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ON INTRABEAM SCATTERING IN A HEAVY ION STORAGE RING

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Based on the intrabeam scattering theory of Bjorken–Mtingwa, the general expressions for the growth rates of the horizontal, vertical and longitudinal emittances are expressed in the simple form of elliptic integrals. Using this analytical approach, the blow-up of a bunched heavy ion beam in the proposed HIRFL storage ring is simulated. The time evolution of transverse emittances and momentum spread is obtained. A comparison of the results with those obtained from the extended Piwinski theory exhibits a maximum discrepancy of 30%.

Keywords: Intrabeam scattering; Emittance growth rates; Bjorken–Mtingwa expressions; Elliptic integral

1 INTRODUCTION

Intrabeam scattering (IBS) often dominates the resolution limits of a cooled ion beam in a storage ring.[1] In order to perform reliable theoretical forecasts, the original Piwinski[2] IBS theory was extended to include the variations of betatron functions and momentum dispersion function along the lattice by Sacherer, Möhl, Piwinski and Martini,[3] whose results were summarized in the paper by Martini.[3] Meanwhile, Bjorken and Mtingwa[4] (hereafter abbreviated B–M) also worked out the general formulae through a scattering matrix formalism, considering the variations of betatron and dispersion functions. The sum of the three

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growth rates derived by B–M was expressed in the form of elliptic integrals. However, the growth rate in each of the 3 dimensions was still expressed in complicated integral forms. The peculiarities of the integrands bring difficulties for a practical application, although approximations were made for ultra-relativistic energy[4] ($\gamma \geq 10$) and lower energy[5] ($\gamma \approx 3.77$) beams respectively.

In this paper, following B–M, we extend the derivations of IBS growth rate formulae. We find the expression of the growth rate in each of the 3 dimensions in the form of elliptic integrals which can be evaluated quite easily in practical applications. These formulae are adopted to simulate the blow-up process for a bunched beam of Ar^{18+} of 30 MeV/u in the proposed HIRFL Cooler-Storage Ring (HIRFL-CSR).[6] The continuous enlargements of beam emittances and momentum spread obtained from the simulations are shown as a function of time, and are compared with predictions[7] based upon the extended Piwinski theory.

2 INTRABEAM SCATTERING GROWTH RATES

For convenience, we summarize the B–M growth rates for the transverse emittances (1σ) $\varepsilon_{\rm h}$, $\varepsilon_{\rm v}$ and the longitudinal phase ellipse area (1σ) $\varepsilon_{\rm l}$ [see Eq. (3.4) of Ref. 2]

$$\frac{1}{\tau_j} = \frac{1}{\varepsilon_j}\frac{\mathrm{d}\varepsilon_j}{\mathrm{d}t} = \Gamma\left\langle \int_0^\infty \frac{\lambda^{1/2}\cdot \mathrm{d}\lambda}{|L+\lambda I|^{1/2}}\left\{\left[\mathrm{Tr}(L_j)\right]\cdot \mathrm{Tr}\left(\frac{1}{L+\lambda I}\right) - 3\mathrm{Tr}\left[L_j\cdot\left(\frac{1}{L+\lambda I}\right)\right]\right\}\right\rangle, \quad j = \mathrm{h}, \mathrm{v}, \mathrm{l}. \tag{1}$$

Here the brackets $\langle\cdots\rangle$ denote an average around the ring, Tr denotes the trace of the matrix, and $|L+\lambda I|$ denotes the determinant of the matrix L (defined below) plus λ times the unit matrix, where λ is the variable of the integration.

For an unbunched beam,

$$\Gamma = \frac{6.24\times 10^{18}\cdot I_{\rm i}\cdot r_{\rm i}^2\cdot L_{\rm c}}{4\sqrt{\pi} q_{\rm i}\beta_{\rm i}^4\gamma_{\rm i}^4\cdot \varepsilon_{\rm h}\varepsilon_{\rm v}\sigma_p}, \tag{2}$$

in which $r_{\rm i} = (q_{\rm i}^2/A_{\rm i})\cdot r_p$ is the classical radius of the ion with charge state $q_{\rm i}$ and mass number $A_{\rm i}$, $r_p = 1.547\times 10^{-18}$ m is the classical proton radius, σ_p is the rms relative momentum spread, $\beta_\mathrm{i}, \gamma_\mathrm{i}$ are the usual relativistic factors, I_i is the beam current in Ampere, and L_c is the Coulomb logarithm (taken as $L_\mathrm{c} = 20$ throughout our calculation).

For a bunched beam,

$$\Gamma = \frac{N_\mathrm{i} \cdot r_\mathrm{i}^2 \cdot c \cdot L_\mathrm{c}}{8\pi\beta_\mathrm{i}^3\gamma_\mathrm{i}^4 \cdot \varepsilon_\mathrm{h}\varepsilon_\mathrm{v}\sigma_p\sigma_\mathrm{s}}. \tag{3}$$

In Eq. (3) N_i is the number of particles in the bunch, c is the speed of light, and σ_s is the rms bunch length.

The matrices L_j for the horizontal, vertical and longitudinal dimensions respectively are given as follows

$$L_\mathrm{h} = \frac{\beta_\mathrm{h}}{\varepsilon_\mathrm{h}} \begin{pmatrix} 1 & -\gamma_\mathrm{i}\phi & 0 \\ -\gamma_\mathrm{i}\phi & \dfrac{\gamma_\mathrm{i}^2 D_\mathrm{h}^2}{\beta_\mathrm{h}^2} + \gamma_\mathrm{i}^2\phi^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_\mathrm{v} = \frac{\beta_\mathrm{v}}{\varepsilon_\mathrm{v}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_\mathrm{l} = \frac{m\gamma_\mathrm{i}^2}{\sigma_p^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

and L is given by the sum $L = L_\mathrm{h} + L_\mathrm{v} + L_\mathrm{l}$. Here $m = 1$ for an unbunched beam and $m = 2$ for a bunched beam, $\phi = D_\mathrm{h}' - \beta_\mathrm{h}' D_\mathrm{h}/(2\beta_\mathrm{h})$, D_h is the horizontal dispersion, $\beta_\mathrm{h}' = \mathrm{d}(\beta_\mathrm{h})/\mathrm{d}s$ and $D_\mathrm{h}' = \mathrm{d}(D_\mathrm{h})/\mathrm{d}s$ are derivatives of β_h and D_h with respect to the longitudinal coordinate s respectively. The vertical dispersion is neglected.

A simple expression for the total growth rate obtained by summation over $j = \mathrm{h, v, l}$ and expressed already in the form of elliptic integrals was given by B–M. However, the growth rate in each of the 3 dimensions is of major concern in our application, so we start the derivations from Eq. (1).

Due to symmetry, the matrices L_j and L may be written as

$$L_j = \begin{pmatrix} S_{1j} & S_{2j} & 0 \\ S_{2j} & S_{3j} & 0 \\ 0 & 0 & S_{4j} \end{pmatrix},$$

$$L = L_h + L_v + L_l = \begin{pmatrix} T_1 & T_2 & 0 \\ T_2 & T_3 & 0 \\ 0 & 0 & T_4 \end{pmatrix}. \tag{5}$$

Accordingly, eigenvalues of the matrix L are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[(T_1 + T_3) + \sqrt{(T_1 - T_3)^2 + 4T_2^2}\right], \\ \lambda_2 &= \frac{1}{2}\left[(T_1 + T_3) - \sqrt{(T_1 - T_3)^2 + 4T_2^2}\right], \\ \lambda_3 &= T_4 \end{aligned} \tag{6}$$

and then Eq. (1) can be written in the form

$$\frac{1}{\tau_j} = \Gamma\left\langle \int_0^\infty \frac{\left(F_{1j}\lambda^{3/2} + F_{2j}\lambda^{1/2}\right) \cdot \mathrm{d}\lambda}{(\lambda + \lambda_1)^{3/2}(\lambda + \lambda_2)^{3/2}(\lambda + \lambda_3)^{3/2}} \right\rangle. \tag{7}$$

In Eq. (7) the F_{1j}, F_{2j} are related to the matrix elements by

$$\begin{aligned} F_{1j} &= 2(S_{1j} + S_{3j} + S_{4j})(T_1 + T_3 + T_4) \\ &\quad - 3(S_{1j}T_3 + S_{1j}T_4 - 2S_{2j}T_2 + S_{3j}T_1 + S_{3j}T_4 + S_{4j}T_1 + S_{4j}T_3), \\ F_{2j} &= (S_{1j} + S_{3j} + S_{4j})(T_3T_4 + T_1T_4 + T_1T_3 - T_2^2) \\ &\quad - 3(S_{1j}T_3T_4 - 2S_{2j}T_2T_4 + S_{3j}T_1T_4 + S_{4j}T_1T_3 - S_{4j}T_2^2). \end{aligned} \tag{8}$$

Whenever $\phi \neq 0$, the eigenvalues λ_n $(n = 1, 2, 3)$ cannot be all equal. This happens at almost all points around a strong-focusing lattice. For the most general case, let $\lambda_1 > \lambda_2 > \lambda_3$, we can express analytically the integrals in terms of elliptic integrals [see Appendix]

$$\int_0^\infty \frac{\lambda^{3/2}\,\mathrm{d}\lambda}{(\lambda + \lambda_1)^{3/2}(\lambda + \lambda_2)^{3/2}(\lambda + \lambda_3)^{3/2}} = A_1 \cdot E(\xi, k) + B_1 \cdot F(\xi, k) + C_1, \tag{9}$$

$$\int_0^\infty \frac{\lambda^{1/2}\,\mathrm{d}\lambda}{(\lambda + \lambda_1)^{3/2}(\lambda + \lambda_2)^{3/2}(\lambda + \lambda_3)^{3/2}} = A_2 \cdot E(\xi, k) + B_2 \cdot F(\xi, k) + C_2, \tag{10}$$

where $E(\xi, k)$ and $F(\xi, k)$ are the elliptic integrals of the first and second kinds, i.e.

$$E(\xi, k) = \int_0^{\xi} \sqrt{1 - k^2 \sin^2 \theta} \cdot \mathrm{d}\theta, \qquad F(\xi, k) = \int_0^{\xi} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \cdot \mathrm{d}\theta \tag{11}$$

and the coefficients A_n, B_n and C_n $(n = 1, 2)$ are expressed as below:

$$\begin{aligned}
A_1 &= \frac{2V}{\lambda_1^2 \cdot U^{3/2} \cdot (1 - V)^{7/2}} \cdot \frac{(1 + V)k^4 + (V - 3)k^2 + 2}{k^4(1 - k^2)^2}, \\
B_1 &= \frac{2V}{\lambda_1^2 \cdot U^{3/2} \cdot (1 - V)^{7/2}} \cdot \frac{(2 - V)k^2 - 2}{k^4(1 - k^2)}, \\
C_1 &= \frac{2V}{\lambda_1^2 \cdot U^{3/2} \cdot (1 - V)^{7/2}} \cdot \frac{(-V)(k^2 + 1)}{k^2(1 - k^2)^2} \cdot \sqrt{\frac{1 - V}{U}}, \\
A_2 &= \frac{2}{\lambda_1^3 \cdot U^{3/2} \cdot (1 - V)^{7/2}} \cdot \frac{(-2)\left[(V^2 - V + 1)k^4 - (2 - V)k^2 + 1\right]}{k^4(1 - k^2)^2}, \\
B_2 &= \frac{2}{\lambda_1^3 \cdot U^{3/2} \cdot (1 - V)^{7/2}} \cdot \frac{(V^2 - 2V + 1)k^4 + (2V - 3)k^2 + 2}{k^4(1 - k^2)}, \\
C_2 &= \frac{2}{\lambda_1^3 \cdot U^{3/2} \cdot (1 - V)^{7/2}} \cdot \frac{V[(2V - 1)k^2 + 1]}{k^2(1 - k^2)^2} \cdot \sqrt{\frac{1 - V}{U}}
\end{aligned}$$

in which

$$U = \frac{\lambda_2}{\lambda_1}, \qquad V = \frac{\lambda_3}{\lambda_1},$$

$$k^2 = \frac{\lambda_1(\lambda_2 - \lambda_3)}{\lambda_2(\lambda_1 - \lambda_3)}, \qquad \xi = \arcsin \sqrt{1 - V} = \arcsin \sqrt{1 - \frac{\lambda_3}{\lambda_1}}. \tag{12}$$

Substituting Eqs. (9) and (10) into Eq. (7), we finally obtain the growth rates in 3 dimensions:

$$\begin{aligned}
\frac{1}{\tau_j} = \Gamma \Big\langle &(F_{1j}A_1 + F_{2j}A_2)E(\xi, k) + (F_{1j}B_1 + F_{2j}B_2)F(\xi, k) \\
&+ (F_{1j}C_1 + F_{2j}C_2) \Big\rangle.
\end{aligned} \tag{13}$$

In the special case of $\lambda_1 > \lambda_2 = \lambda_3$, we can also analytically evaluate

$$\int_0^\infty \frac{\lambda^{3/2}\,d\lambda}{(\lambda+\lambda_1)^{3/2}(\lambda+\lambda_2)^3}$$
$$= \frac{2(1-U)^{-3}}{\lambda_1^2\sqrt{U(1-U)}}\left[\frac{3}{2}U\xi + \frac{3}{8}\xi - \frac{13}{8}\sqrt{U(1-U)} - \frac{U}{4}\sqrt{U(1-U)}\right], \tag{14}$$

$$\int_0^\infty \frac{\lambda^{1/2}\,d\lambda}{(\lambda+\lambda_1)^{3/2}(\lambda+\lambda_2)^3}$$
$$= \frac{2(1-U)^{-3}}{\lambda_1^3 U\sqrt{U(1-U)}}\left[-U(1+U)\xi + \frac{1}{8}\xi + \frac{1}{8}\sqrt{U(1-U)} + \frac{7U}{4}\sqrt{U(1-U)}\right]. \tag{15}$$

3 APPLICATIONS

As a first check, we apply the calculation to the Antiproton Accumulator ($\bar{p}$A) at Fermilab using the beam and lattice parameters given in Ref. 4, especially

$$\varepsilon_h = \varepsilon_v = 0.42 \times 10^{-6}\,\text{m}, \qquad \sigma_p = 1.2 \times 10^{-4},$$
$$I_i = 0.041\,\text{A}, \qquad \gamma_i = 9.53.$$

The resulting growth rates (Eq. (13)) at 24 lattice locations are listed in Table I (labeled by "Present Values"), together with the results given by B–M. We obtain consistently faster growth rates than B–M by 22% because B–M neglected two terms in their results.

As the main application, a computer program has been written to simulate evolution of an ion beam in a storage ring under the influence of IBS. It has been used to study the case of a bunched beam of Ar^{18+} at 30 MeV/u in the proposed HIRFL-CSR. The results are compared with predictions obtained from the INTRAB program[7] which is based upon the extended Piwinski model.

TABLE I Comparison of IBS growth rates

Lattice locations	τ_h^{-1} (h^{-1})		τ_v^{-1} (h^{-1})		τ_l^{-1} (h^{-1})	
	B–M values	*Present values*	*B–M values*	*Present values*	*B–M values*	*Present values*
1	0.252	0.308	−0.00267	−0.00326	0.920	1.125
2	0.235	0.287	−0.00275	−0.00336	0.857	1.048
3	0.231	0.283	−0.00273	−0.00334	0.842	1.030
4	0.226	0.277	−0.00271	−0.00332	0.823	1.007
5	0.202	0.247	−0.00261	−0.00319	0.731	0.894
6	0.188	0.230	−0.00314	−0.00383	0.683	0.836
7	0.188	0.230	−0.00322	−0.00393	0.683	0.836
8	0.191	0.233	−0.00316	−0.00386	0.695	0.850
9	0.194	0.238	−0.00310	−0.00379	0.708	0.866
10	0.126	0.154	−0.00311	−0.00380	0.758	0.928
11	0.127	0.155	−0.00309	−0.00378	0.762	0.932
12	0.0692	0.0846	−0.00314	−0.00384	0.801	0.980
13	0.0701	0.0858	−0.00312	−0.00381	0.810	0.991
14	0.0703	0.0860	−0.00313	−0.00383	0.812	0.994
15	0.0709	0.0868	−0.00320	−0.00391	0.818	1.001
16	0.0187	0.0229	−0.00320	−0.00391	0.828	1.013
17	0.0187	0.0229	−0.00318	−0.00389	0.828	1.013
18	−0.00287	−0.00351	−0.00287	−0.00351	0.825	1.009
19	−0.00281	−0.00344	−0.00284	−0.00347	0.823	1.008
20	−0.00293	−0.00358	−0.00290	−0.00355	0.827	1.011
21	−0.00322	−0.00394	−0.00325	−0.00397	0.874	1.070
22	−0.00315	−0.00385	−0.00321	−0.00392	0.889	1.088
23	−0.00269	−0.00329	−0.00282	−0.00344	1.060	1.300
24	0.347	0.424	−0.00209	−0.00256	1.270	1.550
Average values	0.117	0.143	−0.00297	−0.00363	0.830	1.016

Figure 1 shows the time evolution of transverse emittances, momentum spread and bunch length for the Ar^{18+} beam obtained from the formulae given in this paper, and from the extended Piwinski theory. The lattice taken is described in Ref. 8, and the initial parameters of the ion beam are

$$\varepsilon_{h0} = \varepsilon_{v0} = 1.0\pi\,\mathrm{mm\cdot m\,rad}, \quad \frac{\Delta p}{p} = \pm 1.0 \times 10^{-4},$$

$$\text{bunch full length} = 3.84\,\mathrm{m}, \quad N_i = 5 \times 10^8\,\text{ions/bunch}.$$

The transverse emittances, momentum spread and bunch length are defined as 1σ distribution, so that,

$$\varepsilon_{h,v} = \frac{\sigma_{h,v}^2}{\beta_{h,v}}, \qquad \frac{\Delta p}{p} = \sigma_p, \qquad \text{bunch length} = \sigma_s.$$

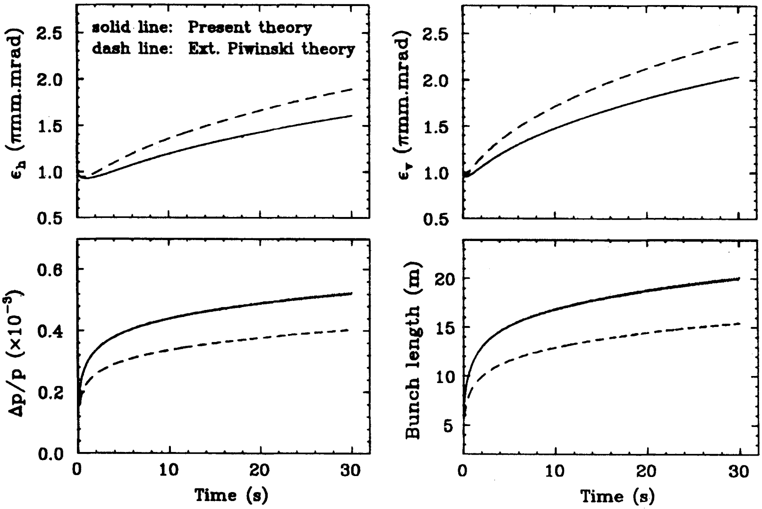

FIGURE 1 Time evolution of beam emittances, momentum spread and bunch length due to the IBS.

It can be seen from the Figure 1 that increase of the transverse emittances based upon the B–M theory is approximately 20% lower than that obtained from the extended Piwinski model, whereas for the longitudinal degree of freedom, the B–M approach shows higher growth by roughly 30%. The fact that the Coulomb logarithm was considered as independence of the beam characteristics and taken as a constant 20 throughout the B–M derivations may be one source of this discrepancy. However, both theories show that the beam does not tend to an equilibrium, but displays a continuous blow-up in all three planes. This is explained by the transfer of beam kinetic energy into transverse and longitudinal energy spreads.

4 CONCLUSION

We have shown that the general expressions of B–M can be brought into analytical form expressed in terms of the elliptic integrals which can readily be resolved. Comparison of results from these formulae with those obtained from the extended Piwinski theory exhibits a maximum deviation of 30%.

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APPENDIX

We consider an analytical evaluation of the integral

$$Int = \int_0^\infty \frac{\lambda^{3/2} \cdot d\lambda}{(\lambda+\lambda_1)^{3/2}(\lambda+\lambda_2)^{3/2}(\lambda+\lambda_3)^{3/2}}$$

for the general case $\lambda_1 > \lambda_2 > \lambda_3$. Setting $U = \lambda_2/\lambda_1$, $V = \lambda_3/\lambda_1$ and $x = \lambda/\lambda_1$, we get

$$Int = \frac{1}{\lambda_1^2}\int_0^\infty \frac{x^{3/2} \cdot dx}{(x+1)^{3/2}(x+U)^{3/2}(x+V)^{3/2}}.$$

Letting $x = V\tan^2\phi$ and $1 - V/U = k^2(1-V)$, we have

$$\begin{aligned} Int &= \frac{2V}{\lambda_1^2 \cdot U^{3/2}}\int_0^{\pi/2} \frac{\sin^4\phi \cdot \cos^3\phi \cdot d\phi}{[1-(1-V)\sin^2\phi]^{3/2} \cdot [1-k^2(1-V)\sin^2\phi]^{3/2}} \\ &= \frac{2V \cdot (1-V)^2 \cdot \sqrt{1-V}}{\lambda_1^2 \cdot U^{3/2}} \\ &\quad \times \int_0^{1/\sqrt{1-V}} \frac{x^4[1-x^2(1-V)] \cdot dx}{[1-x^2(1-V)^2]^{3/2} \cdot [1-k^2x^2(1-V)^2]^{3/2}} \\ &= \frac{2V}{\lambda_1^2 \cdot U^{3/2}}\int_0^1 \frac{y^4 \cdot (1-y^2) \cdot dy}{[1-(1-V)y^2]^{3/2} \cdot [1-k^2(1-V)y^2]^{3/2}}. \end{aligned}$$

Letting $y = \sin\theta/\sqrt{1-V}$, $\sin\xi = \sqrt{1-V}$, we get

$$Int = \frac{2V \cdot (1-V)^{-7/2}}{\lambda_1^2 \cdot U^{3/2}}\left[\int_0^\xi \frac{\sin^4\theta \cdot d\theta}{(1-k^2\sin^2\theta)^{3/2}} - V\int_0^\xi \frac{\sin^2\theta \cdot \tan^2\theta \cdot d\theta}{(1-k^2\sin^2\theta)^{3/2}}\right].$$

Through partial integration, we have

$$\int_0^{\xi} \frac{\sin^4\theta \cdot d\theta}{(1-k^2\sin^2\theta)^{3/2}} = \frac{1}{k^2}\frac{\sin^3\xi}{\cos\xi}\frac{1}{\sqrt{1-k^2\sin^2\xi}} - \frac{2}{k^2}\int_0^{\xi}\frac{\sin^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} - \frac{1}{k^2}\int_0^{\xi}\frac{\tan^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

$$\int_0^{\xi} \frac{\sin^2\theta\cdot\tan^2\theta \cdot d\theta}{(1-k^2\sin^2\theta)^{3/2}} = \frac{1}{k^2}\frac{\sin^3\xi}{\cos\xi}\frac{1}{\sqrt{1-k^2\sin^2\xi}} - \frac{3V}{k^2}\int_0^{\xi}\frac{\tan^4\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} - \frac{3V}{k^2}\int_0^{\xi}\frac{\tan^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

so that

$$Int = \frac{2V(1-V)^{-7/2}}{\lambda_1^2\cdot U^{3/2}}\left[-\frac{2}{k^2}\int_0^{\xi}\frac{\sin^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} - \frac{1-3V}{k^2}\int_0^{\xi}\frac{\tan^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} + \frac{3V}{k^2}\int_0^{\xi}\frac{\tan^4\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}}\right]. \quad \text{(A1)}$$

We can express the integrals in terms of the elliptic integrals of the first and second kinds[9] by

$$\int_0^{\xi}\frac{\sin^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} = E(\xi,k) - F(\xi,k), \quad \text{(A2)}$$

$$\int_0^{\xi}\frac{\tan^2\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{1-k^2}\left[\sqrt{1-k^2\sin^2\xi}\cdot\tan\xi - E(\xi,k)\right], \quad \text{(A3)}$$

$$\int_0^{\xi}\frac{\tan^4\theta\cdot d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{3(1-k^2)^2}\left\{2(2-k^2)E(\xi,k) - (1-k^2)F(\xi,k) + \tan\xi\cdot\sqrt{1-k^2\sin^2\xi}\cdot\left[\frac{1-k^2}{\cos^2\xi} - 2 - 2(1-k^2)\right]\right\}. \quad \text{(A4)}$$

Substituting Eqs. (A2), (A3) and (A4) into (A1), and combining the corresponding coefficients of $E(\xi, k)$ and $F(\xi, k)$, we eventually obtain

$$\int_0^\infty \frac{\lambda^{3/2}\,\mathrm{d}\lambda}{(\lambda+\lambda_1)^{3/2}(\lambda+\lambda_2)^{3/2}(\lambda+\lambda_3)^{3/2}} = A_1 \cdot E(\xi, k) + B_1 \cdot F(\xi, k) + C_1$$

in which

$$A_1 = \frac{2V}{\lambda_1^2 \cdot U^{3/2} \cdot (1-V)^{7/2}} \cdot \frac{(1+V)k^4 + (V-3)k^2 + 2}{k^4(1-k^2)^2},$$

$$B_1 = \frac{2V}{\lambda_1^2 \cdot U^{3/2} \cdot (1-V)^{7/2}} \cdot \frac{(2-V)k^2 - 2}{k^4(1-k^2)},$$

$$C_1 = \frac{2V}{\lambda_1^2 \cdot U^{3/2} \cdot (1-V)^{7/2}} \cdot \frac{(-V)(k^2+1)}{k^2(1-k^2)^2} \cdot \sqrt{\frac{1-V}{U}}.$$

In a similar manner, we can also obtain Eq. (10).

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