outcome of our work is that we have shown that this gauge is particularly well suited to obtain combinatorial expressions for Vassiliev invariants. These are much simpler than the integral expressions obtained in covariant gauges or the ones leading to Kontsevich integrals which emerge in the light-cone gauge.

One of the crucial ingredients of our work is the observation that in the temporal gauge all the signature-dependent parts of the invariant can be easily extracted. In fact we have obtained an explicit general expression for the leading signature-dependent terms. These terms are the ones in the expansion (3.8) and constitute the kernels of the Vassiliev invariants. The kernels are not Vassiliev invariants. Different kernels may belong to the same knot, but they are well defined on knot projections. As an order-n Vassiliev invariant, an order-n kernel vanish in signed sums of order $n+1$. The kernel is the only part of the order-n Vassiliev invariant that in general does not vanishes for signed sums of order less than $n+1$. In other words, an order-n kernel differs from an order-n Vassiliev invariant by terms which vanish in signed sums of order n.

The kernels contain a large amount of information about the Vassiliev invariants. We have shown how the full invariants can be reconstructed from them. The two main ingredients of the reconstruction procedure are the factorization theorem and the structure of the perturbative series expansion of the vacuum expectation value of the Wilson loop in the temporal gauge. The key observation of the reconstruction procedure is that combinatorial expressions can be obtained without actually performing any of the D-integrals. All these integrals are solved in terms of the kernels using the series of relations provided by the factorization theorem.

In our analysis we have carried along the unknown function $b(\mathcal{K})$, obtaining a series of consistency relations for it. These relations are necessary conditions to have knot invariants. We have shown that these relations possess a simple solution, similar to the one that must be introduced in the light-cone gauge. It would be very helpful to understand the origin of this function in the context of Chern-Simons gauge theory, and to prove that, indeed, our ansatz is correct. The same type of problem has not been solved in the light-cone gauge.

The reconstruction procedure has been performed up to order four. We have obtained known combinatorial expressions at orders two and three, and new ones for the two primitive Vassiliev invariants present at order four. The form of these combinatorial expressions suggests some general structure. For example, it seems that at even orders the quantities that enter the combinatorial expressions are paired, one of the terms for the diagram associated to $\mathcal{K}$, and another for the corresponding ascending diagram with opposite sign. In addition, the terms involved in the splitting of a knot are evaluated on the ascending diagram. For odd orders, crossing numbers seem not to be accompanied by their ascending-diagram counterparts. However, as in the even-order cases, the terms involved in the splitting are evaluated in the ascending diagram.

We have successfully applied the reconstruction procedure up to order four, obtaining new combinatorial expressions for Vassiliev invariants. The question to ask now is if the procedure can be generalized to higher orders. We conjecture that this can be done. Certainly, the complexity of the combinatorial expressions will increase with the order, but it would be very important to establish if the procedure would work at any order. In other words, it would be very important to possess a reconstruction theorem which would guarantee that from the kernels (3.8) and the factorization theorem, we can solve for all the D-integrals present at each order. Provided we know a basis of primitive group factors, this would imply that there exists a systematic algorithm to obtain combinatorial expressions for Vassiliev invariants at any order.

Acknowledgements

We would like to thank M. Alvarez for helpful discussions. This work was supported in part by DGICYT under grant PB97-0960, and by the EU Commission under TMR grant FMAX-CT96-0012.

APPENDIX

In this appendix we present the values of the primitive Vassiliev invariants at orders two, three and four for all prime knots up to nine crossings. These have been computed, with the aid of a Mathematica algorithm, using the formulae (4.23), (4.50), (4.92) and (4.93). In the tables 1 and 2 we present the value of these invariants once their value for the unknot has been substracted. In other words, the $\alpha_{ij}(K)$ shown in the tables are the result of the replacement:

$$\alpha_{ij}(K) \longrightarrow \alpha_{ij}(K) - \alpha_{ij}(U).$$

The values for the unknot primitive invariants up to order four are, in the normalization and basis that we used:

$$\begin{array}{ll} \alpha_{21}(U) = -\frac{1}{6}, & \alpha_{31}(U) = 0, \\ \alpha_{42}(U) = \frac{1}{360}, & \alpha_{43}(U) = -\frac{1}{360}. \end{array}$$

Knot	α_{21}	α_{31}	α_{42}	α_{43}
3_1	4	8	62/3	10/3
4_1	-4	0	34/3	14/3
5_1	12	40	174	26
5_2	8	24	268/3	44/3
6_1	-8	-8	116/3	52/3
6_2	-4	-8	34/3	38/3
6_3	4	0	14/3	$-14/3$
7_1	24	112	684	100
7_2	12	48	222	34
7_3	20	88	1510/3	242/3
7_4	16	64	1016/3	184/3
7_5	16	64	968/3	136/3
7_6	4	16	158/3	34/3
7_7	-4	8	$-14/3$	$-10/3$
8_1	-12	-24	66	38
8_2	0	-8	0	24
8_3	-16	0	520/3	200/3
8_4	-12	8	114	54
8_5	-4	-24	$-62/3$	86/3
8_6	-8	-24	68/3	100/3
8_7	8	-16	124/3	$-28/3$
8_8	8	-8	124/3	$-4/3$
8_9	-8	0	212/3	124/3
8_{10}	12	-24	110	10
8_{11}	-4	-16	$-14/3$	62/3
8_{12}	-12	0	82	30
8_{13}	4	-8	14/3	$-38/3$
8_{14}	0	0	16	16
8_{15}	16	56	776/3	112/3
8_{16}	4	-8	14/3	$-38/3$
8_{17}	-4	0	82/3	62/3
8_{18}	4	0	62/3	34/3
8_{19}	20	80	1270/3	170/3
8_{20}	8	16	172/3	20/3
8_{21}	0	-8	-16	8

Table 1: Primitive Vassiliev invariants up to order four for all prime knots up to eight crossings.

Knot	α_{21}	α_{31}	α_{42}	α_{43}
9_1	40	240	5660/3	820/3
9_2	16	80	1304/3	184/3
9_3	36	208	1578	246
9_4	28	152	3122/3	502/3
9_5	24	120	780	140
9_6	28	144	2834/3	382/3
9_7	20	96	1654/3	218/3
9_8	0	16	48	16
9_9	32	176	3760/3	560/3
9_{10}	32	176	3856/3	656/3
9_{11}	16	−72	1160/3	208/3
9_{12}	4	24	302/3	58/3
9_{13}	28	144	2930/3	478/3
9_{14}	−4	16	−110/3	−34/3
9_{15}	8	−40	508/3	92/3
9_{16}	24	112	668	84
9_{17}	−8	0	116/3	28/3
9_{18}	24	120	748	108
9_{19}	−8	8	68/3	4/3
9_{20}	8	32	412/3	68/3
9_{21}	12	−48	238	50
9_{22}	−4	8	34/3	−10/3
9_{23}	20	88	1462/3	194/3
9_{24}	4	16	110/3	34/3
9_{25}	0	8	64	24

Knot	α_{21}	α_{31}	α_{42}	α_{43}
9_{26}	0	8	−32	−8
9_{27}	0	8	16	8
9_{28}	4	0	−34/3	−14/3
9_{29}	4	−16	62/3	−14/3
9_{30}	−4	−8	82/3	38/3
9_{31}	8	16	172/3	20/3
9_{32}	−4	16	−62/3	14/3
9_{33}	4	−8	62/3	10/3
9_{34}	−4	0	34/3	14/3
9_{35}	28	144	3026/3	574/3
9_{36}	12	−56	270	42
9_{37}	−12	8	82	22
9_{38}	24	112	684	100
9_{39}	8	−32	460/3	116/3
9_{40}	−4	−8	34/3	38/3
9_{41}	0	8	−48	−24
9_{42}	−8	0	164/3	76/3
9_{43}	4	16	254/3	82/3
9_{44}	0	8	0	−8
9_{45}	8	−32	412/3	68/3
9_{46}	−8	−24	20/3	52/3
9_{47}	−4	−16	−110/3	−34/3
9_{48}	12	−40	190	42
9_{49}	24	112	700	116

Table 2: Primitive Vassiliev invariants up to order four for all prime knots with nine crossings.

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