Invariant Box–Parameterization of Neutrino Oscillations

Thomas J. Weiler[1] and DJ Wagner[2]
[1]*Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235*
[2]*Department of Physics and Astronomy, Angelo State University, San Angelo, TX 76909*

The model-independent "box" parameterization of neutrino oscillations is examined. The invariant boxes are the classical amplitudes of the individual oscillating terms. Being observables, the boxes are independent of the choice of parameterization of the mixing matrix. Emphasis is placed on the relations among the box parameters due to mixing–matrix unitarity, and on the reduction of the number of boxes to the minimum basis set. Using the box algebra, we show that CP-violation may be inferred from measurements of neutrino flavor mixing even when the oscillatory factors have averaged. General analyses of neutrino oscillations among $n \geq 3$ flavors can readily determine the boxes, which can then be manipulated to yield magnitudes of mixing matrix elements.

I. INTRODUCTION

If neutrinos have mass and are non-degenerate, then their flavors may oscillate as they propagate. Resonant oscillations for the sun [1], oscillations for the atmosphere [2], and the LSND data [3] each require a different neutrino mass-squared difference if neutrino oscillations are to account for all features of the data [4]. Since three-neutrino models can have at most two independent mass-squared differences, a sterile neutrino is apparently needed to reconcile all the data while retaining consistency with LEP measurements of $Z \to \nu\overline{\nu}$ [5]. Several four-neutrino analyses appear in the literature [4,6]. It is also possible that some data will turn out to have an explanation other than neutrino oscillations, in which case three-neutrino oscillations may be sufficient. So our task is to examine the physics of neutrino oscillations with three or more mixed flavors.

Oscillation probabilities depend on products of four mixing-matrix elements. Several parameterizations of the mixing matrix in terms of rotation angles have been introduced, beginning with the pioneering work of Kobayashi and Maskawa [7]. With three or more neutrino generations, the oscillation probabilities are complicated functions of the neutrino mixing angles. But oscillations are observable and therefore parameterization-invariant. One must ask if there is not a better description of oscillations which avoids the arbitrariness of angular-parameterization schemes. Recently, we introduced a "box" parameterization of neutrino mixing valid for any number of neutrino generations [8]. Oscillation probabilities are linear in the boxes, enabling a straighforward description of oscillation data. Here we present the algebra of the boxes and the unitarity constraints on that algebra. Then we illustrate the boxes' reduction to a basis in the case of three generations, thereby setting the framework for a future phenomenological analysis.

The probability for a neutrino to oscillate from ν_α to ν_β is given by the square of the transition amplitude:

$$P_{\nu_\alpha \to \nu_\beta}(x) = \left| \sum_{i=1}^{n} V_{\alpha i} V_{\beta i}^{*} e^{-i\phi_i} \right|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (V_{\alpha i}\, V_{\beta i}^{*}\, V_{\alpha j}^{*}\, V_{\beta j}) e^{-i(2\Phi_{ij})}, \tag{1}$$

where n is the number of neutrino generations,

$$\Phi_{ij} \equiv \frac{1}{2}(\phi_i - \phi_j) = \frac{1}{2}(E_i t_i - p_i x_i - E_j t_j + p_j x_j), \tag{2}$$

and $V_{\alpha i}$ is the mixing–matrix element which connects the α^{th} charged lepton mass eigenstate and the i^{th} neutrino mass eigenstate. For relativistic neutrinos, Φ_{ij} is given by

$$\Phi_{ij} \approx \frac{\Delta m_{ij}^2}{4p} x, \quad \text{where} \quad \Delta m_{ij}^2 \equiv m_i^2 - m_j^2. \tag{3}$$

With a little bit of algebra, the oscillation probability may be brought into the form

$$\begin{aligned} P_{\nu_\alpha \to \nu_\beta}(x) = & -2 \sum_i \sum_{j \neq i} \mathrm{Re}(V_{\alpha i} V_{\beta i}^{*} V_{\alpha j}^{*}\, V_{\beta j}) \sin^2(\Phi_{ij}) \\ & + \sum_i \sum_{j \neq i} \mathrm{Im}(V_{\alpha i} V_{\beta i}^{*} V_{\alpha j}^{*}\, V_{\beta j}) \sin(2\Phi_{ij}) + \delta_{\alpha\beta}. \end{aligned} \tag{4}$$

The probability for an antineutrino to oscillate from $\overline{\nu}_\alpha$ to $\overline{\nu}_\beta$ is obtained by replacing V with V^*. This is equivalent to changing the sign of Φ_{ij}, or the second term in equation (4).

With the familiar case of two neutrino flavors, the mixing matrix V has the simple form of a rotation matrix (phases cancel in oscillation probabilities for Majorana neutrinos, and may be absorbed into the definitions of Dirac fermion fields):

$$V = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5}$$

The oscillation probability in the two-flavor case is simply

$$\underset{\nu_\alpha \to \nu_\beta}{P}(x) = \delta_{\alpha\beta} + \sin^2 2\theta \ \sin^2\left(\frac{\Delta m^2_{12}}{4p}x\right), \quad n = 2. \tag{6}$$

The mixing-angle parameterization is a natural choice in the two-flavor situation.

The formalism becomes more complicated with three flavors. An arbitrary 3×3 unitary matrix has three real degrees of freedom and six phases, but $2n - 1 = 5$ phases may be absorbed into field redefinitions. The original choice of the four remaining parameters, due Kobayashi and Maskawa to describe quark mixing, is [7]

$$\begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}, \tag{7}$$

where $c_a \equiv \cos\theta_a$, and $s_a \equiv \sin\theta_a$. There is arbitrariness associated with the placement of the phase, since we absorb five relative phases into the field definitions. Because of this arbitrariness, the phases of individual matrix elements are not observable.

The observable oscillation probabilities are quite complicated functions of the angle-based parameterizations. As an example, consider the product $V_{22}V^*_{23}V^*_{32}V_{33}$ appearing in the $\nu_\mu \to \nu_\tau$ oscillation probability:

$$\begin{aligned} V_{22}V^*_{23}V^*_{32}V_{33} = {} & c_3^2 s_3^2 \left[s_2^2 c_2^2 (s_1^4 + 6c_1^2 + 2c_1^2 \cos 2\delta) - c_1^2\right] \\ & + \frac{\mathcal{J}}{s_1^2}(1 + c_1^2)(c_2^2 - s_2^2)(s_3^2 - c_3^2)\cot\delta + i\mathcal{J}, \quad n = 3. \end{aligned} \tag{8}$$

where the Jarlskog invariant $\mathcal{J}$ [9] has the form $\mathcal{J} = c_1 s_1^2 c_2 s_2 c_3 s_3 \sin\delta$ in this parameterization.

The expression (but not its value) on the right-hand side of equation (8) is convention-dependent, as well as being unwieldy. Our development of a model-independent parameterization is motivated by the arbitrariness and complexity of this traditional approach.

II. THE BOX PARAMETERIZATION

The immeasurability of the individual complex mixing–matrix elements in the quark sector has been addressed by numerous authors [9–13]. Measurable quantities include only the magnitudes of mixing matrix elements, the products of four mixing-matrix elements appearing in the oscillation probabilities, and particular higher-order functions of mixing-matrix elements [11,14]. As evidenced in equations (1) and (4), neutrino oscillation probabilities depend linearly on the fourth-order objects,

$$^{\alpha i}\square_{\beta j} \equiv V_{\alpha i}V^\dagger_{i\beta}V_{\beta j}V^\dagger_{j\alpha} = V_{\alpha i}V^*_{\alpha j}V^*_{\beta i}V_{\beta j}, \tag{9}$$

which we call "boxes" since each contains as factors the corners of a submatrix, or "box," of the mixing matrix. For example, the upper left 2×2 submatrix elements produce the box

$$^{11}\square_{22} = V_{11}V^*_{12}V^*_{21}V_{22}. \tag{10}$$

The name "box" also seems appropriate in light of the Feynman box–diagram which describes the oscillation process. Examination of equation (9) reveals a few symmetries in the indexing:

$$^{\alpha i}\square_{\beta j} = {}^{\beta j}\square_{\alpha i} = {}^{\beta i}\square^*_{\alpha j} = {}^{\alpha j}\square^*_{\beta i}. \tag{11}$$

If the order of either set of indices is reversed (*id est*, $j \leftrightarrow i$ or $\beta \leftrightarrow \alpha$), the box turns into its complex conjugate; if both sets of indices are reversed, the box returns to its original value [10]. And if V is replaced by $V^\dagger$, then ${}^{\alpha i}\Box_{\beta j} \ \rightarrow \ {}^{i\alpha}\Box^*_{j\beta}$.

Boxes with $\alpha = \beta$ or $i = j$, are real, given from equation (9) as

$$ {}^{\alpha i}\Box_{\alpha j} = |V_{\alpha i}|^2 |V_{\alpha j}|^2, \qquad {}^{\alpha i}\Box_{\beta i} = |V_{\alpha i}|^2 |V_{\beta i}|^2, \quad \text{and} \quad {}^{\alpha i}\Box_{\alpha i} = |V_{\alpha i}|^4. \tag{12} $$

We call boxes with one and two repeated indices "singly-degenerate" and "doubly-degenerate," respectively. Boxes with $\alpha \neq \beta$ and $i \neq j$ are called "nondegenerate". As can be seen from equation (4), singly-degenerate boxes with repeated flavor indices enter into the formulae for flavor-conserving survival probabilities, but not for flavor-changing transition probabilities. Degenerate boxes with repeated mass indices (including the doubly-degenerate boxes) do not appear in any oscillation formula. Degenerate boxes may be expressed in terms of the nondegenerate boxes, as will be shown shortly. This possibility and the symmetries expressed in equation (11) allow us to express combinations of boxes in terms of only the nondegenerate "ordered" boxes for which $\alpha < \beta$ and $i < j$.

Using the symmetries expressed in equation (11), the oscillation probabilities (4) in terms of boxes become

$$ P_{\nu_\alpha \to \nu_\beta}(x) = \delta_{\alpha\beta} - 2 \sum_{i=1}^{n} \sum_{j>i} \left[2\ {}^{\alpha i}\mathrm{R}_{\beta j} \sin^2 \Phi_{ij} - \ {}^{\alpha i}\mathrm{J}_{\beta j} \sin 2\Phi_{ij} \right], \tag{13} $$

where we have defined the shorthand ${}^{\alpha i}\mathrm{R}_{\beta j} \equiv \mathrm{Re}\left({}^{\alpha i}\Box_{\beta j} \right)$ and ${}^{\alpha i}\mathrm{J}_{\beta j} \equiv \mathrm{Im}\left({}^{\alpha i}\Box_{\beta j} \right)$. From equation (11) we deduce that the Js are antisymmetric in both flavor indices and mass indices; Rs are symmetric in both. Survival probabilities $P_{\nu_\alpha \to \nu_\alpha}(x) = 1 - \sum_{\beta \neq \alpha} P_{\nu_\alpha \to \nu_\beta}(x)$ are more simply expressed in terms of degenerate boxes, or $|V|$s, rather than nondegenerate boxes. From equations (13) and (12), they are

$$ P_{\nu_\alpha \to \nu_\alpha}(x) = 1 - 4 \sum_{i=1}^{n} \sum_{j>i} {}^{\alpha i}\Box_{\alpha j} \ \sin^2 \Phi_{ij} = 1 - 4 \sum_{i=1}^{n} \sum_{j>i} |V_{\alpha i}|^2 |V_{\alpha j}|^2 \sin^2 \Phi_{ij}. \tag{14} $$

Interchanging $\alpha \leftrightarrow \beta$ in equation (13) gives the time-reversed reactions $P_{\nu_\beta \to \nu_\alpha}(x)$:

$$ P_{\nu_\beta \to \nu_\alpha}(x) = \delta_{\alpha\beta} - 2 \sum_{i=1}^{n} \sum_{j>i} \left[2\ {}^{\alpha i}\mathrm{R}_{\beta j} \sin^2 \Phi_{ij} + \ {}^{\alpha i}\mathrm{J}_{\beta j} \sin 2\Phi_{ij} \right]. \tag{15} $$

Ignoring possible CP-violating phases in the mixing matrix, the number of real parameters determining V is the number of rotational planes available in n–dimensions, $N \equiv \frac{1}{2}n(n-1)$. Determining these N parameters determines the complete mixing matrix. Conveniently, there are N transition probabilities $P_{\nu_\alpha \to \nu_\beta}(x) = P_{\nu_\beta \to \nu_\alpha}(x)$. Thus, all of the information in the mixing matrix is contained in the N transition probabilities. In this sense, they form a convenient basis for determining all oscillation parameters. Of course, if the same transition probability is measured at two or more different distances, then all N transition probabilities may not be needed to determine V.

Allowing CP-violation in the mixing matrix, there are N real parameters and $\frac{1}{2}(n-1)(n-2)$ phases, for a total of $(n-1)^2$ parameters. With CP-violation, however, there are $2N = n(n-1)$ independent transition probabilities $P_{\nu_\alpha \to \nu_\beta}(x)$. The number of transition probabilities exceeds the number of independent parameters, so they again form a convenient basis for determining the mixing matrix. In reality, only the three flavor indices e, μ, τ are easily accessible. Moreover, some of the N parameters in the mixing matrix, namely those which rotate sterile states for $n \geq 5$, are not accessible at all, which complicates the counting.

The transition probabilities for which $\alpha \neq \beta$ in equation (13) may be conveniently expressed in matrix form. The matrix of boxes is an $N \times N$ matrix. For three flavors, we have

$$ \mathcal{P}(n=3) \equiv \begin{pmatrix} P_{\nu_e \to \nu_\mu}(x) \\ P_{\nu_\mu \to \nu_\tau}(x) \\ P_{\nu_e \to \nu_\tau}(x) \end{pmatrix} = -4\ \mathrm{Re}(\mathcal{B})\ S^2(\Phi) + 2\ \mathrm{Im}(\mathcal{B})\ S(2\Phi), \tag{16} $$

where

$$ \mathcal{B} \equiv \begin{pmatrix} {}^{e1}\Box_{\mu 2} & {}^{e2}\Box_{\mu 3} & {}^{e1}\Box_{\mu 3} \\ {}^{\mu 1}\Box_{\tau 2} & {}^{\mu 2}\Box_{\tau 3} & {}^{\mu 1}\Box_{\tau 3} \\ {}^{e1}\Box_{\tau 2} & {}^{e2}\Box_{\tau 3} & {}^{e1}\Box_{\tau 3} \end{pmatrix}, \quad \text{and} \quad S^k(\Phi) \equiv \begin{pmatrix} \sin^k \Phi_{12} \\ \sin^k \Phi_{23} \\ \sin^k \Phi_{13} \end{pmatrix}, \quad n = 3. \tag{17} $$

For the time–reversed channels, or for the antineutrino channels, the sign of the $\mathrm{Im}(\mathcal{B})$ term is reversed. The box parameterization is especially well-suited for considering higher numbers of generations. The matrix $\mathcal{B}$ merely acquires extra columns when new flavors are introduced; extra rows are not accessible at energies below new charged-lepton thresholds. Furthermore, oscillation probabilities are linear in boxes, no matter how many generations.

Neutrino oscillation experiments will directly measure the boxes in equation (13), not the individual mixing matrix elements, $V_{\alpha i}$. But one would like to obtain the fundamental $V_{\alpha i}$ from the measured boxes. We develop here an algebra relating boxes and mixing matrix elements.

Some tautologous relationships between the degenerate and nondegenerate boxes are easily confirmed using equation (9); they hold for any number of generations:

$$|V_{\alpha i}|^2|V_{\alpha j}|^2 = {}^{\alpha i}\Box_{\alpha j} = \frac{{}^{\alpha i}\Box^*_{\eta j}\ {}^{\alpha i}\Box_{\lambda j}}{{}^{\eta i}\Box_{\lambda j}}, \qquad (\eta \neq \lambda \neq \alpha), \tag{18}$$

$$|V_{\alpha i}|^2|V_{\beta i}|^2 = {}^{\alpha i}\Box_{\beta i} = \frac{{}^{\alpha i}\Box^*_{\beta x}\ {}^{\alpha i}\Box_{\beta y}}{{}^{\alpha x}\Box_{\beta y}}, \qquad (x \neq y \neq i), \quad \text{and} \tag{19}$$

$$\frac{|V_{\alpha i}|^2}{|V_{\beta j}|^2} = \frac{{}^{\alpha i}\Box^*_{\eta j}\ {}^{\alpha i}\Box_{\beta x}}{{}^{\alpha j}\Box_{\beta x}\ {}^{\beta i}\Box^*_{\eta j}}, \qquad (\eta \neq \alpha \neq \beta, \text{ and } x \neq i \neq j). \tag{20}$$

Equations (18) and (19) are themselves special cases of the more general

$$\begin{aligned} {}^{\alpha i}\Box_{\beta j}\ {}^{\gamma i}\Box_{\delta j} &= \left[V_{\alpha i}V^*_{\alpha j}V_{\beta j}V^*_{\beta i}\right]\left[V_{\gamma i}V^*_{\gamma j}V_{\delta j}V^*_{\delta i}\right] \\ &= \left[V_{\alpha i}V^*_{\alpha j}V_{\delta j}V^*_{\delta i}\right]\left[V_{\gamma i}V^*_{\gamma j}V_{\beta j}V^*_{\beta i}\right] = {}^{\alpha i}\Box_{\delta j}\ {}^{\gamma i}\Box_{\beta j}\ , \end{aligned} \tag{21}$$

and the analogous relation ${}^{\alpha i}\Box_{\beta j}\ {}^{\alpha k}\Box_{\beta l} = {}^{\alpha i}\Box_{\beta l}\ {}^{\alpha k}\Box_{\beta j}$. The relations above hold for both degenerate boxes and nondegenerate boxes.

Due to the symmetry ${}^{\alpha i}\Box_{\beta j} \rightarrow {}^{i\alpha}\Box^*_{j\beta}$ when $V \rightarrow V^\dagger$, there will generally be analogous but distinct pairing of our box equations, differing only in whether the degeneracy or sum is over a flavor index or a mass index. In the following we will mainly show only one equation per analogous pair, for reasons of space limitations in this proceeding.

We may express $|V_{\alpha i}| = \left({}^{\alpha i}\Box_{\alpha i}\right)^{\frac{1}{4}}$ in terms of three singly-degenerate boxes by setting $\alpha = \beta$ in equation (19). Then, using equation (18) to substitute for the singly-degenerate boxes yields an expression for the doubly-degenerate box in terms of nine nondegenerate boxes:

$$|V_{\alpha i}|^4 = {}^{\alpha i}\Box_{\alpha i} = \frac{{}^{\alpha i}\Box_{\alpha x}\ {}^{\alpha i}\Box_{\alpha y}}{{}^{\alpha x}\Box_{\alpha y}} = \frac{{}^{\alpha x}\Box_{\tau i}\ {}^{\alpha i}\Box_{\sigma x}\ {}^{\alpha y}\Box_{\rho i}\ {}^{\alpha i}\Box_{\zeta y}\ {}^{\omega x}\Box_{\mu y}}{{}^{\tau i}\Box_{\sigma x}\ {}^{\rho i}\Box_{\zeta y}\ {}^{\alpha y}\Box_{\omega x}\ {}^{\alpha x}\Box_{\mu y}}\ , \tag{22}$$

where the index constraints are $\tau \neq \sigma \neq \alpha$, $\zeta \neq \rho \neq \alpha$, $\mu \neq \omega \neq \alpha$, and $x \neq y \neq i$. In the three-generation case, equation (22) is uniquely specified by the index constraints. For example,

$$|V_{11}|^4 = \frac{{}^{11}\Box_{22}\ {}^{11}\Box^*_{23}\ {}^{11}\Box_{33}\ {}^{11}\Box^*_{32}\ {}^{22}\Box_{33}}{{}^{12}\Box^*_{23}\ {}^{12}\Box_{33}\ {}^{21}\Box_{33}\ {}^{21}\Box^*_{32}} \tag{23}$$

holds with any number of generations, but it is the unique 5 on 4 box representation of $|V_{11}|^4$ in three generations.

We note that all of the relationships in this section follow from the definitions of the boxes in equation (9) and so are valid for any matrix, unitary or otherwise. The constraints of unitarity will provide us with expressions for $|V_{\alpha i}|^4$ which are easier to manage than the expression in (22) above.

III. UNITARITY RELATIONS AMONG THE BOXES

Unitarity requires that

$$\sum_{\eta=1}^{n} V_{\eta i}V^*_{\eta j} = \delta_{ij}, \quad \text{and} \quad \sum_{y=1}^{n} V_{\alpha y}V^*_{\beta y} = \delta_{\alpha\beta}. \tag{24}$$

Multiplying the first equation in (24) by $V^*_{\lambda i}V_{\lambda j}$ and the second by $V^*_{\alpha x}V_{\beta x}$ gives the unitarity constraints for the boxes:

$$\sum_{\eta=1}^{n} \;{}^{\eta i}\Box_{\lambda j} \;= \sqrt{\;{}^{\lambda i}\Box_{\lambda i}}\,\delta_{ij}\,, \tag{25}$$

and its analogue. Isolating the manifestly degenerate boxes from the nondegenerate boxes, equation (25) becomes

$$\sum_{\eta\neq\lambda} \;{}^{\eta i}\Box_{\lambda j} \;= \sqrt{\;{}^{\lambda i}\Box_{\lambda i}}\,\delta_{ij} - \;{}^{\lambda i}\Box_{\lambda j}\;. \tag{26}$$

Summing equation (25) over λ in the $i\neq j$ case, we find

$$0=\sum_{\lambda=1}^{n}\sum_{\eta=1}^{n} \;{}^{\eta i}\Box_{\lambda j} \;= \sum_{\lambda=1}^{n} \;{}^{\lambda i}\Box_{\lambda j} \;+2\sum_{\lambda=1}^{n}\sum_{\eta<\lambda} \;{}^{\eta i}\mathrm{R}_{\lambda j}. \tag{27}$$

The double sum is over Rs only, since the first term is manifestly real. The resulting conditions on the Js are found in equation (29) below. Comparison of equation (27) with equations (17) and (18) reveals an interesting property of the matrix $\mathcal{B}$:

$$\sum_{\text{column}\,of\,\mathcal{B}} \mathrm{Re}\,(\mathcal{B}) = -\frac{1}{2}\sum_{\lambda=1}^{n}|V_{\lambda i}|^2|V_{\lambda j}|^2, \tag{28}$$

where the sum is over a column of $\mathcal{B}$ specified by fixed i and j. There is an analogue relation for the sum over a row of $\mathcal{B}$.

The unitarity constraint (25) holds independently for the real and imaginary parts of the sum. We will first explore the implications of these constraints for the imaginary parts of boxes, before turning to the more complicated constraints for the real parts. The right-hand side of (25) is manifestly real, so the imaginary constraints are simply

$$\sum_{\eta\neq\lambda} \;{}^{\eta i}\mathrm{J}_{\lambda j}=0, \quad \text{and} \quad \sum_{y\neq x} \;{}^{\alpha y}\mathrm{J}_{\beta x}=0\,. \tag{29}$$

Equation (11) indicates that ${}^{\eta i}\mathrm{J}_{\lambda j}$ is an antisymmetric matrix in the indices η and λ for fixed i and j, and vice versa. Equation (29) shows that the sum of elements along any row or column of that antisymmetric matrix equals zero, whether the sum is over mass indices or flavor indices.

Summing the first equation in (29) over λ gives zero trivially since a sum of all elements of an antisymmetric matrix vanishes by definition. Hence, for fixed (i,j), the first equation in (29) expresses $n-1$ constraints. Thus, the number of independent flavor pairs on Js after implementing the constraints of equation (29) is $N-(n-1)=\frac{1}{2}(n-1)(n-2)$. Ditto for independent mass pairs, so the number of independent Js after implementing both sets of constraints is the product $\frac{1}{4}\,(n-1)^2\,(n-2)^2$.

In three generations, this number of independent Js is one. Each sum in equation (29) contains only two terms, leading to

$$\mathrm{Im}(\mathcal{B})=\begin{pmatrix} \mathcal{J} & \mathcal{J} & -\mathcal{J} \\ \mathcal{J} & \mathcal{J} & -\mathcal{J} \\ -\mathcal{J} & -\mathcal{J} & \mathcal{J} \end{pmatrix}, \quad n=3, \tag{30}$$

with $\mathcal{J}\equiv\;{}^{11}\mathrm{J}_{22}$ [9]. One consequence of the equality of all $|J|$s in three generations is that if any one $V_{\alpha i}$ is zero, then all ${}^{\alpha i}\mathrm{J}_{\beta j}$ vanish and there can be no CP-violation.

We now consider the real parts of the constraint (25), focusing first on the homogeneous constraint for which the Kronecker delta is zero. This constraint gives the singly-degenerate boxes as sums of ordered boxes:

$$|V_{\lambda i}|^2|V_{\lambda j}|^2 = \;{}^{\lambda i}\Box_{\lambda j} \;= -\sum_{\eta\neq\lambda} \;{}^{\eta i}\Box_{\lambda j} \;= -\sum_{\eta\neq\lambda} \;{}^{\eta i}\mathrm{R}_{\lambda j}, \qquad i\neq j. \tag{31}$$

This linear relation complements the relation expressed in equation (18). For three generations, each of the sums contains two terms, allowing us to express the singly-degenerate boxes in terms of two nondegenerate boxes which are measurable in neutrino appearance oscillation experiments.

The real unitarity constraint (31) greatly simplifies our expressions for a doubly-degenerate box ${}^{\alpha i}\Box_{\alpha i} \;= |V_{\alpha i}|^4$:

$$ {}^{\alpha i}\Box_{\alpha i} = \frac{{}^{\alpha i}\Box_{\alpha x}\ {}^{\alpha i}\Box_{\alpha y}}{{}^{\alpha x}\Box_{\alpha y}} = \frac{\left(-\sum_{\eta\neq\alpha}\ {}^{\alpha i}\mathrm{R}_{\eta x}\right)\left(-\sum_{\lambda\neq\alpha}\ {}^{\alpha i}\mathrm{R}_{\lambda y}\right)}{\left(-\sum_{\tau\neq\alpha}\ {}^{\alpha x}\mathrm{R}_{\tau y}\right)}, \qquad x\neq y\neq i, \tag{32} $$

where the first equality is due to equation (18) with $j=i$. Applying equation (32) to three generations, one finds that doubly-degenerate boxes are expressible in terms of the real parts of six ordered boxes, rather than the nine complex boxes used in equation (22). For example,

$$ |V_{11}|^4 = {}^{11}\Box_{11} = \frac{-\left({}^{11}\mathrm{R}_{22} + {}^{11}\mathrm{R}_{32}\right)\left({}^{11}\mathrm{R}_{23} + {}^{11}\mathrm{R}_{33}\right)}{{}^{12}\mathrm{R}_{23} + {}^{12}\mathrm{R}_{33}}, \quad n=3. \tag{33} $$

In cyclic coordinates α,β,γ, and with $x\neq y\neq i$,

$$ {}^{\alpha i}\Box_{\alpha i} = |V_{\alpha i}|^4 = \frac{-\left({}^{\alpha i}\mathrm{R}_{\beta x} + {}^{\alpha i}\mathrm{R}_{\gamma x}\right)\left({}^{\alpha i}\mathrm{R}_{\beta y} + {}^{\alpha i}\mathrm{R}_{\gamma y}\right)}{{}^{\alpha x}\mathrm{R}_{\beta y} + {}^{\alpha x}\mathrm{R}_{\gamma y}}, \quad n=3. \tag{34} $$

When considering $n>3$, each sum has more terms, but all terms in the numerator in equation (32) always contain Rs to the second order, while the denominator terms contain only the first order of Rs. Thus these expressions will be much more manageable than equation (22) which exhibits the fifth order of complex boxes in the numerator and the fourth order in the denominator.

For fixed (i,j) in equation (31), λ can take n possible values, implying n constraint equations. N ordered nondegenerate boxes appear in these n equations. Thus, for $N\leq n$, which is true for $n\leq 3$, the unitarity constraint (31) may be inverted to find a nondegenerate box in terms of singly-degenerate boxes. Manipulation of equation (31) gives an expression in term of the flavor triad (α,β,γ):

$$ {}^{\alpha i}\mathrm{R}_{\beta j} = -\frac{1}{2}\left(|V_{\alpha i}|^2|V_{\alpha j}|^2 + |V_{\beta i}|^2|V_{\beta j}|^2 - |V_{\gamma i}|^2|V_{\gamma j}|^2\right), \quad n=3. \tag{35} $$

It is known that knowledge of four $|V|$s completely specifies the three-generation mixing matrix, provided no more than two $|V|$s are taken from the same row or same column [15]. Here, we can use three-generation unitarity and equation (35) to re-write ${}^{\alpha i}\mathrm{R}_{\beta j}$ in terms of just four $|V|$s. The result is

$$ {}^{\alpha i}\mathrm{R}_{\beta j} = \frac{1}{2}\left[1 - |V_{\alpha i}|^2 - |V_{\alpha j}|^2 - |V_{\beta i}|^2 - |V_{\beta j}|^2 + |V_{\alpha i}|^2|V_{\beta j}|^2 + |V_{\alpha j}|^2|V_{\beta i}|^2\right], \tag{36} $$

which for $n=3$ expresses the real part of the box $\mathrm{Re}\left[V_{\alpha i}V^*_{\alpha j}V_{\beta j}V^*_{\beta i}\right]$ in terms of the magnitudes of the four complex Vs which define the box. Three-generation unitarity may be used again to replace the first five terms on the right-hand side of equation (36) with $-|V_{\gamma k}|^2$.

Summing equation (31) over $j\neq i$ yields another expression for $|V_{\lambda i}|^2$ in terms of nondegenerate boxes, which further complements equations (32) and (22):

$$ |V_{\lambda i}|^2 \sum_{j\neq i} |V_{\lambda j}|^2 = |V_{\lambda i}|^2\left(1 - |V_{\lambda i}|^2\right) = -\sum_{j\neq i}\sum_{\eta\neq\lambda} {}^{\eta i}\mathrm{R}_{\lambda j}. \tag{37} $$

The explicit solution of this equation, valid for any number of generations, is,

$$ |V_{\lambda i}|^2 = \frac{1}{2}\left[1 \pm \sqrt{1 + 4\sum_{j\neq i}\sum_{\eta\neq\lambda} {}^{\eta i}\mathrm{R}_{\lambda j}}\right], \tag{38} $$

which yields $|V_{\lambda i}|^2$ in terms of $(n-1)^2$ Rs, but subject to a two-fold ambiguity.

We may use the real homogeneous unitarity condition (31) along with the tautology (18) to obtain constraints between nondegenerate boxes, thereby reducing the number of real degrees of freedom. Substituting the tautology (18) into the unitarity constraint (31) gives

$$ {}^{\eta i}\Box_{\alpha j}\ {}^{\alpha i}\Box_{\lambda j} + {}^{\eta i}\Box_{\lambda j} \sum_{\tau\neq\alpha} {}^{\tau i}\mathrm{R}_{\alpha j} = 0. \tag{39} $$

This unitarity constraint interrelates imaginary and real parts of n different boxes for any number of generations. For example, taking the imaginary part of equation (39) leads to

$$ {}^{\eta i}\mathrm{J}_{\alpha j}\ {}^{\alpha i}\mathrm{R}_{\lambda j} + {}^{\alpha i}\mathrm{J}_{\lambda j}\ {}^{\eta i}\mathrm{R}_{\alpha j} + {}^{\eta i}\mathrm{J}_{\lambda j} \sum_{\tau\neq\alpha} {}^{\tau i}\mathrm{R}_{\alpha j} = 0, \quad \eta \neq \lambda \neq \alpha, \quad i \neq j. \tag{40} $$

Taking the real part of equation (39) leads to

$$ {}^{\eta i}\mathrm{R}_{\alpha j}\ {}^{\alpha i}\mathrm{R}_{\lambda j} + {}^{\eta i}\mathrm{R}_{\lambda j} \sum_{\tau\neq\alpha} {}^{\tau i}\mathrm{R}_{\alpha j} = {}^{\eta i}\mathrm{J}_{\alpha j}\ {}^{\alpha i}\mathrm{J}_{\lambda j}, \quad \eta \neq \lambda \neq \alpha, \quad i \neq j. \tag{41} $$

One may also use the pairs of equations (40) and (41) to eliminate the sums and isolate a single R:

$$ {}^{\alpha i}\mathrm{R}_{\beta j} = \frac{{}^{\alpha i}\mathrm{J}_{\beta j}\ {}^{\alpha i}\mathrm{R}_{\lambda j}\ {}^{\beta i}\mathrm{R}_{\lambda j} + {}^{\alpha i}\mathrm{J}_{\beta j}\ {}^{\alpha i}\mathrm{J}_{\lambda j}\ {}^{\beta i}\mathrm{J}_{\lambda j}}{{}^{\alpha i}\mathrm{J}_{\lambda j}\ {}^{\beta i}\mathrm{R}_{\lambda j} - {}^{\alpha i}\mathrm{R}_{\lambda j}\ {}^{\beta i}\mathrm{J}_{\lambda j}}, \tag{42} $$

with $\beta \neq \lambda \neq \alpha$, $i \neq j$. Input from these unitarity relations among Rs and Js is necessary to establish the minimum set of independent box parameters.

IV. INDIRECT MEASUREMENT OF CP-VIOLATION

Suppose CP is conserved. Then ${}^{\alpha i}\mathrm{J}_{\beta j} = 0$ for all index choices. The inference from equation (41) is that

$$ {}^{\alpha i}\mathrm{R}_{\eta j}\ {}^{\alpha i}\mathrm{R}_{\lambda j} + {}^{\eta i}\mathrm{R}_{\lambda j} \sum_{\tau\neq\alpha} {}^{\alpha i}\mathrm{R}_{\tau j} = 0, \qquad (\eta \neq \lambda \neq \alpha, \text{ and } i \neq j). \tag{43} $$

If this relation is violated, then so is CP.

For three generations, ${}^{\eta i}\mathrm{J}_{\alpha j} = {}^{\alpha i}\mathrm{J}_{\lambda j}$ by equation (29), and equation (41) may be solved for $\mathcal{J}^2$ directly:

$$ \mathcal{J}^2 = {}^{\alpha i}\mathrm{R}_{\beta j}\ {}^{\beta i}\mathrm{R}_{\lambda j} + {}^{\alpha i}\mathrm{R}_{\beta j}\ {}^{\alpha i}\mathrm{R}_{\lambda j} + {}^{\alpha i}\mathrm{R}_{\lambda j}\ {}^{\beta i}\mathrm{R}_{\lambda j}, \quad n = 3. \tag{44} $$

Equation (44) says that the three real elements in any row (or any column in the analogue equation) of the matrix $\mathcal{B}$ may be summed in their three pairwise products to yield the CP-violating invariant $\mathcal{J}^2$. These real elements on the right-hand side of this equation are measurable with CP-conserving averaged neutrino oscillations. Thus, even if CP violating asymmetries are not directly observable in an experiment, the effects of CP violation may be seen through the relationships among the real parts of different boxes, which are determinable from averaged flavor–mixing measurements! Note that if CP is conserved and $\mathcal{J}$ is zero, then equation (44) also tells us that all three Rs in any row (or column) cannot have the same sign.

V. INHOMOGENEOUS UNITARITY CONSTRAINTS AND A BOX–BASIS

The inhomogeneous unitarity constraints with the Kronecker delta nonzero in equation (25) are necessary to provide the desired normalization of the $V_{\alpha i}$ or the boxes. The inhomogeneous constraints are functions of degenerate boxes and therefore purely real:

$$ {}^{\alpha i}\Box_{\alpha i} + \sum_{\eta\neq\alpha} {}^{\alpha i}\Box_{\eta i} = \sqrt{{}^{\alpha i}\Box_{\alpha i}}, \tag{45} $$

This equation can be rewritten strictly in terms of nondegenerate boxes by using the homogeneous unitarity constraints (31) to replace the singly-degenerate boxes, and equation (32) to replace the doubly-degenerate box:

$$ \Sigma_\lambda \Sigma_\sigma + \sqrt{-\Sigma_\lambda \Sigma_\sigma \Sigma_\tau} + \Sigma_\tau \Sigma_{\eta z} = 0, \tag{46} $$

with $\Sigma_\lambda \equiv \sum_{\lambda\neq\alpha} {}^{\alpha i}\mathrm{R}_{\lambda x}$, $\Sigma_\sigma \equiv \sum_{\sigma\neq\alpha} {}^{\alpha i}\mathrm{R}_{\sigma y}$, $\Sigma_\tau \equiv \sum_{\tau\neq\alpha} {}^{\alpha x}\mathrm{R}_{\tau y}$, $\Sigma_{\eta z} \equiv \sum_{\eta\neq\alpha}\sum_{z\neq i} {}^{\alpha z}\mathrm{R}_{\eta i}$, and $x \neq y \neq i$. These inhomogeneous unitarity constraints do not involve the Js. Isolating the square root and squaring the equation, we get polynomial equations of degrees three and four in the Rs, each relating $n(n-1)$ Rs.

We provide here an example of a basis construction, obtained by substituting in the unitarity equations derived above. The unitarity constraints among the Js, given in equation (29), are linear and therefore the simplest to implement. These constraints may be used first to reduce the number of independent Js to $\frac{1}{4}(n-1)^2(n-2)^2$. Further reduction to independent Js and Rs requires the nonlinear constraints. The homogeneous constraints (40) and (41) are much simpler than the inhomogeneous constraints (46), but the inhomogeneous constraints must be invoked at least once (Otherwise, the the boxes and the matrix element magnitudes $|V|$ could not be normalized.).

For three generations, one begins with nine Rs and one J, and seeks a basis of just four elements. Rearranging the three-generation equation (44) yields expressions for one R in terms of two other Rs and $\mathcal{J}$. This equation may be used three times to eliminate ${}^{12}\mathrm{R}_{23}$, ${}^{11}\mathrm{R}_{32}$ and ${}^{21}\mathrm{R}_{33}$. The utility of the analogue equation is exhausted to eliminate ${}^{12}\mathrm{R}_{33}$ and ${}^{11}\mathrm{R}_{33}$. As expected, one must next turn to the inhomogeneous constraints (46) to eliminate the last degree of freedom. We are left with a constraint which is quartic in all five of its parameters $A \equiv {}^{11}\mathrm{R}_{22}$, $B \equiv {}^{11}\mathrm{R}_{23}$, $C \equiv {}^{21}\mathrm{R}_{32}$, $D \equiv {}^{22}\mathrm{R}_{33}$, and $\mathcal{J}^2$:

$$\begin{aligned}0 &= (A+B)^2 (A+C)^2 \left(BC+BD-CD+\mathcal{J}^2\right)^2 \\ &+ \left(AD+BD-AB+\mathcal{J}^2\right)^2 \left[(A+B)(C+D)+C^2+\mathcal{J}^2\right]^2 \\ &+ (A+B)(A+C)\left(BC+BD-CD+\mathcal{J}^2\right)\left(AD+BD-AB+\mathcal{J}^2\right) \\ &\times \left[C+D+2\left((A+B)(C+D)+C^2+\mathcal{J}^2\right)\right].\end{aligned} \tag{47}$$

We may eliminate any one parameter by either algebraic or numerical means, leaving us with the desired four parameters as the basis.

VI. SUMMARY

Neutrino physics has entered a golden age of research. New experiments all over the globe promise an unequaled amount of data from the sun, the atmosphere, accelerators, supernovae, and other cosmic sources. The latest data suggests that more than three neutrino flavors may participate in neutrino oscillations [4]. Analyzing such refined data requires a consistent, model-independent approach which may be easily applied, and easily extended to higher generations. Here we have discussed such an approach, wherein one works directly with the observable coefficients of the oscillating terms. From unitarity of the mixing matrix, we derived relations among these CP–conserving and CP–violating coefficients for the various oscillation channels. One result which we view as particularly noteworthy is that high-statistics data on *averaged* oscillations are sufficient to determine the conservation or non-conservation of CP in the lepton mixing matrix. This indirect test of CP can be traced back to unitarity of the mixing matrix, but in the present formulation there is no need to even mention the mixing matrix.

Acknowledgements: This work was supported in part by the U.S. Department of Energy, Division of High Energy Physics, under Grant No. DE-F605-85ER40226, and the Vanderbilt University Research Council.

---

[1] J. Bahcall and M. H. Pinsonneault, Rev. Mod. Phys. **67**, 781 (1995); J.N. Bahcall, S. Basu, and M.H. Pinsonneault, astro-ph/9805135; B. T. Cleveland, *et al.*, Nucl. Phys. B (Proc. Suppl.) **38**, 47 (1995); Kamiokande collaboration, Y. Fukuda *et al.*, Phys. Rev. Lett, **77**, 1683 (1996); GALLEX collaboration, W. Hampel *et al.*, Phys. Lett. **B388**, 384 (1996); SAGE collaboration, J. N. Abdurashitov *et al.*, Phys. Rev. Lett. **77**, 4708 (1996).

[2] Super–Kamiokande Collaboration, Y. Fukuda et al., hep–ex/9803006 and Phys. Lett. **B**, to appear; hep–ex/9805006; talks by Y. Suzuki and Y. Totsuka at *Neutrino–98*, Takayama, Japan, June 1998; Kamiokande collaboration, K. S. Hirata *et al.*, Phys. Lett. **B280**, 146 (1992); Y. Fukuda *et al.*, Phys. Lett. **B335**, 237(1994); IMB collaboration, R. Becker-Szendy *et al.*, Nucl. Phys. Proc. Suppl. **38B**, 331 (1995); Soudan-2 collaboration, W. W. M. Allison *et al.*, Phys. Lett. **B391**, 491 (1997); J. G. Learned, S. Pakvasa, and T. J. Weiler, Phys. Lett. **B207**, 79 (1988); V. Barger and K. Whisnant, Phys. Lett. **B209**, 365 (1988); K. Hidaka, M. Honda, and S. Midorikawa, Phys. Rev. Lett. **61**, 1537 (1988); M. C. Gonzalez-Garcia, H. Nunokawa, O. Peres, T. Stanev, and J. W. F. Valle, hep-ph/9801368.

[3] Liquid Scintillator Neutrino Detector (LSND) collaboration, C. Athanassopoulos *et al.*, Phys. Rev. Lett. **75**, 2650 (1995); *ibid.* **77**, 3082 (1996); nucl-ex/9706006.

[4] V. Barger, T.J. Weiler, and K. Whisnant, hep-ph/9712495; V. Barger, S, Pakvasa, T.J. Weiler, and K. Whisnant, hep-ph/9806328.

[5] LEP Electroweak Working Group and SLD Heavy Flavor Group, D. Abbaneo *et al.*, CERN-PPE-96-183, December 1996.

[6] D.O. Caldwell and R.N. Mohapatra, Phys. Rev. **D 48**, 3259 (1993); R. Foot and R.R. Volkas, Phys. Rev. **D 52**, 6595 (1995); J.J. Gomez-Cadenas and M.C. Gonzalez-Garcia, Z. Phys. **C 71**, 443 (1996); S.M. Bilenky, C. Giunti, and W. Grimus, hep-ph/9711416; R.N. Mohapatra, hep-ph/9711444.

[7] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1972).
[8] D.J. Wagner and T.J. Weiler, hep-ph/9801327, and Phys. Rev. **D**, to appear.
[9] C. Jarlskog, Z. Phys. C **29**, 491 (1985); C. Jarlskog, Phys. Rev. D **35**, 1685 (1987); here the maximum value for $J = c_1 s_1^2 c_2 s_2 c_3 s_3$ in three generations is easily found to be $\pm\frac{1}{6\sqrt{3}}$.
[10] J. F. Nieves and P. B. Pal, Phys. Rev. D **36**, 315 (1987).
[11] Dan-di Wu, Phys. Rev. D **33**, 860 (1986).
[12] J. Bjorken and I. Dunietz, Phys. Rev. D **37**, 2109 (1987).
[13] D. Du, I. Dunietz, and Dan-di Wu, Phys. Rev. D **34**, 3414 (1986); O. W. Greenberg, Phys. Rev. D **32**, 1841 (1985); I. Dunietz, O. W. Greenberg, and D. Wu, Phys. Rev. Lett. **55**, 2935 (1985); C. Hamzaoui and A. Barroso, Phys. Lett. **154B**, 202 (1985).
[14] A. Kusenko and R. Shrock, Phys. Rev. D **50**, R30 (1994); Phys. Lett. B **323**, 18 (1994).
[15] C. Hamzaoui, Phys. Rev. Lett. **61**, 35 (1988); C. Jarlskog, in *CP Violation*, edited by C. Jarlskog (World Scientific, Singapore, 1989).