CERN-9894, LPTHE-9818, NYU-9809
hep-th/9807024
June 16, 1998

Renormalizable Non-Covariant Gauges and Coulomb Gauge Limit

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Abstract

To study "physical" gauges such as the Coulomb, light-cone, axial or temporal gauge, we consider "interpolating" gauges which interpolate linearly between a covariant gauge, such as the Feynman or Landau gauge, and a physical gauge. Lorentz breaking by the gauge-fixing term of interpolating gauges is controlled by extending the BRST method to include not only the local gauge group, but also the global Lorentz group. We enumerate the possible divergences of interpolating gauges, and show that they are renormalizable, and we show that the expectation value of physical observables is the same as in a covariant gauge. In the second part of the article we study the Coulomb-gauge as the singular limit of the Landau-Coulomb interpolating gauge. We find that unrenormalized and renormalized correlation functions are finite in this limit. We also find that there are finite two-loop diagrams of "unphysical" particles that are not present in formal canonical quantization in the Coulomb gauge. We verify that in the same limit, the Gauss-BRST Ward identity holds, which is the functional analog of the operator statement that a BRST transformation is generated by the Gauss-BRST charge. As a consequence, gA_0 is invariant under renormalization, whereas in a covariant gauge, no component of the gluon field has this property.

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§Work supported in part by National Science Foundation Grant no. PHY9520978

1 Introduction

Although different gauges are formally equivalent, some are simpler than others, or may have attractive properties. Covariant gauges are well adapted to perturbative expansion and renormalization. However in QCD we are interested in confinement and eventually in bound-state problems which are inherently non-perturbative. For such puposes, non-covariant gauges such as the Coulomb gauge, the Weyl gauge, the axial gauge or the light-front gauge may be attractive. These gauges are considered "physical" in the sense that the space of states is believed to be unitary and does not involve ghosts. (For a discussion of various gauges see [1].) Indeed non-covariant gauges such as the Coulomb gauge and the light-front gauge have recently been used to investigate confinement in QCD [2, **?**].

However it is a fact that at the level of quantum field theory, the well established, renormalizable gauges for QCD are, on the one hand, covariant, and on the other, involve "unphysical" particles. These are the fermi-ghosts that are needed to cancel the unphysical gluon degrees of freedom. One would like to know whether or not the physical gauges really exist in the sense of perturbatively renormalizable quantum field theories, and whether they are really unitary in the sense that they may be expressed without ghosts, in terms of the two transverse degrees of freedom of the gluon. We shall see that for the Coulomb gauge, the answer to the first question is "yes" and to the second, a slightly qualified "no".

The point of view which we adopt in the present article is that the BRST formulation provides a reliable method of quantizing and perturbatively renormalizing non-Abelian gauge theories. (For a review see [4] and [5].) The existence and properties of physical or canonical gauges will be investigated deductively starting from the BRST formulation. To be sure, this inverts the historical order in which gauge theories were first canonically quantized, and subsequently the BRST method was found; however the canonical method has remained heuristic, and to this day does not allow systematic renormalization.

There are two different problems raised by the commonly used "physical gauges": (i) the breaking of Lorentz covariance and (ii) an arbitrariness due to incomplete gauge fixing. For example the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$ obviously breaks Lorentz invariance. It is also an incomplete gauge-fixing in the sense that it leaves a one-parameter family of gauge transformations arbitrary, namely gauge transformations $g(t)$ that are independent of the spatial coordinate $\vec{x}$, but may depend on the time t. Similarly, the Weyl gauge condition $A_0 = 0$ leaves arbitrary a 3-parameter family of gauge transformations, $g(\vec{x})$. We call the dimension σ of this parameter space the "degree of arbitrariness" of the gauge, and we have $\sigma = 1$ for the Coulomb gauge, and $\sigma = 3$ for the Weyl, the axial, and the light-front gauges. Not surprisingly, the degree of arbitrariness of the gauge determines the dimension of the divergences of Feynman integrals that are not controlled by usual ultraviolet regularization.

Strictly speaking, incomplete gauge fixing with $\sigma > 0$ implies that the correlation functions of charged fields actually vanish at generic space-time separation. For example in the Coulomb gauge, the arbitrariness under $g(t)$ implies that the correlation function of two charged fields vanishes at unequal times, $\langle \psi(\vec{x}, t)\psi^*(0,0)\rangle = 0$ for $t \neq 0$, even in abelian gauge theory. This vanishing of correlation functions due to gauge arbitrariness is not what one has in mind by a 'physical' gauge, and it is usually overcome in continuum gauge theory by additional gauge fixing by more or less explicit prescriptions.[1] Incomplete gauge fixing would appear to be the origin of ambiguities that occur in higher loop diagrams [6], and which make the formal Coulomb gauge, defined by canonical quantization after elimination of the Coulomb-gauge constraints, not particularly well-defined. Consequently it is very misleading to speak of *the* Coulomb gauge, as in the question, 'What are the Feynman rules for *the* Coulomb gauge?'. Unless one is willing to accept the vanishing of correlation functions of charged fields at unequal times, this question cannot have a unique answer without further stipulation of the gauge condition. This applies to all gauges with $\sigma > 0$.

We shall deal with both problems, Lorentz breaking and gauge arbitrariness, by the device of an "interpolating" gauge. For example the gauge condition $-a\partial_0 A_0 + \vec{\nabla} \cdot \vec{A} = 0$, with $0 \leq a \leq 1$ interpolates between the Landau gauge, $a = 1$, and the Coulomb gauge, $a = 0$. For $a > 0$ the gauge condition is regular, in the sense that the degree of arbitrariness vanishes, $\sigma = 0$, but Lorentz invariance is broken for $a \neq 1$. This allows one to first address the problem of Lorentz breaking in a regular gauge, and then to see if the singular limit $a \to 0$ yields finite correlation functions. In the present article we shall use and extended BRST symmetry to control the violation of Lorentz invariance, and we shall then study the Coulomb gauge limit of the Landau-Coulomb interpolating gauge.

Use of an interpolating gauge and an extension of BRST symmetry to control the violation of Lorentz invariance, was reviewed by Piguet [7], particularly for the interpolating light-cone gauge. Doust [8] used a gauge which interpolates between the Coulomb and Feynman gauge to regularize the Coulomb gauge, and showed that extra terms in

[1]In lattice gauge theory, gauge-fixing is frequently left incomplete.

the Feynman rules which he obtained in the Coulomb-gauge limit correspond to an additional potential term obtained by Christ and Lee from an operator ordering of their Coulomb Hamiltonian [10]. Difficulties with renormalization in the Coulomb gauge were exhibited by Doust and Taylor [9]. The Weyl gauge ($A_0 = 0$) has been studied by Rossi and Testa [11], and by Cheng and Tsai [12].

As commonly used in non-Abelian gauge theories, BRST-invariance provides a substitute for invariance under local gauge transformations which is broken by the gauge-fixing term. In Lorentz-covariant gauges, one uses the BRST method to enumerate the independent divergent counter-terms necessary to ensure finitness of the renormalized theory, while preserving all requirments of gauge invariance for physical quantities. It is a powerful algebraic method of great generality, relying as it does on the simplicity of invariance under a generator s that is nil-potent $s^2 = 0$.

In the first part of this article, we develop an extension of the BRST method that also provides a substitute for invariance under global Lorentz rotations when the gauge-fixing term breaks global Lorentz invariance as well as local gauge invariance. The method is of considerable generality in that it does not rely on particular properties of the symmetry which is broken by the gauge-fixing term, but only that the symmetry operations form a Lie group, and it allows us to explicitly enumerate all counter-terms.

For the class of interpolating gauges, defined by $(\alpha\partial)^\mu A_\mu = f$, with α a non-singular matrix, the partition function is formally given by the Faddeev-Popov formula

$$Z = \int dA\delta[(\alpha\partial)^\mu A_\mu - f] \det[(\alpha\partial)^\mu D_\mu(A)] \exp[-S_{YM}], \tag{1}$$

where S_{YM} is the Euclidean Yang-Mills action. Feynman graphs contain denominators of the form $k_\mu\alpha^{\mu\nu}k_\nu$ and k^2. As long as α is non-singular, these denominators provide the same degree of convergence in all directions in k-space as the corresponding denominator k^2 in covariant gauges. Consequently in this class of interpolating gauges, power counting of graphs is exactly the same as in Lorentz-covariant gauges. The problem of renormalizability is reduced to an algebraic one of enumerating the form of possible local divergent terms, which we control by extended BRST-invariance. On the contrary, because of gauge arbitrariness in the limiting cases of the Coulomb, light-cone or other singular gauges, the degree of convergence depends on the direction in k-space, and a more detailed analysis is required to determine if the limit is finite.

In the second part of the present article, we analyse the singular Coulomb gauge limit, $a \to 0$, from the Landau-Coulomb interpolating gauge. For this purpose we express the partition function Z as a functional integral in phase space, and then make a linear shift in the field variables in order to exhibit a symmetry (r-symmetry) between the fermi and bose unphysical degrees of freedom. Individual closed fermi-ghost loops and closed unphysical bose loops diverge like $a^{-1/2}$, but they cancel pairwise by virtue of the r-symmetry. Consequently the correlation functions are finite in the limit $a \to 0$ from the Landau-Coulomb gauge. This remains true for the renormalized correlation functions. (See remark 1 at the end of sect. 9.) However we also find that there are one-loop graphs that vanish like $a^{1/2}$, and that are missing in the formal ($a = 0$) Coulomb gauge, but which cannot be neglected because they give a finite contribution when inserted into the graphs that diverge like $a^{-1/2}$. It remains a logical possibility that these two-loop graphs, that are missing in the formal Coulomb gauge, are mere gauge artifacts that decouple from expectation values of all gauge-invariant quantities such as a Wilson loop. However there is at the moment no argument to show that they do. Indeed unless for some reason these two-loop graphs decouple from all physical amplitudes, then the ghosts do not decouple in the Coulomb gauge limit, and the Coulomb gauge is not unitary in the usual sense of being a canonical theory of the transverse gluon degrees of freedom.

Nevertheless we find that correlation functions of the Coulomb-gauge limit of the Landau-Coulomb interpolating gauge do exist, and moreover they display a kind of simplicity that is absent from covariant gauges. A certain Gauss-BRST Ward identity holds in the Coulomb gauge limit which implies, among other things, that the time-time component of the gluon propagator $g^2 D_{00}$ is a renormalization-group invariant and thus depends only on a physical mass, Λ_{QCD}, but not on the ultra-violet cut-off, Λ, nor the renormalization mass, μ, which may make it a useful order parameter for color confinement. No component of the propagator has this property in a covariant gauge.

2 Interpolating Gauges

In this section we introduce interpolating gauges for various familiar classical gauges. The Landau and Coulomb gauges are defined by $-\partial_0 A_0 + \vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \cdot \vec{A} = 0$. The Weyl and axial gauges are frequently defined by $A_0 = 0$ and $A_3 = 0$ respectively. However if periodic boundary conditions are introduced in time or space, the conditions $A_0 = 0$ and $A_3 = 0$ are too strong, and cannot be maintained. For they fix to unity the values of straight-line Wilson loops $\mathrm{tr} P \exp(\int dx^\mu A_\mu)$ that close by periodicity, which however are gauge-invariant objects. We take instead as the Weyl and axial gauge conditions the weaker conditions $\partial_0 A_0 = 0$ and $\partial_3 A_3 = 0$. In momentum space these read $k_0 \tilde{A}_0(k) = 0$ and $k_3 \tilde{A}_3(k) = 0$, so the weaker conditions differ from the stronger ones by zero modes only. Similarly for the light-front gauge condition, instead of $-A_0 + A_3 = 0$ we take $(\partial_0 + \partial_3)(-A_0 + A_3) = 0$.

All these gauge conditions have the linear form $(P\partial) \cdot A = 0$, where in the various cases P_μ^ν is the projector

$$
\begin{aligned}
\text{Landau} : P_\mu^\nu &= \delta_\mu^\nu = \mathrm{diag}(1,1,1,1) \\
\text{Coulomb} : P_\mu^\nu &= \mathrm{diag}(0,1,1,1) \\
\text{Weyl} : P_\mu^\nu &= \mathrm{diag}(1,0,0,0) \\
\text{axial} : P_\mu^\nu &= \mathrm{diag}(0,0,0,1) \\
\text{light}-\text{front} : P_3^3 &= P_3^0 = P_0^3 = P_0^0 = 1/2 \quad \text{and} \;\; P_\mu^\nu = 0 \;\; \text{otherwise}.
\end{aligned} \tag{2}
$$

These projectors have a null space of dimension $\sigma = 0$, 1, 3, 3, and 3 respectively, where σ is the degree of arbitrariness of the gauge, as defined in the Introduction.

To separate the problem of violation of Lorentz invariance by the gauge-fixing condition from the problem of the arbitrariness of the classical gauges, we introduce an interpolating gauge defined by the condition $(\alpha\partial) \cdot A = 0$. Here α is the numerical matrix

$$\alpha \equiv P + aQ, \tag{3}$$

where P is one of the above projectors, $Q \equiv (1 - P)$ is the orthogonal projector, and a is real, in the interval $0 \leq a \leq 1$. These gauges interpolate between the Landau gauge, at $a = 1$, and any one of the above singular classical gauges, which is achieved at $a = 0$.

For the quantum field theory we consider the slightly more general gauge condition $(\alpha\partial) \cdot A = f$. By the usual Faddeev-Popov argument, the partition function, eq. (1), is expressed in terms of the local Faddeev-Popov action,

$$S_{\mathrm{FP}}(A, c, \bar{c}) \equiv S_{\mathrm{YM}}(A) + \int d^4x \; \{ \; (2\beta)^{-1}[(\alpha\partial) \cdot A]^2 + (\alpha\partial)\bar{c} \; \cdot D(A)c \; \} \tag{4}$$

where $D(A)$ is the gauge-covariant derivative $[D_\mu(A)c]^a \equiv \partial_\mu c^a + f^{abd} A_\mu{}^b c^d$, and β is a gauge parameter.

From this action, one reads off the ghost propagator

$$G = -i \; (k \cdot k')^{-1}, \tag{5}$$

where

$$k' \equiv \alpha k = Pk + aQk. \tag{6}$$

Similarly the gluon propagator D is obtained from the quadratic part of the gluon action $(1/2)(A, KA)$ by $K^{\lambda\mu} D_{\mu\nu} = -i\delta^{\lambda}_{\nu}$. From the Faddeev-Popov action we have

$$K^{\mu\nu} = k^2 g^{\mu\nu} - k^\mu k^\nu + \beta^{-1} k'^\mu k'^\nu, \tag{7}$$

and one easily verifies that the gluon propagator is given by

$$D_{\mu\nu} = -i\ k^{-2}[g_{\mu\nu} - (k \cdot k')^{-1}(k_\mu k'_\nu + k'_\mu k_\nu) + (k \cdot k')^{-2}(\beta k^2 + k'^2)k_\mu k_\nu]\ . \tag{8}$$

As long as α is a non-singular matrix, namely for $a > 0$, convergence of Feynman integrals is independent of direction in momentum space. The familiar power counting arguments hold, and Feynman integrals may be regularized by dimensional regularization.

We now consider some special cases. A Landau-type interpolating gauge is obtained at $\beta = 0$. In this case the propagator satisfies the generalized transversality condition $k'^\mu D_{\mu\nu} = 0$. For $a = 1$, we have the Landau-gauge propagator, so this gauge interpolates smoothly between the Landau gauge and the classical singular gauges.

A Feynman-'tHooft type gauge is obtained by choosing β so that the double pole becomes a simple pole. For Coulomb, Weyl and axial gauges, the projector P commutes with the metric tensor g = diag(-1, 1, 1, 1), and we have $(Pk) \cdot (Qk) = 0$. In these gauges the double pole is eliminated by setting $\beta = a$, for we have

$$\beta k^2 + k'^2 = a[(Pk)^2 + (Qk)^2] + (Pk)^2 + a^2(Qk)^2 = (1 + a)k \cdot k', \tag{9}$$

which gives the propagator

$$D_{\mu\nu} = -i\ k^{-2}\{g_{\mu\nu} + (k \cdot k')^{-1}[-k_\mu k'_\nu - k'_\mu k_\nu + (1 + a)k_\mu k_\nu]\}\ . \tag{10}$$

This propagator has the attractive 'tHooft-type property that it is block diagonal in the P-Q subspaces.

On the other hand, for the light-front gauge $(Pk) \cdot (Qk) \neq 0$, but Pk is a null vector, $(Pk)^2 = 0$. In this case the double pole is eliminated by setting $\beta = a^2$, for we have

$$\begin{aligned}\beta k^2 + k'^2 = a^2[2(Pk) \cdot (Qk) + (Qk)^2] + 2a(Pk) \cdot (Qk) + a^2(Qk)^2 \\ = 2ak \cdot k',\end{aligned} \tag{11}$$

which gives the propagator

$$D_{\mu\nu} = -i\ k^{-2}\{g_{\mu\nu} + (k \cdot k')^{-1}[-k_\mu k'_\nu - k'_\mu k_\nu + 2ak_\mu k_\nu]\}\ . \tag{12}$$

In the last two expressions for $D_{\mu\nu}$, the Feynman gauge is obtained at $a = 1$, so these gauges interpolate smoothly between the Feynman gauge and the classical singular gauges.

We write these expressions explicitly for interpolating Coulomb gauges. In this case we have $k' = Pk + aQk = (ak_0, \vec{k})$, and the ghost propagator is given by

$$G = -i \, \frac{1}{\vec{k}^2 - ak_0^2} \, . \tag{13}$$

For the gauge which interpolates between the Landau and the Coulomb gauges, the gluon propagator is given by

$$i \, D_{ij} = \frac{1}{k^2}\left(g_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) - \frac{k_i k_j}{\vec{k}^2}\frac{a^2 k_0^2}{(\vec{k}^2 - ak_0^2)^2} \tag{14}$$

$$i \, D_{0i} = -\frac{ak_0 k_i}{(\vec{k}^2 - ak_0^2)^2} \tag{15}$$

$$i \, D_{00} = -\frac{\vec{k}^2}{(\vec{k}^2 - ak_0^2)^2} \, . \tag{16}$$

These expressions are obtained by partial fractionation, and there is no singularity at $\vec{k} = 0$ for $a > 0$.

It is easy to understand intuitively how the Coulomb-gauge limit from the Landau-Coulomb gauge fixes the gauge arbitrariness of the Coulomb gauge discussed in the Introduction. Under the residual gauge freedom of the Coulomb gauge, A_0 transforms according to $A_0 \to g^\dagger(t) A_0 g(t) + g^\dagger(t)\partial_0 g(t)$, where the inhomogeneous term is $\vec{x}$-independent. With periodic boundary conditions, the Landau-Coulomb gauge condition $a\partial_0 A_0 = \vec{\nabla}\cdot\vec{A}$ for $a > 0$ gives $\partial_0 \int d^3x A_0 = 0$. However, as one sees from the above expression for the $A_0 - A_0$ propagator, D_{00} vanishes at $\vec{k} = 0$ for all finite a, so the stronger condition $\int d^3x A_0 = 0$ in fact holds in the Landau-Coulomb gauge for all finite a. This provides the additional gauge-fixing condition needed to make the limit $a \to 0$ well defined. By contrast D_{00} in the Feynman-Coulomb gauge, given below, becomes ill defined at $\vec{k} = 0$ with periodic boundary conditions, in the limit $a \to 0$.

For the gauge which interpolates between the Feynman and the Coulomb gauges one has

$$i \, D_{ij} = \frac{1}{k^2}\left(g_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) + \frac{k_i k_j}{\vec{k}^2}\frac{a}{\vec{k}^2 - ak_0^2} \tag{17}$$

$$i \, D_{0i} = 0 \tag{18}$$

$$i\ D_{00} = -\frac{1}{\vec{k}^2 - a k_0^2}\ . \tag{19}$$

There is no mixing of space and time components of the gluon propagator in this gauge.

These expressions for the propagators are quite illuminating. The transverse part of D_{ij} is the Coulomb gauge propagator. The parameter a acts as a regulator for simultaneity in the Coulomb gauge. These expressions imply exact compensations between the "unphysical" contributions in internal loops between gluon and ghost propagators. The main thing is of course that these compensations occur because the poles of the propagators of the unphysical fields with opposite statistics sit at the same point, i.e, at $\vec{k}^2 - a k_0^2 = 0$.

For completeness, we indicate the form of propagators in the interpolating gauges for the light-front gauge quoted in [3], with $k'_\mu = (\alpha_{\rm LFG} k)_\mu$ given by

$$k'_1 = a k_1, \quad k'_2 = a k_2, \quad k'_3 = \frac{(1+a)}{2} k_3 + \frac{(1-a)}{2} k_0, \quad k'_0 = \frac{(1-a)}{2} k_3 + \frac{(1+a)}{2} k_0. \tag{20}$$

We have $k \cdot k' = a(k_1^2 + k_2^2) + \frac{(1+a)}{2}(k_3^2 - k_0^2)$. For the gauge which interpolates between the light-front and Feynman gauges, namely, with $\beta = a^2$ which eliminates the double pole in the gluon propagator, the above expression for the gluon propagator reads

$$\begin{aligned}
D_{ij} &= -i\ \frac{\delta_{ij}}{k^2} \\
D_{i-} &= 0 \\
D_{i+} &= i\ (1-a)\ \frac{k_i (k_3 + k_0)}{k^2\ k \cdot k'} \\
D_{--} &= 0 \\
D_{+-} &= \frac{-2ia}{k \cdot k'} \\
D_{++} &= 2\ i\ (1-a)\ \frac{(k_3 + k_0)^2}{k^2\ k \cdot k'},
\end{aligned} \tag{21}$$

where $i, j = 1, 2$, $D_{\mu\pm} = D_{\mu 3} \pm D_{\mu 0}$, and $D_{\pm\pm} = D_{3\pm} \pm D_{0\pm}$. We observe that $D_{\mu -} = 0$ at $a = 0$, and the light-front gauge condition is satisfied.

3 BRST symmetry for local gauge and global Lorentz invariance

Suppose that we have a Lie algebra with basis X_i and structure constants f_{ijk}, so $[X_i, X_j] = f_{ijk} X_k$. According to the BRST method, for each generator X_i we introduce a corresponding Grassmann or ghost variable c_i. The BRST operator s acts on these variables according to

$$sC_i = -\frac{1}{2} f_{ijk} C_j C_k. \tag{22}$$

It is nilpotent, $s^2 = 0$. The preceding relation is isomorphic to the action of Cartan's exterior differential operator d acting on the Maurer-Cartan form $\omega = \omega_i t^i = g^{-1}dg$ of the Lie group.

We wish to apply this method to the Lie group which consists of local gauge transformations and global Lorentz transformations. The structure constants of this group are given by

$$\begin{aligned}
[G^a(x), G^b(y)] &= f^{abc}\delta(x-y)G^c(x) \\
[H_{\mu\nu}, G^a(x)] &= -(x_\mu \partial_\nu - x_\nu \partial_\mu) G^a(x) \\
[H_{\kappa\lambda}, H_{\mu\nu}] &= g_{\lambda\mu}H_{\kappa\nu} - g_{\kappa\mu}H_{\lambda\nu} - g_{\lambda\nu}H_{\kappa\mu} + g_{\kappa\nu}H_{\lambda\mu}.
\end{aligned} \tag{23}$$

According to the method described above, corresponding to the local generators $G^a(x)$ we introduce the usual anti-commuting Grassmann field variables $c^a(x)$ and corresponding to the $H_{\mu\nu}$ and we introduce the global generators $V_{\mu\nu} = -V_{\nu\mu}$, so $C_i = (c^a(x), V_{\mu\nu})$. In 4-dimensional space-time there are 6 independent generators $V_{\mu\nu}$. For the structure constants just found, the BRST operator s acts according to

$$sc^a(x) = -\frac{1}{2} f^{abc} c^b(x) c^c(x) + \ z\, V_\mu{}^\nu x^\mu \partial_\nu c^a(x) \tag{24}$$

$$sV_\lambda{}^\nu = -V_\lambda{}^\mu V_\mu{}^\nu, \tag{25}$$

where the parameter z will be determined shortly. Because $V_{\mu\nu}$ is a Grassmann variable, $(V^2)_{\kappa\nu} = V_{\kappa\lambda} g^{\lambda\mu} V_{\mu\nu}$ is an anti-symmetric matrix, $(V^2)_{\mu\nu} = -(V^2)_{\nu\mu}$. Equation 24 determines the normalization of the ghost field $c^a(x)$, and eq. 25 determines the normalization of Grassmann variables $V_{\mu,\nu}$. The parameter z is most easily determined by requiring that s be nil-potent, $s^2 = 0$, which gives $z = -1$. We could as easily have derived the corresponding result for the Poincare group.[2]

The BRST operator associated to the Lie algebra just defined is of the form $s = s_g + s_L$, where s_g and s_L satisfy $(s_g)^2 = (s_L)^2 = s_g s_L + s_L s_g = 0$. On the fields $c^a(x)$ and $V_\mu{}^\nu$ they act according to

$$\begin{aligned}
s_g c^a(x) &= -\tfrac{1}{2} f^{abc} c^b(x) c^c(x) \\
s_L c^a(x) &= V_\mu{}^\nu x^\mu \partial_\nu c^a(x) \\
s_g V_\lambda{}^\nu &= 0 \\
s_L V_\lambda{}^\nu &= -(V^2)_\lambda{}^\nu.
\end{aligned} \tag{26}$$

[2] Use of a "large" BRST operator in the present context was suggested to us by Massimo Porrati

The BRST oprator s defined here may be viewed as a "large" BRST operator, which is the usual BRST operator s_g for the local gauge group extended by the BRST operator s_L for the global Lorentz group.[3]

To determine the action of the BRST operator on the connection $A_\mu{}^a(x)$, we could start with the familiar transformation law of the connection under local gauge and global Lorentz transformation,

$$\begin{aligned} G(\omega)A_\mu{}^a(x) &= \partial_\mu \omega^a(x) + f^{abc}A_\mu{}^b(x)\omega^c(x) \\ H(\epsilon)A_\mu{}^a(x) &= \epsilon_\kappa{}^\lambda x_\kappa \partial_\lambda A_\mu{}^a(x) + \epsilon_\mu{}^\nu A_\nu{}^a(x). \end{aligned} \tag{27}$$

A more economical way is to construct the most general expression $sA_\mu{}^a = (s_g + s_L)A_\mu{}^a$ which satisfies $s^2 = 0$. Suppose that s_L acts according to

$$s_L A_\mu{}^a(x) = z_1 V_\kappa{}^\lambda x^\kappa \partial_\lambda A_\mu{}^a(x) + z_2 V_\mu{}^\nu A_\nu{}^a(x). \tag{28}$$

Here z_1 and z_2 are parameters that are determined by the condition $(s_L)^2 = 0$. From eq. (26) one obtains $z_1 + (z_1)^2 = 0$ and $z_2 + (z_2)^2 = 0$. We take $z_1 = -1$ because $z_1 = 0$ gives a trivial field transformation law. The components of A_μ transform either like scalars ($z_2 = 0$) or a vector ($z_2 = -1$). We take the vector case and obtain

$$s_L A_\mu{}^a(x) = -V_\kappa{}^\lambda x^\kappa \partial_\lambda A_\mu{}^a(x) - V_\mu{}^\nu A_\nu{}^a(x). \tag{29}$$

Finally, suppose that s_g acts on A according to

$$s_g A_\mu{}^a = z_{1\mu}{}^\nu \partial_\nu c^a + z_{2,\mu}{}^\nu f^{abc} A_\nu{}^b c^c. \tag{30}$$

We obtain from $(s_g)^2 = 0$ that $z_{2,\mu}{}^\nu = \delta_\mu{}^\nu$, and $z_{1,\mu}{}^\nu$ remains arbitrary. The condition $s_g s_L + s_L s_g = 0$ gives $z_1 V = V z_1$. Because $V_\mu{}^\nu$ is arbitrary, z_1 is of the form $z_{1\mu}{}^\nu = z\delta_\mu{}^\nu$. We write $A_\mu{}^a \equiv zA'_\mu{}^a$, and obtain for the BRST operator $s = s_g + s_L$,

$$sA_\mu{}^a = \partial_\mu c^a + f^{abc}A_\mu{}^b c^c - V_\kappa{}^\lambda x^\kappa \partial_\lambda A_\mu{}^a(x) - V_\mu{}^\nu A_\nu{}^a(x), \tag{31}$$

where we have dropped the prime on A'. This completes the determination of the action of the BRST operator on the basic fields A and C.

[3]The Lorentz rotations are a subset of general reparametrization. If we define the vector $\xi^\mu = V^\mu_\nu x^\nu$, out of the constant ghosts V and the coordinates x^μ, we have $s\xi^\mu = \xi^\rho \partial_\rho \xi^\mu$ and $s_L c = \xi^\mu \partial_\mu c$.

4 Extended action

The partition function in eq. (1) may be expressed in terms of the local Faddeev-Popov action,

$$S_{\rm FP}(\Phi) \equiv S_{\rm YM}(A) + \int d^4x[-(\alpha\partial)^\mu b A_\mu + (\alpha\partial)^\mu \bar{c} D(A)_\mu c + \frac{\beta}{2} b^2], \tag{32}$$

where $D(A)$ is the gauge-covariant derivative $[D_\mu(A)c]^a \equiv \partial_\mu c^a + f^{abc} A_\mu{}^b c^c$, and Φ represents the set of fields $\Phi = (A, c, \bar{c}, b)$. We introduce a corresponding set of sources, $J = (J_A, J_c, J_{\bar{c}}, J_b)$, and write

$$Z(J) = \int d\Phi \exp[-S_{\rm FP}(\Phi) + (\Phi, J)], \tag{33}$$

where $d\Phi \equiv dA dc d\bar{c} db$, and

$$(\Phi, J) \equiv \int d^4x (A \cdot J_A + c \cdot J_c + \bar{c} \cdot J_{\bar{c}} + b \cdot J_b). \tag{34}$$

The Faddeev-Popov action is not invariant under Lorentz transformations because of the appearance of the numerical matrix $\alpha^{\mu\nu}$. Consider instead the extended action

$$S_{\rm ext}(\Phi, V) \equiv S_{\rm YM}(A) - s \int d^4x[(\alpha\partial)^\mu \bar{c} A_\mu - \frac{\beta}{2} \bar{c} b], \tag{35}$$

where s is the "large" BRST operator that expresses the substitute gauge and Lorentz transformations. Its action on A, c and V is defined in eqs. (31), (24) and (25), and its action on $\bar{c}$ and b is defined by $s\bar{c} = b$ and $sb = 0$, which preserves $s^2 = 0$. Because the Yang-Mills action $S_{\rm YM}(A)$ is both gauge and Lorentz invariant, it is invariant under the "large" BRST operator $sS_{\rm YM}(A) = 0$, and consequently so is the extended action, $sS_{\rm ext}(\Phi, V) = 0$. The extended action differs from the Faddeev-Popov action by terms linear in the global Grassmann variables V introduced in the preceding section,

$$S_{\rm ext}(\Phi, V) = S_{\rm FP}(\Phi) - \int d^4x (\alpha\partial)^\mu \bar{c} (V_\kappa{}^\lambda x^\kappa \partial_\lambda A_\mu + V_\mu{}^\nu A_\nu). \tag{36}$$

We treat the variables $V_{\mu\nu}$ as external sources,[4] and define the extended partition function

$$Z(J, V) \equiv \int d\Phi \exp[-S_{\rm ext}(\Phi, V) + (\Phi, J)]. \tag{37}$$

The original partition function is obtained from it by $Z(J) = Z(J, 0)$. Because there are 6 independent global Grassmann variables V, there are, in all, 2^6 terms in the expansion of $Z(J, V)$ in powers of V. They are related by the symmetry generated by the large s.

[4] Equivalently we may treat the vector $\xi_\mu = V_\nu^\mu x^\nu$ as an extended source.

The usual argument that the expectation values of gauge-invariant observables are independent of the gauge parameters must be slightly modified because the variable V is not integrated over. We consider only s-invariant observables W are indepndent of V. We shall show that $\langle W \rangle$ is independent of the α matrix when the external source V is set to 0. We also set all sources J to 0, and we have

$$\begin{aligned}\partial\langle W\rangle/\partial\alpha^{\mu\nu} &= \int d\Phi\ W \int d^4x\ s(\partial_\nu\bar{c}\ A_\mu)\exp(-S_{\rm ext})|_{V=0} \\ &= \int d^4x\ \int d\Phi\ s[W\ \partial_\nu\bar{c}\ A_\mu\exp(-S_{\rm ext})\ ]|_{V=0},\end{aligned} \tag{38}$$

where we have used $sW = sS_{\rm ext} = 0$. At $V = 0$ we have $s = s_g$, where s_g is a derivative with respect to the variables of integration $\Phi = (A, c, \bar{c}, b)$. This gives $\partial\langle W\rangle/\partial\alpha^{\mu\nu} = 0$, as asserted. We conclude that for physical observables, the interpolating gauges gives the same expectation values as the covariant gauges. In particular they are independent of the gauge parameter α, and similarly for β.

5 Quantum Effective Action

To exploit BRST symmetry in renormalization theory, it is helpful to also introduce sources for the BRST transforms that are non-linear in the fields. We therefore define the (fully extended) action

$$\begin{aligned}\Sigma(\Phi, V, K, L, M) &\equiv S_{\rm ext} + (K, sA) + (L, sc) + M\cdot sV, \\ &= S_{\rm ext} + s[-(K, A) + (L, c) + M\cdot V\ ]\end{aligned} \tag{39}$$

where $K_\mu{}^a(x)$ and $L^a(x)$ are the usual sources for $sA_\mu{}^a(x)$ and $sc^a(x)$, and we have introduced a corresponding source $M^{\mu\nu} = -M^{\nu\mu}$ for $sV = -V^2$, with $M\cdot sV = \frac{1}{2}M^{\mu\nu}sV_{\mu\nu}$. These sources are not acted on by s, $sK = sL = sM = 0$. The action Σ is invariant under the "large" BRST operator,$s\Sigma = 0$.

We define the corresponding partition function

$$Z(J, V, K, L, M) \equiv \int d\Phi \exp[-\Sigma + (\Phi, J)]. \tag{40}$$

It satisfies

$$\frac{\delta Z}{\delta M^{\mu\nu}} = -(V^2)_{\mu\nu}Z. \tag{41}$$

The BRST operator s has been defined as a linear differential operator that acts on (and mixes) the variables $\Phi = (A, c, \bar{c}, b)$ and V. Because only the Φ variables are integrated over, it is convenient to decompose s according to $s = s_\Phi + s_V$, where s_Φ acts only on the Φ variables, and s_V only on V, so $s_\Phi V = s_V \Phi = 0$. The explicit form of s_V is

$$s_V \equiv (sV) \cdot \frac{\delta}{\delta V} = -(V^2)_{\mu\nu} \frac{\delta}{\delta V_{\mu\nu}} \tag{42}$$

By the invariance of Σ with respect to $s = s_\Phi + s_V$, we have

$$(sV) \cdot \frac{\delta Z}{\delta V} = [-(J_A, \frac{\delta}{\delta K}) - (J_c, \frac{\delta}{\delta L}) - (J_{\bar{c}}, \frac{\delta}{\delta J_b})\ ]Z \tag{43}$$

The free energy $W(J, V, K, L, M) \equiv \ln Z(J, V, K, L, M)$, satisfies the corresponding equations

$$\frac{\delta W}{\delta M^{\mu\nu}} = -sV_{\mu\nu} \tag{44}$$

and

$$(J_A, \frac{\delta W}{\delta K}) + (J_c, \frac{\delta W}{\delta L}) + (J_{\bar{c}}, \frac{\delta W}{\delta J_b}) + (sV) \cdot \frac{\delta W}{\delta V} = 0. \tag{45}$$

We make a Legendre transformation from the variables $J = (J_A, J_c, J_{\bar{c}}, J_b)$, and the free energy W to the external field variables $\Phi = (A, c, \bar{c}, b)$, and the quantum effective action Γ,

$$\Gamma(\Phi, V, K, L, M) = (\Phi, J) - W(J, V, K, L, M) \tag{46}$$

where

$$A_\mu = \frac{\delta W}{\delta J_{A_\mu}} \quad c = \frac{\delta W}{\delta J_c} \quad \bar{c} = \frac{\delta W}{\delta J_{\bar{c}}} \quad b = \frac{\delta W}{\delta J_b} \tag{47}$$

$$J_{A_\mu} = \frac{\delta \Gamma}{\delta A_\mu} \quad J_c = \frac{\delta \Gamma}{\delta c} \quad J_{\bar{c}} = \frac{\delta \Gamma}{\delta \bar{c}} \quad J_b = \frac{\delta \Gamma}{\delta b} \tag{48}$$

and

$$\frac{\delta \Gamma}{\delta V} = -\frac{\delta W}{\delta V}; \quad \frac{\delta \Gamma}{\delta K} = -\frac{\delta W}{\delta K}; \quad \frac{\delta \Gamma}{\delta L} = -\frac{\delta W}{\delta L}; \quad \frac{\delta \Gamma}{\delta M} = -\frac{\delta W}{\delta M}; \tag{49}$$

Here and elsewhere, all derivatives with respect to fermionic variables are left derivatives. In terms of Γ, eqs. (44) and (45) give

$$\frac{\delta \Gamma}{\delta M^{\mu\nu}} = sV_{\mu\nu} = -(V^2)_{\mu\nu}. \tag{50}$$

and

$$(\frac{\delta\Gamma}{\delta A},\frac{\delta\Gamma}{\delta K})+(\frac{\delta\Gamma}{\delta c},\frac{\delta\Gamma}{\delta L})+(\frac{\delta\Gamma}{\delta\bar{c}},b)+\frac{\delta\Gamma}{\delta V}\cdot\frac{\delta\Gamma}{\delta M}=0\ . \tag{51}$$

This type of equation, which was introduced in [13], now includes a V-M term. Here is it assumed that there is no gauge anomaly. No Lorentz anomaly can occur in $D=4$ dimensions.

Because the gauge condition is linear, we may solve the equations of motion to obtain the dependence of Γ on the Lagrange multiplier fields $\bar{c}$ and b. As this is standard, we simply give the result [14],

$$\Gamma(A,K,c,L,\bar{c},b,V,M)=\int d^4x[-(\alpha\partial)^\mu bA_\mu+(\frac{\beta}{2})b^2]+\tilde{\Gamma}(A,K+(\alpha\partial)\bar{c},c,L,V,M) \tag{52}$$

The property that the K and $\bar{c}$ dependences are only through the combination $K_\mu+\alpha\partial_\mu\bar{c}$ can be imposed as a Ward identity in the class of linear gauges that we consider. This plays an important role in the renormalisation program.

The master equation satisfied by $\tilde{\Gamma}(A,K,c,L,V,M)$ is symmetric in the pair V,M and the other variables,

$$(\frac{\delta\tilde{\Gamma}}{\delta A},\frac{\delta\tilde{\Gamma}}{\delta K})+(\frac{\delta\tilde{\Gamma}}{\delta c},\frac{\delta\tilde{\Gamma}}{\delta L})+\frac{\delta\tilde{\Gamma}}{\delta V}\cdot\frac{\delta\tilde{\Gamma}}{\delta M}=0. \tag{53}$$

$\tilde{\Gamma}$ has the simple dependence on M given by

$$\frac{\partial\tilde{\Gamma}}{\partial M^{\mu\nu}}=sV_{\mu\nu}=-(V^2)_{\mu\nu}. \tag{54}$$

6 Form of Divergences

The new $V-M$ term has the same structure as the other terms, so we may use familiar arguments, which we now sketch, to determine the form of possible divergences to each order in $\hbar$, when using a regulator that preserves Lorentz and gauge symmetries. We make a loop or $\hbar$ expansion of $\tilde{\Gamma}$, using any suitable regularization for divergences.

$$\tilde{\Gamma}=\sum_n\tilde{\Gamma}^n. \tag{55}$$

To find $\tilde{\Gamma}^0$, we observe that Σ, eq. (39), is of the form

$$\Sigma(A,K,c,L,\bar{c},b,V)=\int d^4x[-(\alpha\partial)^\mu bA_\mu+\frac{1}{2}\beta b^2]+\tilde{\Sigma}(A,K+(\alpha\partial)\bar{c},c,L,V,M) \tag{56}$$

where

$$\tilde{\Sigma}(A,K,c,L,V) \equiv S_{\rm YM} + (K, sA) + (L, sc) + M \cdot sV. \tag{57}$$

This gives

$$\tilde{\Gamma}^0 = \tilde{\Sigma}. \tag{58}$$

We will impose that $\tilde{\Gamma}$ is renormalized while satisfying the master equation (53). From the s-invariance of $S_{\rm YM}$, and from $sA = \delta\tilde{\Sigma}/\delta K$, $sc = \delta\tilde{\Sigma}/\delta L$, and $sV = \delta\tilde{\Sigma}/\delta M$, we have

$$(\frac{\delta\tilde{\Sigma}}{\delta A}, \frac{\delta\tilde{\Sigma}}{\delta K}) + (\frac{\delta\tilde{\Sigma}}{\delta c}, \frac{\delta\tilde{\Sigma}}{\delta L}) + \frac{\delta\tilde{\Sigma}}{\delta V} \cdot \frac{\delta\tilde{\Sigma}}{\delta M} = 0 \tag{59}$$

so the master equation (53) is satisfied by $\tilde{\Gamma}^0 = \tilde{\Sigma}$. We define the star product

$$\tilde{\Gamma}_a * \tilde{\Gamma}_b \equiv (\frac{\delta\tilde{\Gamma}_a}{\delta A}, \frac{\delta\tilde{\Gamma}_b}{\delta K}) + (\frac{\delta\tilde{\Gamma}_a}{\delta c}, \frac{\delta\tilde{\Gamma}_b}{\delta L}) + \frac{\delta\tilde{\Gamma}_a}{\delta V} \cdot \frac{\delta\tilde{\Gamma}_b}{\delta M} \tag{60}$$

To each order n in $\hbar$, eq. (53) reads

$$\sum_{p+q=n} \tilde{\Gamma}^p * \tilde{\Gamma}^q = 0. \tag{61}$$

We assume that renormalization has been done to order $n-1$, so that $\tilde{\Gamma}^p$ for $p = 0, \cdots (n-1)$ is finite, and that eq. (61) is satisfied to order $n-1$. We separate the regular and divergent parts of order n,

$$\tilde{\Gamma}^n = \tilde{\Gamma}^n_R + \tilde{\Gamma}^n_{\rm div}, \tag{62}$$

where the first term is the renormalized part of the n-th order effective action and is finite. By hypothesis, the only divergence in eq. (61) comes from $\tilde{\Gamma}^n_{\rm div}$. The divergent part must satisfy eq. (61) separately, namely

$$\sigma\tilde{\Gamma}^n_{\rm div} = 0, \tag{63}$$

where the linear operator σ, defined by

$$\sigma\Gamma \equiv \tilde{\Sigma} * \Gamma + \Gamma * \tilde{\Sigma}, \tag{64}$$

has the explicit expression

$$\sigma = \int d^4x \left(\frac{\delta\tilde{\Sigma}}{\delta K}\frac{\delta}{\delta A} + \frac{\delta\tilde{\Sigma}}{\delta A}\frac{\delta}{\delta K} + \frac{\delta\tilde{\Sigma}}{\delta L}\frac{\delta}{\delta c} + \frac{\delta\tilde{\Sigma}}{\delta c}\frac{\delta}{\delta L} + \frac{\delta\tilde{\Sigma}}{\delta M}\frac{\delta}{\delta V} + \frac{\delta\tilde{\Sigma}}{\delta V}\frac{\delta}{\delta M} \right) \tag{65}$$

It is nilpotent $\sigma^2 = 0$. Here σ represents the symmetry of the "large" BRST operator, with the obvious decomposition into local gauge and global Lorentz parts, $\sigma = \sigma_g + \sigma_L$, that corresponds to $s = s_g + s_L$. From eq. (54) we have

$$\frac{\delta \tilde{\Gamma}^n}{\delta M} = 0; \quad n \geq 1, \tag{66}$$

so $\tilde{\Gamma}^n$ is independent of M.

Consistent with the last equation, with locality of divergent terms, with global color invariance, with the ghost quantum numbers $(0, -1, 1, -2, 1)$ and dimensions $(1, 2, 1, 2, 1)$ of the variables $(A_\mu^a, K^{a,\mu}, c^a, L^a, V_{\mu\nu})$ on which $\tilde{\Gamma}^n$ depends, eq. (63) has the solution

$$\tilde{\Gamma}^n_{\text{div}} = \int d^4x \left[c_1 \frac{1}{4} F^2_{\mu\nu} + \sigma \left(K^{a\mu} c_{2,\mu}{}^{\nu} A^a_\nu + c_3 L^a C^a \right) \right], \tag{67}$$

where c_1, $c_{2,\mu}{}^{\nu}$ and c_3 are divergent constants of order $\hbar^n$. The operator $x_\mu \partial_\nu - x_\nu \partial_\mu$ may appear in $\tilde{\Gamma}^n$ in the combination $V^{\mu\nu} x_\mu \partial_\nu$. However $x_\mu \partial_\nu - x_\nu \partial_\mu$ is dimensionless and carries no ghost or global color quantum number so it does not affect our counting arguments, which exclude the explicit appearance of V in the last equation. However a V dependence is introduced into $\tilde{\Gamma}^n$ from the definition of σ, so that V appears in the expansion of the σ-exact term.

With this result we have achieved our goal of limiting the number of possible divergences, by maintaining invariance under the larger group of substitute gauge and Lorentz invariance. Indeed only the combination $\int d^4x F^2_{\mu\nu}$ is invariant under σ without being exact, of the form σX. ($\int d^4x F^2_{\mu\nu}$ is said to be the cohomology of the operator σ.) For if only invariance under s_g or σ_g were enforced, then the most general cohomology would be $\int d^4x (c_E E^2 + c_B B^2)$, if ordinary rotational invariance is preserved by the gauge fixing, where c_E and c_B are *independent* renormalization constants. Indeed, E^2 and B^2 are separately invariant under s_g, and in [2], it was necessary to *assume* $c_E = c_B$. This is now established for the gauges considered here. On the other hand the breaking of Lorentz invariance by the gauge fixing does lead to the Lorentz non-invariant divergent terms $\sigma K^{a,\mu} c_{2\mu}{}^{\nu} A^a_\mu$, which however are exact σ-forms.

If ordinary rotational invariance is maintained by the gauge fixing, then $c_{2,\mu}{}^{\nu}$ is a diagonal tensor with $c_{2,1}{}^1 = c_{2,2}{}^2 = c_{2,3}{}^3 \neq c_{2,4}{}^4$. In Lorentz-type gauges, defined by $\beta = 0$ in eqs. (32) or (35), the (possibly) divergent constant c_3 vanishes, $c_3 = 0$, by virtue of the factorization of the external ghost momentum, as it does in the Landau gauge[15].

7 Multiplicative Renormalization

In previous sections we had implicitly absorbed the coupling constant g into the SU(N) structure constant f^{abc}. Since we are interested in the perturbative expansion we now make the coupling constant explicit by the substitution $f^{abc} \to g f^{abc}$.

We define $\Sigma_{\mathrm{r}}(\Phi, K_\Phi, V)_r$ as the *local* part of $\Gamma = \Gamma_R + \Gamma_{\mathrm{div}}$. We call $\Sigma_{\mathrm{r}}(\Phi, K_\Phi, V)$ the renormalized action since by inserting $\exp\Sigma_{\mathrm{r}}(\Phi, K_\Phi, V)$ in the path integral over the Φ, all relevant local divergent counterterms are present to determine finite Green functions of the fields $A, c, \bar{c}, b$ and of their BRST transformations which satisfy the BRST master equations.

The result found in eq. (67), proves that the renormalized action $\Sigma_{\mathrm{r}}(\Phi, K_\Phi, V)_r$ has the following form:

$$\begin{aligned}
\Sigma_{\mathrm{r}}(\Phi, K_\Phi, V)_r = \int d^4x \{ & \frac{1}{4} | \, F_{\mu\nu}(Z^\nu_{A\mu} A_\nu, Z_g g) \, |^2 \\
& + Z_c [\, (\alpha\partial)^\mu \bar{c} + K^\mu] Z^{-1\nu}_{A\mu} (\, \partial_\nu c + Z_g g [\, Z^\rho_{A\mu} A_\rho \, , \, c] \,) \\
& - \frac{1}{2} Z_c Z_g g [\, c, \, c \,] \, L + \frac{\beta}{2} b^2 + b(\alpha\partial)^\mu A_\mu - \frac{1}{2} M^{\mu\nu} (V^2)_{\mu\nu} \, \}
\end{aligned} \tag{68}$$

For the sake of notational simplicity, we use the graded commutator notation, $[X, Y]^a = f^a_{bc} X^b Y^c$, and $F_{\mu\nu}(A, g) = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$.

The relation between the renormalization constants Z and the constants c appearing in eq. (67) is

$$\begin{aligned}
Z^\nu_{A\mu} &= \delta^\mu_\nu (1 + \frac{c_1}{2}) + c^\nu_{2\mu} \\
Z_c &= 1 + c_3 + \frac{c_1}{2} \\
Z_g &= 1 - \frac{c_1}{2}
\end{aligned} \tag{69}$$

Thus, the effect of renormalisation, constrained by the BRST invariance, can be seen as the following redefinitions of fields and parameters:

$$\begin{aligned}
A_\mu &\to Z^\nu_{A\mu} A_\nu \\
c &\to Z_c c \\
g &\to Z_g g \\
K^\mu &\to Z^{-1\mu}_{A\nu} K^\nu \\
L &\to \frac{1}{Z^c} L \\
\bar{c} &\to \bar{c}
\end{aligned}$$

$$
\begin{aligned}
b &\rightarrow b \\
\beta &\rightarrow \beta \\
\alpha^{\mu\nu} &\rightarrow Z^{-1\mu}_{A\rho}\alpha^{\rho\nu} \\
V &\rightarrow V \\
M &\rightarrow M
\end{aligned} \tag{70}
$$

Here A_μ and K^μ transform contragrediently under renormalization, as do c and L, so that the master equation is invariant under renormalization in any finite order.

Equation (70) shows that the renormalization is (matricially) multiplicative for the fields, sources and parameters of the theory and that, as compared to the covariant case, the breaking of Lorentz invariance by the gauge fixing term induces a mixing by the renormalization of the 4 components in A_μ and K_μ. Let us stress that the simplicity of the renormalization of $\bar{c}$ b, β and α^μ_ν, which generalizes that of covariant renormalizable gauges, is a particularity of linear gauges for which one can maintain the K and $\bar{c}$ dependences through the combination $K + \alpha\partial\bar{c}$.

These equations indicate the existence of a renormalized BRST symmetry for the action $\Sigma_{\rm r}(\Phi, K_\Phi, V)$, in eq. (68). We will shortly display its expression. It is however instructive to rederive the renormalized action $\Sigma_{\rm r}(\Phi, K_\Phi, V)$, using the method displayed in [4], which has the advantage of determining at the same time the renormalized BRST invariance of the theory.

In this method, one parametrizes the renormalized action $\Sigma_{\rm r}$, including all relevant couterterms, as

$$
\Sigma_{\rm r} = S_{\rm r}(\Phi, V) + \sum_\Phi (K_\Phi, s_{\rm r}\Phi) + M s_{\rm r} V \tag{71}
$$

Recall that Φ stand for all fields, $A, c, \bar{c}, b$. One has assumed that the dependence on the sources K's of the BRST transformations is linear, which will be checked by self consistency. Then, $s_{\rm r}\Phi$ stand for field polynomials in the fields Φ, which can be expressed as the action on Φ of a yet undetermined graded differential operator $s_{\rm r}$.

One can show that the content of the Ward identities of the BRST symmetry is that (i) $S_{\rm r}$ is invariant under the action of $s_{\rm r}$ and (ii) $s_{\rm r}$ is a nilpotent operator [4]:

$$
s_{\rm r}^2 = 0 \tag{72}
$$

and

$$
s_{\rm r} S_{\rm r} = 0 \tag{73}
$$

To compute the possible action of $s_{\rm r}\Phi$, with $s_{\rm r}^2\Phi = 0$, one uses the results of section (3). Up to inessential overall factors, the only freedom left in determining the action of $s_{\rm r}$ is a matricial redefinition of A_μ, that is, $A_\mu \to Z^\nu_\mu A_\nu$, and the rescaling $g \to Z_g g$ so that the requirement $s_{\rm r}^2 = 0$ implies

$$\begin{aligned} s_{\rm r} V &= -VV \\ s_{\rm r} c &= -\frac{Z_g g}{2}[c, c] - V^\nu_\mu x^\mu \partial_\nu c \\ s_{\rm r} Z^\nu_\mu A_\nu &= \partial_\mu c + Z_g g[Z^\nu_\mu A_\nu, c] - V^\kappa_\lambda x^\lambda \partial_\kappa Z^\nu_\rho A_\nu - V^\rho_\mu Z^\nu_\mu A_\nu \end{aligned} \tag{74}$$

and

$$\begin{aligned} s_{\rm r} \bar{c} &= Z_c^{-1} b \\ s_{\rm r} b &= 0. \end{aligned} \tag{75}$$

This allows one to identify $Z^\mu_{A\nu} = Z^\nu_\mu$. Notice also the freedom in rescaling the field b. In the expression of the action, one furthermore sees that a rescaling of b only amounts to a rescaling of the partition function Z, which is unobservable.

We also remark parenthetically that one can write

$$s_{\rm r} A_\mu = \partial'_\mu c + Z_g g[A_\mu, c] - V'^\kappa_\lambda x'^\lambda \partial'_\kappa A_\mu - V'^\rho_\mu A_\mu \tag{76}$$

where $\partial'_\mu = Z^{-1\nu}_\mu \partial_\nu$, $x'^\mu = Z^\mu_\nu x^\nu$ and $V'^\kappa_\lambda = Z^\kappa_\mu V^\mu_\nu Z^{-1\nu}_\lambda$. This shows another interesting property of the class of non-covariant gauges that we have introduced: the transformation of the components of A implied by breaking of Lorentz invariance, while maintaining BRST invariance, can be absorbed into a transformations of space time coordinates, $x \to x'$, together with redefinitions of constant ghost matrix elements $V \to V'$. (One has $s_{\rm r} V' = -V'V'$.)

The non-trivial part of the cohomology of $s_{\rm r}$ with dimension 4 is $|\ \partial_\mu A^{\rm r}_\nu - \partial_\nu A^{\rm r}_\mu + Z_g[A^{\rm r}_\mu, A^{\rm r}_\nu]\ |^2$, with $A^{\rm r}_\mu = Z^\nu_\mu A_\nu$; the rest of $S_{\rm r}$ can only be $s_{\rm r}$-exact terms with dimension 4. By using the anti-ghost equation of motion as a Ward identity, which implies that no quartic ghost interactions occur in the action, together with the property that the b-dependent part of the action does not need counter-terms, one concludes that $S_{\rm r}$ must be of the form

$$\begin{aligned} S_{\rm r} = \int d^4x \Big(\frac{1}{4} |\ \partial_\mu A^{\rm r}_\nu - \partial_\nu A^{\rm r}_\mu + Z_g[A^{\rm r}_\mu, A^{\rm r}_\nu]\ |^2 + \\ + Z_c s_{\rm r} \{ \bar{c} [\ (\alpha\partial)^\mu A^{\rm r}_\mu + \frac{\beta}{2} b\ ]\ \} \Big) \end{aligned} \tag{77}$$

If we now expand $S_{\rm r}$, using the definition of $s_{\rm r}$, and insert this into eq. (71), we exactly recover the formula giving $\Sigma_{\rm r}(\Phi, K_\Phi, V)$ in eq. (68).

The renormalized action $\Sigma_{\rm r}(\Phi, K_\Phi, V)$, which is suitable for the computation of renormalized Green functions, is thus invariant by construction under the action of the operation $s_{\rm r}$, which is the renormalized expression of BRST symmetry.

8 Gauss-BRST Ward Identity

In the remainder of the article we shall study the Coulomb gauge limit. The preceding results hold in particular for interpolating Coulomb gauges, when the matrix α is diagonal, and $\alpha_{00} = a$ and $\alpha_{ii} = 1$. As compared to the case of covariant gauges, there is just one extra renormalization constant, with $A_i \to Z_{\vec{A}} A_i$ and $A_0 \to Z_{A_0} A_0$, and $Z_{\vec{A}} \neq Z_{A_0}$.

In [2] a Gauss-BRST Ward identity was derived in the formal $a = 0$ Coulomb gauge. This identity is the functional analog of the operator statement that the BRST symmetry transformation is generated by the Gauss-BRST charge. In the present section we shall show that this identity holds in the Coulomb gauge limit $a \to 0$ from the Landau-Coulomb interpolating gauge.

Consider the partition function in the Euclidean theory, with Coulomb type interpolating gauge, $\vec{\nabla} \cdot \vec{A} + a\dot{A}_0 = 0$ (or $= f$). Having assured ourselves of Lorentz invariance, we set $V = M = 0$, and the partition function becomes

$$Z(J, K, L) \equiv \int d\Phi \exp[-\Sigma + (\Phi, J)], \tag{78}$$

where $\Phi = (A, c, \bar{c}, b)$, and

$$\Sigma(\Phi, K, L) \equiv S_{\rm FP}(\Phi) + (K, sA) + (L, sc). \tag{79}$$

With $\Sigma = \int d^4x\ \Lambda$, the Lagrangian density reads

$$\Lambda \equiv (1/4)F_{\mu\nu}^2 + (K_\mu + \partial'_\mu \bar{c}) D(A)_\mu c + L(-g/2) \cdot (c \times c) - \partial'_\mu b A_\mu + \frac{\beta}{2} b^2, \tag{80}$$

where $\partial'_\mu \equiv (\alpha\partial)_\mu \equiv (a\partial_0, \vec{\nabla})$. We are interested in the Coulomb gauge limit $a \to 0$. Because of the gauge arbitrariness of the Coulomb gauge discussed in the Introduction, this limit may be β-dependent, with $\beta = 0$ for the Landau-Coulomb gauge or $\beta = a$ for the Feynman-Coulomb gauge.

The Lagrangian density is BRST-closed, $s\Lambda = 0$. This implies the existence of an identity associated with the corresponding Noether current, which we now derive. For this purpose we make the infinitesimal change of variable of integration corresponding to a space-time dependent BRST transformation

$$\Phi'_\alpha = \Phi_\alpha + \epsilon(x) s\Phi_\alpha, \tag{81}$$

where $\epsilon(x)$ is space-time dependent, and α is an index the runs over all components of all integration variables. This change of variables leaves the measure $d\Phi$ invariant, and so, because $\epsilon(x)$ is arbitrary, it yields the identity

$$0 = \int d\Phi(\partial_\mu j_\mu + sA_\mu J_{A_\mu} + scJ_c + bJ_{\bar{c}}) \exp[-\Sigma + (\Phi, J)], \tag{82}$$

where j_μ is the Noether current of the BRST symmetry of Λ. If we integrate this identity over all space-time, the term $\partial_\mu j_\mu$ is annihilated, and we obtain the Zinn-Justin equation used previously. Instead we integrate over 3-space only, with spatially periodic boundary conditions, and obtain

$$\int d^3x(J_{A_\mu}\frac{\delta Z}{\delta K_\mu} + J_c\frac{\delta Z}{\delta L} - J_{\bar{c}}\frac{\delta Z}{\delta J_b}) = \partial_0 \int d\Phi \ Q \ \exp[-\Sigma + (\Phi, J)]. \tag{83}$$

The conserved BRST charge Q is calculated from

$$Q = \int d^3x[(sA_\mu)\frac{\partial\Lambda}{\partial(\partial_0 A_\mu)} + (sc)\frac{\partial\Lambda}{\partial(\partial_0 c)} + (s\bar{c})\frac{\partial\Lambda}{\partial(\partial_0\bar{c})}], \tag{84}$$

where the fermionic derivatives are left derivatives, which gives

$$Q = \int d^3x[-cD_iF_{0i} - (K_0 + a\partial_0\bar{c})(sc) + abD_0c]. \tag{85}$$

We wish to express the BRST charge in a way which will provide a Ward identity satisfied by the quantum effectve action Γ. For this purpose we observe that Q may be written

$$Q = \int d^3x[-c\frac{\delta\Sigma}{\delta A_0} + K_0\frac{\delta\Sigma}{\delta L}] + Q_a \tag{86}$$

where

$$Q_a \equiv a\int d^3x \ s(bA_0 - \partial_0\bar{c}c) \tag{87}$$

is proportional to a, and is the integral of a BRST-exact density.

The quantity $\frac{\delta\Sigma}{\delta A_0}$ is the left-hand side of Gauss's law. In a canonical formulation, it is also the generator of local gauge transformations, so the first term of Q has the form of the generator of an infinitesimal gauge transformation with generator $-c(x)$. For this reason, the last expression for the BRST charge Q remains correct if coupling to quarks is included in the Lagrangian density, and also in the phase-space representation which we shall introduce in the following section.

From this expression for Q we obtain

$$\int d^3x(J_{A_\mu}\frac{\delta Z}{\delta K_\mu} + J_c\frac{\delta Z}{\delta L} - J_{\bar{c}}\frac{\delta Z}{\delta J_b}) = \partial_0 \int d^3x(J_{A_0}\frac{\delta Z}{\delta J_c} - K_0\frac{\delta Z}{\delta L}) + Z\langle\dot{Q}_a\rangle, \tag{88}$$

The expectation-value $\langle \dot{Q}_a \rangle$ is calculated in the presence of all sources. In terms of the generator of connected correlation functions, $W(J,K,L) = \ln Z(J,K,L)$, this identity reads

$$\int d^3x (J_{A_\mu} \frac{\delta W}{\delta K_\mu} + J_c \frac{\delta W}{\delta L} - J_{\bar{c}} \frac{\delta W}{\delta J_b}) = \partial_0 \int d^3x (J_{A_0} \frac{\delta W}{\delta J_c} - K_0 \frac{\delta W}{\delta L}) + \langle \dot{Q}_a \rangle. \tag{89}$$

We make the Legendre transformation to the quantum effective action $\Gamma(\Phi, K, L)$, which satisfies

$$\int d^3x (\frac{\delta \Gamma}{\delta A_\mu} \frac{\delta \Gamma}{\delta K_\mu} + \frac{\delta \Gamma}{\delta c} \frac{\delta \Gamma}{\delta L} + b \frac{\delta \Gamma}{\delta \bar{c}}) = \partial_0 \int d^3x (c \frac{\delta \Gamma}{\delta A_0} - K_0 \frac{\delta \Gamma}{\delta L}) - \langle \dot{Q}_a \rangle. \tag{90}$$

Because Q_a is proportional to a, one has $Q_a = 0$ in the formal Coulomb gauge $a = 0$. However Feynman integrals diverge in the limit $a \to 0$, so a precise evaluation is required to determine whether or not $\langle \dot{Q}_a \rangle$ really vanishes in the limit $a \to 0$. In the following section we study this limit from the Landau-Coulomb interpolating gauge, $\beta = 0$, by means of a phase-space representation. By power counting of the k_0 integrations, it is found that the correlation functions with dimensional regularization are finite in the limit $a \to 0$. It is found that, although $\langle \dot{Q}_a \rangle$ does not in fact vanish linearly with a, nevertheless it does vanish like

$$\langle \dot{Q}_a \rangle = O(a^{1/2}). \tag{91}$$

in the limit $a \to 0$. (See remark 3 at the end of the following section.)

We now take the limit $a \to 0$, and set $\langle \dot{Q}_a \rangle = 0$. Only first functional derivatives of Γ appear, so the unacceptably singular expression of correlation functions at coincident points is absent, and this identity imposes a constraint on the renormalization constants of the elementary fields. As before, the Lagrangian multiplier fields b and $\bar{c}$ may be eliminated by means of their equations of motion, and the Gauss-BRST identity simplifies to

$$\int d^3x (\frac{\delta \tilde{\Gamma}}{\delta A_\mu} \frac{\delta \tilde{\Gamma}}{\delta K_\mu} + \frac{\delta \tilde{\Gamma}}{\delta c} \frac{\delta \tilde{\Gamma}}{\delta L}) = \partial_0 \int d^3x (c \frac{\delta \tilde{\Gamma}}{\delta A_0} - K_0 \frac{\delta \tilde{\Gamma}}{\delta L}). \tag{92}$$

According to our results on renormalization, the quantum effective action $\tilde{\Gamma}$ is finite when expressed in terms of renormalized quantities,

$$\tilde{\Gamma}(X) = \tilde{\Gamma}_r(X_r), \tag{93}$$

where $X = (A, c, K, L, g, \Lambda)$ and $X_r = (A_r, c_r, K_r, L_r, g_r, \mu)$. Here Λ is the usual ultraviolet regularization parameter, and μ is a renormalization mass. The renormalization constants satisfy

$$Z_{K\,\mu}^{\ \nu} = Z_{A\ \mu}^{-1\,\nu} \qquad Z_L = Z_c^{-1}. \tag{94}$$

Moreover, for the Coulomb gauge, by rotational invariance, the matrix $Z_{A_\mu}^{\ \nu}$ is given by $Z_{A_\mu}^{\ \nu} = \mathrm{diag}(Z_{A_0}, Z_{\vec{A}}, Z_{\vec{A}}, Z_{\vec{A}})$, and $Z_{K_\mu}^{\ \nu} = \mathrm{diag}(Z_{A_0}^{-1}, Z_{\vec{A}}^{-1}, Z_{\vec{A}}^{-1}, Z_{\vec{A}}^{-1})$. Consequently the Gauss-BRST identity reads

$$\int d^3x \Big(\frac{\delta\tilde{\Gamma}_r}{\delta A_{r,\mu}}\frac{\delta\tilde{\Gamma}_r}{\delta K_{r,\mu}} + \frac{\delta\tilde{\Gamma}_r}{\delta c_r}\frac{\delta\tilde{\Gamma}_r}{\delta L_r}\Big) = \frac{Z_c}{Z_{A_0}}\partial_0 \int d^3x \Big(c_r\frac{\delta\tilde{\Gamma}_r}{\delta A_{r,0}} - K_{r,0}\frac{\delta\tilde{\Gamma}_r}{\delta L_r}\Big). \tag{95}$$

Since all other quantities in this equation are finite, the ratio Z_c/Z_{A_0} must also be finite. This implies that in the recursive renormalization procedure described above, the divergent parts of Z_c and Z_{A_0} are equal in each order n. The iterative renormalization may be done so the finite parts are also equal in each order, and the equality

$$Z_{A_0} = Z_c \tag{96}$$

is maintained.

For this purpose we must show that the renormalized action $\tilde{\Sigma}_r$ also satisfies the Gauss-BRST identity. It is instructive to first verify directly that $\tilde{\Sigma}$ satisfies this identity. Indeed by Noether's theorem the variation of Σ under the above space-time dependent BRST transformation is given by

$$\delta\Sigma = -\int d^4x \epsilon(x)\partial_\mu j_\mu \tag{97}$$

where j_μ is the Noether current. On the other hand we have

$$\begin{aligned}\delta\Sigma &= \int d^4x \epsilon(x) s\Phi_i \frac{\delta\Sigma}{\delta\Phi_i} \\ &= \int d^4x \epsilon(x)\Big(\frac{\delta\Sigma}{\delta K_\mu}\frac{\delta\Sigma}{\delta A_\mu} + \frac{\delta\Sigma}{\delta L}\frac{\delta\Sigma}{\delta c} + b\frac{\delta\Sigma}{\delta\bar{c}}\Big).\end{aligned} \tag{98}$$

Since $\epsilon(x)$ is arbitrary, it follows that Σ satisfies,

$$\frac{\delta\Sigma}{\delta K_\mu}\frac{\delta\Sigma}{\delta A_\mu} + \frac{\delta\Sigma}{\delta L}\frac{\delta\Sigma}{\delta c} + b\frac{\delta\Sigma}{\delta\bar{c}} = -\partial_\mu j_\mu. \tag{99}$$

Upon integrating this equation over 3-space and using the above expression for the BRST charge Q, we obtain

$$\int d^3x \Big(\frac{\delta\Sigma}{\delta A_\mu}\frac{\delta\Sigma}{\delta K_\mu} + \frac{\delta\Sigma}{\delta c}\frac{\delta\Sigma}{\delta L} + b\frac{\delta\Sigma}{\delta\bar{c}}\Big) = \partial_0 \int d^3x \Big(c\frac{\delta\Sigma}{\delta A_0} - K_0\frac{\delta\Sigma}{\delta L}\Big) - \dot{Q}_a. \tag{100}$$

We now introduce $\tilde{\Sigma}(A, c, K, L) = \int d^4x\ \tilde{\Lambda}$, where

$$\tilde{\Lambda}(A, c, K, L) \equiv (1/4)F_{\mu\nu}^2 + K_\mu D(A)_\mu c + L(-g/2)\cdot(c\times c), \tag{101}$$

so

$$\Lambda(A, c, \bar{c}, b, K, L) = \tilde{\Lambda}(A, c, K + \partial' \bar{c}, L) - \partial'_\mu b A_\mu + \frac{\beta}{2} b^2. \tag{102}$$

By the above reasoning we conclude that $\tilde{\Sigma}$ satisfies the functional identity

$$\int d^3x (\frac{\delta \tilde{\Sigma}}{\delta A_\mu} \frac{\delta \tilde{\Sigma}}{\delta K_\mu} + \frac{\delta \tilde{\Sigma}}{\delta c} \frac{\delta \tilde{\Sigma}}{\delta L}) = \partial_0 \int d^3x (c \frac{\delta \tilde{\Sigma}}{\delta A_0} - K_0 \frac{\delta \tilde{\Sigma}}{\delta L}). \tag{103}$$

If one makes the change of variables

$$\begin{aligned}
A_\mu &= Z_{A_\mu} A_{r,\mu} \\
K_\mu &= Z_{A_\mu}^{-1} K_{r,\mu} \\
c &= Z_c c_r \\
L &= Z_c^{-1} L_r \\
g &= Z_g g_r \\
\tilde{\Sigma}(A, K, c, L, g) &= \tilde{\Sigma}_r(A_r, K_r, c_r, L_r, g_r),
\end{aligned} \tag{104}$$

with $Z_{A_0} = Z_c$, this identity remains unchanged, so $\tilde{\Sigma}_r$ satisfies the same functional identity as $\tilde{\Sigma}$. This is the required condition for recursive renormalization.

We have taken the limit $a \to 0$ from the Landau-Coulomb interpolating gauge, $\beta = 0$, for which the estimates of the following section hold. In this gauge, as noted at the end of sect. (6), the renormalization constant $c_3 = 0$, so $Z_g Z_c = 1$. We therefore obtain in the $a \to 0$ limit from the Landau-Coulomb interpolating gauge

$$Z_g Z_{A_0} = 1. \tag{105}$$

Consequently the field gA_0 is invariant under renormalization

$$gA_0 = g_r A_{r,0}, \tag{106}$$

as are its correlation functions, including in particular the zero-zero component of the gluon propagator,

$$D_{00}(|\vec{x}|, t) = g^2 \langle A_0(|\vec{x}|, t) A_0(0, 0) \rangle. \tag{107}$$

This quantity is independent of the cut-off Λ and the renormalization mass μ, and consequently it can depend only on physical masses such as Λ_{QCD}. This holds for the instantaneous part of $D_{00}(|\vec{x}|, t)$. However the instantaneous part of $D_{00}(|\vec{x}|, t)$ may not be easy to separate uniquely (for example even in finite orders of perturbation theory), and a more accessible quantity is $U(|\vec{x}|) \equiv -\int dt D_{00}(|\vec{x}|, t)$. It also depends on physical masses only, as does its fourier transform $\tilde{U}(|\vec{k}|)$ which is given simply by $\tilde{U}(|\vec{k}|) = \tilde{D}_{00}(\vec{k}, k_0)|_{k_0=0}$. We write $\tilde{U}(|\vec{k}|) = g_c^2/\vec{k}^2$. Here $g_c = g_c(|\vec{k}|/\Lambda_{QCD})$ is a running coupling constant defined in the Landau-Coulomb gauge that depends only on Λ_{QCD}. Such a quantity cannot be extracted from the gluon propagator in covariant gauges. Indeed to extract it in covariant gauges one must consider the Wilson loop which involves n-point functions of all order n.

9 Coulomb gauge limit

We now turn to a more precise analysis of the behaviour of the correlation functions when the Coulomb-gauge limit $a \to 0$ is taken from the Landau-Coulomb interpolating gauge, characterized by $\beta = 0$. Because the gauge parameter a provides a rescaling of the time, instantaneous interactions appear as a approaches 0.

Consider the partition function in the Euclidean theory, with Landau-Coulomb type interpolating gauge, $\vec{\nabla} \cdot \vec{A} + a\dot{A}_0 = 0$,

$$Z = \int d^4A dc d\bar{c} db \ \exp\{ - \int d^4x \ [(\frac{1}{2})(\vec{E}^2 + \vec{B}^2) \ + ib(\vec{\nabla} \cdot \vec{A} + a\dot{A}_0) \\ + (a\dot{\bar{c}}D_0 c + \vec{\nabla}\bar{c} \cdot \vec{D}c) \,] \,\}, \tag{108}$$

where $t = x_0$ represents Euclidean "time", $E_i \equiv \dot{A}_i - D_i A_0$; $\vec{B} = \vec{B}(\vec{A})$, $D_\mu = D_\mu(A)$. (The i appears in front of b, because b is here integrated over a real instead of imaginary contour.) For simplicity, we have suppressed all sources, and a summation on color indices is understood.

We use the Gaussian identity $\exp[(-\frac{1}{2}) \int d^4x \ \vec{E}^2] = \int d^3P \exp[- \int d^4x \ (i\vec{P} \cdot \vec{E} + (\frac{1}{2})\vec{P}^2]$, to obain the phase-space representation

$$Z = \int d^4A d^3P dc d\bar{c} db \ \exp(-S), \tag{109}$$

where $\vec{A}$ and $\vec{P}$ are canonical variables, and

$$S \equiv \int d^4x \, [\, i\vec{P} \cdot (\dot{\vec{A}} - \vec{D}A_0) + (\frac{1}{2})\vec{P}^2 + (\frac{1}{2})\vec{B}^2 \\ + ib(\vec{\nabla} \cdot \vec{A} + a\dot{A}_0) + \ (a\dot{\bar{c}}D_0 c + \vec{\nabla}\bar{c} \cdot \vec{D}c) \,] \,. \tag{110}$$

The phase-space action is BRST-invariant, with $\vec{P}$ transforming according to $sP_i^a = f^{abd}P_i^b c^d$.

We now make a linear change of field variable in order to diagonalize the gluon propagator, while keeping the action local. We pose

$$A_0 = \vec{\nabla}^2 \psi \tag{111}$$

for which $dA_0 = \text{const}\ d\psi$, and we shift $\vec{A}$ by

$$\vec{A} = \vec{A}' - a\vec{\nabla}\dot{\psi}, \tag{112}$$

for which $d^3A = d^3A'$. This simplifies the Lagrange-multiplier term

$$ib\,(\vec{\nabla}\cdot\vec{A} + a\partial_0 A_0) = ib\,(\vec{\nabla}\cdot\vec{A}') \tag{113}$$

so it imposes the time-independent constraint $\vec{\nabla}\cdot\vec{A}' = 0$, and we have

$$i\vec{P}\cdot(\dot{\vec{A}} - \vec{D}A_0) = i\vec{P}\cdot(\dot{\vec{A}'} - a\vec{\nabla}\ddot{\psi} - \vec{D}A_0), \tag{114}$$

where $\vec{D} = \vec{D}(\vec{A}) = \vec{D}(\vec{A}' - a\vec{\nabla}\dot{\psi})$. We similarly separate $\vec{P}$ into its transverse and longitudinal parts, while keeping the action local, by introducing another lagrange multiplier field by means of the identity,

$$\begin{aligned}\text{const} &= \int d\Omega\ \delta(\vec{\nabla}\cdot\vec{P} + \vec{\nabla}^2\Omega) \\ &= \int d\Omega dv \exp[-i\int d^4x\ v\,(\vec{\nabla}\cdot\vec{P} + \vec{\nabla}^2\Omega)],\end{aligned} \tag{115}$$

which we insert into the partition function. We shift $\vec{P}'$ according to

$$\vec{P} = \vec{P}' - \vec{\nabla}\Omega, \tag{116}$$

under which $d^3P = d^3P'$, so the new Lagrange-multiplier term becomes

$$i\,v\,(\vec{\nabla}\cdot\vec{P} + \vec{\nabla}^2\Omega) = i\,v\,(\vec{\nabla}\cdot\vec{P}'), \tag{117}$$

and enforces the time-independent constraint $\vec{\nabla}\cdot\vec{P}' = 0$. The field Ω represents the color-Coulomb potential.

The partition function now reads

$$Z = \int d^3A'd^3P'dbdvd\psi d\Omega dcd\bar{c}\ \exp(-S') \tag{118}$$

where

$$\begin{aligned}S' \equiv \int d^4x\,[\,i\vec{P}\cdot(\dot{\vec{A}} - \vec{D}A_0) + (\frac{1}{2})\vec{P}^2 + (\frac{1}{2})\vec{B}^2\ + iv\,\vec{\nabla}\cdot\vec{P}' + ib\,\vec{\nabla}\cdot\vec{A}' \\ +(a\dot{\bar{c}}D_0c + \vec{\nabla}\bar{c}\cdot\vec{D}c)],\end{aligned} \tag{119}$$

$\vec{B} = \vec{B}(\vec{A}) = \vec{B}(\vec{A}' - a\vec{\nabla}\dot{\psi})$, and $\vec{P} = \vec{P}' - \vec{\nabla}\Omega$. The first term in S' is given by

$$\begin{aligned}i\vec{P}\cdot(\dot{\vec{A}} - \vec{D}A_0) &= i\vec{P}'\cdot(\dot{\vec{A}'} - a\vec{\nabla}\ddot{\psi} - \vec{D}A_0) \\ &\quad -i\vec{\nabla}\Omega\cdot(\dot{\vec{A}'} - a\vec{\nabla}\ddot{\psi} - \vec{D}A_0).\end{aligned} \tag{120}$$

To cancel cross terms in S' we shift the Lagrange multiplier fields,

$$\begin{aligned} b &= b' + \dot{\Omega} \\ v &= v' - (a\ddot{\psi} + A_0) - i\Omega, \end{aligned} \tag{121}$$

with $dbdv = db'dv'$, and obtain, after integrating by parts in space and time and writing $\vec{\nabla}^2\psi = A_0$,

$$\begin{aligned} S' = \int d^4x \, [\; & i\vec{P}' \cdot (\dot{\vec{A}}' - g\vec{A} \times A_0) + i(a\dot{\Omega}\dot{A}_0 + \vec{\nabla}\Omega \cdot \vec{D}A_0) \\ & + (\frac{1}{2})\vec{P}'^2 + (\frac{1}{2})(\vec{\nabla}\Omega)^2 + (\frac{1}{2})\vec{B}^2 \\ & + iv' \, \vec{\nabla} \cdot \vec{P}' + \; ib' \, \vec{\nabla} \cdot \vec{A}' + (a\dot{\bar{c}}D_0 c + \vec{\nabla}\bar{c} \cdot \vec{D}c) \,]. \end{aligned} \tag{122}$$

The remainder of this section is an analysis of the action S'. The Lagrange multiplier fields b' and v' enforce the time-independent constraints $\vec{\nabla} \cdot \vec{A}' = 0$ and $\vec{\nabla} \cdot \vec{P}' = 0$, on the canonically conjugate variables $\vec{A}'$ and $\vec{P}'$, and we call these "the transverse fields". The bose fields A_0 and Ω form a pair similar to the pair of fermi fields c and $\bar{c}$, and we call this quartet "the scalar fields".

The corresponding free action

$$\begin{aligned} S_0 = \int d^4x \, [\; & i\vec{P}' \cdot \dot{\vec{A}}' + \; (\frac{1}{2})\vec{P}'^2 + (\frac{1}{2})(\epsilon_{ijk}\nabla_j A'_k)^2 + iv' \, \vec{\nabla} \cdot \vec{P}' + \; ib' \, \vec{\nabla} \cdot \vec{A}' \\ & + i\,(a\dot{\Omega}\dot{A}_0 + \vec{\nabla}\Omega \cdot \vec{\nabla}A_0) + (\frac{1}{2})(\vec{\nabla}\Omega)^2 \\ & + (a\dot{\bar{c}}\dot{c} + \vec{\nabla}\bar{c} \cdot \vec{\nabla}c) \,]. \end{aligned} \tag{123}$$

determines the free propagators. In momentum space the propagators of the transverse fields are given by

$$\begin{aligned} D_{A'_i A'_j} &= (\delta_{ij} - \hat{k}_i\hat{k}_j)(k_0^2 + \vec{k}^2)^{-1} \\ D_{P'_i P'_j} &= (\delta_{ij}\vec{k}^2 - k_i k_j)(k_0^2 + \vec{k}^2)^{-1} \\ D_{P'_i A'_j} &= ik_0(\delta_{ij} - \hat{k}_i\hat{k}_j)(k_0^2 + \vec{k}^2)^{-1}, \end{aligned} \tag{124}$$

whereas the propagators of the scalar fields are given by

$$\begin{aligned} D_{A_0\Omega} &= (ak_0^2 + \vec{k}^2)^{-1} \\ D_{\Omega\Omega} &= 0 \\ D_{A_0 A_0} &= \vec{k}^2(ak_0^2 + \vec{k}^2)^{-2} \\ D_{c\bar{c}} &= (ak_0^2 + \vec{k}^2)^{-1}. \end{aligned} \tag{125}$$

Propagators of the ψ field are obtained from $\psi = (\vec{\nabla}^2)^{-1} A_0$. The new fields have conveniently diagonalized the gluon propagator by separating the 3-dimensionally transverse and scalar parts. The transverse propagators have denominators $(k_0^2 + \vec{k}^2)$, whereas the scalar propagators have denominators $(ak_0^2 + \vec{k}^2)$. Thus the scalar fields have a reaction time of order $a^{1/2}$ which is very rapid as a aproaches 0. Consequently it is natural to integrate out, if possible, the scalar fields and obtain an effective theory for the transverse degrees of freedom.

To study the limit $a \to 0$, we separate the action into 4 terms,

$$S' = S_r + S_X + S_Y + S_Z. \tag{126}$$

Here S_r is the free action S_0 plus all vertices that are independent of a,

$$\begin{aligned} S_r \equiv \int d^4x \, [\; & i\vec{P}' \cdot (\dot{\vec{A}}' - g\vec{A}' \times A_0) + (\frac{1}{2})\vec{P}'^2 + (\frac{1}{2})\vec{B}'^2 \\ & + i\, v' \, \vec{\nabla} \cdot \vec{P}' + \; ib' \, \vec{\nabla} \cdot \vec{A}' \\ & + i\,(a\dot{\Omega}\dot{A}_0 + \vec{\nabla}\Omega \cdot \vec{D}' A_0) + (\frac{1}{2})(\vec{\nabla}\Omega)^2 \\ & + (a\dot{\bar{c}}\dot{c} + \vec{\nabla}\bar{c} \cdot \vec{D}' c) \,], \end{aligned} \tag{127}$$

where $\vec{B}' \equiv \vec{B}(\vec{A}')$ and $\vec{D}' \equiv \vec{D}(\vec{A}')$. We shall see that S_r has graphs that diverge as $a \to 0$, but that they cancel by virtue of an r-invariance. The term S_X consists of all vertices with 3 scalar fields and one power of a,

$$S_X = ag \int d^4x [-i\nabla_i\Omega(\nabla_i\dot{\psi} \times A_0) + \dot{\bar{c}}(A_0 \times c) - \nabla_i\bar{c}(\nabla_i\dot{\psi} \times c)]. \tag{128}$$

The term S_Y consists of a vertex with one power of a,

$$S_Y = agi \int d^4x P_i'(\nabla_i\dot{\psi} \times A_0)]. \tag{129}$$

There remains

$$S_Z = \int d^4x (1/2)[\vec{B}^2(\vec{A}' - a\vec{\nabla}\dot{\psi}) - \vec{B}^2(\vec{A}')]. \tag{130}$$

We first discuss the theory defined by S_r, temporarily ignoring the vertices S_X, S_Y and S_Z that vanish with a. The action S_r is at most quadratic in the scalar fields. Its vertices contain no powers of a and no time derivatives, so in momentum space there are no factors of k_0 at the vertices of S_r. Consider a closed loop that consists entirely of scalar propagators with denominators $(ak_0^2 + \vec{k}^2)$. It is controlled by a time scale of order $a^{1/2}$. The loop integral on k_0 is effected by the change of variable $k_0 = a^{-1/2}k_0'$, which effectively eliminates a from the denominators, but the volume element of the loop integral changes by $dk_0 = a^{-1/2} dk_0'$. We conclude that each closed loop that consists entirely of scalar propagators and vertices of S_r diverges like $a^{-1/2}$.

Nevertheless the theory defined by S_r is finite as $a \to 0$, as we now show. We write

$$S_r = S_{r,1} + S_{r,2} \tag{131}$$

where $S_{r,1}$ consists of all terms that contain only transverse fields and their Lagrange multipliers,

$$\begin{aligned} S_{r,1} \equiv \int d^4x \, [\, i\vec{P}' \cdot \dot{\vec{A}}' + (\frac{1}{2})\vec{P}'^2 + (\frac{1}{2})\vec{B}'^2 \\ + i\, v' \, \vec{\nabla} \cdot \vec{P}' + \, ib' \, \vec{\nabla} \cdot \vec{A}']. \end{aligned} \tag{132}$$

It is independent of a. The remainder $S_{r,2}$ also depends on the scalar fields and on a. It is helpful to express $S_{r,2}$ in terms of the color charge density of the transverse fields $\rho \equiv gP_i' \times A_i'$, and the Faddeev-Popov operator $M \equiv -a\partial_0^2 - \vec{\nabla} \cdot \vec{D}'$ characteristic of S_r. We have, in an obvious notation,

$$S_{r,2} = -i(\rho, A_0) + i(\Omega, MA_0) + (1/2)(\vec{\nabla}\Omega, \vec{\nabla}\Omega) + (\bar{c}, Mc). \tag{133}$$

If one integrates out the ghost fields c and $\bar{c}$, one obtains the Faddeev-Popov determinant $\det M$. If one next integrates out A_0, one obtains $\delta(M\Omega - \rho)$, which expresses the form of Gauss's law appropriate to S_r. Finally the integral on $d\Omega$ absorbs the Faddeev-Popov determinent

$$\int d\Omega \det M \;\, \delta(M\Omega - \rho) \exp[-(1/2)(\vec{\nabla}\Omega, \vec{\nabla}\Omega)] = \text{const.} \times \exp(-S_{\text{coul}}), \tag{134}$$

where

$$S_{\text{coul}} \equiv (1/2)(\vec{\nabla}M^{-1}\rho, \vec{\nabla}M^{-1}\rho), \tag{135}$$

depends on the transverse fields only. It represents the non-local color-Coulomb interaction, regularized however by the finite value of a. Thus the theory described by the local action $S_r = S_{r,1} + S_{r,2}$ that contains the scalar fields is equivalent to the theory with transverse degrees of freedom only, described by the non-local action $S_{r,1} + S_{\text{coul}}$.

Moreover $S_{r,1} + S_{\text{coul}}$ at finite a provides a regularized version of the canonical action,

$$S_{\text{can}} = S_{r,1} + S_{\text{coul}}|_{a=0}. \tag{136}$$

The canonical action S_{can} is a function of the canonical variables which are the transverse fields $\vec{A}'$ and $\vec{P}'$. It is obtained by formal canonical quantization in the Coulomb gauge, in which one solves the constraints to eliminate the so-called unphysical degrees of freedom.

To show that the theory described by the local action S_r, or equivalently by $S_{r,1}+S_{\rm coul}$, is finite in the limit $a \to 0$, we observe that the perturbative expansion of $S_{\rm coul}$ produces ladder graphs, in which the instantaneous parts are the horizontal rungs, corresponding to the instantaneous color-Coulomb interaction. Since these ladder graphs do not contain any instantaneous closed loops, they are finite in the limit $a \to 0$. To summarize: in the theory described by S_r, each closed loop of bose and fermi scalars diverges like $a^{-1/2}$, but they precisely cancel to give a result that is finite as $a \to 0$.

It is helpful to exhibit the cancellation between bosons and fermions in the theory described by S_r by means of an r-symmetry. We express the action $S_{r,2}$ in terms of the field $\bar{\Omega} \equiv \Omega - M^{-1}\rho$,

$$S_{r,2} = i(\bar{\Omega}, MA_0) + (\bar{c}, Mc) + (1/2)(\vec{\nabla}\bar{\Omega}, \vec{\nabla}\bar{\Omega}) + (\vec{\nabla}\bar{\Omega}, \vec{\nabla}M^{-1}\rho) + S_{\rm coul}. \tag{137}$$

Let r be a BRST-type transformation that acts on the scalar fields according to

$$\begin{aligned} rA_0 &= c & rc &= 0 \\ r\bar{c} &= -i\bar{\Omega} & r\bar{\Omega} &= 0, \end{aligned} \tag{138}$$

and that annihilates the transverse fields and their Lagrange multipliers, $r\vec{A}' = r\vec{P}' = rb' = rv' = 0$. It is nil-potent, $r^2 = 0$. The action $S_{r,2}$ may be written

$$S_{r,2} = S_{\rm coul} + r\Psi, \tag{139}$$

where

$$\Psi = -\ (\bar{c}, MA_0) + (i/2)(\vec{\nabla}\bar{c}, \vec{\nabla}\Omega') + i(\vec{\nabla}\bar{c}, \vec{\nabla}M^{-1}\rho), \tag{140}$$

and we have

$$S_r = S_{r,1} + S_{\rm coul} + r\Psi. \tag{141}$$

The first 2 terms depend on the transverse fields only, and are thus r-invariant, $rS_{r,1} = rS_{\rm coul} = 0$. The last term $r\Psi$, which contains all the dependence on the scalar fields, is r-exact. We have $rS_r = 0$, and r is indeed a symmetry of the theory defined by S_r. Now consider the integral over the scalar fields while the transverse fields and their Lagrange multipliers are held fixed. The effective action for the scalar fields is $r\Psi$, which is r-exact. A theory whose action is exact under a BRST-type transformation is called "topological", and has the property that the partition function,

$$\int dA_0 d\Omega dc d\bar{c}\ \exp(-r\Psi), \tag{142}$$

is constant under continuous variations of the external parameters, namely the transverse fields and the parameter a. We have obtained the previous result, with the understanding that the cancellation of bose and fermi loops that diverge in the limit $a \to 0$ is preserved by the r-symmetry of S_r. The r-symmetry which transforms A_0 into c explains the equality of renormalization constants that holds in the limit $a \to 0$, $Z_{A_0} = Z_c$, which was established in the last section .

We now come to the remaining vertices of S', namely S_X, S_Y and S_Z. These vertices formally vanish in the limit $a \to 0$, and they would not appear in formal canonical quantization in the Coulomb gauge. However because, as we have seen, there are closed loops in the expansion of S_Z that are of order $a^{-1/2}$ (and that cancel pairwise), we must verify whether insertions into these loops of the vertices S_X, S_Y or S_Z may give a finite result. These vertices are not r-invariant, so if there are such contributions there is no reason to expect that they cancel.

Consider first the vertices of S_X which we call X-vertices. (Similarly we call r-vertices the vertices of S_r etc.) The X-vertices are linear in a. They also contain one time derivative, so in momentum space they contain one power of $k_0 = a^{-1/2}k_0'$. Thus overall when an X-vertex is inserted into a closed loop of scalar propagators it gives a contribution of order $a^{1/2}$. As we have observed from $dk_0 = a^{-1/2}dk_0'$, the volume element for a closed loop consisting of scalar propagators is of order $a^{-1/2}$. Thus the presence of a single X-vertex in a closed loop of scalar propagators and r-vertices would give a finite limit, except for the fact that such a loop is odd in k_0' at large k_0', and consequently a closed loop of scalar propagators with a single X-vertex is reduced to order $a^{1/2}$. By the same reasoning, a closed loop of scalar propagators that contains two X-vertices (and is thus even in k_0') is also of order $a^{1/2}$. Thus a single closed loop with one or two X-vertices vanishes like $a^{1/2}$ as $a \to 0$. However a closed scalar loop with two X-vertices has two external scalar lines, because each X-vertex is trilinear in the scalar fields. Consequently such a loop may be inserted into a closed scalar loop whose remaining vertices are all r-vertices. [See fig. (1).] This gives a two-loop graph, with two X-vertices, each of order $a^{1/2}$, and two closed loops of scalar propagators, each of order $a^{-1/2}$. This is finite in the limit $a \to 0$. (Further insertion of X-vertices gives a vanishing contribution in the limit.) We conclude that scalar bose or fermi closed loops do not decouple as $a \to 0$, but give a finite two-loop graph. This contribution is missing in formal canonical quantization in the Coulomb gauge.

The analysis of the vertices of S_Y is similar. Each Y-vertex contains two scalar fields and one $\vec{P}'$ field. It also contains one power of a and one time derivative, so a Y-vertex is also of order $a^{1/2}$. Again, insertion of single Y-vertex into a scalar closed loop would be finite except that it is odd in k_0'. We cannot connect up two Y-vertices by an additional scalar propagator because Y-vertices are bilinear in the scalar fields. However the Y-vertex contains the $\vec{P}'$ field which has the $P_i' - A_j'$ propagator $P_{ij}(\hat{k})k_0(k_0^2 + \vec{k}^2)^{-1}$ that contains k_0 in the numerator. (It is the only propagator with k_0 in the numerator.) Now consider a closed loop that consists of scalar propagators and one $\vec{P}' - \vec{A}'$ propagator. All the vertices are r-vertices except for one Y-vertex at one end of the $P_i' - A_j'$ propagator. [See fig. (2).] When the loop momentum k_0 is of order $a^{-1/2}$, the Y-vertex is of order $a^{1/2}$, the $P_i' - A_j'$ propagator is of order $a^{1/2}$, and the volume element of the loop integral is of order $a^{-1/2}$, so overall this closed loop is of order $a^{1/2}$. However it has two scalar external lines that emerge from the two ends of the $\vec{P}' - \vec{A}'$ propagator. Consequently this closed loop, which is of order $a^{1/2}$, may be inserted into in a scalar closed loop consisting of S_r vertices which is of order $a^{-1/2}$. [See fig. (3).] This again gives a finite two-loop contribution that is missing in canonical quantization in the formal Coulomb gauge. (Further insertions of Y-vertices give a vanishing contribution in the limit.)

Finally, the vertices of S_Z give vanishing contribution in the limit $a \to 0$, because when they contain 2 or 3 scalar fields they also contain 2 or 3 powers of a respectively.

We summarize the results of this section: (1) The diagrams for which the k_0 integrations would diverge in the Coulomb-gauge limit, $a \to 0$, have been have been shown to cancel at finite a. The remaining diagrams are finite in this limit by power counting of the k_0 integration. (2) There are two-loop graphs of the scalar particles A_0-Ω and $c - \bar{c}$ that are finite in the limit $a \to 0$, and that are missing from canonical quantization in the formal Coulomb gauge. It remains a logical possibility that these graphs are mere gauge artifacts that do not contribute to a gauge-invariant expectation-value such as a Wilson loop. However there is at the moment no argument to show that this is true.

Remarks

1. The correlation functions that do not involve the field P are the same as in the configuration-space representation, so the finiteness of the unrenormalized correlation functions in the Coulomb-gauge limit of the Landau-Coulomb interpolating gauge also holds in each order n for the configuration-space correlation functions. This implies that the configuration-space generating functionals Z, W, Γ and $\tilde{\Gamma}$ are also finite in the limit $a \to 0$. Here an ultraviolet dimensional regulator ϵ is understood to be in place. For the diagrams we have examined, the a-dependence at small a is given by $a^{-m/2}$, where m is a non-negative integer. (The terms with negative powers of $a^{1/2}$ cancel.) These powers are ϵ-independent, and so cause no trouble in the $\epsilon \to 0$ limit (as would,

for example, terms like a^ϵ). This is because the terms that diverge with a come from divergences in the one-dimensional k_0 integrations and are not affected by dimensional regularization which is a continuation in the number of spatial dimensions. Likewise the cancellation of terms that diverge as $a \to 0$ is assured by r-invariance, and is also dimension-independent. Moreover $\tilde{\Gamma}^n_{\rm div}(a, \epsilon)$, eq. (67), has a simple pole structure in ϵ. Consequently the finiteness of $\tilde{\Gamma}^n(a, \epsilon) = \tilde{\Gamma}^n_R(a, \epsilon) + \tilde{\Gamma}^n_{\rm div}(a, \epsilon)$ as $a \to 0$ implies that the residue of $\tilde{\Gamma}^n_{\rm div}(a, \epsilon)$ and $\tilde{\Gamma}^n_R(a, \epsilon)$ are separately finite as $a \to 0$. Although we have not made an exhaustive examination of all diagrams, we expect that the remaining diagrams behave similarly, and thus that the renormalized correlation functions are finite in the Coulomb-gauge limit of the interpolating Landau-Coulomb gauge.

2. We may regard the finite value of the 2-loop scalar graphs that are missing in the formal Coulomb gauge, $a = 0$, as an anomaly of the r-symmetry; for the action $S'(a)$ is r-invariant at $a = 0$, $rS'(0) = 0$, but not at finite a, and the symmetry is not regained in the limit $a \to 0$. This comes about because individual graphs diverge in this limit, and they combine with subgraphs containing r-noninvariant vertices which vanish in the limit, to give a finite result. However the divergent graphs result from a part of the action $r\Psi(a)$ that is r-exact at finite a, and thus topological. This assures that the divergent graphs cancel each other, so that the limit is finite. It also preserves the equality, $Z_{A_0} = Z_c$, among the limiting renormalization constants found in the last section, eq. (96), which would hold if the transformation $rA_0 = c$ were actually a symmetry of the limiting theory.

3. To establish the Gauss-BRST Ward identity of the last section, there remains to verify that $\langle \dot{Q}_a \rangle = 0$ in the limit $a \to 0$, where the expectation value is calculated in the presence of all sources. Here Q_a, is the part of the total BRST charge Q defined in eq. (87),

$$Q_a = a \int d^3x [bD_0c - \partial_0 bc + \partial_0 \bar{c}(-g/2)c \times c]. \tag{143}$$

To evaluate $\langle \dot{Q}_a \rangle$, one makes a diagrammatic expansion of each term by the method of the present section, using $b = b' + \dot{\Omega}$. The only non-zero propagator of the b' field is the $b' - A'_i$ propagator, $k_i(ak_0^2 + \vec{k}^2)^{-1}$. For example, consider the contribution of the term $\dot{b}^d c^d = \dot{b}'^d c^d + \ddot{\Omega}^d c^d$. The term $\ddot{\Omega}^d c^d$ looks dangerous because it contains two powers of k_0. However the only non-zero propagator of the Ω field is the $\Omega - A_0$ propagator, and the vertex where A_0 is absorbed is proportional to a. A typical graph representing the contribution of $\ddot{\Omega}^d c^d$ to the fourier transform $\int dt \exp(ip_0 t) \langle \dot{Q}_a(t) \rangle$ is illustrated in fig 4. (Note that Q_a is a color scalar, so it must decay into at least two quanta, namely c and

A_i in fig. 4.) This graph contributes

$$ap_0\ (2\pi)^{-4} \int d^4k\ k_0^2(ak_0^2 + \vec{k}^2)^{-1}\ a(k_0 + q_0)[a(k_0 + q_0)^2 + (\vec{k} + \vec{q})^2]^{-1}$$
$$\times k_i[a(k_0 + p_0)^2 + \vec{k}^2]^{-1}. \qquad (144)$$

We rescale the variable of integration $dk_0 = a^{-1/2}dk_0'$, and obtain a contribution of leading order $a^{1/2}$, keeping in mind that terms that are asymptotically odd in k_0' are suppressed by $a^{1/2}$. The other terms are evaluated similarly. One finds that each term of Q_a gives a contribution to the expectation-value of order $a^{1/2}$. QED.

4. We we have seen that a closed loop of unphysical particles, with propagators $(k_1^2 + k_2^2 + k_3^2 - ak_0^2)^{-1}$, is of order $a^{-1/2}$ as the Coulomb-gauge limit of the Landau-Coulomb interpolating gauge is approached. This behavior came from the rescaling, $k_0 = a^{-1/2}k_0'$, that makes the loop integral of order $dk_0 = a^{-1/2}dk_0'$. However in the light-front or axial gauge, the unphysical propagator is $[a(k_1^2 + k_2^2) + (1 + a)(k_3^2 - k_0^2)/2]^{-1}$ or $[a(k_1^2+k_2^2-k_0^2)+k_3^2]^{-1}$, and the required rescaling, $(k_1, k_2) = a^{-1/2}(k_1', k_2')$ or $(k_0, k_1, k_2) = a^{-1/2}(k_0', k_1', k_2')$, gives an uphysical closed loop integral of order a^{-1} or $a^{-3/2}$. Thus the light-cone and axial gauge limits appear to be more singular than the Coulomb-gauge limit, and additional cancellations would be required to give finite correlation functions.

10 Conclusion

We briefly review our results. We have addressed the problem of the existence of "physical gauges", by the device of interpolating gauges which interpolate linearly between a covariant gauge, such as the Feynman or Landau gauge and a physical gauge such as the Coulomb or light-cone gauge. For example, the interpolating Landau-Coulomb interpolating gauge is defined by the gauge condition $a\partial_0 A_0 + \vec{\nabla}\cdot\vec{A} = 0$, which gives the Landau gauge for $a = 1$, and the Coulomb gauge is achieved in the singular limit $a \to 0$. More generally an interpolating gauge is defined by the condition $\alpha^{\mu\nu}\partial_\nu A_\mu = 0$ (or $= f$), where α is a non-singular numerical matrix, and a "physical" gauge is a limiting case in which α becomes singular.

In general the interpolating gauge breaks Lorentz invariance as well as local gauge invariance. Nevertheless we are able to establish the existence of the perturbative expansion and perturbative renormalizability of the interpolating gauges in full generality, by extending the BRST method to include the Lorentz group in addition to the usual local gauge group. This extension is necessary to control the form of divergences, for example to show that the divergent coefficients of the term $c_E\vec{E}^2 + c_B\vec{B}^2$ are equal, $c_E = c_B$.

The enumeration of the possible divergence terms that are BRST-invariant is not substantially more difficult than for Lorentz-covariant gauges. Moreover the matrix α is a gauge-parameter in the sense that the expectation values of physical observables are independent of α, as long as α is non-singular. Thus the interpolating gauges are strictly equivalent to the covariant gauges.

However the singular limit to a physical gauge is quite subtle. It is analyzed in the present article for the Coulomb gauge limit, $a \to 0$, from the Landau-Coulomb interpolating gauge. There are closed bose and fermi-ghost loops that become instantaneous in the limit $a \to 0$ and that individually diverge like $a^{-1/2}$. We we use a phase space representation and a linear shift of field variables to exhibit the cancellation of loops that diverge like $a^{-1/2}$, and to show by power counting of the k_0 integrals that this limit gives finite correlation functions.

An important aspect of this limit is that there are also closed bose and fermi one-loop graphs that are not present in the formal Coulomb gauge ($a = 0$). Although they vanish like $a^{1/2}$, they cannot be neglected because, when these one-loop graphs are inserted into the above-mentioned closed loops that diverge like $a^{-1/2}$, they give a finite contribution. Consequently the closed bose and fermi-ghost loops do not decouple in the Landau-Coulomb gauge limit, but give a finite two-loop contribution.

One logical possibility is that these two-loop ghost contributions are merely a gauge artifact that do not actually contribute to expectation-values of gauge-invariant objects such as Wilson loops. However there is at present no argument in hand to show this. If these two-loop bose and fermi-ghost graphs do contribute to physical expectation values, then the traditional picture of the Coulomb gauge would have to be revised. The state space would not be simply describable in terms of transverse gluons. In the latter case the Coulomb gauge is not more unitary than other gauges, in the sense that it cannot be simply described in terms of the classical dynamical variables that remain after the constraints are solved. Indeed we are unable to provide a set of Feynman rules to be used in the Coulomb gauge at $a = 0$, although we have shown that both the unrenormalized and renormalized correlation functions are finite in the limit $a \to 0$ of the Landau-Coulomb interpolating gauge.

Nevertheless there is a reward to be gained by taking this limit. For we have shown that the Gauss-BRST Ward identity holds in the Coulomb gauge limit of the Landau-Coulomb interpolating gauge. This identity is the functional analog of the operator statement that the BRST symmetry transformation is generated by the Gauss-BRST charge. Among other things, it implies that gA_0 is invariant under renormalization,

$gA_0 = g^{(r)} A_0^{(r)}$. This means that all correlation functions of gA_0 are renormalization-group invariants, including in particular the time-time component of the gluon propagator $g^2 D_{00}$. It depends only on physical masses such as Λ_{QCD}, but is independent of the cut-off or the renormalizaion mass. Thus the Coulomb-gauge limit of the Landau-Coulomb gauge provides direct access to renormalization-group invariant quantities, whereas no component of the gluon propagator has this property in covariant gauges. Indeed in covariant gauges one must go to the Wilson loop, which involves gluon correlation functions of all orders, to obtain a renormalization-group invariant quantity. For this reason the Coulomb gauge may prove advantageous for non-perturbative formulations. In particular, the instantaneous part of $g^2 D_{00}$ may be a confining color-Coulomb potential that may serve as an order parameter for confinement of color [2].

Acknowledgements: We wish to thank the Newton Institute for its hospitality, where part of this work was done. One of us (LB) wishes to thank SISSA. One of us (DZ) recalls with pleasure stimulating conversations with Massimo Porrati and Martin Schaden.

11 Figure Captions

1. Diagram with 2 scalar loops and 2 X-vertices, and any number of r-vertices.
2. One-loop diagram with a single $P' - A'$ propagator and an r-vertex.
3. Insertion of graph of fig. 2 into a closed scalar loop of r-vertices.
4. A typical graph contributing to $\langle \dot{Q}_a \rangle$.

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