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Quantum Fluctuations in Open Pre-Big Bang Cosmology

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Abstract

We solve exactly the (linear-order) equations for tensor and scalar perturbations over the homogeneous, isotropic, open pre-big bang model recently discussed by several authors. We find that, in spite of claims to the contrary, vacuum quantum fluctuations, although parametrically amplified, remain negligible throughout the perturbative pre-big bang phase.

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I. INTRODUCTION

The question of whether, in the presence of spatial curvature, the pre-big bang (PBB) scenario [1–3] needs a very large amount of fine-tuning is still a subject of debate [4–10]. Without addressing this issue directly, we discuss here a related objection to the PBB idea recently raised by Kaloper et al. [8]. These authors have claimed that, even assuming that the two classical moduli of the open ($\mathcal{K} = -1$), homogeneous, isotropic cosmological solution [11,5] lie deeply inside the perturbative region, the unavoidable existence of vacuum quantum fluctuations, and their cosmological amplification (for a review, see [12]), modifies so drastically the classical behaviour as to prevent the occurrence of an appreciable amount of inflation.

In this paper we will check this claim by carrying out a detailed study of quantum fluctuations around the $\mathcal{K} = -1$ solution of [11,5]. It is well known [13] that quantum fluctuations in a non-spatially flat background are considerably harder to study than the corresponding ones in a flat Universe. Nevertheless, somewhat to our surprise, the corresponding equations can still be integrated *exactly* in terms of hypergeometric functions. This allows us to conclude that, contrary to what was stated in [8], quantum fluctuations remain small through the whole perturbative PBB phase and do not impede the occurrence of PBB inflation. Complementary arguments reaching the same conclusion were recently given in [10].

We will first recall the explicit form of the homogeneous, isotropic, $\mathcal{K} = -1$ PBB background we shall be dealing with and derive the general, covariant form of the action to second order in the perturbations. We then solve, successively, the equations for tensor and scalar perturbations. Finally, we discuss the physical implications of our results.

II. THE BACKGROUND AND THE SECOND-ORDER ACTION

Our conventions are such that (after reduction to $D = 4$) the (normalized) string-frame action takes the form

$$\hbar^{-1} S^{(s)} = \frac{1}{2\ell_s^2} \int d^4x \sqrt{-G} e^{-\phi} (R(G) + G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \ldots) \, , \tag{2.1}$$

where $G_{\mu\nu}$ is the string-frame metric, ϕ is the ($D = 4$) dilaton, ℓ_s is the fundamental length scale of string theory, and the dots indicate other fields (e.g. a Kalb–Ramond axion field) that will be set to zero hereafter. The above action allows for classical homogeneous, isotropic solutions of the standard Friedmann–Robertson–Walker (FRW) type

$$ds^2 = a_s^2(\eta) \left(-d\eta^2 + \frac{dr^2}{1 - \mathcal{K} r^2} + r^2 d\Omega^2 \right) . \tag{2.2}$$

As usual there are both post- and pre-big bang solutions coming from a singularity, or going towards it, respectively. For $\mathcal{K} = -1$, the PBB-type solution was first given in [11] and then rederived and discussed in [5]. It reads:

$$\begin{aligned} a_s(\eta) &= L (\cosh \eta)^{\frac{1+\sqrt{3}}{2}} (-\sinh \eta)^{\frac{1-\sqrt{3}}{2}} \\ \phi(\eta) &= -\sqrt{3} \ln(-\tanh \eta) + \phi_{\rm in} \, , \quad \eta < 0 \, , \end{aligned} \tag{2.3}$$

where L and $\phi_{\rm in}$ are a dimensional and a dimensionless integration constant, respectively.

The arbitrariness of L and $\phi_{\rm in}$ reflects the symmetries of the classical problem under a constant shift of the dilaton ϕ and a constant rescaling of the metric $G_{\mu\nu}$. These are precisely the two parameters to be chosen in an appropriate (fine-tuned [4–10]?) range in order to ensure a sufficient amount of PBB inflation. Indeed, Eq. (2.3) describes a universe that is almost trivial (Milne-like) from $-\infty < \eta < \mathcal{O}(-1)$, and then inflates with an initial curvature $\mathcal{O}(L^{-2})$ and initial coupling $\mathcal{O}(\exp(\phi_{\rm in}/2))$ till it meets, eventually, the strong curvature and/or strong coupling regimes at $\eta \sim \eta_1$. The critical value η_1 is easily determined in terms of the integration constants L and $\phi_{\rm in}$:

$$(-\eta_1) = \max\left(e^{\phi_{\rm in}/\sqrt{3}}, (\ell_s/L)^{1+1/\sqrt{3}}\right) . \tag{2.4}$$

It is well known [1] that the study of perturbations is technically simpler in the so-called Einstein frame, defined by $g_{\mu\nu} = \exp(\phi_{\rm today} - \phi) G_{\mu\nu}$, and, correspondingly, by the action:

$$\hbar^{-1} S^{(E)} = \frac{1}{2\ell_P^2} \int d^4x \sqrt{-g} \left(R(g) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) , \tag{2.5}$$

where $\phi_{\rm today}$ is the present value of the dilaton and $\ell_P \equiv \sqrt{8\pi G\hbar} = \exp(\phi_{\rm today}/2)\ell_s \sim 0.1\ell_s$ is the present value of Planck's length. We will compute perturbations in the Einstein frame and then convert the results back to the original string frame for a physical interpretation.

In the Einstein frame the background equations for a generic FRW universe are given by[1]:

$$\begin{aligned} \mathcal{H}' &= -\frac{1}{6}\phi'^2, \quad \text{where} \quad \mathcal{H} = \frac{a'}{a} \\ \mathcal{H}^2 + \mathcal{K} &= \frac{1}{12}\phi'^2, \qquad \phi'' + 2\mathcal{H}\phi' = 0 , \end{aligned} \tag{2.6}$$

where a prime denotes differentiation with respect to the conformal time η. For $\mathcal{K} = -1$ the solution is just given by rewriting (2.3) in the Einstein frame:

$$\begin{aligned} a(\eta) &= \ell\, (-\sinh\eta \cosh\eta)^{\frac{1}{2}} \\ \phi(\eta) &= -\sqrt{3}\ \ln(-\tanh\eta) + \phi_{\rm in} , \quad \eta < 0 , \end{aligned} \tag{2.7}$$

where the new modulus ℓ, given by $\ell^2 = L^2 \exp(\phi_{\rm today} - \phi_{\rm in})$, replaces the string-frame classical modulus L.

To estimate quantum fluctuations around (2.7) we first go over to isotropic spatial coordinates (x, y, z) defined by

$$r = R \left(1 + \frac{\mathcal{K}}{4} R^2 \right)^{-1} , \qquad \text{where} \quad R^2 = x^2 + y^2 + z^2 , \tag{2.8}$$

[1] Although we restrict our attention to the case $\mathcal{K} = -1$, we will occasionally keep $\mathcal{K}$ in the formulae for an easy comparison with the spatially-flat case.

and by the obvious identification of the angular coordinates. In these coordinates the FRW metric takes the generic form

$$ds^2 = a^2(\eta)\left(-d\eta^2 + \gamma_{ij}dx^i dx^j\right), \qquad \text{where} \quad \gamma_{ij} = \delta_{ij}\left(1 + \frac{\mathcal{K}}{4}R^2\right)^{-2}, \ i,j = 1,2,3 \ , \tag{2.9}$$

and generic perturbations are defined by

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \ , \quad \phi = \phi^{(0)} + \delta\phi \ , \tag{2.10}$$

where a superscript (0) denotes the background solution.

We now consider the form of the action (2.5) up to second-order terms in the fluctuations. The calculations are long but straightforward. After using the background equations (2.6), and after dropping irrelevant boundary terms (total divergences), the result can be expressed covariantly in the form:

$$\begin{aligned}
\delta^{(2)}S = \frac{1}{2\ell_P^2}\int d^4x\,\sqrt{-g}\,\Big[&-\frac{1}{2}g^{\mu\nu}g^{\alpha\beta}g^{\lambda\sigma}\Big(\nabla_\lambda\delta g_{\beta\mu}\nabla_\sigma\delta g_{\nu\alpha} - \nabla_\sigma\delta g_{\mu\nu}\nabla_\lambda\delta g_{\alpha\beta} \\
&+ 2\nabla_\alpha\delta g_{\mu\nu}\nabla_\sigma\delta g_{\beta\lambda} - 2\nabla_\lambda\delta g_{\beta\mu}\nabla_\nu\delta g_{\alpha\sigma}\Big) - g^{\mu\nu}\partial_\mu\delta\phi\partial_\nu\delta\phi \\
&+ g^{\mu\nu}g^{\lambda\sigma}\partial_\lambda\phi\delta\phi\nabla_\sigma\delta g_{\mu\nu} - 2g^{\mu\lambda}g^{\nu\sigma}\partial_\lambda\phi\delta\phi\nabla_\sigma\delta g_{\mu\nu} - 2g^{\mu\lambda}g^{\nu\sigma}\nabla_\sigma\partial_\lambda\phi\delta\phi\delta g_{\mu\nu}\Big] ,
\end{aligned} \tag{2.11}$$

where, to this order, we can replace $g_{\mu\nu}$ and ϕ by their background expression (2.7), and all covariant derivatives are to be evaluated with respect to the background metric.

III. SOLVING THE PERTURBATION EQUATIONS

A. Tensor perturbations

Since tensor metric perturbations are automatically gauge-invariant, and decouple from dilatonic perturbations, they are easier to study. They can be defined by

$$\delta g^{(\mathrm{T})}_{\mu\nu} = \mathrm{diag}(0, a^2 h_{ij}) \ , \tag{3.1}$$

where the symmetric three-tensor h_{ij} satisfies the transverse-traceless (TT) conditions

$$\nabla^i h_{ij} = 0, \qquad h^i{}_i = 0 \ , \tag{3.2}$$

with ∇^i denoting the covariant derivative with respect to γ_{ij}. Inserting (3.1) into Eq. (2.11), and using (2.6), we easily find:

$$\delta^{(2)}S^{(T)} = \frac{1}{4\ell_P^2}\int d^4x\sqrt{\gamma}\,a^2\left(h'^{ij}h'_{ij} - \nabla^l h^{ij}\nabla_l h_{ij} - 2\mathcal{K}h^{ij}h_{ij}\right) \ . \tag{3.3}$$

For $\mathcal{K} = -1$, tensor perturbations h_{ij} can be expanded in TT tensor pseudospherical harmonics [14] as

$$h_{ij}(\eta,\mathbf{x}) = \int dn \sum_{l=2}^{\infty}\sum_{m=-l}^{l} h_{nlm}(\eta)(G_{ij}(\mathbf{x}))^n_{lm} \ , \tag{3.4}$$

where the tensor harmonics $(G_{ij})^n_{lm}$ satisfy the eigenvalue equation

$$\nabla^2(G_{ij}(\mathbf{x}))^n_{lm} = -(n^2+3)(G_{ij}(\mathbf{x}))^n_{lm} \ . \tag{3.5}$$

Choosing their normalization so that:

$$\int d^3x\sqrt{h}(G_{ij}(\mathbf{x}))^n_{lm}(G_{ij}(\mathbf{x}))^{n'}_{l'm'} = \delta(n-n')\delta_{ll'}\delta_{mm'} \ , \tag{3.6}$$

and inserting (3.4) in (3.3), we obtain

$$\delta^{(2)}S^{(T)} = \frac{1}{4\ell_P^2}\int d\eta\ dn\ a^2\sum_{l,m}\left[(h'_{nlm})^2-(n^2+1)h^2_{nlm}\right]. \tag{3.7}$$

Introducing finally the canonical variable

$$u_{nlm} = a h_{nlm} \ , \tag{3.8}$$

and using the background equations (2.6), we get:

$$\delta^{(2)}S^{(T)} = \frac{1}{4\ell_P^2}\int d\eta\ dn\ \sum_{l,m}\left[(u'_{nlm})^2-(n^2+\frac{1}{12}\phi'^2)u^2_{nlm}\right], \tag{3.9}$$

yielding for u_{nlm} the simple equation

$$u''_{nlm} + \left(n^2+\frac{1}{12}\phi'^2\right)u_{nlm} = 0 \ . \tag{3.10}$$

Luckily, for the background (2.7), Eq. (3.10) can be exactly solved in terms of the standard hypergeometric function $F \equiv {}_2F_1$ [15] by

$$\begin{aligned} u_N(\eta) = {} & C_1\left[\mathrm{csch}^2(2\eta)\right]^{-\frac{in}{4}}F\left[\frac{1-in}{4},\frac{1-in}{4},\frac{2-in}{2},-\mathrm{csch}^2(2\eta)\right] \\ & + C_2\left[\mathrm{csch}^2(2\eta)\right]^{\frac{in}{4}}F\left[\frac{1+in}{4},\frac{1+in}{4},\frac{2+in}{2},-\mathrm{csch}^2(2\eta)\right] \ , \end{aligned} \tag{3.11}$$

where N stands for the collection of indices (nlm) and $C_{1,2}$ are (classically arbitrary) integration constants. In order to correctly normalize the tensor perturbations, the action (3.9) has to be quantized. At early times, $n^2 \gg \phi'^2$, and thus u is a free canonical field. Hence we impose, as $\eta \to -\infty$,

$$u_N(\eta) \to u_N^{-\infty}(\eta) \equiv \frac{2\ell_P}{\sqrt{n}}e^{-in\eta}. \tag{3.12}$$

Using $F[a,b,c,0]=1$, Eq. (3.12) fixes the integration constants as $|C_1| = 2\ell_P/\sqrt{n}, C_2 = 0$. The deviation from a trivial plane-wave behaviour can easily be computed from the small argument limit of F. We find

$$u_N(\eta) = u_N^{-\infty}(\eta)\left(1 + \alpha_n\, e^{4\eta - i\beta_n}\right) , \tag{3.13}$$

where α_n, β_n are n-dependent constants fixed from the Taylor expansion of the hypergeometric function. We note that the correction to the vacuum amplitude dies off as $e^{4\eta}$, i.e. as t^{-4} in terms of cosmic time $t \sim -e^{-\eta}$.

We finally estimate the behaviour of the solution near the singularity, i.e. for $\eta \to 0$, using [15]

$$F\left[a, a, c, -\mathrm{csch}^2(2\eta)\right] \simeq \frac{\Gamma(c)}{\Gamma(a)\Gamma(a+\frac{1}{2})}\left[-2^{2a+1}|\eta|^{2a}\ln|\eta|\right] . \tag{3.14}$$

Then, by virtue of the small η behaviour $a \simeq \ell|\eta|^{1/2}$ and of Eq. (3.8), we find

$$|h_N| \simeq 2\sqrt{\frac{2}{\pi}}\frac{\ell_P}{\ell}\sqrt{\coth\left(\frac{n\pi}{2}\right)}\ln|\eta| . \tag{3.15}$$

We shall come back to this result after deriving a similar expression for scalar perturbations.

B. Scalar perturbations

Consider now scalar metric-dilaton perturbations defined by [12]

$$\delta\phi, \qquad \delta g_{\mu\nu}^{(\mathrm{S})} = -a^2(\eta)\begin{pmatrix} 2\varphi & \nabla_i B \\ \nabla_i B & 2(\psi\gamma_{ij} + \nabla_i\nabla_j E)\end{pmatrix} . \tag{3.16}$$

Inserting (3.16) in Eq. (2.11), and making use of (2.6), we find

$$\begin{aligned}\delta^{(2)}S^{(S)} = & \frac{1}{2\ell_P^2}\int d^4x\, a^2(\eta)\sqrt{\gamma}\Big[(\delta\phi')^2 - \nabla\delta\phi\cdot\nabla\delta\phi + 6\phi'\delta\phi\psi' - 2\varphi\phi'\delta\phi' - 2\phi'\delta\phi\nabla^2(B - E') \\ & - 12\psi'^2 - 8\nabla\varphi\cdot\nabla\psi + 4(\nabla\psi)^2 - 24\mathcal{H}\varphi\psi' + 12\mathcal{K}(\varphi^2 - \psi^2 + 2\varphi\psi) - 8\nabla\psi'\cdot\nabla B \\ & - 8\mathcal{H}\nabla\varphi\cdot\nabla B - 8\mathcal{H}\varphi\nabla^2 E' - 8\psi'\nabla^2 E' + 4\mathcal{K}(B - E')\nabla^2(B - E')\Big] . \end{aligned} \tag{3.17}$$

In (3.17) the variables B, φ do not have time derivatives and thus act as Lagrange multipliers, which provide constraints. These are:

$$\begin{aligned} 0 &= \mathcal{C}_{\mathcal{B}} \equiv \phi'\delta\phi - 4\psi' - 4\mathcal{H}\varphi - 4\mathcal{K}(B - E') \\ 0 &= \mathcal{C}_{\varphi} \equiv \phi'\delta\phi' - 12\mathcal{K}\varphi + 12\mathcal{H}\psi' - 4(\nabla^2 + 3\mathcal{K})\psi - 4\mathcal{H}\nabla^2(B - E') . \end{aligned} \tag{3.18}$$

Following [13], we introduce the gauge-invariant variable Ψ by

$$\Psi = \frac{4}{\phi'}[\psi + \mathcal{H}(B - E')] , \tag{3.19}$$

and, after inserting the constraints, we recast the action (3.17) in the convenient form

$$\delta^{(2)}S^{(S)} = \frac{1}{2\ell_P^2}\int d^4x\, a^2\sqrt{\gamma}(\nabla^2 + 3\mathcal{K})\Psi\left[\partial_\eta^2 - \nabla^2 + 2(\mathcal{H}' + \mathcal{K})\right]\Psi . \tag{3.20}$$

One can now make use of the constraints to eliminate the variable $(B - E')$ from the action (3.20) in terms of φ, ψ and $\delta\phi$. The latter variables are not independent either, being related by a linear combination of the two constraints $\mathcal{C}_\varphi, \mathcal{C}_\mathcal{B}$. After its implementation the action (3.20) contains only true degrees of freedoms.

In analogy with the case of tensor perturbations, we introduce a canonical field Ψ_c and expand it as

$$\Psi_c \equiv a\Psi = \int dn \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Psi_{nlm}(\eta) Q_{nlm}(\mathbf{x}), \tag{3.21}$$

where $Q_{nlm}(\mathbf{x})$ are the scalar pseudospherical harmonics, satisfying [14]:

$$\begin{aligned} \nabla^2 Q_{nlm}(\mathbf{x}) &= -(n^2+1) Q_{nlm}(\mathbf{x}) \\ \int d^3x \sqrt{\gamma}\, Q_{nlm}(\mathbf{x}) Q_{n'l'm'}(\mathbf{x}) &= \delta(n-n')\delta_{ll'}\delta_{mm'} \ . \end{aligned} \tag{3.22}$$

As a result, (3.20) becomes

$$\delta^{(2)} S^{(S)} = \frac{1}{2\ell_P^2} \int d\eta\, dn \left[(\bar{\Psi}'_N)^2 - (n^2 - \frac{1}{4}\phi'^2)\bar{\Psi}_N^2 \right], \quad N = (nlm)\ , \tag{3.23}$$

where $\bar{\Psi}_N \equiv \sqrt{n^2+4}\Psi_N$. The quantity $\bar{\Psi}_N$ enters the action in a canonical way and therefore its vacuum fluctuations, like those of u, are easily normalized. The equation for $\bar{\Psi}_N$ is simply

$$\bar{\Psi}''_N + (n^2 - \frac{1}{4}\phi'^2)\bar{\Psi}_N = 0\ , \tag{3.24}$$

so that we must impose, as $\eta \to -\infty$,

$$\begin{aligned} \bar{\Psi}_N(\eta) &\to \bar{\Psi}_N^{-\infty}(\eta) \equiv \frac{\ell_P}{\sqrt{n}} e^{-in\eta} \\ \bar{\Pi}_N(\eta) &\to \bar{\Pi}_N^{-\infty}(\eta) \equiv -i\frac{\sqrt{n}}{\ell_P} e^{-in\eta}\ . \end{aligned} \tag{3.25}$$

As was the case for tensor perturbations, also Eq. (3.24) can be transformed (for the background (2.6)) into a hypergeometric equation. We find, specifically,

$$\begin{aligned} \bar{\Psi}_N(\eta) &= \tilde{C}_1\, [\mathrm{csch}^2(2\eta)]^{-\frac{in}{4}} F\,[\frac{-1-in}{4}, \frac{3-in}{4}, \frac{2-in}{2}, -\mathrm{csch}^2(2\eta)] \\ &+ \tilde{C}_2\, [\mathrm{csch}^2(2\eta)]^{\frac{in}{4}} F\,[\frac{-1+in}{4}, \frac{3+in}{4}, \frac{2+in}{2}, -\mathrm{csch}^2(2\eta)]\ , \end{aligned} \tag{3.26}$$

where, as before, we have to take $|\tilde{C}_1| = \ell_P/\sqrt{n}, \tilde{C}_2 = 0$. Corrections to the free plane-wave can be easily computed and, again, are suppressed by four powers of $1/t$:

$$\bar{\Psi}_N(\eta) = \bar{\Psi}_N^{-\infty}(\eta) \left(1 + \tilde{\alpha}_n\, e^{4\eta - i\tilde{\beta}_n}\right)\ , \tag{3.27}$$

where $\bar{\Psi}_N^{-\infty}$ is given by (3.25) and $\tilde{\alpha}_n, \tilde{\beta}_n$ are n-dependent constants fixed from the expansion of the hypergeometric function.

To estimate the behaviour of (3.26) near $\eta \simeq 0$, we use the formula [15]

$$F\left[a, a+1, c, -\mathrm{csch}^2(2\eta)\right] \simeq \frac{\Gamma(c)}{\Gamma(a+1)\Gamma(c-a)}\left[\; -2^{2a+3}a(a-c+1)|\eta|^{2(a+1)}\ln|\eta| \right. \\ \left. + 2^{2a}|\eta|^{2a}\right] , \tag{3.28}$$

and obtain:

$$|\bar{\Psi}_N| \simeq \ell_P\sqrt{\frac{n^2+1}{2\pi}}\sqrt{\coth\left(\frac{n\pi}{2}\right)}\left(-|\eta|^{3/2}\ln|\eta| + \frac{2}{n^2+1}|\eta|^{-1/2}\right) . \tag{3.29}$$

IV. DISCUSSION

In order to discuss the physical significance of our results it is useful to choose a convenient gauge. In the spatially flat case it was found [16] that the so-called off-diagonal gauge [17] [16] was particularly useful in order to suppress the large gauge artefacts present in the more commonly used [12] longitudinal gauge. The off-diagonal gauge is defined by setting $\psi = E = 0$ in Eq. (3.16). We shall now see how one can reconstruct the scalar field fluctuation from Ψ in this gauge.

We first note that, in this gauge, the variables Ψ and B are related through (3.19) as:

$$\Psi = \frac{4\mathcal{H}B}{\phi'} . \tag{4.1}$$

Using Eq. (3.24) for $\bar{\Psi}_N$, as well as (4.1), we can derive the evolution equation for B:

$$B'' - \nabla^2 B + \left(2\mathcal{H} - \frac{4\mathcal{K}}{\mathcal{H}}\right)B' - (4\mathcal{H}^2 + 12\mathcal{K})B = 0 , \tag{4.2}$$

which agrees with Ref. [16] for $\mathcal{K} = 0$. To relate $\delta\phi$ and Ψ we first observe that the first of the two constraints (3.18) provides the relation

$$\phi'\delta\phi = 4(\mathcal{H}\varphi + \mathcal{K}B) , \tag{4.3}$$

while, eliminating $\delta\phi$ from the two constraints (3.18) and using (4.2), we arrive at a second relation

$$\varphi = B' + 2\mathcal{H}B . \tag{4.4}$$

Combining (4.4) and (4.3), and making use of (4.1), we are finally able to express $\delta\phi$ directly in terms of Ψ as

$$\delta\phi = \Psi' + \frac{\mathcal{K} - \mathcal{H}'}{\mathcal{H}}\Psi , \tag{4.5}$$

implying that $\delta\phi$ represents, in this gauge, a gauge-invariant object.

It is instructive to compare the $\mathcal{K} = -1$ case with the spatially flat one, where the relevant gauge-invariant variable, given by

$$\psi^{(\mathrm{gi})} = \psi + \frac{H}{\phi'}\delta\phi \, , \tag{4.6}$$

becomes $\delta\phi$ itself in the off-diagonal gauge. The *canonical field*, given by $v = a\delta\phi$, satisfies the well-known equation [12]:

$$v'' + \left(n^2 - \frac{z''}{z}\right) v = 0, \qquad \text{where} \quad z = \frac{a\phi'}{\mathcal{H}}. \tag{4.7}$$

Even in the presence of spatial curvature, the field v still plays the role of the canonical field in the far past, when η is large and negative. This can be checked by computing the equation of motion for v in the presence of curvature. The explicit form of the equation for v is given by

$$\begin{aligned} v'' + A_1 v' + A_2 v = 0, \qquad & A_1 = -\mathcal{K}^2\phi'^2 \left[\mathcal{H}^3(n^2 - \mathcal{K} + \frac{3\mathcal{K}^2}{\mathcal{H}^2})\right]^{-1} , \\ & A_2 = n^2 + \frac{\phi'^2}{12}(1 - \frac{12\mathcal{K}}{\mathcal{H}^2}) - (\mathcal{H} + \frac{3\mathcal{K}}{\mathcal{H}})A_1 \, . \end{aligned} \tag{4.8}$$

Thus, as long as we are interested in the early-time regime, A_1 is exponentially small, $A_2 \to n^2$, and v can be treated as the canonical field.

Using Eq. (4.5), the behaviour of v in the far past follows directly from that of $\bar{\Psi}_N$, given in Eqs. (3.25), (3.27):

$$\begin{aligned} v^{-\infty}(\eta) &\equiv \frac{\ell_P}{\sqrt{n}}\sqrt{\frac{2-in}{2+in}}e^{-in\eta} \\ \pi_v^{-\infty}(\eta) &\equiv -i\frac{\sqrt{n}}{\ell_P}\sqrt{\frac{2+in}{2-in}}e^{-in\eta} \, , \end{aligned} \tag{4.9}$$

with corrections again suppressed as t^{-4}, i.e.

$$v(\eta) = v^{-\infty}(\eta)\left(1 + \hat{\alpha}_n \, e^{4\eta - i\hat{\beta}_n}\right) \, , \tag{4.10}$$

where $\hat{\alpha}_n, \hat{\beta}_n$ are n-dependent constants.

We can study how other variables behave near $\eta \simeq 0$ by using their relation to Ψ in this gauge and the behaviour of Ψ, Eq. (3.29). We easily find:

$$|B_N| \simeq \frac{\ell_P}{\ell}\sqrt{\frac{n^2+1}{2\pi}}\sqrt{\frac{\coth(\frac{n\pi}{2})}{n^2+4}}\left(-|\eta|\ln|\eta| + \frac{2}{n^2+1}|\eta|^{-1}\right) \, , \tag{4.11}$$

while

$$|\delta\phi_N| \simeq \frac{\ell_P}{\ell}\sqrt{\frac{n^2+1}{2\pi}}\sqrt{\frac{\coth(\frac{n\pi}{2})}{n^2+4}}\ln|\eta| \, . \tag{4.12}$$

Let us finally compare the energy contained in the quantum fluctuations of the dilaton and that in the classical solution, near the singularity. Note that the expansion (3.28) can be trusted only up to some maximum n for which $1 \ll n_{\max} \sim 1/|\eta|$. Consequently, the ratio of the kinetic energy densities near $|\eta| \simeq 0$ (up to constant prefactors of $\mathcal{O}(1)$) becomes

$$\frac{\mathcal{E}_{\mathrm{Q}}}{\mathcal{E}_{\mathrm{C}}} = \frac{\int d^3x\sqrt{\gamma}\, a^2(\delta\phi')^2}{\int d^3x\sqrt{\gamma}\, a^2\phi'^2} \simeq \frac{\ell_P^2}{\ell^2} \int^{n_{\max}} \frac{dn}{n}\, n^3 \; . \tag{4.13}$$

We can express the above result in terms of the value of the physical Hubble parameter $H(\eta) \equiv \mathcal{H}/a$ at horizon crossing of the scale n, $H_{\mathrm{HC}}(n)$, which is easily computed as

$$H_{\mathrm{HC}}(n) \sim \frac{1}{\eta a}(\eta \sim 1/n) \sim n^{3/2}/\ell \; . \tag{4.14}$$

Thus (4.13) takes the suggestive form

$$\frac{\mathcal{E}_{\mathrm{Q}}}{\mathcal{E}_{\mathrm{C}}} = \ell_P^2 \int^{n_{\max}} \frac{dn}{n}\, H_{\mathrm{HC}}^2(n) \; . \tag{4.15}$$

In general, in order to draw physical conclusions, we should transform the results back to the string frame. However, in our case, this is hardly necessary. Indeed, concerning the far past behaviour of tensor and scalar quantum fluctuations, this can be evaluated in either frame, since the dilaton is approximately constant in the far past. Our results, expressed in Eqs. (3.13) and (4.10), show that quantum fluctuations die off as $e^{4\eta} \sim t^{-4}$, i.e. much faster than the classical perturbation (over a constant dilaton). They also die much faster than the classical inhomogeneities discussed in [4], [7] and [10]. There is a physical reason for this result: quantum fluctuations are not amplified in a trivial (Minkowski or Milne) background. Hence, their amplification should vanish as fast as the (non-trivial component of the) dilaton, rather than as $\mathcal{H}$ itself.

Coming now to the importance of vacuum fluctuations at later times, we observe that the final result (4.15) expresses the relative importance of quantum and classical fluctuations near the singularity in terms of a frame-independent quantity, the ratio of the (effective, dilaton-dependent) Planck length to the size of the horizon. Note that the latter is the same in either frame, up to factors $\mathcal{O}(1)$. Since, by definition of the perturbative dilaton phase, the latter is always larger than the string scale, we find that the relative importance of quantum fluctuations is always bounded by the ratio ℓ_P/ℓ_s, which is always less then 1 in the perturbative phase.

In conclusion, the pre-big bang scenario appears to pass another test of self-consistency: having assumed that the initial state is deeply inside the small-coupling, small-curvature region, we can safely neglect quantum corrections, until the string-curvature scale *and* the strong-coupling regime are simultaneously met.

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