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GAUGE FIELDS AND SINGLETONS OF AdS$_{\mathbf{2p+1}}$

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ABSTRACT. We show that p-forms on AdS_{2p+1} describe both singletons and massless particles. On the $2p$-dimensional boundary the singleton p-form Lagrangian reduces to the conformally invariant functional $\int F^2$. All the representations, singletons as well as massless, are zero center modules and involve a vacuum mode. Two- and three-form singleton fields are required by supersymmetry in AdS$_5$ and AdS$_7$ supergravity respectively.

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1. Introduction

It is well known that spin-0 singletons on AdS_{d+1} can be extended to "super singletons" by introducing spin-1/2 particles [1]. This fact can be related to the property of the massless Klein-Gordon and Dirac equations, on the boundary at infinity, of being conformally invariant. In this note we point out that whenever the space-time dimension d is even, Maxwell's equations for the $p = d/2$-form field strength are also conformally invariant, and therefore one expects $(p-1)$-form potentials to describe degrees of freedom of the singleton type on AdS_{d+1}. [2],[3],[4].

This is a generalization of what is already known for AdS_5 and AdS_7 in which case supersymmetry does indeed require one- [3] and two-form gauge potentials respectively [4].

The extension of $2p$-dimensional Maxwell theory to a singleton theory in AdS_{2p+1} is a generalization of the circumstance that superconformal invariance at the boundary of AdS_5 and AdS_7 requires the singletons to be accompanied by 1-form and 2-form potentials, respectively, in the same super multiplet. The extension of 4-dimensional Maxwell theory to a singleton theory on AdS_5 was described in two papers by the same authors [5],[6]. There it was pointed out that the Maxwell field in four dimensions transforms as zero-center module of $SO(4,2)$. Other important examples of zero-center modules are in [7].

In terms of $SO(4,2)$ irreducibles, as usual denoted $D(E_0, J_1, J_2)$, the complete non-decomposable representation carried by the Maxwell potential in four dimensions is [8]

$$D(1, \tfrac{1}{2}, \tfrac{1}{2}) \rightarrow \big(D(2,1,0) \oplus D(2,0,1) + \mathrm{id}\big) \rightarrow D(1, \tfrac{1}{2}, \tfrac{1}{2}),$$

where $D(2,1,0)$ is associated with the self-dual part of the field strength. The quantum numbers E_0, J_1, J_2 refer to the highest weight, a weight of $O(2) \otimes SO(4)$.

In the six-dimensional conformal field theory the field representations can be denoted $D(E_0, a_1, a_2, a_3)$, where a_1, a_2, a_3 are $SO(6)$ Dynkin labels. The three-form (self-dual) field strength carries $D(3,2,0,0)$. The nondecomposable representation is in this case

$$D(2,1,0,1) \rightarrow \big(D(3,2,0,0) \oplus D(3,0,0,2) \oplus D(1,0,1,0)\big) \rightarrow D(2,1,0,1).$$

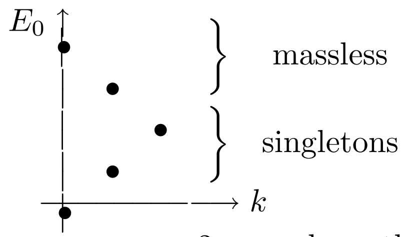
This is again a zero-center module, of $SO(6,2)$. The representations that are carried by the extensions of these boundary conformal fields to AdS_{d+1} are more complicated, though the central, physical subquotients are the same. The non-decomposable character of the representation is of course the characteristic property of gauge theories.

It is worth while to notice that these are precisely the forms that can be "self-dual", in any even-dimensional space,* and that can be coupled to $(p-1)$-brane dyons. This leads to the following conjecture, that will be proved in this paper. Conformal field strengths of degree $p = d/2$ on the boundary of AdS_{d+1} are zero-center modules for any even d.

Actually, the hierarchy of q-forms that carry zero-center $SO(2p,2)$ modules goes from $q = 0$ to $q = p$, with conformal degree q for singletons and conformal degree $p+q$ for representations that are associated with "massless" field theories in the bulk. The space of

* Self-dual "real" p-forms exist for $p = 4k+2$, for $p = 4k$ the forms must be complexified.

$(p-1)$-form singleton potential field modes carries all of these representations. The scalar field (conformal dimension $2p$) and the vector potential (conformal dimension $2p-1$) are of the type that occur in supergravities, the others seem to belong to the same category and they will therefore be referred to as "massless" in this paper. These forms, of rank higher than 1, do not appear in spectra of higher dimensional supergravities; they represent something that is new and perhaps interesting.



In this diagram, drawn for the case $p = 2$, we show the zero-center modules $D(p \pm k, w_{p-k})$ (w_k is a highest weight of $SO(2p)$) as dots at the points $E_0 = p \pm k$. We show that there are extensions between neighbour representations, for all p.

In Section 2 we study gauge theories of the conventional type on AdS_{2p+1}. These potentials are $(p-1)$-forms of conformal degree $p+1$, and we call them massless. They are massless merely in the sense that they have a familiar gauge structure, but not in the sense that they appear in supergravities. Instead, what appear in supergravities (in the bulk) are certain $(p-1)$-forms (with $E_0 = 2p-1$) that satisfy self-duality constraints of the type $mA =^* dA$ [4]. These forms occur for $p = 2, 3$ in AdS_{2p+1} supergravity and can be thought of as "composite" boundary operators of a scalar singleton and a p-form singleton field strength.

In Section 3 we concentrate on $(p-1)$-form potentials of conformal degree $p-1$ and show that they have gauge sectors of the singleton type (and of conventional type as well). All these representations are zero-center modules. What is even more remarkable is that the massless representations appear in the gauge sector of the singleton representations. Section 4 summarizes the conclusions and the Appendix gives an alternative derivation of some of the results.

2. Spaces, Groups and Highest Weight representations.

2.1. The space $M = AdS_{2p+1}$ is a space-time with one time and $2p$ space dimensions, endowed with an anti-De Sitter metric. The conformal completion $\bar{M}$ of this space includes a boundary M_∞ with a Minkowski metric, signature $1, 2p-1$.

This paper is a study of a field theory on M_∞, the principal field a $(p-1)$-form A_∞ (the potential), with associated field strength $F_\infty = dA_\infty$ and action $\int F_\infty^2$, and mainly the extension of this theory to a singleton topological gauge theory on $\bar{M}$. We suppose that F_∞ has conformal dimension p, making the boundary action conformally invariant.

Let r be a radial coordinate on $\bar{M}$. The extension A of A_∞ to $\bar{M}$ is a $(p-1)$-form on $\bar{M}$ that is related to A_∞ by [9][5]

$$\lim_{r \to \infty} r^N A = A_\infty \ , \tag{2.1}$$

for some real number N. This boundary condition on A determines the conformal degree of A_∞; it is $-N$. Hence $N = 1-p$. (Of course, the identification (2.1) implies a projection of the left-hand side from the tangent space of $\bar{M}$ to the tangent space of M_∞.)

We shall show that the field A on $\bar{M}$, $M = AdS_{2p+1}$, is a gauge theory, in two ways. The logic of the investigation leads us to include a study of p-forms of conformal dimension $p+2$ as well. These fields describe massless particles on AdS_{2p+1}.

2.2. Notation.

The space AdS_{2p+1} is the homogeneous space $SO(2p,2)/SO(2p,1)$. For calculational purposes we represent it as the hyperboloid

$$y^2 = 1 \ , \quad y^2 := y_0^2 + y_{00}^2 - \sum_{i=1}^{2p} y_i^2 \ . \tag{2.2}$$

A considerable simplification results from extending our fields on M to the domain

$$M_+ = \{y \in \mathbf{R}^{2p+2}, \ y^2 > 0\} \ .$$

This is done by fixing the degree of homogeneity of the extended fields. Thus A becomes a homogeneous field on M_+. In fact, we extend A to a homogeneous $(p-1)$-form on M_+. This can be done by stipulating that this $(p-1)$-form be transverse, or otherwise as the situation favors it. We continue to use the same letter A for this extended field. If the degree of homogeneity of A coincides with the number N in (2.1), then $A_\infty = A|_{y^2=0}$. The boundary cone $y^2 = 0$ of M_+ can thus be identified with the boundary M_∞ of M [10].

The principal advantage of this trick is that the vector fields $\partial_\alpha = \partial/\partial y^\alpha$, $\alpha = 0, 00, 1, \ldots, 2p$, are now well defined. The degree N of A is given by

$$y^\alpha \partial_\alpha A = y \cdot \partial_y A = NA \ . \tag{2.3}$$

For the time being we look at N as a free parameter that we can choose to suit our convenience.

2.3. Tensor Gauge Theory.

We try to impose conditions on the $(p-1)$-form A so as to make it the carrier of an irreducible representation of $SO(2p,2)$:

$$y \cdot A = 0 \ , \tag{2.4}$$

$$(y^2 \partial^2 + \kappa) A = 0 \ , \tag{2.5}$$

$$\partial \cdot A = 0 \ , \tag{2.6}$$

$$^*(dA) = \pm dA \ . \tag{2.7}$$

Explanation. The vector field $y \cdot \partial$ is invariant, Eq. (2.4) makes A transverse to this radial field and assures its interpretation as a $(p-1)$-form on M. Because of (2.3), Eq. (2.5) is equivalent to the covariant wave equation on M. We have

$$\Box = y^2\partial^2 - N(N+2p) \ , \quad \partial^2 = \partial_\alpha \partial^\alpha \ . \tag{2.8}$$

Hence (2.5) is the same as

$$(\Box + m^2)A = 0, \quad m^2 := \kappa - N(N+2p) \ . \tag{2.9}$$

We choose N so as to make $\kappa = 0$. Finally, in (2.7), $^*(dA)$ is the Hodge dual. Since we shall have no use for it we skip the definition.

The representation of $SO(2p,2)$ that is induced on the space of solutions of all these equations is not reducible, but it may admit an invariant subspace of gauge fields of the form

$$A = y^2 d\Phi + ay \wedge \Phi \ , \tag{2.10}$$

Φ a $(p-2)$-form. A straightforward calculation gives the result that this happens if and only if

$$N + p = \pm 1 \ , \quad a = 3 - N - p \ . \tag{2.11}$$

The appearance of this invariant subspace is characteristic of massless representations. There is also, when $N = 1 - p$, an invariant subspace of field modes of the form $d\Phi$, * as well as an invariant subspaces of the form $y^2\Phi$; this last is what characterizes singletons. For a more systematic approach, that encompasses all the possibilities at once, see Appendix.

2.4. Highest Weight Representations, Massless Representations.

The second-order Casimir operator $\mathcal{C}$ of $SO(2p,2)$ takes the form

$$\mathcal{C}A = \big(-y^2\partial^2 + N(N+2p) + p^2 - 1\big)A, \tag{2.12}$$

when A is constrained by (2.3), (2.4) and (2.6). Using (2.5) we obtain

$$\mathcal{C}A = [(N+p)^2 - 1]A \ ,$$

and when we require the existence of gauge modes, hence Eq. (2.11),

$$\mathcal{C}A = 0.$$

On a highest weight $D(E_0, w)$, where E_0 is the energy L_{05} and w is the highest weight of a representation of $SO(2p)$, $\mathcal{C}$ takes the value

$$\mathcal{C} \to E_0(E_0 - 2p) + c(w), \tag{2.13}$$

* We recall that the boundary M_∞ of M can be identified with the boundary cone $y^2 = 0$ of M_+. Since the fields are required to be defined on $\bar{M}$, no inverse powers of y^2 are permitted in (2.10). This why this case could not be included in (2.10).

where $c(w)$ is the value of the second-order Casimir operator of $SO(2p)$ on the representation with highest weight w. If this representation is the k^{th} alternating power of the fundamental representation (real representation of dimension $2p$) then it is

$$c(w_k) = k(2p-k) \ . \tag{2.14}$$

The following set of representations,

$$D(k, w_k), \quad k = 0, 1, \cdots, p \ , \tag{2.15}$$
$$D(p+k, w_{p-k}), \quad k = 1, \ldots, p \ , \tag{2.16}$$

all have $\mathcal{C} = 0$. In fact, they are all zero-center modules; that is, all the Casimirs operators are zero on these representations. The proof is in the Appendix.

We shall show here the existence of an extension, within the field module, of the type

$$D(p+1, w_{p-1}) \rightarrow D(p+2, w_{p-2}) \ . \tag{2.17}$$

Let $z = \{z_\alpha\}_{\alpha=0,00,1,\ldots,2p}$ be an auxiliary Grassmannian vector "coordinate", and represent k-forms on TM_+ as polynomials of order k in z. For example, the gauge field (2.10) beomes

$$y^2 z \cdot \partial_y \Phi + ay \cdot z\Phi \ . \tag{2.18}$$

A very simple ground state with $N = -p-1$ is

$$y_+^{-2-p}(y_+ - z_+ y \cdot \partial_z)\Psi \ , \tag{2.19}$$

with Ψ a polynomial of order $p-1$ in $z_1, \ldots, z_{2p}$ only, and

$$y_+ = y_0 + iy_{00} \propto re^{it} \ , \quad z_+ = z_0 + iz_{00} \ .$$

It is a highest weight vector for the representation $D(p+1, w_{p-1})$, where w_{p-1} is the highest weight of the $(p-1)^{\text{th}}$ exterior power of the fundamental representation D_1 of $SO(2p)$. According to (2.13) and (2.14), the Casimir takes the value 0.

We shall now show that the space generated from (2.19) (the Harish-Chandra module) contains an invariant subspace that is generated from a singular vector of the form (2.18).

Let $\{L_1^i\}_{i=1,\ldots,2p}$ be a set of step-up operators for the energy; they span a space of dimension $2p$ that, under the restriction of the adjoint action to $SO(2p)$, carries the fundamental representation D_1 of $SO(2p)$. Applying these operators to (2.19) we obtain an $SO(2p)$ p-form that can be contracted to a $(p-2)$-form with the $SO(2p)$ metric, which gives us a vector with weight $(p+2, w_{p-2})$. The representation $D(p+2, w_{p-2})$ has $\mathcal{C} = 0$ (see above) and this state is thus a candidate for being the highest weight vector of an invariant submodule.

For completeness, we present the details of the calculation. The energy raising operators are

$$L_+^i = y_i \partial_+ + y_- \partial_i + z_i \partial'_+ + z_- \partial'_i, \quad i = 1, \ldots, 2p \ .$$

The following function is a highest weight vector with weight $(p+1, w_{p-1})$,

$$y_+^{-p-2} A_{i_1 \dots i_{p-2}}, \quad A_{i_1 \dots i_k} = (y_+ - z_+ \vec{y} \cdot \vec{\partial}_z) z_{i_1} \dots z_{i_k} \ . \tag{2.20}$$

We find

$$L_+^i \left[y_+^{-p-2} A_{i i_1 \dots i_{p-2}} \right] = \frac{p+2}{4} (y^2 z \cdot \partial_y + 4 y \cdot z) \left[y_+^{-p-3} A_{i_1 \dots i_{p-2}} \right] \ .$$

This shows that the extension (2.17) actually occurs in the space generated by the ground state (2.20), among the modes of a $(p-1)$-form potential.

2.5. Highest Weight Representations: Singletons.

We turn to an analysis of the case that is of main interest to us, the case $N + p = 1$ in (2.11), so we are now dealing with $(p-1)$ forms of degree $p-1$ and field strengths of degree p, as required by the conformal invariance of $\int F^2$ on the $2p$-dimensional boundary.

This case is quite different, for an extension of the type (2.17), linking $D(p-1, w_{p-1})$ to $D(p, w_{p-2})$, is not possible, the two representations having different values of $\mathcal{C}$.

Instead, there is the extension

$$D(p, w_p) \rightarrow D(p-1, w_{p-1}) \ . \tag{2.21}$$

The "ground state" of $D(p, w_p)$ is

$$y_+^{-p} y \wedge \psi \ , \quad \psi_{i_1 \dots i_{p-1}} = z_{i_1} \dots z_{i_{p-1}} \ . \tag{2.22}$$

Applying the lowering operators we find

$$L_-^i (y_+^{-p} y \wedge \psi)_{i i_p \dots i_{p-1}} \propto y_+^{-p} (y_+ - z_+ \vec{y} \cdot \vec{\partial}_z) \psi_{i_1 \dots i_{p-1}} \ . \tag{2.23}$$

This is the ground state of $D(p-1, w_{p-1})$. Pushing up again, one obtains a mode with symmetry different from that of (2.22), orthogonal to it. Hence (2.22) is a ground state modulo the invariant subspace generated from (2.23).

This case is indeed very different. There is a gauge subspace generated by (2.23). The cyclic vector for (2.21) is (2.22), which is not the highest weight, so (2.21) is not a Harish-Chandra module.

The extension (2.21) does not reveal the full extent of the field multiplet carried by the $(p-1)$-form quantum field operator. We shall show, in Section 3.2, that there are extensions to the massless representation $D(p+1, w_{p-1})$ and to $D(p-2, w_{p-2})$.

3. The Quantum Field Operator.

The degree of homogeneity of the $(p-1)$-form A can be chosen so that each component satisfies the scalar wave equation

$$\partial^2 A = 0 \quad \text{or} \quad \partial^2 A_{\alpha_1 \dots \alpha_{p-1}} = 0 \ .$$

Choose a space of solutions of the scalar wave equation, and let D_0 be the highest weight representation of $SO(2p,2)$ induced on this space. Then the representation carried by the field A is a subquotient of the direct product

$$D_0 \otimes D_{p-1}\ , \quad D_{p-1} = D_1^{\wedge(p-1)},$$

where D_1 is the fundamental (vector) representation of $SO(2p,2)$. Now suppose that $D_0 = D(E_0,0)$, then for E_0 big enough, this product is completely reducible. But if $E_0 = p \pm 1$, then this is not the case.

The strategy that will uncover all possibilities is thus to calculate the structure of the representations

$$D(p \pm 1, 0) \otimes D_{p-1}\ .$$

But first we need to recall a special property of the representation $D(p-1,0)$.

There is a scalar field that carries the representation $D(p+1,0)$, but there is no scalar field that carries just $D(p-1,0)$. This phenomenon has already been explained several times, in dimensions 3, 4, 5 and even in the general case. See [10], and [2] for a review. Therefore, let us just recall the fact that the modes of $D(p-1,0)$ appear in the scalar quantum field operator inside a complete Gupta-Bleuler triplet, namely

$$D(p+1,0) \to D(p-1,0) \to D(p+1,0) =: D_S\ . \tag{3.1}$$

The ground state for this representation is the highest weight of $D(p-1,0)$, namely*

$$y_+^{1-p}.$$

It is easy to show that

$$\sum_i (L_+^i)^2 y_+^{1-p} \ \propto\ y^2 y_+^{-1-p}\ ,$$

hence $D(p+1,0)$ appears as a subrepresentation of the Harish-Chandra representation generated from the ground state of $D(p-1,0)$. Consequently, we shall have to investigate

$$D(p+1,0) \otimes D_{p-1},\ \text{(massless fields)} \quad \text{and} \quad D_S \otimes D_{p-1}\ \text{(singletons)}.$$

The first case is a type of generalized electrodynamics (in the bulk). The second case leads to tensor singleton fields; they are gauge fields both in the traditional sense and (in the bulk) in the topological singleton sense.

The massless case is quite simple: we give the result without proof:

$$D(p+1,0) \otimes D_{p-1} = D(p+2,0) \oplus D(p,0) \oplus D_{\text{massless}}$$

* The ground state of the left factor of (3.1), which is cyclic for the whole representation, is $py^2 y_+^{-p-1} \ln y_+ + 2y_+^{-p} y_-$.

(omit the first term when $p=2$), with

$$D_{\text{massless}} = D(p+2, w_{p-2}) \to D(p+1, w_{p-1}) \to D(p+2, w_{p-2}). \tag{3.2}$$

The direct summands can be eliminated by requiring the field operator to satisfy the constraint $(y^2 \partial_y \cdot \partial_z + 2y \cdot \partial_z)A = 0$. The Lorentz condition is $y \cdot A = 0$, and the invariant subspace of gauge modes consists of potentials of the form $(y^2 z \cdot \partial_y + 4y \cdot z)\Phi$.***

The propagators of the massless field that carries the entire product representation $D(p+1,0) \otimes D_1^{\wedge(p+1)}$ is

$$(z \cdot z')^{p+1} K_1(y \cdot y'), \quad K(y \cdot y') \propto {}_2F_3($$

There is another propagator, for the transverse vector field, that carries the representation (3.2) only.**

3.2 Singleton gauge theory, the field module. This is a single Gupta-Bleuler triplet,

$$D_S \otimes D_1 = D_{\text{scalar}} \to D_{\text{physical}} \to D_{\text{gauge}}, \tag{3.3}$$

with the two outer terms equivalent. No covariant constraint can be imposed on the quantum field operator and the propagator is simply $(z \cdot z')^{p-1} K(y, y')$ where K is the propagator for D_S, see below. The gauge sector is very complicated. The Lorentz condition projects the theory on the boundary, where it is a conformal formulation of ordinary electrodynamics—Maxwell theory to be precise. The best way to formulate the theory is in terms of a dipole field [11] that satisfies $(\Box^2 + m^2)^2 A = 0$, or

$$(\Box + m^2)A = B\ , \quad (\Box + m^2)B = 0\ .$$

Then the Lorentz condition includes the constraints $B = 0$ and $y \cdot A = 0$.

We shall now justify these remarks, and at the same time demonstrate that the field module includes an extension of the singleton representation to the massless representation; that is, we show that the extension

$$D(p, w_p) \to D(p+1, w_{p-1})$$

actually occurs. The calculation is presented here for the case $p = 2$ only.

The ground state $y_+^{-2}(y \wedge z)$ of $D(2, w_2)$ is a relative one that pushes down to the gauge mode $y_+^{-2}(y_+ \vec{z} - \vec{y} z_+) = z \cdot \partial_y (y_+^{-1} \vec{y})$. Examination of the propagator at levels 0 and 1 shows that the conjugate of this mode is $y_+^{-1}\vec{z}$, modulo gauge. This one pushes down to

* The ground states of the three subquotients, reading from right to left, are as follows, $y_+^{-3} z_+$, $y_+^{-3} z_- - y_+^{-4} \vec{y} \cdot \vec{z} + (y_+^{-5} \vec{y}^{\,2} - y_+^{-4} y_-) z_+$ and $2y_+^{-3} z_- - y_+^{-4} \vec{y} \cdot \vec{z}$.

** This is not the only way, and perhaps not the best way.

** Give it, and the hypergeometric above and below.

the vacuum mode z_+/y_+, which shows that the latter is part of the physical sector. It also pushes up, and at level 3 we find in particular the mode

$$L_{+i}L^j_+(y_+^{-1}z_j) - L^j_+L_{+j}(y_+^{-1}z_i) = 6y^2y_+^{-4}(y_+z_i - y_iz_+),$$

which is the ground state of $D(3, w_1)$. The complete reduction of the singleton field module is thus

$$D_S \oplus D_1 = D(1, w_1) \to \left[D_{\text{massless}} \oplus D(2, 0) \oplus \text{id}\right] \to D(1, w_1).$$

The singleton propagator is

$$(z \cdot z')^{p-1}{}_3F_2($$

The hypergeometric function is of logarithmic type.

4. Conclusions.

In this paper we have shown that conformally invariant p-form field strengths, of conformal degree p, and the associated $(p-1)$-form potentials, in $2p$-dimensional Minkowski space, can be extended to a topological singleton field theory in AdS_{2p+1}. The representations of $SO(2p, 2)$ carried by these fields are all zero-center modules, just as is the case for conformal electrodynamics in 4 dimensions (the case $p = 2$).

Field strengths of conformal degree $p+2$ describe massless fields in the bulk; they too are zero-center modules. In fact, these representations appear as part of the gauge sector in the singleton field module.

Fields of this kind are known to be required by superconformal symmetry [6],[2],[3] in the case $p = 2, 3$; they may play a role for $p = 4, 5$ as well, though simple super extensions of AdS_9 and AdS_{11} do not exist. One may point out that the self-dual five form field strength of the $10d, IIB$ string is a "singleton" candidate for AdS_{11} in the sense of this paper. This may have some implications for attempts to construct a fundamental framework for strings or M-theory.

Appendix.

Here it will be shown that the following highest weight representations of $SO(2p, 2)$,

$$D(k, w_k), \quad k = 0, ..., p,$$

are zero-center modules; that is, the center of the enveloping algebra is represented by zero. The demonstration consists of exhibiting equivariant maps from $D(k, w_k)$ to $D(k+1, w_{k+1})$, for $k = 0, ..., p-1$.

This proof will extend to higher forms, of degree $p = d/2$, the same singleton behaviour of $4d$ electrodynamics when extended to the AdS_{d+1} bulk.

Let V^k be the Harish-Chandra module with highest weight (lowest energy) (k, w_k), let v_0 be a highest weight vector in V^k, and let V^k_0 be the subspace generated by the subalgebra $so(2p)$ from v_0. This subspace can be identified with the space $Z^k \otimes \mathbf{C}v_0$, where Z^k is level k in the Grassmann algebra generated by $z_1, ...z_{2p}$. Let ψ denote a vector in this space. Apply the energy raising operators to obtain the vectors

$$\{L_{+i}\psi\}_{\psi \in Z^k \otimes \mathbf{C}v_0, i=1,...,2p}.$$

There is an action of $so(2p)$ on this space. Project out the irreducible component with highest weight w_{k+1}, spanned by the vectors

$$(L_{+i} - L_{+j} z_i \partial_j)\psi. \tag{A.1}$$

Now apply the energy lowering operators to these vectors, using the commutation relations

$$[L_{-m}, L_{+i}] = L_{im} + k\delta_{im}, \quad [L_{ij}, z_m] = \delta_{jm} z_i - \delta_{im} z_j.$$

The result

$$L_{-m}(L_{+i} - L_{+j} z_i \partial_j)\psi = 0, \quad i, m = 1, ..., 2p,$$

is immediate. This shows that the Harish-Chandra module V^k contains V^{k+1} as an invariant subspace, and that there is a unique equivariant map from one to the other, that sends the vectors (A.1) in V^k to the highest level in V^{k+1}.

Next, consider the representations $D(p+k, w_{p-k}), k = 0, ..., p$. Let U^k be the Harish-Chandra module with highest weight $(p+k, w_{p-k}$. Apply the energy raising operators to $\psi \in U_0^k = Z^{p-k} \oplus \mathbf{C}u_0$, u_0 the highest weigp-ht vector in U^k. Again $SO(2p)$ acts on this space. The subspace that contains the component with the highest weight w_{k-1} consists of the vectors

$$\sum_{i=1}^{2p} L_+^i z_i \tilde{\psi}, \quad \tilde{\psi} \in Z^{p-k-1} \otimes \mathbf{C}u_0.$$

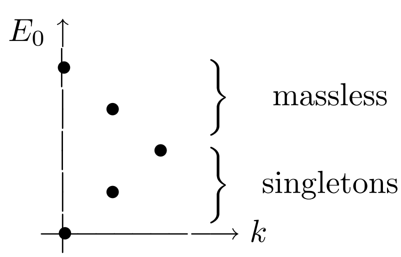
Apply the lowering operators, using the commutation relations as above, to get

$$L_{-j} \sum L_+^i z_i \tilde{\psi} = \big(L_{ij} + \delta_{ij}(p+k)\big)\tilde{\psi} = 0, \quad j = 1, ..., 2p.$$

This shows that there is an extension $D(p+k, w_{p-k}) \to D(p+k+1, w_{p-k-1})$. We conclude that all the representations

$$D(p \pm k, w_{p-k}), \quad k = 0, ..., p,$$

are zero-center modules.



In the diagram, drawn for the case $p = 3$, we show the zero-center modules $D(p \pm k, w_{p-k})$ as dots at the points $E_0 = p \pm k$. We have shown that there are extensions between neighbour representations. In addition, we have seen that, as a field theory, the singleton $D(2, w_1)$ carries the massless representation $D(3, w_1)$ in its gauge subspace. We are confident that this is but an example of a general phenomenon.

It is easy to show that all the extensions between neighbours are realized on field modules. To do this we have only to generalize the calculation of Section 2.3, replacing the formula (2.10) for the (candidate) gauge modes by

$$A = d\phi_1 + y \wedge \Phi_2 + y^2 \Phi_3.$$

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