Grand Unification at Intermediate Mass Scales through Extra Dimensions

Keith R. Dienes[1][*], Emilian Dudas[1,2][†], and Tony Gherghetta[1][‡]

[1] *CERN Theory Division, CH-1211 Geneva 23, Switzerland*
[2] *LPTHE, Univ. Paris-Sud, F-91405, Orsay Cedex, France*

Abstract

One of the drawbacks of conventional grand unification scenarios has been that the unification scale is too high to permit direct exploration. In this paper, we show that the unification scale can be significantly lowered (perhaps even to the TeV scale) through the appearance of extra spacetime dimensions. Such extra dimensions are a natural consequence of string theories with large-radius compactifications. We show that extra spacetime dimensions naturally lead to gauge coupling unification at intermediate mass scales, and moreover may provide a natural mechanism for explaining the fermion mass hierarchy by permitting the fermion masses to evolve with a power-law dependence on the mass scale. We also show that proton-decay constraints may be satisfied in our scenario due to the higher-dimensional cancellation of proton-decay amplitudes to all orders in perturbation theory. Finally, we extend these results by considering theories without supersymmetry; experimental collider signatures; and embeddings into string theory. The latter also enables us to develop several novel methods of explaining the fermion mass hierarchy via D-branes. Our results therefore suggest a new approach towards understanding the physics of grand unification as well as the phenomenology of large-radius string compactifications.

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[*]E-mail address: keith.dienes@cern.ch
[†]E-mail address: emilian.dudas@cern.ch
[‡]E-mail address: tony.gherghetta@cern.ch

1   Introduction and basic idea

One of the most widely investigated proposals for physics beyond the Minimal Supersymmetric Standard Model (MSSM) is the possible appearance of a grand unified theory (GUT). There are several profound attractions to the idea of grand unification. Perhaps the most obvious is that GUT's have the potential to unify the diverse set of particle representations and parameters found in the MSSM into a single, comprehensive, and hopefully predictive framework. For example, through the GUT symmetry one might hope to explain the quantum numbers of the fermion spectrum, or even the origins of fermion mass. Moreover, by unifying all $U(1)$ generators within a non-abelian theory, GUT's would also provide an explanation for the quantization of electric charge. Furthermore, because they generally lead to baryon-number violation, GUT's have the potential to explain the cosmological baryon/anti-baryon asymmetry. By combining GUT's with supersymmetry in the context of SUSY GUT's, it might then be possible to realize the attractive features of GUT's simultaneously with those of supersymmetry in a single theory. Indeed, there is even a bit of "experimental" evidence for the idea of grand unification, for the three MSSM gauge couplings appear to unify when extrapolated towards higher energies within the framework of the MSSM, with a unification scale $M_{\mathrm{GUT}} \approx 2 \times 10^{16}$ GeV. The phenomenon of gauge coupling unification thus sets the natural energy scale for grand unification.

Unfortunately, because this energy scale is so high, it proves rather difficult to probe the physics of grand unification directly at low energies. There are, of course, numerous *indirect* probes of such high-scale GUT physics, most notably through rare decays such as proton decay. However, all of these tests are ultimately limited by the remoteness of the GUT scale.

In this paper, we shall investigate whether it is possible to bring grand unification down towards more accessible energy scales by *lowering the unification scale itself*. Of course, it hardly seems possible to consider a unification of the MSSM gauge groups at any lower scale $M \ll M_{\mathrm{GUT}}$ at which their couplings have not yet unified. Therefore, in order to sensibly contemplate the possibility of an intermediate-scale GUT, we are forced to consider whether it is possible to achieve gauge coupling unification at scales $M \ll M_{\mathrm{GUT}}$. It is immediately clear that adding arbitrary extra matter states to the MSSM cannot achieve the desired effect, for in general such extra matter not only tends to raise (rather than lower) the unification scale, but also tends to drive the theory towards strong coupling. What we require, therefore, is a different mechanism.

The underlying reason that gauge coupling unification is delayed until such a high energy scale is essentially that the one-loop MSSM gauge couplings run only logarithmically with energy scale μ (or linearly versus log μ). Thus, given the different values of these couplings at the weak scale, one must extrapolate upwards over many orders of magnitude in energy before they have a chance of unifying. Clearly, if there were a way to change the running of the gauge couplings so that they ran more

quickly (*e.g.*, exponentially rather than linearly), we would have a chance to achieve a more rapid unification.

What physical effect could cause the running of gauge couplings to be exponential rather than linear? Remarkably, there does exist a simple way in which such an exponential running can arise: the appearance of extra spacetime dimensions. Since extra spacetime dimensions are naturally predicted in string theory (both through the need to compactify as well as through various non-perturbative effects), we expect that such a scenario might find a natural home within the context of string theory. However, as we shall see, such a scenario can be discussed in purely field-theoretic terms.

Of course, there is one subtle complication with this naïve picture: in field theory, extra spacetime dimensions lead to a loss of renormalizability. Thus, strictly speaking, quantities such as gauge couplings do not "run" in the usual sense. However, our basic idea is nevertheless correct, and as we shall see, the "exponential running" is more correctly described as an exponential dependence on the cutoff pertaining to high-scale physics (such as the appearance of a fundamental string theory) at energy scales beyond those we shall be considering. Moreover, as we shall demonstrate, even though our theory is non-renormalizable, there exists a *renormalizable* theory which, for our purposes, is essentially equivalent to our non-renormalizable theory. Thus, our basic intuitive idea of exponential "running" remains intact.

Given this motivation, in this paper we shall undertake a general analysis of the effects of extra spacetime dimensions on the MSSM. We shall begin, in Sect. 2, with a discussion of various issues that arise when attempting to extrapolate the MSSM to higher dimensions. Then, in Sect. 3, we shall discuss the effects of extra spacetime dimensions on ordinary gauge coupling unification. Quite remarkably, we shall find that within the MSSM, the appearance of *any* number of extra spacetime dimensions at *any* intermediate scale always preserves gauge coupling unification — indeed, we shall find that the effect of the extra dimensions is simply to shift the unification scale downwards towards lower energies, as desired. Thus, within the MSSM, we find that the appearance of extra spacetime dimensions naturally leads to an intermediate-scale grand unified theory.

In Sect. 4, we shall then consider the effects of extra spacetime dimensions of the proton-lifetime problem, and propose that to all orders in perturbation theory, proton-decay amplitudes are exactly cancelled as the result of new Kaluza-Klein selection rules corresponding to the extra spacetime dimensions. This is therefore an intrinsically *higher-dimensional* solution to the proton-decay problem.

In Sect. 5, we shall then turn our attention to the effects of extra spacetime dimensions on the evolution of the fermion Yukawa couplings. We shall find that extra dimensions cause the Yukawa couplings to run exponentially as well, and thereby show that this exponential running for the Yukawa couplings has the potential to explain the fermion mass hierarchy.

In Sect. 6, we shall then consider the effects of extra dimensions on the non-

supersymmetric Standard Model, and show that once again it is possible to obtain gauge coupling unification at relatively low energy scales, even without supersymmetry. Thus, if the unification scale is sufficiently low, we can actually avoid the *gauge* hierarchy problem.

In Sect. 7, we shall then discuss how our scenario can be embedded into string theory, and make some general remarks concerning the manner in which the appearance of extra spacetime dimensions and intermediate-scale grand unification can be incorporated and interpreted in terms of the mass scales expected within string theory. We shall also consider various non-perturbative D-brane realizations of our scenario. Then, in Sect. 8, we shall revisit the fermion mass hierarchy problem using some of these non-perturbative D-brane insights, and we shall propose several new D-brane methods for addressing the fermion mass hierarchy problem which do not rely on the *ad hoc* introduction of extra low-energy matter states or flavor-dependent couplings. The mechanisms we propose in this section might therefore be of general use for the phenomenology of Type I model-building.

In Sect. 9, we shall discuss how our our work relates to prior work in the literature, and in Sect. 10 we shall discuss some of the experimental consequences of our scenario. As we shall see, our scenario can be expected to lead to numerous exciting collider signals; it can also have important cosmological implications. We will then conclude in Sect. 11 with a summary of our results, and as well as future prospects and ramifications. Three Appendices contain ancillary calculations which justify the basic approach that we shall be following in this paper. Note that an abbreviated discussion of some of the ideas in this paper can also be found in Ref. [1].

2   Preliminaries

How can we incorporate extra spacetime dimensions into a field-theoretic analysis of the MSSM? In this section, we shall provide a discussion of some of the issues that arise, including the appearance and proper treatment of Kaluza-Klein modes as well as the resulting lack of renormalizability that afflicts higher-dimensional field theories. Throughout, we shall focus on taking a "bottom-up" approach, and seek the "minimal" scenarios that consistently embed the MSSM into higher dimensions. Thus, our approach will necessarily be part of any larger structure that may ultimately be derived from a more complete high-energy theory such as string theory.

2.1   Incorporating extra dimensions into the MSSM

Given that the observed low-energy world consists of only four flat dimensions, the only rigorous way in which to discuss the appearance of extra spacetime dimensions is to assume that they are compactified. For this purpose, we shall begin by assuming that they are simply compactified on a circle of a certain fixed radius R, where R^{-1} exceeds presently observable energy scales. Thus $\mu_0 \equiv R^{-1}$ sets the mass scale at

which the extra dimensions become significant.

The appearance of extra dimensions of radius R implies that a given complex quantum field $\Phi(x)$ now depends on not only the usual four-dimensional spacetime coordinates $\mathbf{x} \equiv (x_0, x_1, x_2, x_3)$, but also the additional spacetime coordinates $\mathbf{y} \equiv (y_1, y_2, ..., y_\delta)$ where $\delta \equiv D - 4$ is the number of additional dimensions. We shall denote these coordinates collectively as $x = (\mathbf{x}, \mathbf{y})$. Demanding periodicity of $\Phi(x)$ under

$$y_i \ \rightarrow \ y_i + 2\pi R \qquad\qquad (2.1)$$

then implies that $\Phi(x)$ takes the form

$$\Phi(x) \ = \ \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \cdots \sum_{n_\delta = -\infty}^{\infty} \Phi^{(\mathbf{n})}(\mathbf{x}) \exp(i\mathbf{n} \cdot \mathbf{y}/R) \qquad\qquad (2.2)$$

where $\mathbf{n} = (n_1, n_2, ..., n_\delta)$, with $n_i \in \mathbb{Z}$. The "four-dimensional" fields $\Phi^{(\mathbf{n})}(\mathbf{x})$ are the so-called *Kaluza-Klein modes*, and n_i are the corresponding Kaluza-Klein excitation numbers (with $p_i \equiv n_i/R$ serving as the Kaluza-Klein momenta). In general, the mass of each Kaluza-Klein mode is given by

$$m_n^2 \ \equiv \ m_0^2 \ + \ \frac{\mathbf{n} \cdot \mathbf{n}}{R^2} \qquad\qquad (2.3)$$

where m_0 is the mass of the zero-mode. At energies far below R^{-1}, we expect our extra dimensions to be unobservable. However, in this limit, none of the non-zero Kaluza-Klein modes can be excited, and we will observe only the zero-mode field $\Phi^{(0)}(\mathbf{x})$. Thus, the zero-mode field corresponds to the usual four-dimensional state, and the appearance of the extra spacetime dimensions is felt through the appearance of an infinite tower of associated Kaluza-Klein states of increasing mass.

How then do we incorporate extra spacetime dimensions into the MSSM? It might initially seem that for every MSSM state of mass m_0, there will exist a corresponding infinite tower of Kaluza-Klein states with masses given by (2.3), each of which exactly mirrors the zero-mode MSSM ground state. Since R^{-1} is presumed to exceed presently observable energy scales, we are free to neglect m_0 in (2.3).

However, it turns out that not every MSSM state can have Kaluza-Klein excitations that exactly mirror the MSSM state itself. This complication arises because it is necessary for Kaluza-Klein excitations to fall into representations that permit suitable Kaluza-Klein mass terms to be formed. This issue is particularly important for us because a chiral MSSM state (such as a quark or lepton) by itself cannot be given a Kaluza-Klein mass. Thus, we cannot have an infinite tower of chiral Kaluza-Klein excitations. Instead, we have two choices: either a given chiral MSSM representation will *not* have Kaluza-Klein excitations, or the corresponding Kaluza-Klein excitation will consist of not only the original MSSM representation but also its chiral-conjugate mirror. Both cases are also consistent with the situation in string theory.

In this paper we shall consider a variety of different cases. Specifically, we shall let the variable η denote the number of generations of MSSM chiral fermions that we

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shall permit to have Kaluza-Klein excitations. Of course, the simplest scenario is one for which $\eta = 0$ — *i.e.*, no chiral MSSM fermions having Kaluza-Klein excitations. Thus, in this scenario, only the *non-chiral* MSSM states (namely the gauge bosons and Higgs fields) will have Kaluza-Klein excitations, while the quark and lepton representations will be assumed not to have Kaluza-Klein excitations. We shall refer to this as the "minimal scenario", and as we shall see it has a number of special properties. However, in the expectation that string theory will ultimately give rise to various mixtures of configurations, in this paper we shall also consider the cases with $\eta = 1, 2, 3$.

It is important to consider the exact form of the Kaluza-Klein excitations that the Higgs fields, gauge bosons, and chiral fermions will have. In the case of the Higgs fields, for each set of non-zero Kaluza-Klein momenta $\{n_i\}$ the corresponding Kaluza-Klein state will be a chiral $N = 1$ multiplet $\mathcal{H}^{(n)} \equiv (H^{(n)}, \psi_H^{(n)})$. Note that the structure of these chiral multiplets does not depend on whether they are massless or massive. Since the MSSM contains two separate Higgs fields, we thus find that at each Kaluza-Klein mass level there will be two massive chiral $N = 1$ supermultiplets $\mathcal{H}_{1,2}$. As we shall see, it will prove convenient to combine these two $N = 1$ supermultiplets to form a single $N = 2$ hypermultiplet H:

$$H^{(n)} \;=\; \begin{pmatrix} H_1^{(n)} & H_2^{(n)} \\ \psi_{H_1}^{(n)} & \psi_{H_2}^{(n)} \end{pmatrix} \tag{2.4}$$

where we have suppressed gauge and Lorentz indices.

A similar situation exists for the gauge bosons. An ordinary massless gauge boson is an $N = 1$ vector supermultiplet. However, a massive gauge boson is represented by a massive $N = 1$ vector supermultiplet, which is equivalent to an $N = 1$ massless vector supermultiplet $\mathcal{A} \equiv (A, \lambda)$ plus an additional $N = 1$ chiral supermultiplet $\mathcal{A}' \equiv (\phi, \psi)$. Together, these form a massive $N = 2$ vector supermultiplet:

$$V^{(n)} \;=\; \begin{pmatrix} A^{(n)} & \phi^{(n)} \\ \lambda^{(n)} & \psi^{(n)} \end{pmatrix} \;. \tag{2.5}$$

One of the real scalar fields in the chiral supermultiplet $\mathcal{A}'$ becomes the longitudinal component of the massive gauge boson, while the other real scalar field and the Weyl fermion remain in the spectrum at the massive level. Thus, once again, we see that our Kaluza-Klein towers of states are effectively $N = 2$ supersymmetric.*

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*Strictly speaking, the extra Kaluza-Klein towers of states will effectively be $N = 2$ supersymmetric only for $\delta = 1$ or 2. For higher values of δ, the situation can be more complicated — *e.g.*, for $\delta = 6$ we naïvely expect our Kaluza-Klein towers of states to be $N = 4$ supersymmetric. In general, the enhanced supersymmetry for the excited Kaluza-Klein arises because the minimum number of supersymmetries in higher spacetime dimensions (as counted in terms of four-dimensional gravitino spinors) grows with the spacetime dimension. Thus, since the higher number of supersymmetries must be always restored in the limit $R \to \infty$, we see that our Kaluza-Klein towers must exhibit a

Finally, in the cases $\eta \geq 1$ for which we allow certain chiral fermions to have Kaluza-Klein excitations, these excitations will have the form

$$F^{(n)} = \begin{pmatrix} \phi_1^{(n)} & \phi_2^{(n)} \\ \psi_1^{(n)} & \psi_2^{(n)} \end{pmatrix} \tag{2.6}$$

where $\mathcal{F}_1^{(n)} \equiv (\phi_1^{(n)}, \psi_1^{(n)})$ are the Kaluza-Klein excitations of the original fermion field $\mathcal{F}$, and where $\mathcal{F}_2^{(n)} \equiv (\phi_2^{(n)}, \psi_2^{(n)})$ are the Kaluza-Klein excitations of the corresponding mirror fermion. Together, (2.6) forms an $N = 2$ hypermultiplet (like the Higgs field).

For the purposes of this paper, it will not often be necessary to use $N = 2$ language to describe these towers of Kaluza-Klein states. But the main point is that we shall give Kaluza-Klein excitations to the gauge bosons, to the Higgs fields, and to only η generations of the MSSM fermions, where $\eta = 0, 1, 2, 3$. Thus, the effects of the extra dimensions are restricted to only the corresponding subset of the MSSM.

At first glance, this may seem to be an inconsistent situation because of two worries. First, how can we have Kaluza-Klein towers of gauge-boson states that fall into $N = 2$ representations when we know that their zero-modes (their corresponding observable states at low energies) are only $N = 1$ supersymmetric? How is it possible to "decouple" the gauge-boson zero-modes from the excited states in this way? Second, we may also ask how can we have Kaluza-Klein excitations for some fields, while forbidding them for other fields. How can this be reconciled with the presence of additional spacetime dimensions, which are presumed to apply to the entire theory at once?

To understand both of these points, let us consider the case of a single additional dimension (*i.e.*, $\delta = 1$) for simplicity, so that our spacetime coordinates are $x \equiv (\mathbf{x}, y)$ where $\mathbf{x} \equiv (x_0, x_1, x_2, x_3)$. Note that we can recast (2.2) into the form $\Phi(x) = \Phi_+(x) + i\Phi_-(x)$, where

$$\Phi_+(x) = \sum_{n=0}^{\infty} [\Phi^{(n)}(\mathbf{x}) + \Phi^{(-n)}(\mathbf{x})] \cos(ny/R)$$

$$\Phi_-(x) = \sum_{n=1}^{\infty} [\Phi^{(n)}(\mathbf{x}) - \Phi^{(-n)}(\mathbf{x})] \sin(ny/R) . \tag{2.7}$$

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higher number of supersymmetries than the ground states, even at finite R. However, by making suitable choices of orbifolds (as we shall see will be necessary in any case), it is always possible to project the relevant Kaluza-Klein towers down to representations of $N = 2$ supersymmetry, even if $\delta > 2$. Hence, without loss of generality, we shall consider $N = 2$ supersymmetric Kaluza-Klein towers for arbitrary values of δ.

In this context, we also remark that (2.4) is not the only way in which we might have constructed an $N = 2$ supermultiplet from the MSSM Higgs fields. Rather than combining the two MSSM Higgs fields into a single $N = 2$ multiplet, another possibility would have been to augment each Higgs field separately into its own $N = 2$ multiplet. This would therefore have required the introduction of even more fields. We shall adopt the "minimal" approach of (2.4) in this section, and defer a discussion of the remaining possibilities to Sect. 7.

Note that $\Phi_\pm(x)$ are generally complex fields. Of course, if $\Phi(x)$ were a real field, then only Φ_+ would be non-zero. However, even if $\Phi(x)$ is complex, it is possible to distinguish between Φ_+ and Φ_- through their properties under the $\mathbb{Z}_2$ transformation

$$y \;\to\; -\,y \;. \tag{2.8}$$

Specifically, we have

$$\begin{aligned}
\Phi_+(\mathbf{x},-y) &=& +\,\Phi_+(\mathbf{x},y) \\
\Phi_-(\mathbf{x},-y) &=& -\,\Phi_-(\mathbf{x},y) \;.
\end{aligned} \tag{2.9}$$

What is particularly useful about the decomposition (2.7) is that $\Phi_-(x)$ lacks a zero-mode. Thus, even though our Kaluza-Klein tower of states for the gauge bosons are $N = 2$ supersymmetric, as in (2.5), we can ensure that their corresponding zero-mode is only $N = 1$ supersymmetric (as appropriate for the MSSM) by additionally demanding that A and λ transform as even functions under (2.8), while ϕ and ψ transform as odd functions.

What are the implications of making such additional requirements? By demanding that our wavefunctions exhibit certain symmetry properties under the transformation (2.8), we are not, strictly speaking, compactifying on a circle. Instead, we are implicitly making the additional $\mathbb{Z}_2$ identification $y \approx -y$. Such an identification changes the circle into a so-called $\mathbb{Z}_2$ *orbifold*, so what we are really doing is compactifying on a $\mathbb{Z}_2$ orbifold rather than on a circle. The fact that we are compactifying on an orbifold is what allows us to demand specific symmetry properties under the orbifold relation (2.8). Such orbifold choices are completely natural from the point of view of string theory, and are therefore completely consistent with an ultimate embedding of our scenario into string theory.[†]

Let us now consider our second question: in the scenarios with $\eta < 3$, how can we ensure that $3 - \eta$ generations of MSSM fermions lack Kaluza-Klein excitations altogether? Once again, it is the fact that we are compactifying on an orbifold which provides the explanation. For such an orbifold compactification, we see that there are two special points, $y^{(A)} = 0$ and $y^{(B)} = \pi R$, which are invariant under the orbifold relation (2.8) in conjunction with the circle relation (2.1). Such special points are called *fixed points* of the orbifold. The existence of such fixed points implies that rather than having a mode expansion of the form (2.2), a perfectly consistent alternative mode-expansion would be

$$\Phi(x) \;=\; \Phi^{(A)}(\mathbf{x})\,\delta(y) \;+\; \Phi^{(B)}(\mathbf{x})\,\delta(y - \pi R) \;. \tag{2.10}$$

[†]In this regard, it is also important to note that in this paper we are considering only the Kaluza-Klein *momentum* states for which $m_n \sim n/R$. These are the states which are appropriate for a field-theoretic treatment of extra spacetime dimensions. In string theory, however, there are also Kaluza-Klein *winding-mode* states for which $m_w \sim w M_{\mathrm{string}}^2 R$, where w is the Kaluza-Klein winding number and M_{string} is the string scale. This will cause no inconsistency for us because we will ultimately take $R^{-1} \ll M_{\mathrm{string}}$ in our scenario. Thus, winding-mode states will play no role in the field-theoretic limit.

Generalizations to higher spacetime dimensions are obvious. Note that such mode-expansions also respect the symmetries of the orbifold. However, such mode-expansions do not give rise to infinite Kaluza-Klein towers because such states exist only at the fixed points of the orbifold. Thus, it is possible to ensure that a given MSSM fermion will have no Kaluza-Klein excitations by requiring that it be situated only at the fixed points of the orbifold. Once again, this is completely natural from the point of view of string theory. In string theory, the act of "twisting" by the orbifold element (2.8) naturally gives rise to so-called "twisted string sectors", and the physical states that arise in such twisted sectors will precisely be of the "fixed point" variety. Note that it will not matter whether these fermions are located at $y = 0$ or $y = \pi R$.

Strictly speaking, we remark that this orbifold "fixed point" mechanism is valid only for *closed* string theories (such as the heterotic string). For open string theories, by contrast, an analogous mechanism will involve D-branes and will be discussed in Sect. 7.

Thus, putting the pieces together, we see that our higher-dimensional MSSM must be described in terms of a spacetime consisting of four flat dimensions along with a certain number of extra dimensions compactified on orbifolds of radius R. At the massless (zero-mode) level, our particle content will consist of the full MSSM. At the higher Kaluza-Klein levels, however, we will have infinite towers of Kaluza-Klein states associated with the Higgs field, the gauge-boson states, and η generations of the chiral MSSM fermions, where $\eta = 0, 1, 2, 3$. The remaining $3 - \eta$ generations of chiral fermions will not have Kaluza-Klein states, and will instead be restricted to the fixed points of the orbifold.

Let us now consider how this system behaves at different energy scales. At energy scales much smaller than $\mu_0 \equiv R^{-1}$, the energy of the system is less than the mass of the lowest Kaluza-Klein excitations, and the existence of the Kaluza-Klein states (and indeed that of the extra dimensions) can be ignored. Thus, in this limit, our theory reduces to the usual four-dimensional MSSM. For $\mu \gg \mu_0$, by contrast, excitations of many Kaluza-Klein modes become possible, and the contributions of these Kaluza-Klein states must be included in all physical calculations. For example, these contributions must be included in the running of gauge couplings, and they tend to accelerate this running, ultimately changing the scale-dependence of the gauge couplings from logarithmic to power-law as a function of μ. This reflects the fact that beyond the scale R^{-1}, a certain subset of the MSSM is essentially higher-dimensional, and the effective radius R of these extra dimensions appears to be infinite relative to the energy scale μ. Indeed, in this limit, the new spacetime dimensions that appear are effectively flat. Thus, we see that the effect of the extra Kaluza-Klein excitations is essentially to make the spacetime appear to be D-dimensional rather than four-dimensional for the appropriate subset of the MSSM.

At certain points in this paper, we shall need to make recourse to a slightly modified description of the extra dimensions. Strictly speaking, we know that our

theory contains infinite towers of Kaluza-Klein states. However, it is clear that only the lowest-lying Kaluza-Klein states can possibly play an important role in the physics because the contributions of the very heavy Kaluza-Klein states are suppressed by their large masses. Thus, in some cases it will prove useful to retain only a *finite* number of low-lying Kaluza-Klein states in the theory. The point at which the Kaluza-Klein towers are truncated will ultimately depend on the energy scale at which we wish our theory to apply, but for our purposes it will always be possible to choose such a fixed truncation point. We shall refer to this as the "truncated" Kaluza-Klein description of the extra spacetime dimensions. As we shall see in Sect. 3, this truncated description will prove very useful.

The appearance of extra spacetime dimensions has an important effect on the gauge couplings of the theory. In four spacetime dimensions, the gauge couplings g_i are quantities of zero mass dimension (*i.e.*, pure numbers). In D spacetime dimensions, however, the gauge couplings $\tilde{g}_i$ accrue a classical mass dimension

$$[\tilde{g}_i] \;=\; 2 - \frac{D}{2} \quad \Longrightarrow \quad [\tilde{\alpha}_i^{-1}] \;=\; D - 4 \;=\; \delta \tag{2.11}$$

where $\delta \equiv D - 4$. It is therefore important to understand the connection between the higher- and lower-dimensional couplings. However, since our extra spacetime dimensions have a fixed radius R, we can follow the standard compactification procedure to find that the four- and D-dimensional gauge couplings are related to each other via

$$\alpha_i \;=\; R^{-\delta}\, \tilde{\alpha}_i \;. \tag{2.12}$$

2.2 Renormalizability and the interpretation of cutoffs

Finally, we conclude this section with a few important comments regarding the renormalizability of these higher-dimensional theories.

As is well-known, higher-dimensional field theories are non-renormalizable because of their enhanced divergence structure. This non-renormalizability stems from the presence of infinite towers of non-chiral Kaluza-Klein states which circulate in the loops of all quantum-mechanical processes. Even if we choose to ignore these Kaluza-Klein states by treating the non-chiral sector of the MSSM as being in D flat spacetime dimensions, the resulting description is still non-renormalizable because of the need to integrate over D dimensions' worth of uncompactified loop momenta in such sectors.

Given the non-renormalizable nature of such higher-dimensional field theories, it therefore makes no sense to talk of a "running" of gauge couplings as a function of a floating energy scale μ. Instead, for a non-renormalizable theory, we must introduce an explicit *cutoff* parameter Λ. Consequently, strictly speaking, the values of physical parameters such as gauge couplings do not "run" — they instead receive *finite quantum corrections* whose magnitudes depend explicitly on the value of this cutoff

parameter. Therefore, in the language appropriate to a non-renormalizable field theory, we do not seek to calculate the "running" of the gauge or Yukawa couplings; we instead seek to calculate the one-loop-corrected values of the gauge couplings $\alpha_i(\Lambda)$ as functions of the value of this cutoff parameter Λ.

In many cases, this mathematical dependence on the cutoff is identical to the scale-dependence that we would have naïvely calculated if the theory had been renormalizable. Therefore, we will occasionally continue to use words such as "running", even in our non-renormalizable context, to describe the dependence on the cutoff.

There is, however, one profound distinction that arises due to the fact that our theory is non-renormalizable. Since our theory is non-renormalizable, it can only be viewed as an *effective* theory, valid up to some even higher mass scale M. Thus, throughout this paper, our higher-dimensional theory will be interpreted in precisely this way, as an effective theory requiring the emergence of an even more fundamental theory (such as a string theory) at an even higher energy scale. Indeed, given our results, we shall see that this interpretation will be particularly natural.

In itself, this is not a problem. However, this then broaches the question: just how can we interpret the cutoff parameter Λ which will appear throughout our calculations? It is important to resolve this issue because such a cutoff Λ is not a physical parameter with intrinsic meaning. Indeed, such a cutoff ultimately depends on the form of the regulator in which it is presumed to appear and thereby on the normalization that is used for defining Λ within this regulator.

It might seem natural, of course, to associate the cutoff parameter Λ with the physical mass scale M at which we presume new physics to appear *beyond* our higher-dimensional non-renormalizable theory. This would seem to make sense because, as we stated above, we must ultimately assume that our higher-dimensional non-renormalizable theory is only an *effective* description of physics for energy scales below some new fundamental mass scale M. In general, such an association works well. However, this is not always the case. For example, it has been pointed out [2] that in certain situations one may obtain misleading results, essentially due to poor choices of cutoff and regulator variables. In such a case, the value of the cutoff Λ gives no information about the underlying mass scale M. Moreover, it is not even straightforward to recognize the situations in which such misleading results will arise.

Fortunately, in our case we will be able to completely sidestep all of these issues by making a crucial observation. As we have stated, the lack of renormalizability can be attributed to the fact that our towers of Kaluza-Klein states are *infinite*. However, for *calculational* purposes it is often unnecessary to include *all* of the Kaluza-Klein states — indeed, one may often truncate the tower at a suitable energy level without seriously altering the results of a given calculation. However, because this tower of states is truncated, this description of the physics has the potential to give rise to a *completely renormalizable field theory*. This issue will be discussed in more detail in Appendix B. In such cases, we then have a remarkable situation: Although our full underlying theory is non-renormalizable, there will exist a fully renormalizable

field theory which gives essentially the same results for certain calculations. This in turn implies that any ambiguities or uncertainties that might arise in relating Λ to M in the full theory can be completely resolved by making recourse to the fully renormalizable approximation. Therefore, by making recourse to our renormalizable approximate description of the physics, we shall be able to formulate a clear relation between the cutoff parameter Λ and the corresponding physical mass scale M. This will thereby enable us to interpret our cutoff Λ, and likewise determine our new fundamental mass scale M, without ambiguity.

3   Extra dimensions and gauge coupling unification

Let us now begin by considering how extra spacetime dimensions affect the Standard Model gauge couplings and their unification.

In ordinary four-dimensional field theory, gauge couplings g_i are dimensionless quantities and their evolution as a function of the mass scale μ is given by the usual one-loop renormalization group equation (RGE)

$$\frac{d}{d \ln \mu} \, \alpha_i^{-1}(\mu) \;=\; -\,\frac{b_i}{2\pi} \tag{3.1}$$

for which the solution is given by

$$\alpha_i^{-1}(\mu) \;=\; \alpha_i^{-1}(M_Z) \;-\; \frac{b_i}{2\pi} \, \ln \frac{\mu}{M_Z} \; . \tag{3.2}$$

This is the usual logarithmic running of the gauge couplings. Here $\alpha_i \equiv g_i^2/4\pi$, the b_i are the MSSM one-loop beta-function coefficients

$$(b_Y, b_2, b_3) \;=\; (11, 1, -3) \; , \tag{3.3}$$

and we have taken the Z-mass $M_Z \equiv 91.17$ GeV as an arbitrary low-energy reference scale. At this scale (and within the $\overline{\text{MS}}$ renormalization group scheme), the gauge couplings are given by

$$
\begin{aligned}
\alpha_Y^{-1}(M_Z)|_{\overline{\text{MS}}} &\equiv\; 98.29 \pm 0.13 \\
\alpha_2^{-1}(M_Z)|_{\overline{\text{MS}}} &\equiv\; 29.61 \pm 0.13 \\
\alpha_3^{-1}(M_Z)|_{\overline{\text{MS}}} &\equiv\; 8.5 \pm 0.5
\end{aligned}
\tag{3.4}
$$

and we shall henceforth define $\alpha_1 \equiv (5/3)\alpha_Y$ and $b_1 \equiv (3/5)b_Y$. As is well-known, an extrapolation of these low-energy couplings according to (3.2) leads to the celebrated unification relation

$$\alpha_1(M_{\text{GUT}}) \;=\; \alpha_2(M_{\text{GUT}}) \;=\; \alpha_3(M_{\text{GUT}}) \;\approx\; \frac{1}{24} \tag{3.5}$$

at the unification scale

$$M_{\mathrm{GUT}} \approx 2 \times 10^{16} \text{ GeV} .\tag{3.6}$$

Let us now consider how this picture is modified in the presence of extra spacetime dimensions. Recall that, as discussed in Sect. 2, we shall take our spacetime to consist of four flat spacetime dimensions and δ additional dimensions of radius R. This results in the appearance of infinite towers of Kaluza-Klein states of masses $m_n \approx n\mu_0$, $n \in \mathbb{Z}^+$, where $\mu_0 \equiv R^{-1}$. Recall that, in general, these Kaluza-Klein states will be assumed to appear for the gauge bosons, the Higgs fields, and η generations of chiral fermions of the MSSM, where we shall consider the cases $\eta = 0, 1, 2, 3$. By contrast, the remaining $3 - \eta$ generations of MSSM quarks and leptons will be assumed not to have Kaluza-Klein excitations.

Below μ_0, we may ignore the effects of the Kaluza-Klein states, and simply treat our higher-dimensional theory as equivalent to the ordinary four-dimensional MSSM. We shall show in Appendix A that this is an excellent approximation. Thus, for $\mu \leq \mu_0$, we may assume that the gauge couplings run in the usual fashion described above.

Above μ_0, however, we must take into account the full spectrum of Kaluza-Klein states. Because these towers of Kaluza-Klein states are infinite, our higher-dimensional theory is non-renormalizable. Thus, for $\mu \geq \mu_0$, we cannot talk of the "running" of these gauge couplings. Rather, as we discussed in Sect. 2, the gauge couplings instead receive finite corrections which depend on the appropriate cutoffs in the theory. In the present case, we have both an infrared cutoff μ_0 (below which the physics is effectively described by the usual four-dimensional MSSM with no Kaluza-Klein states), and an ultraviolet cutoff Λ (which marks the scale at which some new physics beyond our higher-dimensional theory is presumed to emerge).

The corresponding corrections to these gauge couplings can then be calculated in the usual manner by evaluating the same one-loop diagrams (particularly the wave-function vacuum polarization diagram) that renormalize the usual gauge couplings in four dimensions. Denoting the value of this diagram as $\Pi_{\mu\nu}(k)$ where k is the overall momentum flowing through this diagram, and using gauge invariance to write

$$\Pi_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu} k^2)\, \Pi(k^2) ,\tag{3.7}$$

we then find the approximate one-loop result

$$\Pi(0) = \frac{g_i^2}{8\pi^2} \left[(b_i - \tilde{b}_i) \ln \frac{\Lambda}{\mu_0} + \tilde{b}_i\, \mu_0^{-\delta}\, \frac{X_\delta}{\delta} \left(\Lambda^\delta - \mu_0^\delta \right) \right] .\tag{3.8}$$

In this expression, g_i is the original, uncorrected gauge coupling, $\delta \equiv D - 4$, and the numerical coefficient X_δ will be discussed below. The beta-function coefficients b_i are those of the usual MSSM given in (3.3), which correspond to the zero-mode states, while the new beta-function coefficients $\tilde{b}_i$ are given by

$$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = (3/5, -3, -6) + \eta\, (4, 4, 4) .\tag{3.9}$$

These beta-function coefficients correspond to the contributions of the appropriate Kaluza-Klein states at each massive Kaluza-Klein excitation level. As discussed in Sect. 2, at each mass level these Kaluza-Klein states consist of two Higgs chiral $N = 1$ supermultiplets as well as the massive gauge-boson multiplets, each of which consists of one vector and chiral $N = 1$ supermultiplet. They also include η generations of chiral MSSM fermions, along with their appropriate mirrors.

It is easy to interpret the form of the one-loop correction (3.8). The second term reflects the contributions from the infinite towers of Kaluza-Klein states corresponding to the MSSM states that feel the extra dimensions, with beta-function coefficients $\tilde{b}_i$. If the states in the excited levels of these Kaluza-Klein towers had matched the zero-mode states in our theory, this term (which effectively combines both the zero-modes and all the excited modes) would have been sufficient. However, as we have seen, the states at the zero-mode level of our theory actually differ from those at the excited levels. The first term in (3.8) therefore compensates for this difference between the zero-mode states and the excited states. Moreover, as we discussed above, at energy scales above μ_0 we may treat the sector of the MSSM which has Kaluza-Klein excitations as being effectively in D flat spacetime dimensions. The powers of Λ and μ_0 that appear in the second term of (3.8) thus result from the higher-dimensional loop-momentum integration. Equivalently, these powers may be viewed as the "classical scaling" behavior that we expect the gauge couplings to experience due to their enhanced classical mass dimensions in (2.11). The factor δ^{-1} in (3.8) ensures that the formal $\delta \to 0$ limit of the second term reproduces the expected logarithmic behavior, and the overall factor of $\mu_0^{-\delta}$ is required on dimensional grounds.

In Appendix A, we shall provide a rigorous expression for $\Pi(0)$ which does not make the approximation of using D flat dimensions for the appropriate subsector of the MSSM, but which instead incorporates the effects of infinite towers of Kaluza-Klein states at all energy scales. This will enable us to explicitly demonstrate that our approximation (3.8) for $\Pi(0)$ is an excellent one.

Perhaps the most subtle feature of (3.8) is the coefficient X_δ that appears in the second term. Note that this coefficient can be interpreted as providing a *normalization* for the cutoffs Λ and μ_0 that appear in the D-dimensional integrations. Thus, it becomes immediately apparent that within the context of our non-renormalizable field theory, the coefficient X_δ is essentially cutoff- and regulator-dependent. Even if we assume that our cutoff Λ is to be associated with an underlying physical mass scale M, as discussed in Sect. 2, the relation between Λ and M would still involve an overall unknown proportionality constant which would pollute any value of X_δ that we might otherwise calculate via phase-space arguments. Thus, in some sense, all of the uncertainties inherent in working with a non-renormalizable field theory can ultimately be embodied in our inability to determine a precise value for X_δ.

Fortunately, as we discussed in Sect. 2, we can circumvent this problem completely by making recourse to our "equivalent" truncated Kaluza-Klein theory. In this way

we are then able to make a direct comparison between the two theories in order to precisely calculate X_δ in (3.8). As we shall show in Appendix B, this leads to the result

$$X_\delta \; = \; \frac{\pi^{\delta/2}}{\Gamma(1 + \delta/2)} \; = \; \frac{2\pi^{\delta/2}}{\delta\Gamma(\delta/2)} \tag{3.10}$$

where Γ is the Euler gamma function satisfying $\Gamma(n) = (n-1)!$, $\Gamma(1) = 1$, and $\Gamma(1/2) = \sqrt{\pi}$. Thus, $X_0 = 1$ (as expected), while $X_1 = 2$, $X_2 = \pi$, $X_3 = 4\pi/3$, and so forth. In fact, X_δ is nothing but the volume V_δ of a δ-dimensional unit sphere! Thus, in the remainder of this paper, we shall use this precise value for X_δ. This will enable us to interpret Λ literally as the physical mass scale M at which new physics appears beyond our effective D-dimensional field theory.

Our next step is to resum these vacuum polarization diagrams in order to obtain the full one-loop-corrected gauge couplings $g_i(\Lambda)$. This resummation proceeds as in the usual case, yielding

$$g_i(\Lambda) \; = \; \left(\frac{1}{1 - \Pi(0)} \right)^{1/2} g_i \; , \tag{3.11}$$

or equivalently

$$\alpha_i^{-1}(\Lambda) \; = \; \alpha_i^{-1} \; - \; \frac{b_i - \tilde{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} \; - \; \frac{\tilde{b}_i X_\delta}{2\pi\delta} \left[\left(\frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] \; . \tag{3.12}$$

The second and third terms on the right side are the finite one-loop corrections which depend explicitly on the cutoff Λ.

Finally, we impose our matching condition that α_i, the uncorrected value of the effective four-dimensional coupling, must agree with the value of the true four-dimensional coupling $\alpha_i(\mu_0)$ at the scale μ_0. Substituting for $\alpha_i^{-1}(\mu_0)$ from (3.2) then yields our final result, valid for all $\Lambda \geq \mu_0$:

$$\alpha_i^{-1}(\Lambda) \; = \; \alpha_i^{-1}(M_Z) \; - \; \frac{b_i}{2\pi} \ln \frac{\Lambda}{M_Z} \; + \; \frac{\tilde{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} \; - \; \frac{\tilde{b}_i X_\delta}{2\pi\delta} \left[\left(\frac{\Lambda}{\mu_0} \right)^\delta - 1 \right] \; . \tag{3.13}$$

Although this result bears a resemblance to a renormalization group equation, we stress that its physical interpretation is entirely different. Rather than describe the higher-dimensional "running" of a higher-dimensional gauge coupling, this equation instead expresses the dependence that such a coupling exhibits on the value of the cutoff Λ. Thus, given any values for μ_0, δ, and η (which are the parameters that describe the underlying non-renormalizable theory), and given any value for Λ (a parameter which is external to the theory and which is interpreted as the energy scale at which a new theory is presumed to appear), $\alpha_i^{-1}(\Lambda)$ is the value of the one-loop corrected effective four-dimensional coupling.

It is clear from (3.13) that the presence of the extra dimensions clearly has a profound effect on the values of these gauge couplings. Remarkably, however, it turns out that there always exists a value of Λ for which the gauge couplings unify! Indeed, this property is robust, occurring independently of the number δ of extra dimensions, independently of the scale μ_0 at which they appear, and independently of the number η of chiral MSSM generations that feel these extra dimensions!

This unification is illustrated in Figs. 1, 2, 3, and 4. For $\mu < \mu_0$, we are plotting the usual running four-dimensional gauge couplings. For $\mu > \mu_0$, however, we are treating μ as the cutoff Λ and plotting the values of the gauge couplings as functions of this cutoff. It is our matching condition at μ_0 which guarantees that this procedure results in continuous curves of dimensionless numbers. We see that below μ_0, the gauge couplings run in the usual logarithmic fashion. Above μ_0, by contrast, the appearance of the extra spacetime dimensions causes this "running" to accelerate, quickly leading to a unification.

It is natural to wonder if this unification might simply be an artifact of our approximation of treating the physics as being in four flat dimensions below the scale μ_0 and in D flat dimensions above this scale, all while ignoring Kaluza-Klein modes. However, it is possible to do a rigorous calculation which assumes only four flat dimensions at all energy scales and which explicitly allows the complete infinite towers of Kaluza-Klein states to circulate in the one-loop wavefunction renormalization diagram. The details of this calculation are given in Appendix A, and the results are virtually identical to those in Figs. 1 and 2. Thus, we see that our conclusion is unaltered: gauge coupling unification continues to occur.

It is possible to understand physically why this gauge coupling unification occurs. Let us first imagine that it had been the case that $\tilde{b}_i = b_i$ for all i. This would have occurred, for example, if *all* of the MSSM states had had Kaluza-Klein excitations that exactly matched the zero-modes, with masses m_n given in (2.3). If this had been the case, then each energy level $\{n_i\}$ would have effectively provided a heavier duplicate copy of the entire chiral MSSM particle content. However, within the MSSM, gauge coupling unification is independent of the number of generations because each generation provides extra matter in complete $SU(5)$ multiplets. Therefore, we would have found that gauge coupling unification is preserved regardless of the number of extra dimensions or the scale at which they appear.

Of course, in the present case we do not have Kaluza-Klein excitations for all of the MSSM states, and consequently $\tilde{b}_i \neq b_i$. However, in order to preserve unification, we need not demand that $\tilde{b}_i = b_i$: we simply need to demand that the ratios

$$B_{ij} \equiv \frac{\tilde{b}_i - \tilde{b}_j}{b_i - b_j} \tag{3.14}$$

be independent of i and j. In other words, we want

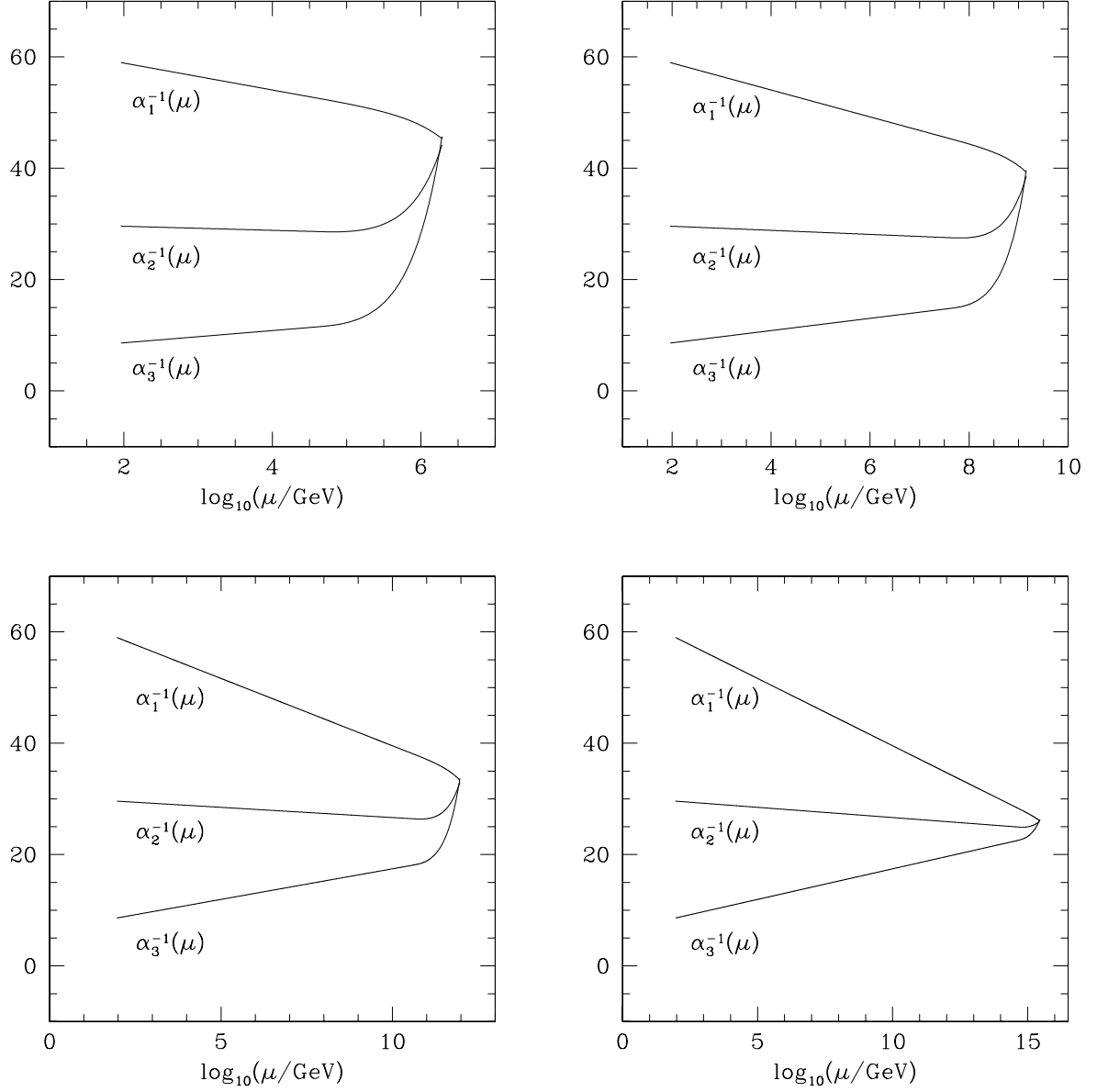$$\frac{B_{12}}{B_{13}} = \frac{B_{13}}{B_{23}} = 1 \ . \tag{3.15}$$

Figure 1: Unification of gauge couplings in the presence of extra spacetime dimensions. We consider four representative cases: $\mu_0 = 10^5$ GeV (top left), $\mu_0 = 10^8$ GeV (top right), $\mu_0 = 10^{11}$ GeV (bottom left), and $\mu_0 = 10^{15}$ GeV (bottom right). In each case we have taken $\delta = 1$ and $\eta = 0$.
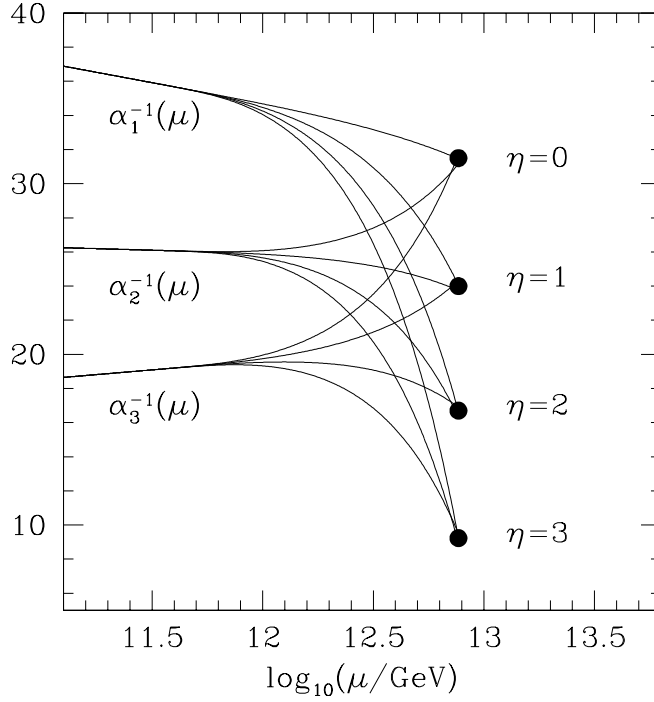
16

Figure 2: Unification of gauge couplings in the presence of extra spacetime dimensions. Here we fix $\mu_0 = 10^{12}$ GeV, $\delta = 1$, and we vary η. For this value of μ_0, we see that the unification remains weakly coupled for all η.

It is easy to check that although these relations are not satisfied *exactly* in our case, they are nevertheless *approximately* satisfied:

$$\frac{B_{12}}{B_{13}} = \frac{72}{77} \approx 0.94 \,, \qquad \frac{B_{13}}{B_{23}} = \frac{11}{12} \approx 0.92 \,. \qquad (3.16)$$

This remains true independently of the value of η, which shifts all $\tilde{b}_i$ by a fixed amount.* Thus, we expect that gauge coupling unification will continue to hold to a good degree of accuracy. In fact, it is apparent from Fig. 1 that the unification is quite precise. Thus, given the large experimental uncertainties in the measured value of $\alpha_3(M_Z)$ quoted in (3.4), we see that the unification is essentially preserved†

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*We thank C. Wagner for questions and comments that prompted us to investigate the general η scenario.

†In fact, given the well-known (slight) discrepancy between the measured low-energy value of $\alpha_3(M_Z)$ and the value required for unification within the MSSM, we might even be a bit bolder and hypothesize that it is the MSSM beta-function coefficients which lead to a failure of unification, and that our scenario actually *fixes* unification! Similarly, light SUSY threshold effects might also be accommodated more naturally in our scenario than in the usual MSSM.
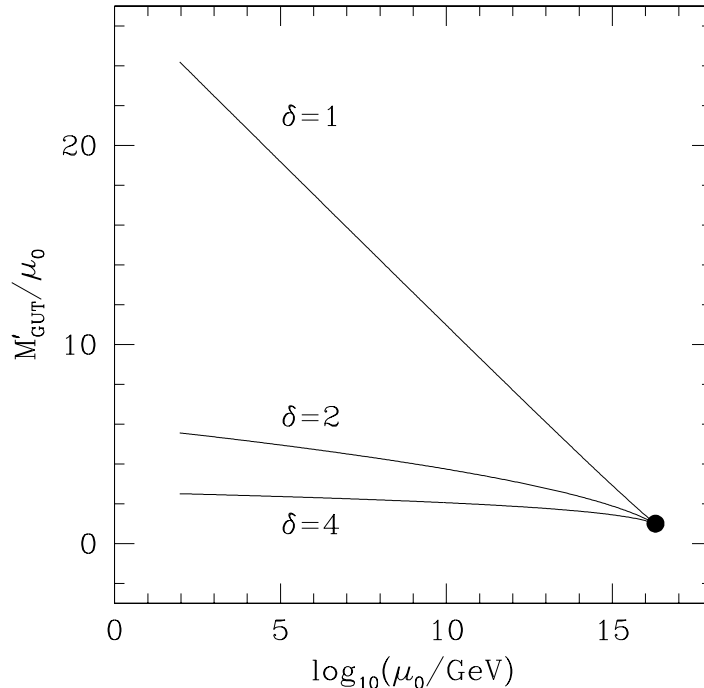
Figure 3: The ratio of the unification scale $M'_{\rm GUT}$ to the scale μ_0 at which δ extra spacetime dimensions appear, as a function of μ_0. This ratio describes the size of the energy range over which our effective higher-dimensional field theory is meant to apply. This curve is independent of the value of η. The limit of the usual four-dimensional MSSM is indicated with a dot.

for all values of μ_0, δ, and η. It is also easy to see why the unification scale is independent of η: increasing the value of η simply amounts to adding extra complete $SU(5)$ multiplets to the spectrum at each excited Kaluza-Klein mass level. However, adding extra complete $SU(5)$ multiplets always preserves the unification scale to one-loop order, and only shifts the unified coupling towards stronger values. This then explains the behavior shown in Fig. 2.

We shall refer to the value of μ or Λ for which this gauge coupling unification occurs as the "new unification scale" $M'_{\rm GUT}$. Indeed, it is natural to interpret such a cutoff Λ for which the gauge couplings unify as the energy scale at which we would expect a more fundamental grand-unified theory to appear. In any case, as we have mentioned, we are always free to pass to a description in terms of our equivalent renormalizable truncated Kaluza-Klein theory. In such a case, these figures can indeed be interpreted as describing the running of gauge couplings in the usual sense, and $M'_{\rm GUT}$ can indeed be interpreted as the scale of unification. Thus, we see that our scenario naturally predicts the emergence of a D-dimensional GUT at the scale
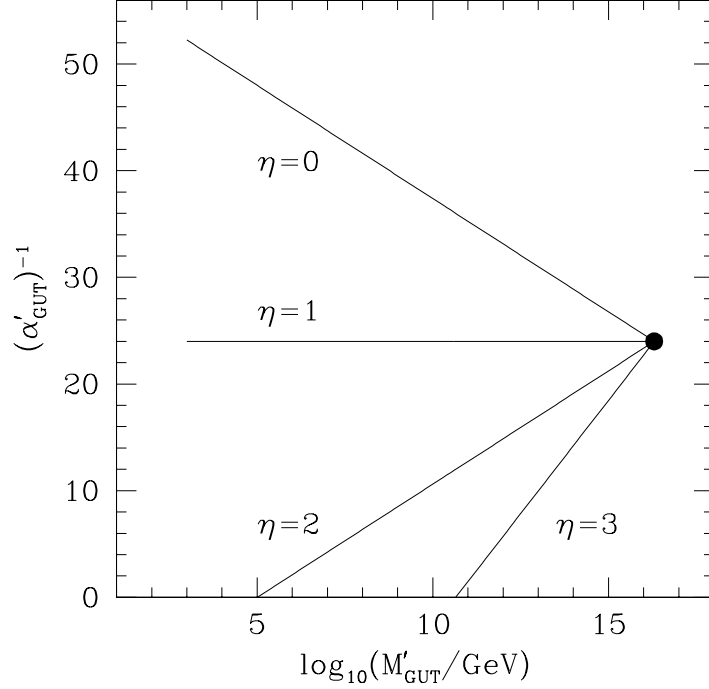
18

Figure 4: The unified coupling $(\alpha'_{\mathrm{GUT}})^{-1}$ as a function of the unification scale M'_{GUT}, for $\eta = 0, 1, 2, 3$. This curve is independent of the number of extra spacetime dimensions, and the limit of the usual four-dimensional MSSM is indicated with a dot. Note that the unified coupling is independent of the unification scale for $\eta = 1$, while it is weaker than the MSSM value for $\eta = 0$ and stronger for $\eta > 1$. It is also clear from this figure that there are natural lower bounds on the possible radii of extra dimensions in the $\eta = 2, 3$ cases if we wish the couplings to unify before diverging. These bounds correspond to $M'_{\mathrm{GUT}} \gtrsim 100$ TeV and $M'_{\mathrm{GUT}} \gtrsim 3 \times 10^{10}$ GeV for $\eta = 2, 3$ respectively.

M'_{GUT}.

Despite the interpretation of M'_{GUT} as a potential grand-unification scale, it is evident from Fig. 1 that this new unification scale M'_{GUT} is generally *not* the usual unification scale $M_{\mathrm{GUT}} \equiv 2 \times 10^{16}$ GeV that appears in the ordinary MSSM. Instead, it is a good deal lower. In order to solve the equations (3.13) for the unification parameters, let us first define

$$ B \equiv \frac{1}{3}(B_{12} + B_{23} + B_{13}) = \frac{233}{336} \approx 0.69 \qquad (3.17) $$

where the B_{ij} are defined in (3.14). We will then make the approximation that $B_{ij} = B$ for all (i, j), as required by our assumption of unification. With this assumption, it is then possible to solve (3.13) at the unification point. For any value of μ_0 and δ,

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we find that $M'_{\rm GUT}$ is approximately given by

$$M'_{\rm GUT} \;\approx\; \mu_0 \, f^{1/\delta} \qquad\qquad (3.18)$$

where f is an exponential enhancement factor, defined as

$$f \;\equiv\; 1 + \frac{\delta}{X_\delta B} \, \ln \frac{M_{\rm GUT}}{\mu_0} \;\geq\; 1 \; . \qquad\qquad (3.19)$$

As $\delta \to 0$ or as $\mu_0 \to M_{\rm GUT}$, we see that $f \to 1$ and $M'_{\rm GUT} \to M_{\rm GUT}$ as well. This is the MSSM limit. In all other cases, however, we see that our new unification scale $M'_{\rm GUT}$ can be reduced relative to the usual $M_{\rm GUT}$, and ultimately depends on the chosen values of δ and μ_0. This behavior is shown in Fig. 3.

The new unified coupling $\alpha'_{\rm GUT}$ also generally differs from its MSSM value $\alpha_{\rm GUT} \approx 1/24$. We find the approximate analytical result

$$(\alpha'_{\rm GUT})^{-1} \;\approx\; \alpha_{\rm GUT}^{-1} \;+\; \frac{2}{\pi B} \, (1 - \eta) \, \ln \frac{M_{\rm GUT}}{M'_{\rm GUT}} \; . \qquad\qquad (3.20)$$

This behavior is shown in Fig. 4. Thus, we see that in the $\eta = 0$ "minimal" scenario, $\alpha'_{\rm GUT}$ is always less than $\alpha_{\rm GUT}$ — *i.e.*, our theory is always more weakly coupled than the usual MSSM. Likewise, for $\eta = 1$, the theory is always exactly as weakly coupled as the usual MSSM, and the unified coupling exhibits an intriguing invariance under changes in the unification scale (a sort of "conformal" symmetry). For $\eta = 2$, the theory is more strongly coupled than the usual MSSM, and in fact the unified coupling becomes infinite near $M'_{\rm GUT} \approx 100$ TeV. Thus, for $\eta = 2$, we see that there is a *natural lower bound* on the radii of the extra dimensions if we wish to have the gauge couplings unify before they diverge. Similarly, for $\eta = 3$, the theory hits an infinite unified coupling at $M'_{\rm GUT} \approx 3 \times 10^{10}$ GeV, which again determines a natural lower bound on the radii of extra dimensions in this scenario. Note that this behavior for $\alpha'_{\rm GUT}$ as a function of $M'_{\rm GUT}$ is independent of δ.

One might worry that our calculation has only been performed to one-loop order, and that higher-loop corrections might destabilize the unification of gauge couplings that we have achieved. Indeed, this worry is particularly significant in the case of our higher-dimensional theory because the gauge couplings have power-law rather than logarithmic behavior. Thus, two- and higher-loop effects might be expected to be particularly large. Moreover, there are also various subtleties involved in assessing the perturbativity of a higher-dimensional theory: although the unified *gauge coupling* is small in our minimal $\eta = 0, 1$ scenarios, the actual expansion parameter of the higher-dimensional perturbation series is typically the gauge coupling multiplied by the effective number of Kaluza-Klein states in the theory. Thus, once again, one might suspect that higher-loop corrections to the unification might be sizable. However, it is easy to see that all higher-order corrections to the gauge couplings are at most logarithmic. This is because the excited Kaluza-Klein states fall into $N = 2$

supermultiplets, thereby guaranteeing that the corresponding power-law corrections to the gauge couplings must vanish identically beyond one-loop order.[‡] Indeed, the only corrections that might exist beyond one-loop order are the logarithmic corrections that come from the zero-modes. Thus, all higher-loop corrections to the gauge couplings will be small (as they are in the usual MSSM), and consequently our gauge coupling unification can be expected to survive beyond one-loop order.

Thus, to summarize this section, we see that the appearance of extra spacetime dimensions offers the interesting possibility of having gauge coupling unification at scales that are reduced, in some cases substantially, relative to the usual GUT scale. In particular, we see that it is no longer necessary for gauge coupling unification to occur at the usual, comfortably remote energy scale M_{GUT}. Moreover, we see that the unified gauge coupling in our "minimal" $\eta = 0$ scenario is always less than its value within the usual MSSM. Thus, for all values of μ_0 and δ, the "minimal" theory is even more weakly coupled than the MSSM. Similarly, the $\eta = 1$ scenario is always exactly as weakly coupled as the MSSM, and even the $\eta = 2$ and $\eta = 3$ scenarios can lead to weak unified couplings for suitable values of μ_0.

Such scenarios can be expected to have a variety of striking implications, ranging from new mechanisms for suppressing proton decay to possible explanations for the fermion mass hierarchy. We shall therefore now turn our attention to these important issues.

4 Extra dimensions and proton decay

Perhaps the most immediate question that arises in our scenario is the question of proton decay. In this section we shall show that for the "minimal" scenario with $\eta = 0$, there is a *higher-dimensional* mechanism involving Kaluza-Klein selection rules which enables us to cancel the usual proton-decay diagrams to all orders in perturbation theory. We shall also show that the non-minimal scenarios with higher values of η also lead to higher-dimensional suppression mechanisms for proton decay.

In our scenario, M_{GUT} is lowered by an amount which depends on μ_0 (the scale of new dimensions) and δ (the number of extra dimensions). However, in our minimal $\eta = 0$ scenario, the unified gauge coupling is weaker. Thus, *a priori*, relative to the usual supersymmetric GUT amplitude, the leading amplitude for proton decay in our scenario is larger by a factor of

$$\left(\frac{\alpha'_{\mathrm{GUT}}}{\alpha_{\mathrm{GUT}}}\right)\left(\frac{M_{\mathrm{GUT}}}{M'_{\mathrm{GUT}}}\right)^2 \;\approx\; f^{-2/\delta}\left[1 + \frac{X_\delta}{12\pi\delta}(f-1)\right]^{-1}\exp\left[\frac{\delta}{X_\delta B}(f-1)\right] \qquad (4.1)$$

[‡]For this purpose, it is necessary to verify that the orbifolding procedure discussed in Sect. 2 actually preserves the $N = 2$ supersymmetry of the excited Kaluza-Klein states. This in turn depends on certain model-dependent details concerning the manner in which the MSSM is embedded into higher dimensions. This issue will be discussed further in Sect. 7. However, as we shall see, it is not difficult to ensure $N = 2$ supersymmetry at the massive Kaluza-Klein levels. We shall therefore assume unbroken $N = 2$ supersymmetry at the massive levels, both here and in subsequent sections.

where $M'_{\rm GUT}$, f, X_δ, and B are defined in (3.18), (3.19), (3.10), and (3.14) respectively. Using the experimental bounds on the proton lifetime, we can then derive bounds on μ_0 and δ. We obtain $\mu_0 \gtrsim 1 \times 10^{14}$ GeV for $\delta = 1$, $\mu_0 \gtrsim 3 \times 10^{14}$ GeV for $\delta = 2$, and $\mu_0 \gtrsim 8 \times 10^{14}$ GeV for $\delta = 3$. Thus, as long as μ_0 is sufficiently large, the usual proton decay bounds can be satisfied in each case.

The above calculation exactly mimics the usual calculation of proton decay, and solves the proton decay problem by pushing the scale of grand unification back up to the usual neighborhood near 10^{16} GeV. However, this would then ruin much of the attractiveness of our scenario. Is there a better way?

One observation is that above the scale μ_0, the physics (and in particular our presumed grand-unified theory) is higher-dimensional. Thus, it may be possible to take a course that cannot be followed within the usual MSSM, namely to find *an intrinsically higher-dimensional solution to the proton-decay problem.*

Let us now see how such a higher-dimensional mechanism might work. As we have already discussed in Sect. 2, we can embed the MSSM into higher dimensions only by compactifying our higher-dimensional theory on an *orbifold*. In the case of a $\mathbb{Z}_2$ orbifold, this requires that we decompose all of our higher-dimensional quantum fields $\Phi(x)$ into even and odd functions $\Phi_\pm(x)$ of these extra coordinates, as in (2.7). Given that such an orbifold is already necessary on the grounds of the MSSM alone, let us now consider how this same orbifold might be exploited in the case of proton decay. Let us begin by separating all of the quantum fields of our grand-unified theory into two groups depending on whether they are present in the MSSM alone, or whether they appear only in the full GUT theory. In a purely schematic notation, we shall denote by $\Phi(x)$ any field which is present in the MSSM alone, and use $\Psi(x)$ to denote any field which only appears at the level of the GUT. Thus, for example, the quarks, leptons, gluons, $W^\pm$, Z, and Higgs doublets are all fields of the $\Phi(x)$ variety, while the $\Psi(x)$ fields include X-bosons and Higgs triplets. It is the appearance of the $\Psi(x)$ fields that leads to proton decay.

We have already discussed in Sect. 2 what the symmetry properties of the Φ fields must be with respect to the compactified coordinates y_i, $i = 1, ..., \delta$. The fact that we do not expect to see the Ψ fields at low energies (for which we demand only the MSSM gauge group and spectrum) suggests that we take the Ψ fields to be odd functions of the y_i, so that we retain only the fields Ψ_- as in (2.7). This choice preserves the gauge symmetries of the MSSM while simultaneously reflecting the breaking of the GUT symmetry below $M'_{\rm GUT}$, and guarantees that the Ψ fields have no zero-modes which could be observable at low energies. Thus, with this choice, all of the fields that mediate proton decay will be odd functions of these extra spacetime coordinates.

This fact has dramatic consequences for proton decay. Let us consider some typical diagrams that can mediate proton decay, as illustrated in Fig. 5. We have already seen in Sect. 2 that in the "minimal" $\eta = 0$ scenario, the MSSM fermions are restricted to the fixed points of the orbifold. However, if the Ψ_- fields are odd functions of the compactified coordinates, then their *wavefunctions vanish at the*
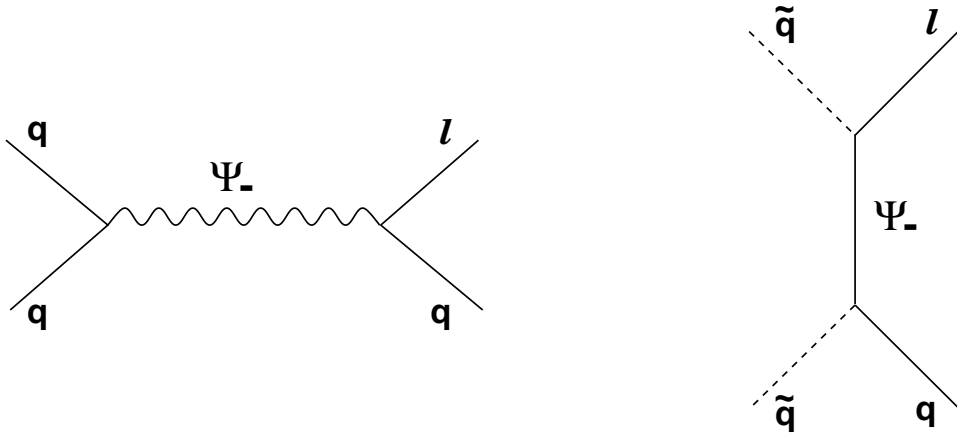
Figure 5: Typical diagrams that can mediate proton decay. Here the external lines correspond to the MSSM (s)quarks and leptons, while the internal Ψ fields correspond to X-bosons (as in the diagram on the left) and Higgsino fields (as in the diagram on the right). We choose the wavefunctions of the Ψ fields to be odd $(-)$ functions of the extra spacetime coordinates. With this choice, all vertices between the Ψ fields and the chiral MSSM fermions vanish identically in the "minimal" $\eta = 0$ scenario, and all possible proton-decay diagrams vanish to all orders in perturbation theory.

orbifold fixed points. Indeed, this property holds for *all* of the Kaluza-Klein modes of the Ψ_- fields. Thus, to all orders in perturbation theory, there is simply no coupling of the Ψ_- fields to the low-energy quarks and leptons of the MSSM. In other words, *all* such perturbative proton-decay diagrams, such as those in Fig. 5, vanish identically. Note that this result holds not only to all orders of perturbation theory, but also independently of the number of extra dimensions or the energy scale at which they appear.

Implicit in this proposal is also a solution to the famous doublet-triplet splitting problem. Rather than make the Higgs triplets much heavier than the Higgs doublets, which is the situation required in the usual GUT scenarios, we instead can allow the Higgs triplets to remain relatively light because they simply do not couple to the chiral MSSM fermions. As with the X-bosons, the Higgs triplets do not couple because their wavefunctions vanish at those locations in the fifth dimension at which the chiral MSSM fermions are located. Thus, no large mass "splitting" is required at all.

It is tempting to think of this suppression mechanism as simply Kaluza-Klein momentum conservation. After all, such an argument would state that the Ψ_- fields have no zero-modes, whereas the MSSM fermions are essentially zero-mode states. Thus, conservation of the Kaluza-Klein momentum at the vertex would seem to imply that any such tree-level vertex must vanish. However, such an argument is ultimately incorrect: we cannot impose Kaluza-Klein momentum conservation at such vertices because we have explicitly broken translational invariance with respect to

the compactified coordinates y_i when we introduced our orbifold relations $y_i \rightarrow -y_i$. Indeed, as we have seen, this action results in fixed points $y_i = 0$ and $y_i = \pi R$ which are not translationally invariant. Moreover, even if we could impose Kaluza-Klein momentum conservation, such a simple mechanism would at best suppress only tree-level proton-decay diagrams; the simplest one-loop box diagram would easily evade such a constraint. Our wavefunction mechanism, by contrast, is more robust, for it completely decouples the low-energy quarks from all of the proton-decay mediating effects of the full GUT theory to all orders in perturbation theory. Indeed, as we shall see in Sect. 7, this mechanism continues to work even in the more general cases of open-string theories.

Thus, we conclude that by making a judicious choice for the modings of the GUT fields with respect to the $\mathbb{Z}_2$ orbifold that produces our $N = 1$ MSSM states in the $\eta = 0$ minimal scenario, it is possible to exploit the higher-dimensional nature of the grand-unified theory in such a way as to completely eliminate proton decay to all orders in perturbation theory. This is clearly a symmetry argument which relies on the presence of extra compactified dimensions, and thus has no analogue in ordinary four-dimensional grand unification.

Finally, let us briefly consider the cases with $\eta \geq 1$. In such cases, not all of the chiral MSSM generations are restricted to orbifold fixed points (or the "three-branes" of an open string theory), and consequently there can be couplings between these fermions and the Ψ_- fields that mediate proton decay. However, for $\eta < 3$, there will typically be new, large, higher-dimensional suppression factors associated with proton-decay amplitudes. For example, let us consider the case $\eta = 1$, and assume that only the third generation has Kaluza-Klein excitations. In this case, the Ψ_- fields can couple only to the fermions of the third generation, and therefore proton decay can proceed only through a higher-order diagram which will be suppressed not only by extra loop factors but also by products of small CKM matrix elements. Thus, the $\eta = 1$ scenario probably does not have a problem with proton decay for $\mu_0 \gtrsim 10^{12}$ GeV. Similar arguments (leading to a more stringent bound) would also apply to the $\eta = 2$ case. In any case, we point out that within the context of string theory, it is also possible to circumvent problems of proton decay through other model-dependent mechanisms, such as the selection rules corresponding to hidden-sector discrete symmetries.

5 Extra dimensions and the fermion mass hierarchy

We now address the question of understanding the fermion mass hierarchy within the context of our extra-dimensional theory. In fact, it is in attempting to explain the fermion mass hierarchy that our scenario is particularly powerful, for (unlike the situation with the gauge couplings) we now are faced with attempting to unify a set of Yukawa couplings whose values at low energies differ by many orders of magnitude. Specifically, if we let y_F ($F = u, d, s, c, b, t, e, \mu, \tau$) denote the different

Yukawa couplings for the quarks and leptons and define $\alpha_F \equiv y_F^2/4\pi$ in analogy with the gauge couplings, we are faced with the task of explaining (*i.e.*, approximately unifying) low-energy values ranging in order of magnitude from $\alpha_t^{-1} \approx 1$ to $\alpha_e^{-1} \approx 10^{12}$. This is extremely difficult to reconcile within the context of the usual grand-unification scenario in which the Yukawa couplings run only logarithmically.

Once again, our fundamental idea is that the presence of extra dimensions causes the Yukawa couplings to evolve exponentially rather than linearly as a function of $\log \mu$. Under the proper conditions, this exponential evolution might therefore be capable of producing a relatively large fermion mass hierarchy over a relatively small energy scale interval.

We begin by recalling how the nine Yukawa couplings y_F (with $F = e, \mu, \tau, u, d, s, c, b, t$) run within the usual four-dimensional MSSM. These Yukawa couplings are defined in relation to the corresponding fermion masses m_F via

$$m_F = y_F \times v \times \begin{cases} \cos \beta & \text{for down-type quarks and leptons} \\ \sin \beta & \text{for up-type quarks} \end{cases} \qquad (5.1)$$

where $v \approx 174$ GeV and where $\tan \beta$ is the usual ratio of up- and down-type Higgs VEV's in the MSSM. These couplings then appear in the superpotential, which takes the generic form

$$W = \sum_F y_F \, F\overline{F} \, H_{u,d} \ . \qquad (5.2)$$

If we define $\alpha_F \equiv y_F^2/4\pi$, then just like the gauge couplings α_i in (3.1), these Yukawa couplings run in the usual MSSM according to a one-loop RGE of the form

$$\frac{d}{d\ln\mu} \, \alpha_F^{-1}(\mu) = -\frac{b_F(\mu)}{2\pi} \ . \qquad (5.3)$$

Indeed, the only difference relative to the gauge couplings is that the one-loop beta-function "coefficients" $b_F(\mu)$ are not constants, but instead depend on the scale μ through extra terms that depend on the couplings themselves. For example, within the usual MSSM, $b_t(\mu)$ is given by:

$$b_t \equiv 6 + \frac{1}{\alpha_t} \left(\alpha_b + 3\alpha_u + 3\alpha_c - \frac{16}{3}\alpha_3 - 3\alpha_2 - \frac{13}{15}\alpha_1 \right) \ , \qquad (5.4)$$

and each of the other $b_F(\mu)$ has a similar form.

It will be important for us to recall how (5.3) is derived. Since the Yukawa coupling y_F describes the Higgs/fermion/anti-fermion interaction term (5.2), the renormalization of α_F depends on the wavefunction renormalization factors Z_i of each of these three fields:

$$\alpha_F^{-1}(\mu) = Z_H Z_F Z_{\overline{F}} \, \alpha_F^{-1}(\mu_0) \ . \qquad (5.5)$$

These three wavefunction renormalization factors Z_i are calculated by evaluating diagrams such as those shown in Fig. 6. In general, we obtain a result of the form

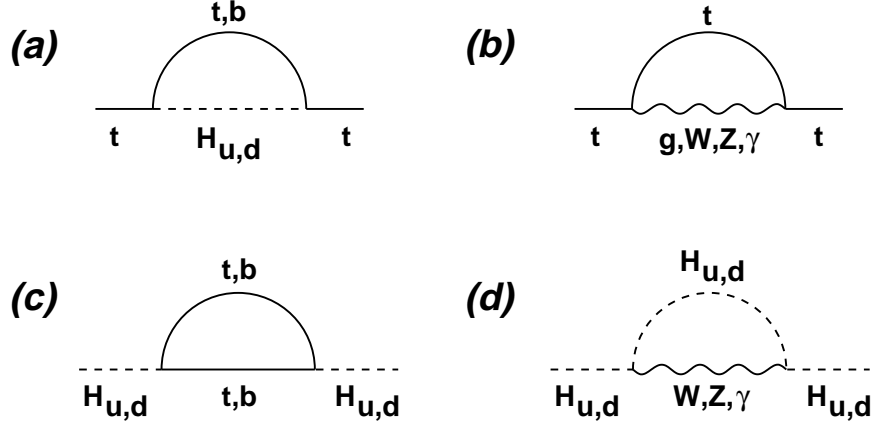$$Z_i = 1 - \frac{\gamma_i}{2\pi} \ln \frac{\mu}{\mu_0} \qquad (5.6)$$

Figure 6: Typical classes of superfield wavefunction renormalization diagrams in the MSSM. Diagrams (a,b) contribute to Z_t and $Z_{\bar{t}}$, while diagrams (c,d) contribute to $Z_{H_{u,d}}$.

where γ_i is the anomalous dimension of the field i. For example, in the case of the Higgs and top quarks, we find

$$
\begin{aligned}
\gamma_t &= \alpha_t + \alpha_b - \frac{1}{30}\alpha_1 - \frac{3}{2}\alpha_2 - \frac{8}{3}\alpha_3 \\
\gamma_{\bar{t}} &= 2\alpha_t - \frac{8}{15}\alpha_1 - \frac{8}{3}\alpha_3 \\
\gamma_{H_u} &= 3\alpha_t + 3\alpha_c + 3\alpha_u - \frac{3}{10}\alpha_1 - \frac{3}{2}\alpha_2 \; .
\end{aligned}
\tag{5.7}
$$

Combining these factors within (5.5) and keeping terms at most linear in the logarithms then yields (5.3), with

$$
\alpha_F\, b_F \;\equiv\; \gamma_F + \gamma_{\overline{F}} + \gamma_{H_i} \; .
\tag{5.8}
$$

Let us now consider how this calculation is modified in the presence of extra spacetime dimensions. As before, we shall assume that a certain number δ of extra spacetime dimensions appear at an energy scale $\mu_0 \equiv R^{-1}$, and for simplicity we shall concentrate on the "minimal" scenario with $\eta = 0$. Below the scale μ_0, the Yukawa couplings run according to (5.3), as in the usual four-dimensional MSSM. Above the scale μ_0, however, the Yukawa couplings instead receive finite one-loop corrections whose size is a function of the cutoff Λ. By comparison with our prior results for the gauge couplings, it is straightforward to write down the expected general form for these corrections:

$$
\alpha_F^{-1}(\Lambda) \;=\; Z_H Z_F Z_{\overline{F}} \, \alpha_F^{-1}(\mu_0)
\tag{5.9}
$$

where now we expect our wavefunction renormalization factors to have the general form

$$
Z_i \;=\; 1 \;-\; \frac{\gamma_i(\mu_0) - \tilde{\gamma}_i(\mu_0)}{2\pi} \ln\frac{\Lambda}{\mu_0} \;-\; \frac{\tilde{\gamma}_i(\mu_0)}{2\pi}\frac{X_\delta}{\delta}\left[\left(\frac{\Lambda}{\mu_0}\right)^\delta - 1\right] \; .
\tag{5.10}
$$

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In this expression for Z_i, the power-law term is expected to arise from the summation over Kaluza-Klein states in the loops of the diagrams in Fig. 6, reflecting the higher-dimensional nature of such contributions. In general, the anomalous dimensions $\tilde{\gamma}_i$ corresponding to the excited Kaluza-Klein modes can differ from the anomalous dimensions γ_i corresponding to the zero-mode ground states. This difference would then give rise to the logarithm term in (5.10).

One important difference relative to the gauge coupling case concerns the scale-dependence of these anomalous dimensions γ_i (or equivalently the "beta-functions" b_F). Note that in (5.9) and (5.10), these coefficients are evaluated at the fixed scale μ_0. This is because we are working within the context of a non-renormalizable theory and calculating corrections to quantities evaluated at the fixed scale μ_0.

Given these general expressions, the task then remains to calculate the functions $\tilde{\gamma}_i$ which parametrize the one-loop contributions from each massive level in the Kaluza-Klein tower. Let us begin by considering a diagram of the form shown in Fig. 6(a). Within the loop, we have both a fermion and a Higgs field. However, as we discussed in Sect. 2, the MSSM fermions do not have Kaluza-Klein towers in the minimal $\eta = 0$ scenario, so the only fermion in the loop is the zero-mode MSSM fermion itself. Furthermore, at each excited level, the Kaluza-Klein tower corresponding to the Higgs fields exactly mirrors the MSSM Higgs (zero-mode) ground state. Consequently, the contribution to $\tilde{\gamma}_F$ from this diagram is exactly the same as it is in the usual four-dimensional MSSM. Turning our attention to diagram in Fig. 6(b), we see that a similar situation exists here too. Only the zero-mode MSSM fermion itself can propagate in the loop. Likewise, although the Kaluza-Klein towers for the gauge bosons are $N = 2$ supersymmetric, *only their $N = 1$ supersymmetric components can couple to the MSSM fermions*. This restriction arises because, as discussed in Sect. 2, the wavefunctions of the additional ϕ and ψ fields which fill out the $N = 2$ gauge-boson Kaluza-Klein tower are *odd* functions of the compactified coordinates; these wavefunctions thus vanish at the orbifold fixed points at which the chiral MSSM fermions are found. Thus, once again, the contribution to the fermion anomalous dimension from this diagram is exactly the same as it is in the usual four-dimensional MSSM. We therefore find

$$\tilde{\gamma}_F \;=\; \gamma_F \;, \qquad \tilde{\gamma}_{\overline{F}} \;=\; \gamma_{\overline{F}} \;. \qquad (5.11)$$

for all fermions F.

Finally, let us consider the anomalous dimensions of the Higgs fields. These diagrams are shown in Figs. 6(c,d). In Fig. 6(c), only the chiral MSSM fermions can propagate in the loop; there are no excited Kaluza-Klein states which can propagate. Thus, this diagram is immune to the effects of the extra dimensions, and makes no contribution to $\tilde{\gamma}_H$. In Fig. 6(d), by contrast, the full $N = 2$ set of Kaluza-Klein states for both the Higgs fields and the gauge bosons can propagate in the loop. Note that we must impose Kaluza-Klein momentum conservation at each of these vertices. Because the appropriate Kaluza-Klein states for this diagram fall into $N = 2$ supermultiplets,

there is again no net contribution to $\tilde{\gamma}_H$. (This is a consequence of the general fact that in unbroken $N = 2$ supersymmetric theories, hypermultiplets do not receive any wavefunction renormalizations.) Consequently, combining our contributions, we find

$$\tilde{\gamma}_{H_i} = 0 . \tag{5.12}$$

The Higgs wavefunction renormalization is therefore completely immune to the effects of the extra spacetime dimensions.

Given the results (5.11) and (5.12), we can now examine the evolution of the Yukawa couplings in the presence of extra spacetime dimensions. Below the scale $\mu_0 \equiv R^{-1}$, the Yukawa couplings run logarithmically, as in the usual four-dimensional MSSM. Above this scale, however, the form of the fermion and anti-fermion Z-factors in (5.10) implies that the Yukawa couplings start evolving with a power-law dependence on the energy scale (cutoff) Λ. Because the anomalous dimensions $\tilde{\gamma}_i$ tend to be dominated by their gauge contributions, we typically find $\tilde{\gamma}_i(\mu_0) < 0$. Thus the Z-factors in (5.9) each tend to be positive and grow quickly with Λ. This in turn drives the Yukawa couplings dramatically towards extremely weak values. Specifically, for $\Lambda \gg \mu_0$ and neglecting the logarithmic contributions to Z_{H_i} in (5.9), we find

$$\frac{\alpha_F(\Lambda)}{\alpha_F(\mu_0)} \approx \frac{4\pi^2}{\gamma_F \gamma_{\overline{F}}} \left(\frac{\delta}{X_\delta}\right)^2 \left(\frac{\Lambda}{\mu_0}\right)^{-2\delta} . \tag{5.13}$$

Thus, we see that the effect of the extra dimensions is to drive all of the Yukawa couplings (including the Yukawa coupling of the top quark) towards weak values.

This effect might be extremely useful for various phenomenological purposes (*e.g.*, avoiding the Landau poles that often arise in the usual MSSM). Unfortunately, however, it is easy to see that this effect cannot be used to explain the fermion mass *hierarchy*. According to (5.13), the *ratio* between any two Yukawa couplings for different fermions evolves as

$$\frac{\alpha_{F_1}(\Lambda)}{\alpha_{F_2}(\Lambda)} \approx \left(\frac{\gamma_{F_2} \gamma_{\overline{F_2}}}{\gamma_{F_1} \gamma_{\overline{F_1}}}\right) \frac{\alpha_{F_1}(\mu_0)}{\alpha_{F_2}(\mu_0)} \tag{5.14}$$

for $\Lambda \gg \mu_0$. Thus, the hierarchy is affected only by scale-independent factors of order one.

In order to do better, let us now consider a slight generalization of the previous scenario. Let us imagine that in addition to the usual MSSM Yukawa terms, there also exists a new $N = 1$ chiral gauge-singlet scalar superfield S which couples to the Higgs fields via a superpotential term of the form

$$W = \lambda H_u H_d S . \tag{5.15}$$

Here λ is a coupling which we can make arbitrarily small. For the purposes of this discussion, we shall consider S to be a light field which lacks Kaluza-Klein excitations

(*e.g.*, this field might arise from a string twisted sector and therefore exist only at orbifold fixed points). We shall also ignore the running of λ, and consider this to be a fixed parameter; this assumption will not affect the qualitative features of our results.
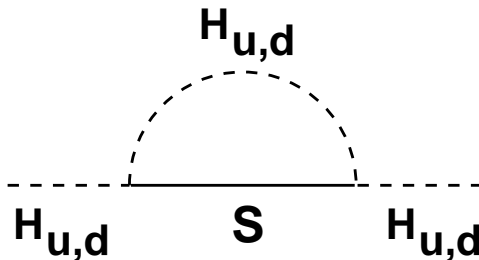

Figure 7: New superfield diagram, where S is a light $N = 1$ scalar singlet superfield. This diagram allows the Higgs wavefunction renormalization to feel the extra spacetime dimensions, which in turn causes the fermion Yukawa couplings to become strong.

Because of the extra dimensions, such a coupling (no matter how weak) has a profound effect. There is a new diagram, as shown in Fig. 7, which must also be considered. Because the scalar field lacks a Kaluza-Klein tower, this component of the diagram is only $N = 1$ supersymmetric. Thus, the Higgs hypermultiplet can now experience wavefunction renormalization. Moreover, thanks to the Kaluza-Klein tower of Higgs fields, this diagram leads to non-zero power-law corrections to Z_{H_i}. Specifically, we now find

$$\tilde{\gamma}_{H_i} = \lambda^2/4\pi . \tag{5.16}$$

Note that $\tilde{\gamma}_{H_i}$ is strictly positive. This means that Z_{H_i} *decreases* from unity, and rapidly vanishes.

This in turn implies that all of the fermion Yukawa couplings quickly become strong rather than weak! We stress that this change in behavior occurs regardless of how small we take λ, since power-law behavior always eventually dominates over logarithmic behavior. Indeed, if we neglect the logarithmic contributions to Z_{H_i} entirely, we see that the power-law contributions to Z_{H_i} have the right signs and magnitudes to bring all of the Yukawa couplings simultaneously to a common Landau pole scale. This can be explicitly seen in Fig. 8, for which we have taken $\mu_0 = R^{-1} = 100$ TeV and $\lambda^2/4\pi \approx 1/4$. It is clear that above the scale μ_0, the power-law term coming from the Kaluza-Klein states dominates the evolution, and the Yukawa couplings tend towards a common large Yukawa coupling (*e.g.*, towards a common Landau pole defined by the equation $Z_{H_i} = 0$). Note that because the power-law corrections for each fermion are not coupled to those of the other fermions, as would have been the case for the usual renormalization group equations, each fermion *independently* tends towards the common Landau pole. Moreover, for appropriate values of the coupling λ, this Yukawa "unification" scale agrees precisely with the scale M'_{GUT} at which the gauge couplings unify.
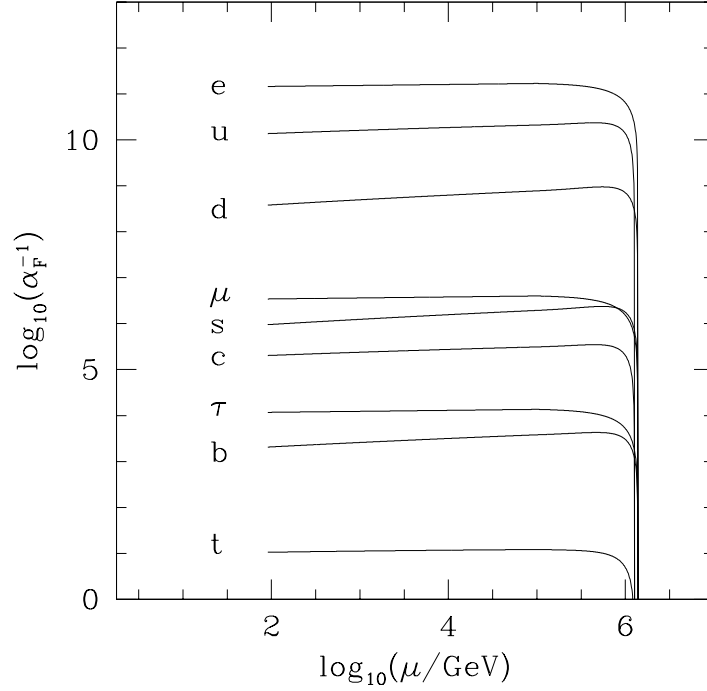
Figure 8: The evolution of the Yukawa couplings $\alpha_F^{-1} \equiv 4\pi/y_F^2$ within the singlet-enhanced MSSM, assuming the presence of a single extra dimension at $\mu_0 = R^{-1} = 100$ TeV. We have taken $m_t = 180$ GeV and $\tan\beta = 3$ as a representative case. Note that we are plotting the Yukawa couplings on a *logarithmic* scale in order to display them all simultaneously. It is evident that all of the Yukawa couplings simultaneously and independently approach a common Landau pole which precisely agrees with the scale at which the gauge couplings unify.

This behavior might also be interesting for a number of phenomenological purposes. However, to what extent does this "unification" actually solve the fermion mass hierarchy problem? Once again, because the behavior of the extra dimensions is universal for all fermions, it might seem that no relative fermion hierarchy can be explained. This is not true, however: certain *partial* hierarchies can indeed be explained. To see this, let us consider the precise positions of the Landau poles. For each fermion, the position of the Landau pole is determined as the solution to the equation

$$Z_{H_i} = 0 \qquad\qquad (5.17)$$

for the appropriate Higgs field. However, thanks to their different logarithmic contributions to Z_{H_i}, the position of the Landau pole for the up-type quarks is slightly shifted relative to that for the bottom-type quarks and leptons. Ordinarily, this might seem to be an insignificant observation. However, *in the presence of extra*

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dimensions, this difference amounts to a huge relative shift in the Yukawa couplings between (up/down)-type pairs of fermions. This allows pairs of up-type and down-type Yukawa couplings to intersect, thereby providing a true unification of pairs of Yukawa couplings and completely eliminating the mass hierarchy between the corresponding fermions. Some of these dramatic unifications are shown in Fig. 9. Note that all of these unifications occur while the corresponding Yukawa couplings are still weak, so perturbation theory remains valid.
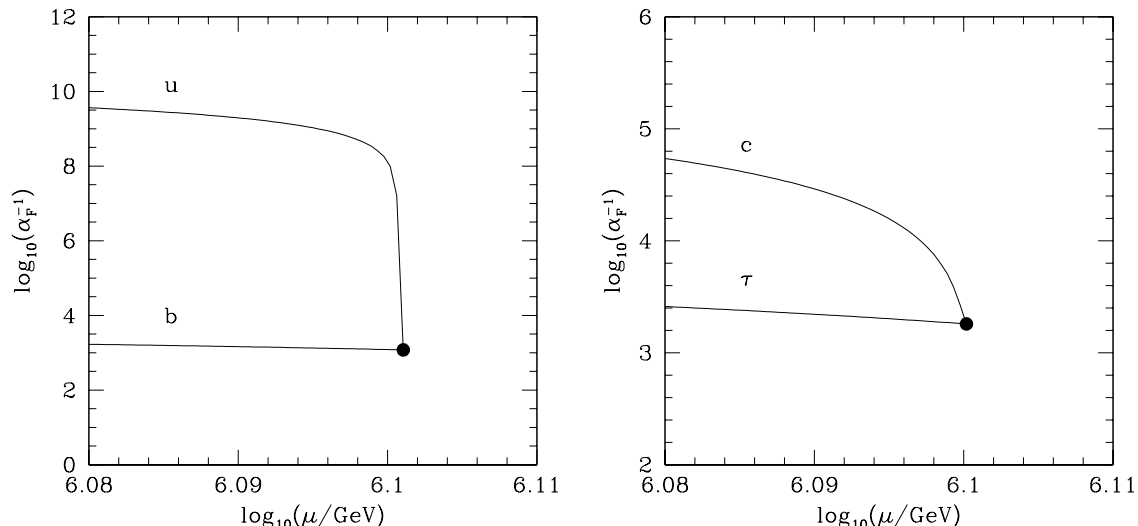

Figure 9: Two representative pairwise unifications of Yukawa couplings within the singlet-enhanced MSSM. In both cases we have taken a single extra dimension at $\mu_0 = R^{-1} = 100$ TeV, with $m_t = 180$ GeV and $\tan\beta = 3$. Once again, we plot the Yukawa couplings on a *logarithmic* scale. It is clear from these plots that extra spacetime dimensions can explain hierarchies of many orders of magnitude, and produce Yukawa coupling unifications while still in the perturbative regime.

Even though we have managed to achieve pairwise Yukawa coupling unifications in this way, we still seek a more complete solution to the fermion mass hierarchy problem. It is clear, of course, that the only true way to explain a fermion mass hierarchy is through a *flavor*-dependent coupling (which we have so far not introduced). Extra dimensions, by themselves, cannot be expected to achieve this since they are universal, and affect all fermions equally in the "minimal" $\eta = 0$ scenario. However, even if we must introduce a flavor-dependent coupling, this flavor-dependence need not be very strong, for the power-law effects of the extra dimensions can easily *enhance* or *amplify* the effects of even a relatively mild flavor-dependence. Thus, by introducing a relatively mild flavor-dependent coupling, we can exploit the effects of extra dimensions in order to achieve a complete explanation of the fermion masses.

In order to illustrate this point, we shall consider one final scenario. Let us go

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back to the usual MSSM, and assume the existence of an additional heavy MSSM-singlet scalar field Φ which modifies the form of the Yukawa interactions from (5.2) to

$$W \;=\; \sum_F \hat{y}_F\, \Phi^{n_F}\, F\overline{F}H_{u,d} \qquad\qquad (5.18)$$

where $n_F \in \mathbb{Z}$. Here $\hat{y}_F$ are a set of *dimensionful* Yukawa couplings, and we shall assume that Φ is endowed with a corresponding $N=2$ supersymmetric tower of Kaluza-Klein states. For the purposes of our discussion, the precise mass of Φ is not important as long as it exceeds observable bounds; for convenience we may assume $m_\Phi \approx \mu_0$. The important point is that the exponents n_F will be assumed to be flavor-dependent, taking different values for different fermions. This will serve as the "input" flavor-dependence that is ultimately required for addressing the fermion mass hierarchy. Although this setup is reminiscent of the Froggatt-Nielsen scenario [3], we will see that the role of Φ is different: for example, we shall make absolutely no assumptions about its vacuum expectation value, and in particular we shall treat Φ as a fully dynamical field above the scale μ_0. Rather, our goal will be to see how extra dimensions themselves can amplify this flavor-dependence into a fermion mass hierarchy.
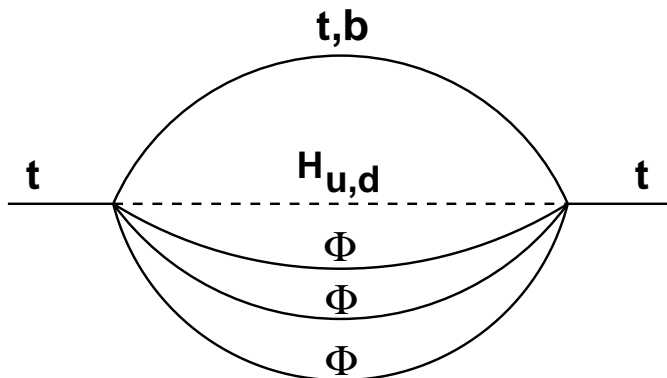

Figure 10: The generalization of Fig. 6(a) in the presence of a superpotential of the form (5.18).

How does the presence of the field Φ affect the analysis? Above the scale μ_0, the usual diagrams in Fig. 6 must be replaced by new diagrams in which the dynamical Φ field also propagates in loops. For example, Fig. 6(a) is replaced with Fig. 10, where there are n_F independent Φ propagators. Because the Φ field is endowed with infinite towers of Kaluza-Klein states, we see that we now have a total of $(n_F + 1)\delta$ *independent towers of Kaluza-Klein states that can propagate in the loop.* These consist of the δ towers for each compactified direction for the Higgs field, and $n_F\delta$ towers for each compactified direction of each of the n_F different Φ propagators. Thus, the *effective number of extra spacetime dimensions* felt by such a diagram is

shifted in a flavor-dependent way:

$$\delta \ \rightarrow \ \Delta_F \ \equiv \ (n_F + 1)\,\delta \ . \tag{5.19}$$

Of all the diagrams in Fig. 6, this is the dominant shift that arises. (For example, there is also a contribution from the generalization of Fig. 6(c), but this contribution has fewer internal Kaluza-Klein propagators.) This in turn implies that the wavefunction renormalization factors Z_F will have the dominant form

$$Z_F \ = \ 1 \ - \ \frac{c_F(\mu_0)}{2\pi}\frac{X_{\Delta_F}}{\Delta_F}\left[\left(\frac{\Lambda}{\mu_0}\right)^{\Delta_F} - 1\right] \ , \tag{5.20}$$

where we have neglected possible logarithmic contributions. Here

$$c_F(\mu_0) \ \sim \ \Lambda^{2n_F} \ [\hat{\alpha}_F(\mu_0) \ + \ ...] \ > \ 0 \tag{5.21}$$

is a (dimensionless) anomalous dimension which must be computed in the presence of the Φ field, and whose precise value is irrelevant for our purposes. Note, however, that $c_F(\mu_0)$ scales as the cutoff Λ^{2n_F}. This is evident from simple power-counting in the $(n_F + 1)$-loop diagram of Fig. 10, since there are $n_F + 1$ internal loop four-momenta p_i and $n_F + 2$ internal superfield propagators (each of which contributes p_i^{-2}).

The result (5.20) implies that the Yukawa couplings $\hat{y}_F$ will evolve according to the general power-law form

$$\hat{\alpha}_F^{-1}(\Lambda) \ = \ \hat{\alpha}_F^{-1}(\mu_0) - \frac{\hat{c}_F}{2\pi}\frac{X_{\Delta_F}}{\Delta_F}\left[\left(\frac{\Lambda}{\mu_0}\right)^{\Delta_F} - 1\right] \tag{5.22}$$

where $\hat{c}_F(\mu_0) \equiv c_F(\mu_0)/\hat{\alpha}_F(\mu_0) \sim \Lambda^{2n_F}(1 + ...)$ is a dimensionful beta-function coefficient and where we are again neglecting possible logarithmic contributions. Thus, we see that the presence of the extra dimensions (*i.e.*, $\delta \neq 0$) is capable of producing a flavor-dependent power-law hierarchy! Specifically, since $c_F(\mu_0)$ is positive, our Yukawa couplings are all once again driven towards a simultaneous Landau pole at which $\hat{\alpha}_F^{-1}(\Lambda) \rightarrow 0$. However, unlike the previous case, these Yukawa couplings approach the Landau pole in a flavor-dependent manner, and have the potential to actually unify on the way. Equivalently, assuming a unification near the Landau pole and solving for $\hat{\alpha}_F^{-1}(\mu_0)$ from (5.22), we find

$$\hat{\alpha}_F^{-1}(\mu_0) \ \approx \ \frac{\hat{c}_F(\mu_0)}{2\pi}\frac{X_{\Delta_F}}{\Delta_F}\left[\left(\frac{\Lambda}{\mu_0}\right)^{\Delta_F} - 1\right] \ . \tag{5.23}$$

Identifying the physical Yukawa coupling $y_F(\mu_0)$ at the scale μ_0 via $y_F(\mu_0) \equiv \mu_0^{n_F}\hat{y}_F(\mu_0)$ then yields the result

$$\alpha_F^{-1}(\mu_0) \ \approx \ \frac{X_{\Delta_F}}{2\pi\Delta_F}\left(\frac{\Lambda}{\mu_0}\right)^{2n_F}\left[\left(\frac{\Lambda}{\mu_0}\right)^{\Delta_F} - 1\right] \ \approx \ \frac{X_{\Delta_F}}{2\pi\Delta_F}\left(\frac{\Lambda}{\mu_0}\right)^{\Delta_F + 2n_F} \ . \tag{5.24}$$

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Thus, we see that as a simple result of having extra spacetime dimensions ($\delta > 0$), the flavor-dependent coupling (5.18) has now been amplified, ultimately producing a large power-law Yukawa mass hierarchy with total exponent

$$\Delta_F + 2n_F = n_F(2 + \delta) + \delta \ . \tag{5.25}$$

This exponent can also be interpreted physically as the mass dimension of the Yukawa coupling $\hat{\alpha}_F^{-1}$ in $4 + \delta$ flat spacetime dimensions.

Note that this scenario differs from the usual Froggatt-Nielsen scenario [3] in that we were not forced to introduce an arbitrary small number to parametrize the VEV of the Φ field. In our scenario, by contrast, the hierarchy is generated purely as a result of a smooth power-law evolution of Yukawa couplings between the scale of the extra dimensions and the unification scale. Indeed, because the Yukawa couplings are driven to strong coupling in this scenario, it is even possible to imagine that more refined results could be obtained using a fixed-point analysis. Finally, we note that because of the naturally large exponent (5.25), the Yukawa coupling hierarchy can be explained without the *a priori* large values of n_F that would have been needed in the usual Froggatt-Nielsen scenario. For example, taking $\delta = 1$ and μ_0 in the TeV range implies (see Fig. 3) that $\Lambda/\mu_0 \approx 20$. We can therefore explain the *entire hierarchy* factor of 10^{12} between the electron and the top quark simply by taking $n_t = 0$ and $n_e = 3$.

Finally, we remark that the possibility of extra spacetime dimensions also opens up new mechanisms for generating a fermion mass hierarchy which do not require the arbitrary introduction of low-energy flavor-dependent couplings, or the introduction of new low-energy matter fields. Such possibilities exist for the "minimal" $\eta = 0$ scenarios as well as the non-minimal $\eta > 0$ scenarios. Such new mechanisms rely on the non-perturbative behavior of open-string theories in the presence of extra large dimensions, and will be discussed in detail in Sect. 8.

6 Unification without supersymmetry

Until this point, our discussion has focused on the effects of extra large spacetime dimensions in theories with supersymmetry. However, given that the observed low-energy world is non-supersymmetric, and given that our extra dimensions can appear at a scale which is comparable (and perhaps even lower) than the superpartner scale, it also makes sense to consider the corresponding *non-supersymmetric* situation. Indeed, two of the primary motivations for introducing supersymmetry at all are the gauge hierarchy problem and the unification of gauge couplings within the MSSM. However, if new dimensions populate the desert between the electroweak scale and the usual GUT scale, then the gauge hierarchy problem is greatly ameliorated, and (in the case of TeV-scale extra dimensions) no longer exists. Thus, it remains to consider whether extra dimensions can also give rise to gauge coupling unification when supersymmetry is absent.

A priori, embedding the Standard Model into higher dimensions is more straightforward than embedding the MSSM. This is because our Kaluza-Klein states no longer need to form $N = 2$ multiplets. As a "minimal" scenario, we shall assume (as we did for the MSSM) that the gauge bosons and Higgs field have Kaluza-Klein excitations, while the chiral Standard Model fermions do not. Note that at each excited Kaluza-Klein level, this requires the introduction of an additional real scalar transforming in the adjoint of each gauge group (in order to make the corresponding gauge bosons massive). This then implies the beta-function coefficients

$$\begin{aligned}
(b_1, b_2, b_3) &= (41/10, -19/6, -7) \\
(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) &= (1/10, -41/6, -21/2)
\end{aligned} \qquad (6.1)$$

where as before we have defined $b_1 \equiv (3/5)b_Y$.

Unfortunately, this simple-minded scenario does not lead to gauge coupling unification. However, it is not hard to find extended scenarios that do. For example, let us imagine adding three extra real scalar fields transforming in the adjoint of $SU(2)$. We shall assume that the wavefunctions of these extra scalar fields are odd functions of the coordinates y_i of the compactified dimensions. Indeed, such sorts of extra states are extremely natural from the point of view of string theory, and the fact that they are odd functions of y_i guarantees that they have no light zero-modes. We then have

$$(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = (1/10, -35/6, -21/2) , \qquad (6.2)$$

which leads to the intermediate-scale gauge coupling unification shown in Fig. 11.

Of course, this scenario is not unique. In principle, there may exist other combinations of extra matter that can also lead to gauge coupling unification without supersymmetry. We may even choose to associate the scale μ_0 with the supersymmetry-breaking scale, *e.g.*, by implementing a Scherk-Schwarz supersymmetry-breaking mechanism [4] using the same orbifold that prevents our chiral fermions from having Kaluza-Klein excitations. One finds that the resulting string spectrum always exhibits a hidden "misaligned supersymmetry" [5] which is responsible for maintaining the fundamental finiteness properties of the string, even without supersymmetry. Indeed, such supersymmetry-breaking scenarios using the $\mathbb{Z}_2$ Scherk-Schwarz mechanism have been investigated in a number of contexts [6, 7]. Moreover, note that just as for the MSSM, we are free to consider non-minimal scenarios with $\eta > 0$, since increasing η does not disturb the unification at $\eta = 0$.

However, the important point is that even without supersymmetry, it is the power-law evolution of the gauge couplings, induced by the extra spacetime dimensions, that enables such extra matter to produce perturbative gauge coupling unification at such low energy scales. Thus, through extra large spacetime dimensions, it may well be possible to contemplate a scenario in which perturbative gauge coupling unification is preserved and the gauge hierarchy problem is eliminated, all without supersymmetry. Indeed, one can even contemplate explaining the Standard Model fermion masses at
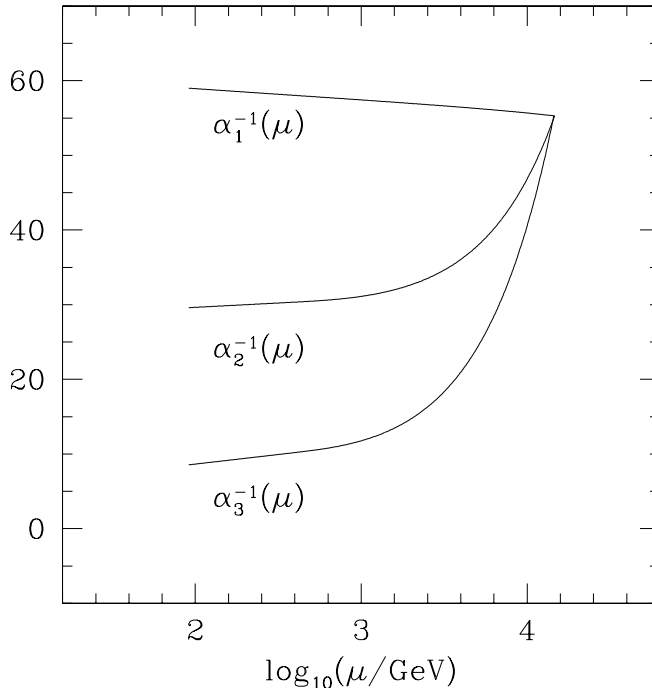
Figure 11: Gauge coupling unification without supersymmetry can also be achieved at very low energy scales. Here we have considered a "minimal" embedding of the non-supersymmetric Standard Model into five dimensions, supplemented with three real scalars transforming in the adjoint of $SU(2)$. We have taken our single extra dimension to have radius $R^{-1} \equiv \mu_0 = 1$ TeV. Thus this scenario also avoids the gauge hierarchy problem.

the same time, again through the power-law behavior induced by the extra spacetime dimensions.

7 Embedding our scenario into string theory

In this paper, we have studied the effects of extra large spacetime dimensions from a strictly *field-theoretic* point of view. Even though we have borrowed certain ideas from string theory (*e.g.*, the notion of orbifolds), we have limited ourselves to questions that are field-theoretic in nature, and for which a purely field-theoretic analysis suffices. For example, we have concentrated on Kaluza-Klein momentum states but neglected winding-mode states (which is a valid approximation in the field-theory limit, since our radii are presumed large); we have not worried about ensuring that the orbifold is a symmetry of the full theory (since we have not constructed a full string theory); and we have not considered any extra states beyond the MSSM (although string theory generically predicts such states). We have not addressed

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these sorts of issues because they tend to be extremely model-dependent.

There are also various questions that are *generic*. For example, one of the attractive features of the conventional GUT paradigm is that the unification scale $M_{\rm GUT} \approx 2 \times 10^{16}$ GeV is close to (though not equal to [8]) the perturbative heterotic string scale $M_{\rm string} \approx 5 \times 10^{17}$ GeV. This suggests that we might identify these two scales, and imagine that our grand unified theory might be directly embedded into string theory. However, in this paper we have seen that extra large spacetime dimensions can lower the unification scale quite substantially. How then do we go about embedding our scenario into string theory?

In this section, we shall address a number of these questions. Rather than attempt to construct a given realistic string model which incorporates our scenario, we shall limit the following discussion to *general* comments concerning various issues that come into play when attempting to embed our scenario into string theory.

7.1   Implementing the orbifold projection

We begin with a general comment concerning the orbifolding projection discussed in Sect. 2. As we have seen, it is necessary to compactify our extra dimensions on orbifolds rather than circles for two reasons: we need the orbifold to break the $N = 2$ supersymmetry of the excited Kaluza-Klein states down to $N = 1$ supersymmetry for the observable MSSM ground states; and we need to ensure, in our "minimal" scenario, that the chiral MSSM fermions do not have Kaluza-Klein towers. Once this scenario is embedded into a full string theory, it becomes necessary to ensure that the $\mathbb{Z}_2$ orbifold action on the compactified coordinates $y_i \to -y_i$ is a symmetry of the full string theory.

It turns out that the "minimal" scenario that we examined in Sects. 2 and 3 does not have this property, and must be extended if it is to embedded into string theory. Specifically, although we have joined the two $N = 1$ supersymmetric MSSM Higgs doublets together in (2.4) to form a single $N = 2$ supermultiplet, it is actually necessary for one of these Higgs fields to be *odd* under the orbifold action $y_i \to -y_i$ in order for this action to be a symmetry of the full theory. This then prevents one of the Higgs fields from having a light zero-mode. Of course, this need not cause any logical inconsistency, for it may well be that the remaining MSSM Higgs field arises in a "twisted" sector. Alternatively, one of the MSSM Higgs fields might arise as the first excited Kaluza-Klein state of the odd tower, with mass $m \approx \mu_0$. Gauge coupling unification would then be slightly altered, though not significantly damaged. In any case, string theory also predicts new so-called threshold corrections [9, 8] which can easily accommodate such minor discrepancies.

A third alternative would be to introduce a new, separate, odd-transforming Higgs field for each of the MSSM Higgs fields. This would then imply the existence of *four* Higgs fields at each excited Kaluza-Klein level, while maintaining only two Higgs doublets at the massless level. Unlike the previous case, this choice would have a

considerable effect on the unification of the gauge couplings. However, since this is essentially a model-dependent choice that would be dictated by the string model in question, it is quite reasonable to expect that string theory will provide even further states that can once again restore gauge coupling unification. We stress, however, that these are model-dependent questions that can be addressed only in the context of a fully constructed, realistic string model.

As a separate issue related to the orbifold, we remark that in our "minimal" scenario, the MSSM chiral fermions were taken not to have Kaluza-Klein towers. The orbifold manages to accomplish this because, in any closed string theory, modular invariance must be preserved. Thus, modding out by the orbifold action necessarily requires the introduction of so-called "twisted states" whose wavefunctions are restricted to the orbifold fixed points. Hence our assertion that the MSSM chiral fermions have no Kaluza-Klein towers is fully consistent within the context of closed string theory. For an *open* string theory, by contrast, the process of orbifolding generally does not lead to such twisted sectors. This difference arises because modular invariance is not required to be a symmetry of open-string theories. In such cases, we must assume the chiral fermions to be restricted to certain "three-branes" of the open-string theory. This will be discussed further below.

7.2 Including the extra GUT states

The second issue that we shall discuss concerns the appearance and subsequent breaking of the GUT symmetry in string theory. Throughout this paper, we have interpreted the phenomenon of gauge coupling unification as signalling the emergence of a grand unified symmetry at the scale $M'_{\rm GUT}$. Strictly speaking, this terminology is field-theoretic, and implicitly assumes a conventional Higgs mechanism for breaking the GUT symmetry. Indeed, in the terminology appropriate for the Higgs mechanism, the GUT symmetry can be said to exist above the scale of unification, and to be broken below this scale. However, as is clear from the discussion in Sect. 4, in this paper we are actually imagining that the GUT symmetry is broken in a *string-theoretic manner*, namely through the orbifold that acts on the compactified dimensions.

This fact has a number of consequences. The most important of these concerns the extra GUT states that are part of the GUT symmetry but which are not present in the MSSM itself. These include, for example, the X and Y gauge bosons and the colored Higgs triplets; these are the states which we collectively denoted Ψ in Sect. 4. Of course, despite their odd symmetries under $y_i \rightarrow -y_i$, these states still continue to exist in the string spectrum, with masses $m \sim n/R$, $n \in \mathbb{Z}$, with $n \geq 1$. As we stressed in Sect. 4, this does not cause a problem for proton decay because these states do not couple to the chiral MSSM fermions. However, the presence of such states with masses $n\mu_0$ means that these states — and indeed their entire Kaluza-Klein towers — should actually be included in the evolution of the gauge couplings between μ_0 and $M'_{\rm GUT}$.

It may seem very counter-intuitive that we must include the effects of the X and Y bosons and colored Higgs triplets in the running of gauge couplings below the unification scale. However, this is precisely the consequence of breaking the GUT symmetry via an orbifold projection. Indeed, if we break a GUT symmetry in string theory via an orbifold associated with a radius scale $\mu_0 \equiv R^{-1}$, it makes no sense at all to think of the GUT symmetry as being "restored" above M'_{GUT} or broken below it. Rather, our GUT symmetry is actually "restored" at the scale μ_0 in the sense that the states appearing at masses $m \geq \mu_0$ fall into GUT multiplets. Thus, strictly speaking, the scale at which the broken GUT symmetry begins to leave particle remnants in the string spectrum is actually $\mu_0 \equiv R^{-1}$, not M'_{GUT}. These states thus appear as GUT "precursors".

It is easy to see that these precursor states do not upset gauge coupling unification. The zero-mode states consist, as before, of the MSSM spectrum. By themselves, these states are well-known to lead to gauge coupling unification. Starting at the first excited Kaluza-Klein level, however, all additional Kaluza-Klein states appear in complete GUT multiplets. As is well-known, this cannot disturb an already-present unification. Thus, even when we take into account the Kaluza-Klein towers corresponding to the X and Y bosons and Higgs triplets, we see that rapid gauge coupling unification is preserved. Note that this property holds regardless of the GUT group in question, whether a minimal $SU(5)$ or $SO(10)$, or a larger group such as E_6.

7.3 Interpreting the mass scales

Another issue that we face, upon embedding our scenario into string theory, is the origin and interpretation of the different mass scales. How, in particular, might such a small mass scale $\mu_0 = R^{-1}$ arise in string theory, and how can it be implemented when constructing a realistic string model? There are actually two classes of possibilities.

The more traditional class of possibilities is the perturbative one: we simply construct a weakly coupled heterotic string with a large radius of compactification. In such a scenario, the fundamental string scale is tied to the Planck scale, and remains at the perturbative heterotic string value 5×10^{17} GeV. This is essentially the approach taken in Ref. [6] (although the gauge couplings of the specific string models of Ref. [6] do not feel the extra dimensions, and consequently do not evolve with power-law behavior or experience any intermediate-scale gauge coupling unification).

However, if we now attempt to join this perturbative framework with our intermediate-scale grand unification, we cannot explain why the scale of gauge coupling unification should be so much lower than the Planck scale. Indeed, it is well-known [8] that weakly coupled heterotic strings naturally lead to gauge coupling unification near the Planck scale. Moreover, there also remains the question of the dynamical origin of this large scale.

Given our recent understanding of the strong-coupling dynamics of string theory,

however, several more attractive possibilities arise for interpreting the reduced unification scale. Specifically, we might attempt to interpret the unification scale as the *string scale itself*, so that we have a direct embedding into string theory immediately at the reduced unification scale M'_{GUT}! It may seem, at first, that this is impossible, for it is well-known that for weakly coupled heterotic strings, the tree-level string scale M_{string} is irrevocably tied to the Planck scale M_{Planck} through a relation of the form

$$M_{\mathrm{string}} \sim g_{\mathrm{string}} M_{\mathrm{Planck}} \tag{7.1}$$

where g_{string} is the string coupling at unification. This relation holds regardless of the dimensionality of the spacetime (*i.e.*, regardless of how many of the ten spacetime dimensions are compactified), and regardless of their volume (radii) of compactification. For heterotic strings at *strong* coupling, however, this behavior changes. Specifically, it turns out that various (closed) heterotic strings at strong coupling can be equivalently described as (open) Type I strings at weak coupling. Thus, many non-perturbative features of heterotic string theory can be studied by analyzing weakly coupled Type I strings. Remarkably, Type I string theory offers the interesting possibility [10] of lowering the fundamental string scale, for (7.1) no longer continues to apply. Instead, we find the relation

$$M_{\mathrm{string}} \sim e^{\phi/2} g_{\mathrm{gauge}} M_{\mathrm{Planck}} \tag{7.2}$$

where ϕ represents the so-called ten-dimensional dilaton field and where g_{gauge} is the Type I gauge coupling. Thus, simply by adjusting the VEV of the ten-dimensional dilaton, one can lower M_{string} relative to M_{Planck}. Of course, the value of the dilaton indirectly affects the values of the gauge and gravitational couplings, so that g_{gauge} and M_{Planck} themselves change their apparent values when the dilaton is changed. However, using other string relations it is possible to eliminate this dependence algebraically, and relate M_{string} and M_{Planck} directly to each other without exhibiting the dependence on the dilaton. We then find the general relation

$$M_{\mathrm{string}} \sim \sqrt{\frac{1}{\alpha_{\mathrm{gauge}} M_{\mathrm{Planck}}}} \, V^{-1/4} \tag{7.3}$$

where $\alpha_{\mathrm{gauge}} \equiv g_{\mathrm{gauge}}^2/(4\pi)$ and where $(2\pi)^6 V$ is the (normalized) six-dimensional volume of compactification. In writing (7.3), we have ignored (and will continue to ignore) all numerical factors of order one, since our goal will merely be to obtain order-of-magnitude estimates. The relation (7.3) thus represents the non-perturbative counterpart of (7.1).

In order to embed our grand unification scenario into Type I string theory, we shall attempt to identify α_{gauge} with α'_{GUT} and M_{string} with M'_{GUT}. Let us assume, as we have done throughout, that there are δ extra dimensions of radius $R \equiv \mu_0^{-1}$. As usual, these are the dimensions that cause the gauge and Yukawa couplings to experience power-law corrections. Given the relation (7.3), we can now easily determine the

required common radius r for the remaining $6 - \delta$ compactified dimensions. Writing the normalized compactification volume as

$$V \sim R^\delta \, r^{6-\delta} \, , \tag{7.4}$$

we find

$$\frac{M'_{\rm GUT}}{M_{\rm Planck}} \sim \alpha'_{\rm GUT} \, (M'_{\rm GUT} R)^{\delta/2} \, (M'_{\rm GUT} r)^{3-\delta/2} \, . \tag{7.5}$$

Let us begin by considering the case $\delta = 1$. Taking $M'_{\rm GUT} = 10$ TeV, we see from Figs. 3 and 4 that $M'_{\rm GUT} R \approx 20$ and $\alpha'_{\rm GUT} \approx 1/50$. This in turn implies that $M'_{\rm GUT} r \approx 10^{-6}$, which implies that the radius r of the five extra dimensions must be *smaller* than the string length scale! Of course, this is not an inconsistency, but rather a signal that we should pass to a slightly different description (the so-called Type I$'$ description) of the physics. Technically, this procedure of passing from one description to the other is called a T-duality, and in general a Type I theory with a compactified radius r is the T-dual of an equivalent Type I$'$ theory with a compactified radius $r' \equiv (M^2_{\rm string} r)^{-1}$:

$$T\text{-duality:} \qquad M_{\rm string} r \; \leftrightarrow \; (M_{\rm string} r')^{-1} \, . \tag{7.6}$$

We therefore pass to a Type I$'$ description by T-dualizing our five extra dimensions. We then find

$$(r')^{-1} \sim 10^{-6} \, M'_{\rm GUT} \sim 10 \text{ MeV} \, . \tag{7.7}$$

Thus, to summarize our results, we see that we can naturally associate a 10-TeV scale of gauge coupling unification with the string scale of a Type I$'$ theory in which one dimension has radius $R^{-1} \approx 0.5$ TeV and the five remaining dimensions have radii $r' \sim (10 \text{ MeV})^{-1}$. The resulting scenario is sketched in Fig. 12. Note that extra dimensions of this size are not ruled out experimentally, provided that the gauge couplings do not feel their effects [11, 12]. Likewise, a similar calculation for the $\delta = 2$ case yields the result $r' \sim (0.1 \text{ GeV})^{-1}$ for the remaining four dimensions. We note, however, that these results are dependent on the chosen unification scale (string scale) in our scenario. For example, taking $M'_{\rm GUT} = 10^{12}$ GeV instead yields essentially the same result $r' \sim (10^9 \text{ GeV})^{-1}$ for both the $\delta = 1$ and $\delta = 2$ cases.

7.4 D-brane configurations for our scenario

Given these results, it is natural to interpret our scenario in terms of various configurations of the D-branes of the Type I$'$ string theory. To see how this can be done, let us begin by recalling that the original supersymmetric Type I theory contains both nine-branes and five-branes. For the purposes of this discussion, we shall label our ten spacetime coordinates as $\{x_1, ..., x_{10}\}$, with $\{x_1, x_2, x_3, x_4\}$ corresponding to our observed four-dimensional world. We shall concentrate on the $\delta = 1$ case, so that our gauge couplings feel a single extra large dimension with radius R
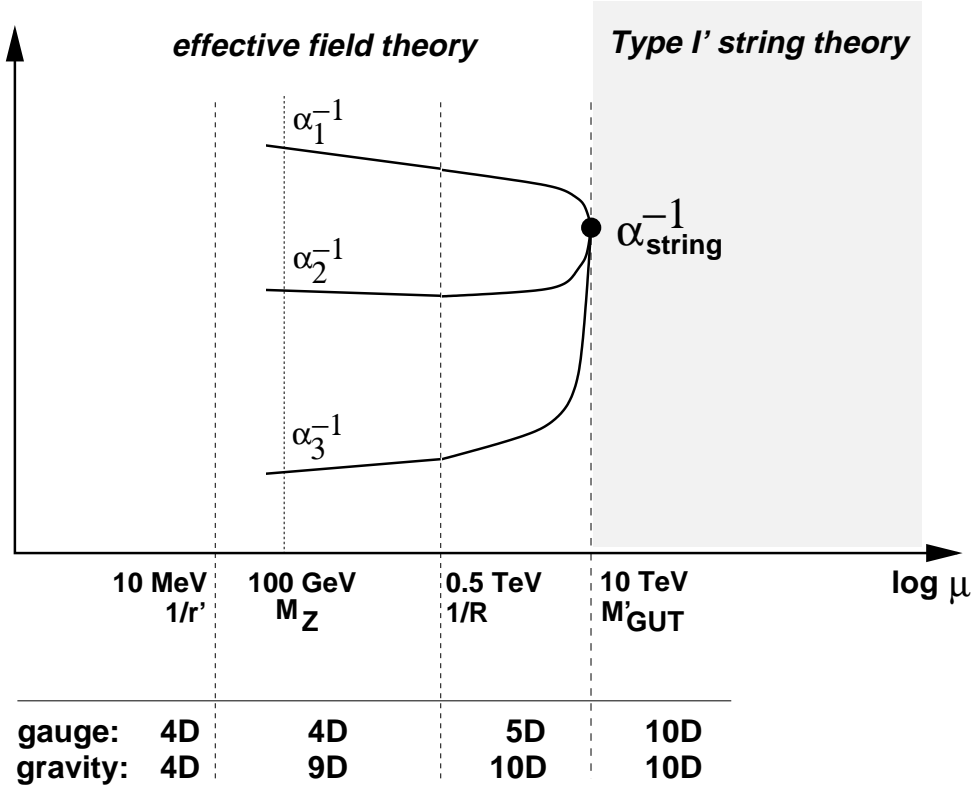
Figure 12: Sketch of the evolution of the gauge couplings within a Type I$'$ realization of our scenario. Below the unification scale, the physics is described by an effective field theory, while the full Type I$'$ string description is appropriate above this scale. Thus, in this realization, the string scale coincides with the new unification scale $M'_{\mathrm{GUT}} \approx 10$ TeV. This is achieved by having the gauge couplings feel a new dimension at $R^{-1} \equiv \mu_0 \approx 0.5$ TeV, while the gravitational coupling feels five new dimensions at $(r')^{-1} \approx 10$ MeV. We have also indicated the corresponding effective number of spacetime dimensions that are felt by the gauge and gravitational couplings at different energy scales.

corresponding to x_5, while the remaining five dimensions $\{x_6, ..., x_{10}\}$ have radius r. The nine-branes of our theory necessarily obey Neumann boundary conditions for all ten dimensions $\{x_1, ..., x_{10}\}$, but the five-branes will (by definition) obey Neumann boundary conditions in only six dimensions and obey Dirichlet boundary conditions in the remaining four dimensions. The choice of which set of dimensions obeys which set of boundary conditions amounts to a choice of the orientation of the five-brane relative to the observed four-dimensional spacetime and the extra large dimension. For reasons that will become clear shortly, we shall take our five-brane to obey Neumann boundary conditions in the $\{x_1, ..., x_4, x_6, x_7\}$ directions, and to obey Dirichlet boundary conditions in the $\{x_5, x_8, x_9, x_{10}\}$ directions.

As we have discussed, the fact that the five dimensions $\{x_6, ..., x_{10}\}$ are compactified with a radius exceeding the Type I string scale implies that we must T-

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dualize these five dimensions. Operationally, this amounts to interchanging Neumann and Dirichlet boundary conditions in the dualized directions. We therefore obtain a Type I' theory with the following branes. First, the nine-brane of the original Type I theory becomes a *four*-brane with Neumann boundary conditions in the directions $\{x_1, ..., x_4, x_5\}$ and Dirichlet boundary conditions in the directions $\{x_6, ..., x_{10}\}$. By contrast, the five-brane of the original Type I theory becomes a *six*-brane, with Neumann boundary conditions in directions $\{x_1, ..., x_4, x_8, x_9, x_{10}\}$ and Dirichlet boundary conditions in directions $\{x_5, x_6, x_7\}$.

How can we interpret this picture? Fortunately, this Type I' brane configuration contains exactly what we require for our scenario. The four-brane, which extends outwards in the directions $\{x_1, ..., x_4, x_5\}$, can be interpreted as the spacetime of the five-dimensional world that the gauge couplings feel at energy scales above $R^{-1} \equiv \mu_0$: the four directions $\{x_1, ..., x_4\}$ are taken to be completely flat, and our fifth dimension $\{x_5\}$ is compactified with radius R. Thus, the MSSM Higgs fields and gauge bosons can be interpreted as living on the four-branes. The six-brane, by contrast, gives rise to a separate *non-perturbative* gauge symmetry whose properties will not concern us here. However, the most important feature is the presence of a non-trivial *intersection* between the four-brane and the six-brane. Note that the joint Neumann directions of this "intersection brane" are only $\{x_1, ..., x_4\}$. Thus, particles localized on this three-brane feel only *four* spacetime directions, regardless of the size of the radii R and r. Such states are typically said to arise in the "46-sector". These states can therefore easily be interpreted as the chiral MSSM fermions, which are required not to have Kaluza-Klein excitations in our minimal scenario! Note that it is crucial that our brane configuration give rise to such a 46-sector, for we must have a way of localizing the chiral MSSM fermions within the context of open-string theories so that they do not feel the extra dimensions. This, then, explains our original choice of the orientation of the Type I five-brane. The resulting brane configuration is illustrated in Fig. 13.

It is straightforward to generalize this brane configuration to the $\delta = 2$ case. In this case we shall take $\{x_5, x_6\}$ as the extra large dimensions of radius R. In the original Type I theory, we shall orient our five-brane so that it satisfies Neumann boundary conditions in the $\{x_1, ..., x_4, x_7, x_8\}$ directions and Dirichlet boundary conditions in the $\{x_5, x_6, x_9, x_{10}\}$ directions. Upon T-dualizing the $\{x_7, x_8, x_9, x_{10}\}$ directions, we then obtain two distinct five-branes. The five-brane that we obtain from the Type I nine-brane satisfies Neumann boundary conditions in the $\{x_1, ..., x_4, x_5, x_6\}$ directions: this is the spacetime that the MSSM Higgs and gauge fields experience at energy scales exceeding $\mu_0 \equiv R^{-1}$. The second five-brane, by contrast, has Neumann boundary conditions in the $\{x_1, ..., x_4, x_9, x_{10}\}$ directions. The intersection of these two different five-branes (*i.e.*, the 55'-sector) once again provides us with an effective "three-brane" which can be associated with our observed four-dimensional world below energy scales $\mu_0 \equiv R^{-1}$. Thus, it is this 55'-sector which is presumed to contain our chiral MSSM fermions.
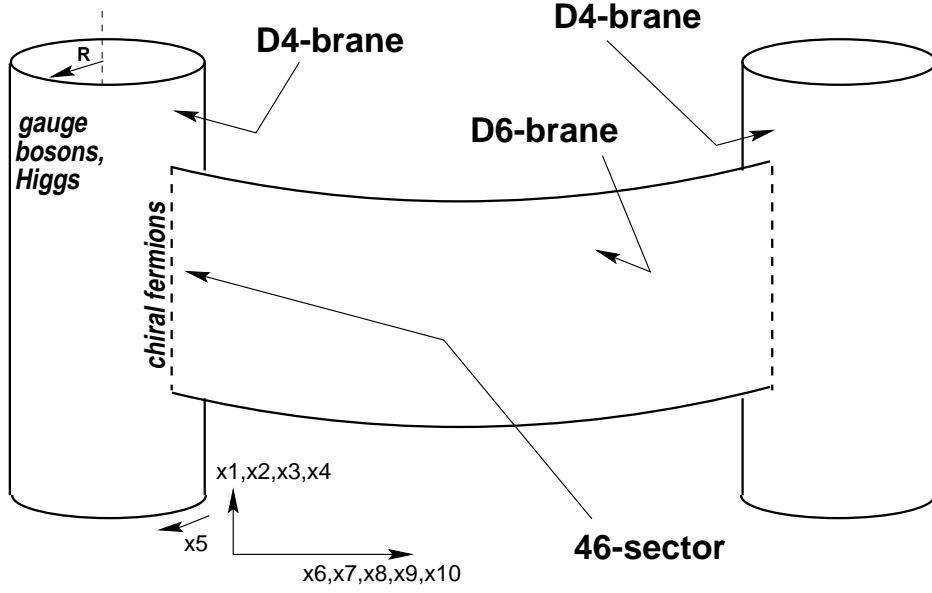
Figure 13: A D-brane configuration which can accommodate our scenario within the context of Type I' string theory. The observed flat four-dimensional world corresponds to the "46-sector" (*i.e.*, the three-brane intersection between the (cylindrical) four-branes and the six-branes). The extra (compactified) direction of the four-branes corresponds to our extra dimension of radius $R \equiv \mu_0^{-1}$. The MSSM Higgs and gauge fields are presumed to lie fully on the four-branes, while in our minimal scenario the chiral MSSM fermions are restricted to the 46-sector and hence have no Kaluza-Klein excitations.

Thus, we see that our intermediate-scale grand unification scenario has a variety of natural interpretations and realizations within string theory, and generalizations to higher values of η are obvious. Hence we conclude that is indeed possible to non-perturbatively lower the string scale in such a way that it directly coincides with our new gauge coupling unification scale. This opens up the exciting possibility that our scenario can be embedded directly into a string theory beyond the scale M'_{GUT}. This embedding into a finite theory such as string theory would then be the "cure" for the non-renormalizability of our effective higher-dimensional field theory between the scales μ_0 and M'_{GUT}.

Of course, the construction of a fully realistic string model which gives rise to these brane configurations is a far more complicated task. Preliminary steps towards such model-building have been taken in Ref. [13]. Therefore, in order to illustrate our scenario as well as the origin of what we have called the "grand unified group" in Sect. 4, let us briefly consider one of the explicit four-dimensional $N = 1$ supersymmetric Type I models constructed in Ref. [13]. This model incorporates a $\mathbb{Z}_3 \times \mathbb{Z}_2$ orbifold defined by the actions g_3 and g_2, where

$$\begin{aligned} g_3(z_1, z_2, z_3) &= (\omega z_1, \omega z_2, \omega z_3) \\ g_2(z_1, z_2, z_3) &= (-z_1, -z_2, z_3) \,. \end{aligned} \tag{7.8}$$

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Here $\omega \equiv e^{2i\pi/3}$, and (z_1, z_2, z_3) are three complex coordinates of the three complex planes formed from the real coordinates $\{x_5, ..., x_{10}\}$ of the six-dimensional compact space. This model contains a background B-field of rank 2 that arises from the NS-NS sector, and contains both nine-branes and five-branes. These are taken to be at the same fixed point. The gauge group from the open-string sector is $[U(2) \times U(2) \times U(4)]^2$, with different massless matter representations coming from the 99, 55, and 59 sectors [13]. The form of the orbifold action (7.8) shows that at the massive level, the first two complex planes (z_1, z_2) give rise to an $N = 4$ supersymmetric spectrum, and the third complex plane (corresponding to z_3) has an $N = 2$ supersymmetric spectrum that can provide power-law corrections to the gauge and Yukawa couplings. Based on the considerations discussed above, we take the compact coordinates corresponding to (z_1, z_2) to be very large (and therefore must T-dualize them, exchanging nine-branes and five-branes in the process). We likewise take the inverse radius of the z_3 plane to be a factor of ten to twenty below the string scale. Close to the string scale, we obtain a unified gauge group $U(8)^2$, one of whose factors to be interpreted as the observable unified group. This gauge group is obtained by "undoing" the $\mathbb{Z}_3$ orbifold action for the massive Kaluza-Klein levels. The matter representations at the massive levels form $N = 2$ supersymmetric representations which are obtained by simply decomposing the massive $U(8)$ representations with respect to a $U(2) \times U(2) \times U(4)$ Pati-Salam subgroup. Given the spectrum arising from the 99 and 95 sectors, the power-law corrections to the gauge couplings as they evolve towards lower energy scales can then be computed as explained in Sect. 3 or Appendix A.

8   Explaining the fermion mass hierarchy via branes

Having explained in the previous section how our intermediate-scale grand unification scenario can be realized within the context of Type I string theory and its associated branes, we now revisit the fermion mass hierarchy problem. In this section, we point out that these sorts of brane configurations can actually provide several entirely new methods of addressing the fermion mass hierarchy problem beyond those considered in Sect. 5. We shall give two examples.

8.1   A scenario with $\eta \neq 0$

Let us begin by considering a situation in which we have two large extra spacetime dimensions $\{x_5, x_6\}$ of radius R. In our original Type I theory, we shall consider our nine-brane along with *two* separate five-branes. The first five-brane will be taken to have Neumann boundary conditions in the directions $\{x_1, ..., x_4, x_5, x_7\}$, while the second will have Neumann boundary conditions in the directions $\{x_1, ..., x_4, x_7, x_8\}$. All unspecified directions will be assumed to satisfy Dirichlet boundary conditions. As before, we must T-dualize the directions $\{x_7, ..., x_{10}\}$. This yields a Type I$'$ theory with the following branes. The Type I nine-brane becomes a Type I five-brane

with Neumann coordinates $\{x_1, ..., x_4, x_5, x_6\}$; the first Type I five-brane becomes a seven-brane with Neumann coordinates $\{x_1, ..., x_4, x_5, x_8, x_9, x_{10}\}$; and the second Type I five-brane becomes a second Type I' five-brane with Neumann coordinates $\{x_1, ..., x_4, x_9, x_{10}\}$. It is the first Type I' five-brane which is to be interpreted as our physical spacetime at energy scales above R^{-1}.

Given this brane configuration, let us now consider the possible locations for the chiral fermion generations of the MSSM. One possibility is that the chiral fermions lie directly in the first Type I' five-brane, so that they have a complete tower of *two dimensions' worth* of Kaluza-Klein excitations. These are the so-called 55 fermions, which we shall collectively denote Ψ_2 (because they feel two dimensions' worth of Kaluza-Klein excitations). A second possibility is that the chiral MSSM fermions arise from the 57-sector. It is clear, given the above configuration, that the fermions arising in the 57-sector feel only *one dimension worth* of Kaluza-Klein excitations. We shall refer to these collectively fermions as Ψ_1. Finally, the third possibility is that the chiral MSSM fermions arise in the 55'-sector. Thus, these fermions have no Kaluza-Klein excitations at all, and will be denoted Ψ_0.

Given these results, and given the power-law dependence that the corresponding Yukawa couplings experience due to the extra dimensions (as discussed in Sect. 5), it is then natural to explain the fermion mass hierarchy by associating one chiral MSSM generation with each of the above sectors. This is therefore an $\eta = 2$ scenario (since two of the chiral MSSM generations have Kaluza-Klein excitations). As we discussed in Sect. 5, in the absence of any additional couplings for the Higgs fields, the dominant contributions to the evolution of the Yukawa couplings come from the diagrams in Figs. 6(a,b). In the case of Fig. 6(b), for any external fermion of type Ψ_i, the internal fermion must also be of type Ψ_i. However, in the case of Fig. 6(a), regardless of the type of the external fermion, the internal fermion can *a priori* be of types $\Psi_{0,1,2}$. Each fermion carries with it an appropriate number of dimensions' worth of Kaluza-Klein modes. The only constraints that govern the counting of modes for each loop are Kaluza-Klein momentum conservation at the vertices: in general, at any vertex, we must impose Kaluza-Klein momentum conservation in the directions for which translational invariance is not broken. In this way, we can generate a whole variety of powers $(\Lambda/\mu_0)^p$, $p = 0, 1, 2, 3, 4$ for the different diagrams. Thus, depending on the fermion Ψ_i in question, the corresponding wavefunction renormalization factors Z_F and $Z_{\overline{F}}$ can feel a different number power-law dependence, and this feature can be used to explain the fermion mass hierarchy.

8.2 A scenario with $\eta = 0$

Clearly, the above scenario relies on the different fermion generations feeling different numbers of spacetime dimensions and therefore having different numbers of Kaluza-Klein excitations. Thus, in the language of Sects. 2 and 3, this is an $\eta \neq 0$ scenario, where η is the number of MSSM fermion generations that feel the extra

dimensions. However, as we stressed in Sect. 4, the minimal scenario with $\eta = 0$ is actually preferable for the purposes of proton decay. It would therefore be interesting to see if there exist brane configurations which have $\eta = 0$ but which nevertheless also lead to fermion mass hierarchies.
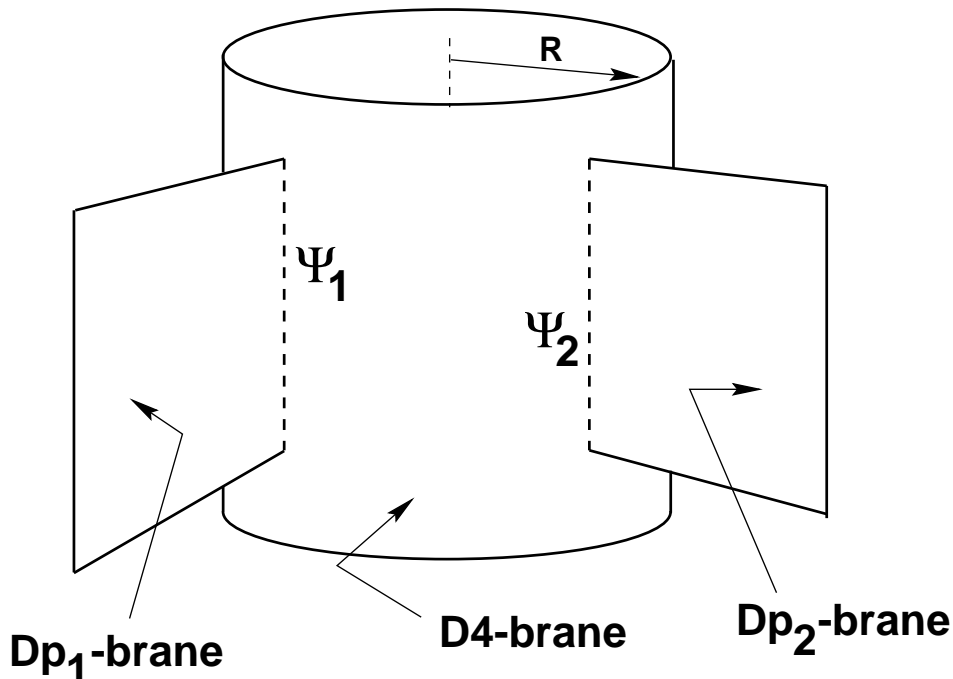

Figure 14: A D-brane configuration which leads to a natural fermion mass hierarchy for the "minimal" scenario with $\eta = 0$ (no Kaluza-Klein excitations for chiral MSSM fermions).

To do this, let us consider the $\delta = 1$ case, and imagine a set of Type I$'$ branes: a four-brane (interpreted as the observable universe at energy scales above R^{-1}); and a variety of other branes of differing dimensions p_i. Let us assume that each of these other branes has a four-dimensional intersection with the four-brane, and let us place our different chiral MSSM fermions at these different intersections, so that they arise in the $4p_i$-sectors. Such a situation is sketched in the case of two additional branes in Fig. 14. Because the fermions all arise in the $4p_i$ sectors, none of them feel extra dimensions directly or have Kaluza-Klein excitations. These are therefore minimal $\eta = 0$ scenarios, as desired. However, because each fermion ψ_i arises in the $4p_i$-sector and is restricted on the intersection of the four-brane with the p_i-brane, in principle it can carry the quantum numbers of not only the perturbative gauge group arising from the four-brane, but also the quantum numbers of the *non*-perturbative gauge group G_i corresponding to the p_i-brane. Note that these extra quantum numbers do not affect the gauge coupling unification that we have already observed in Sect. 3. However, because these fermions have additional quantum numbers in different non-perturbative gauge groups corresponding to different branes of different

dimensionalities, their corresponding wavefunction renormalization factors Z_F and $Z_{\overline{F}}$ will accrue different power-law exponents from the analogues of Fig. 6(b) where we replace the internal Standard Model gauge bosons with the gauge bosons of the non-perturbative gauge group G_i. These different power-law exponents arise because these different non-perturbative gauge bosons each feel a different effective number of dimensions. Thus, in such scenarios, we are able to achieve natural fermion mass hierarchies even while maintaining our "minimal" scenario with $\eta = 0$.

9 Relations to other work

Extra spacetime dimensions and their effects have been studied in the literature from a variety of different perspectives. For completeness, we shall briefly highlight the novel features of our approach and compare it with some others that have been taken.

At the field-theory level, we have seen that extra spacetime dimensions are equivalent to the introduction of infinite towers of Kaluza-Klein states. It may therefore seem that our work is somehow equivalent to prior work in which the effects of possible extra matter beyond the MSSM are analyzed. However, the extra matter that is typically considered in such analyses is vector-like and fills out complete $SU(5)$ or $SO(10)$ multiplets. This then leads to non-perturbative (or at best semi-perturbative) couplings at unification [14], and does not permit a lowering of the unification scale. In our "minimal" scenario, by contrast, we have a unification which remains weakly coupled, even more so than within the MSSM itself; moreover, our unification scale is lowered rather than raised. These differences essentially arise because the prior approaches all entail shifting the one-loop beta-function coefficients b_i by a common fixed finite amount Δb:

$$b_i \;\rightarrow\; b_i' \;\equiv\; b_i \;+\; \Delta b \qquad \text{for all } i \;. \tag{9.1}$$

In our scenario, by contrast, the b_i are not shifted but rather *rescaled*, for at each equally-spaced Kaluza-Klein threshold we are essentially introducing another copy of the MSSM gauge-boson and Higgs representations. Thus, our scenario can essentially be re-interpreted in this language as one in which we continue to have logarithmic running, but with an effective beta-function coefficient that changes with the energy scale μ according to:

$$b_i \;\rightarrow\; b_i'(\mu) \;\equiv\; (b_i - \tilde{b}_i) \;+\; \tilde{b}_i \, X_\delta \left(\frac{\mu}{\mu_0} \right)^\delta \;. \tag{9.2}$$

Hence the b_i' in (9.2) are not all driven in the same direction towards positive values, as they are in (9.1). It is this feature that enables us to achieve rapid gauge coupling unification at low unification scales without encountering large gauge couplings. Moreover, the rescaling factor in (9.2) is itself μ-dependent. This enables us

to generate the *power-law* evolution for the gauge couplings which is the hallmark of our scenario. It is also this feature, for example, which enables us to address the fermion mass hierarchy problem.

Extra spacetime dimensions and Kaluza-Klein towers of states have also been analyzed within the context of string theory. For example, the consequences of extra large (TeV-scale) dimensions have been previously examined in a notable series of papers [6]. However, this analysis was carried out within the context of string models that were deliberately constructed in such a way that the effects of the extra spacetime dimensions were shielded from the evolution of the gauge couplings. In other words, the Kaluza-Klein towers of states were arranged to arise in only certain $N = 4$ supersymmetric sectors of the underlying string model so that they had no effect on the running of the gauge couplings, giving rise to $\tilde{b}_i = 0$. Thus, in these restricted scenarios, the gauge couplings can run only in their usual logarithmic fashion, and unify only at the usual GUT scale $M_{\mathrm{GUT}} \approx 2 \times 10^{16}$ GeV. By contrast, it is precisely the effects of these extra dimensions on the gauge and Yukawa couplings which have been our main focus in this paper — rather than avoid these effects, we have exploited them! Moreover, such effects can be expected to be the generic case in string theory (the existence of certain specially constructed string models notwithstanding).

More recently, extra dimensions have also played a role in understanding the strong-coupling behavior of various string theories. The most famous example of this phenomenon is the ten-dimensional $E_8 \times E_8$ heterotic string: at strong coupling it has been proposed [15] that this string "grows" an eleventh dimension of finite length whose natural size is much larger than the eleven-dimensional Planck length, and in particular is much larger than the presumed size of the six-dimensional manifold on which a subsequent compactification to four dimensions takes place. This leads to a scenario in which our four-dimensional low-energy world should successively look five-dimensional and then ultimately eleven-dimensional as the energy scale is increased. However, the fundamental distinction between this scenario and our own is the effect of this fifth dimension. In our scenario, the extra dimension(s) are universal, affecting both gauge and gravitational couplings. In the $E_8 \times E_8$ case, by contrast, the extra fifth dimension is felt only by the gravitational couplings, and the gauge couplings are again immune to its effects.

Of course, our analysis should be directly applicable to string theories which have generic, large-radius compactifications. Such theories have recently been discussed in a number of theoretical and phenomenological contexts [16].

Finally, there recently appeared a proposal [11, 12] for solving the *gauge* hierarchy problem through the appearance of new *millimeter-scale* extra dimensions! These new dimensions are presumed to affect the gravitational interaction only, and have no effect on the gauge couplings. This proposal is therefore, in some sense, the gravitational counterpart of our proposal, effectively reducing the *Planck* scale. Of course, this proposal differs from ours in essentially the same way that previous proposals have differed: it does not address gauge coupling unification; the Standard

Model particles, unlike the graviton, are presumed to be essentially trapped on a four-dimensional submanifold relative to these extra dimensions; and the Standard Model particles are therefore once again largely immune to the effects of these extra dimensions. Nevertheless, it would be very interesting to combine our scenario (in which the *gauge* part of observable low-energy world feels new extra dimensions as large as a TeV) with the scenario of Ref. [11] (in which the *gravitational* part of the observable low-energy world feels extra dimensions as large as a millimeter). Such a synthesis could proceed along the lines sketched in Sect. 7, and might well lead to a unified picture of gauge *and gravitational* unification, all occurring at around a TeV. Preliminary steps in realizing this possibility within the context of Type I string theory have already been taken in Ref. [13]. Earlier discussions of such "TeV-scale superstrings" can also be found in Ref. [17]. Likewise, recent advances in understanding "the universe as brane" [18] are likely to prove crucial in developing these scenarios at both the string-theoretic and field-theoretic levels. Note that the gauge hierarchy problem has also been addressed within the context of a higher-dimensional field theory in Ref. [19].

10   Collider signals and cosmological implications

As might be expected, the appearance of extra large spacetime dimensions can give rise to many interesting signals for collider experiments (see, *e.g.*, Refs. [20, 21]). They can also have profound implications for cosmology [22]. In this section we will give a short sketch of some of these connections.

If the scale $\mu_0 \equiv R^{-1}$ of the new dimensions is close to the electroweak scale, then future colliders will be able to probe the new dimensions directly. Let us first consider our "minimal" scenario with $\eta = 0$. In this scenario, only the non-chiral MSSM states will have an infinite tower of Kaluza-Klein excitations, and these will be separated by an energy scale μ_0. The importance of such Kaluza-Klein excitations for collider phenomenology or cosmology depends crucially on their transformation properties under the $\mathbb{Z}_2$ orbifold action $y_i \rightarrow -y_i$, where y_i are the coordinates of the new compactified dimensions. Let us first consider the Kaluza-Klein states whose wavefunctions are even with respect to this action. Such states can directly couple to the Standard-Model fermions because these fermions are located either at the orbifold fixed points or on effective "three-branes"; in either case we cannot impose Kaluza-Klein momentum conservation at the vertices because translational invariance in the compactified directions is broken. Therefore, such even Kaluza-Klein states can be directly produced at future colliders if $\mu_0 \simeq \mathcal{O}(\mathrm{TeV})$, with the lowest-lying even Kaluza-Klein mode directly decaying into (s)fermions. Note, on the other hand, that because the gauge bosons and Higgs particles feel the extra dimensions, Kaluza-Klein momentum conservation continues to apply at their vertices. This feature then prevents the lowest-lying even Kaluza-Klein modes from directly decaying into the zero-mode gauge bosons and/or Higgs particles. Of course, the Kaluza-Klein modes

corresponding to the gauge bosons will be more important than the Kaluza-Klein modes corresponding to the Higgs fields because the Higgs-field couplings are usually suppressed by small Yukawa factors.

Given these observations, we see that Kaluza-Klein states corresponding to the gauge bosons might be observable via Drell-Yan production in proton/(anti-)proton collisions; one would seek to identify charged leptons in the final state. The analysis for the lowest-lying neutral gauge boson Kaluza-Klein state is exactly analogous to that for Z' bosons in E_6 superstring-inspired models [23, 21]. Typically, the branching ratio into fermion pairs is reduced by the presence of other supersymmetric channels [24]. Thus, bounds on μ_0 tend to be model-dependent. However, using the results in Ref. [21], we find that recent Fermilab data suggests the simple estimate $\mu_0 \equiv R^{-1} \gtrsim 500$ GeV for the lower bound on the scale of the extra dimensions.

Alternatively, if the scale of extra dimensions is much larger than $\mathcal{O}(\text{TeV})$, the effects of the infinite Kaluza-Klein tower can be seen at low energies via an effective contact interaction. Such an effective contact interaction can arise, for example, from the tree-level exchange of massive gauge-boson Kaluza-Klein modes. Let us assume, for simplicity, that only one extra dimension exists. Then the amplitude for the scattering process $l^+ l^- \to l^+ l^-$ receives a contribution

$$g^2 R^2 \sum_{n=1}^{\infty} \frac{1}{q^2 R^2 + n^2} \tag{10.1}$$

from the infinite tower of even Kaluza-Klein states. At low energy scales $q^2 \ll \mu_0^2$, we find that the above expression is approximately $(\pi^2/6) g^2 R^2$. This then gives rise to an effective four-fermion contact interaction of the form

$$\mathcal{L} \approx \frac{\pi^2}{6} g^2 R^2 (\bar{\Psi} \gamma_\mu \Psi)^2 \tag{10.2}$$

where g is a gauge coupling and where Ψ schematically denotes either quarks or leptons. Recent bounds on four-fermion contact interactions then imply a lower bound $\mu_0 \gtrsim 300$ GeV.

On the other hand, Kaluza-Klein states that are odd under the $\mathbb{Z}_2$ orbifold action do not couple to chiral fermions. Consequently, they can be probed at collider experiments only via higher-loop processes. This implies that there are no significant bounds from collider phenomenology arising from these states. However, since the lowest-lying odd Kaluza-Klein states are stable due to Kaluza-Klein momentum conservation, such states can have important cosmological implications. For example, if the annihilation processes are sufficiently strong [25], such states might serve as ideal dark-matter candidates.

Our "non-minimal" scenarios with $\eta > 0$ (*i.e.*, with Kaluza-Klein excitations for chiral MSSM fermions) can also lead to interesting collider phenomenology, provided that proton decay is not a problem. Proton decay was discussed in Sect. 4. Massive Kaluza-Klein states for the chiral fermions will appear as heavier versions of the

usual zero-mode fermions. If there is at least one generation of chiral fermions with an infinite tower of Kaluza-Klein excitations, then Kaluza-Klein momentum conservation will prevent the first-excited gauge-boson Kaluza-Klein state from decaying into the low-energy fermions associated with the MSSM generation that experiences the extra dimensions. This would therefore be a unique experimental signature of the fact that not all fermion generations have Kaluza-Klein towers (*i.e.*, that $\eta < 3$). Thus, by probing such signatures, one has the possibility of *experimentally* choosing between viable TeV-scale string models or grand unification scenarios! However, if all three fermion generations experience the extra dimensions, then the first-excited even gauge-boson Kaluza-Klein states are now stable and can also serve as suitable dark-matter candidates.

One might worry about the fact that in a more general scenario in which gravity experiences extra dimensions, the thermal regeneration of unstable gravitinos could cause a problem during nucleosynthesis [26]. However, there may be solutions to this difficulty when the effects of extra dimensions are taken into account in analyzing the dynamics of the early universe [22]. *A priori*, there are many effects that come into play in the context of a higher-dimensional cosmology. In addition to the issue surrounding higher-dimensional inflation, additional issues include the effects of extra dimensions on adiabatic density perturbations, on topological defects, and on cosmological phase transitions. Indeed, being slightly bolder, one might even imagine developing a possible explanation for the dimensionality of spacetime (*i.e.*, the number of large dimensions) along the lines of the approach followed in Ref. [27]. All of these issues can be expected to have a profound effect on our understanding of dynamics of the early universe, and are worthy of further study.

11 Conclusions and future prospects

In this paper, we have proposed a new framework in which the physics of conventional grand unification might be brought down to accessible energy scales, perhaps even as low as a TeV. Our fundamental idea involves the appearance of extra large spacetime dimensions. The appearance of extra spacetime dimensions is a natural feature in string theory, and their radii are generally unfixed by string dynamics. Therefore, by postulating the appearance of relatively large extra dimensions, we have shown that the physics of conventional grand unification can be addressed in an entirely new context. Specifically, we have shown that gauge coupling unification is preserved by extra dimensions, and moreover that the unification scale is significantly lowered. This leads to the exciting possibility of intermediate-scale grand unification. We found that proton decay can also be avoided — even with the smaller unification scale — thanks to various new symmetry properties pertaining to the extra spacetime dimensions. Furthermore, we showed that extra dimensions also provide a natural setting in which to address the fermion mass hierarchy problem, for such extra dimensions tend to significantly amplify the effects of relatively small flavor-dependent

couplings. It is also possible to consider the effects of extra spacetime dimensions on the running of the soft SUSY-breaking masses.

Although we primarily applied our scenario to the Minimal Supersymmetric Standard Model (MSSM), the effects of extra dimensions are completely general and can just as easily be applied in a number of other different contexts. To illustrate this, we also considered the role of extra dimensions in the *non-supersymmetric* Standard Model, and found that once again they can lead to gauge coupling unification at very low energy scales. This would then be a non-supersymmetric "solution" to the *gauge* hierarchy problem. We also considered the embedding of our scenario into string theory, and found that there exist several very natural string and D-brane settings in which our scenario can be realized. Moreover, within this context, we also proposed a new method for addressing the fermion mass hierarchy which can be realized in non-perturbative open-string theories and which does not require the *ad hoc* introduction of low-energy flavor-dependent couplings.

Overall, however, we stress that the most exciting aspect of this approach to grand unification is that it permits the predictions of GUT physics (and indeed even of string theory itself) to be brought down to accessible energy scales! Thus, if this framework is correct, we can expect to witness strong and unmistakable signals in the next round of accelerator experiments. Such signals were discussed in Sect. 10, and in fact experimental evidence for extra large spacetime dimensions has "already" been found [28] in the year 2011. Moreover, our scenario should also have important and dramatic implications for cosmology. Taken together, therefore, our results suggest an entirely new approach towards probing — both theoretically and experimentally — the physics of grand unification as well as the phenomenology of large-radius string compactifications.

Acknowledgments

Appendix A

In this Appendix, we shall discuss the precise relation between extra spacetime dimensions and Kaluza-Klein modes. Specifically, we shall exactly calculate the effects of the infinite towers of Kaluza-Klein states on the "running" of the gauge couplings. This will also enable us to determine the extent to which the results of such a calculation can be approximated by the expression given in (3.8).

Before beginning our calculation, let us summarize the main idea. As discussed in Sect. 3, the result (3.8) can be easily obtained by treating the appropriate subset of the MSSM as effectively being in D flat spacetime dimensions, where $D = 4 + \delta$.

Of course, geometrically speaking, a spacetime consisting of four flat dimensions and δ circles of fixed radius $R = \mu_0^{-1}$ is *never* equivalent to a flat $(4 + \delta)$-dimensional spacetime. However, we expect that as the energy scale μ increases, the effective length scale decreases, and consequently the fixed radius R "appears" to become large. Thus, as we shall see, there are essentially two equivalent pictures that can be used to describe the same physics.

Our procedure will be to adopt the strict four-dimensional point of view, and to evaluate the vacuum polarization diagram shown in Fig. 15 where we include the effects of the MSSM particles as well as the appropriate Kaluza-Klein excitations in the loops. Note that parts of our calculation are similar to a calculation in Ref. [29]. For simplicity, we shall begin by performing our calculation in the case of a single Dirac fermion and its corresponding Kaluza-Klein excitations. Since the effects of these Kaluza-Klein excitations will essentially be the same for each particle that has Kaluza-Klein excitations, and since these effects are likewise universal for all theories (whether QED or the MSSM), we can generalize our results to the full MSSM in the final step.
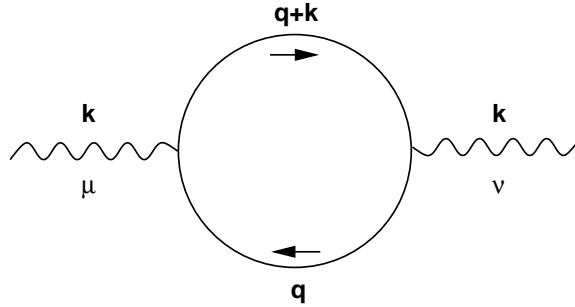

Figure 15: The vacuum polarization diagram. We include the effects of extra Kaluza-Klein modes in the loops.

For a single Dirac fermion with Kaluza-Klein excitations, the vacuum polarization diagram in Fig. 15 is given by

$$
\Pi_{\mu\nu}(k) = -\sum_{n_i=-\infty}^{\infty} g^2 \int_0^{\infty} \frac{d^4q}{(2\pi)^4} \operatorname{Tr}\left(\gamma_\mu \frac{1}{\slashed{q} - m_n} \gamma_\nu \frac{1}{\slashed{k} + \slashed{q} - m_n}\right) \tag{A.1}
$$

where we have used the notation

$$
\sum_{n_i=-\infty}^{\infty} \equiv \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_\delta=-\infty}^{\infty} \tag{A.2}
$$

to represent a summation over all corresponding Kaluza-Klein excitations with masses m_n^2 given in (2.3). In (2.3), m_0 is the energy of the ground state, which we will henceforth take to be zero for simplicity. Restricting our summation to only the $n_i = 0$ term therefore amounts to considering only the original fermionic state without

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its Kaluza-Klein excitations, and opposite values of n_i correspond to Kaluza-Klein states whose i^{th} circle momenta are in opposite directions. The overall sign in (A.1) arises due the fermion loop.

Here and throughout we shall assume the presence of a suitable ultraviolet regulator with cutoff Λ in order to justify our subsequent manipulations. We shall discuss our regulator explicitly when its form becomes crucial for our analysis.

Our initial steps are completely standard. Using gauge invariance to define $\Pi(k^2)$ via (3.7), we contract the Lorentz indices and evaluate the trace to obtain

$$\Pi(k^2) \;=\; -\,\frac{8g^2}{3k^2}\sum_{n_i=-\infty}^{\infty}\int_0^\infty \frac{d^4q}{(2\pi)^4}\left\{\frac{-(k+q)\cdot q + 2m_n^2}{(q^2-m_n^2)\,[(k+q)^2-m_n^2]}\right\}\;. \tag{A.3}$$

Passing to Euclidean momenta, introducing the Feynman x-parameter to combine the propagators, and keeping only terms in the integrand that are even in q then yields

$$\Pi(k^2) \;=\; -\,\frac{8g^2}{3k^2}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx \int_0^\infty \frac{d^4q}{(2\pi)^4}\left\{\frac{q^2 - x(1-x)k^2 + 2m_n^2}{[q^2 + x(1-x)k^2 + m_n^2]^2}\right\}\;. \tag{A.4}$$

Our next step is to rewrite this expression in terms of a Schwinger proper-time parameter t using the identity

$$\frac{1}{A^2} = \int_0^\infty dt\, t\, e^{-At}\;. \tag{A.5}$$

This yields

$$\begin{aligned}
\Pi(k^2) \;=\; &-\frac{8g^2}{3k^2}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx \int_0^\infty dt\, t \int_0^\infty \frac{d^4q}{(2\pi)^4}\,[q^2 - x(1-x)k^2 + 2m_n^2]\;\times \\
&\times\;\exp\left\{-t[q^2 + x(1-x)k^2 + m_n^2]\right\}\;,
\end{aligned} \tag{A.6}$$

and performing the momentum integrations via the identities

$$\int_0^\infty \frac{d^4q}{(2\pi)^4}\, e^{-tq^2} = \frac{1}{16\pi^2 t^2}\;, \qquad \int_0^\infty \frac{d^4q}{(2\pi)^4}\, q^2\, e^{-tq^2} = \frac{1}{8\pi^2 t^3}\;, \tag{A.7}$$

we obtain

$$\begin{aligned}
\Pi(k^2) \;=\; &-\frac{g^2}{6\pi^2 k^2}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx \int_0^\infty \frac{dt}{t}\left[\frac{2}{t} - x(1-x)k^2 + 2m_n^2\right]\;\times \\
&\times\;\exp\left\{-t[x(1-x)k^2 + m_n^2]\right\}\;.
\end{aligned} \tag{A.8}$$

Integrating the first term by parts then yields

$$\Pi(k^2) \;=\; \frac{g^2}{2\pi^2}\sum_{n_i=-\infty}^{\infty}\int_0^1 dx\, x(1-x)\int_0^\infty \frac{dt}{t}\,\exp\left\{-t[x(1-x)k^2 + m_n^2]\right\}\;. \tag{A.9}$$

Let us now perform the summation over the Kaluza-Klein states. In order to do this, we recall the definition of the Jacobi ϑ_3 function:

$$\vartheta_3(\tau) \equiv \sum_{n_i=-\infty}^{\infty} \exp(\pi i \tau n^2) \qquad (A.10)$$

where τ is a complex number. This function has the remarkable property that

$$\vartheta_3(-1/\tau) = \sqrt{-i\tau}\, \vartheta_3(\tau) \qquad (A.11)$$

where one chooses the branch of the square root with non-negative real part. We can thus rewrite our result (A.9) in terms of this function as

$$\Pi(k^2) = \frac{g^2}{2\pi^2} \int_0^1 dx\, x(1-x) \int_0^\infty \frac{dt}{t}\, e^{-tx(1-x)k^2} \left\{ \vartheta_3\left(\frac{it}{\pi R^2}\right) \right\}^\delta , \qquad (A.12)$$

whereupon we find that $\Pi(0)$ is

$$\Pi(0) = \frac{g^2}{12\pi^2} \int_0^\infty \frac{dt}{t} \left\{ \vartheta_3\left(\frac{it}{\pi R^2}\right) \right\}^\delta . \qquad (A.13)$$

At this step, we must introduce our infrared and ultraviolet regulators, along with their corresponding cutoffs. Let us first recall that the ultraviolet and infrared divergences in this expression arise from the $t \to 0$ and $t \to \infty$ limits of integration respectively. Therefore, it is simplest to render this expression finite in both limits by introducing upper and lower cutoffs on the t-integration:

$$\int_0^\infty dt \longrightarrow \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} dt . \qquad (A.14)$$

Here Λ is our ultraviolet cutoff, μ_0 is our infrared cutoff, and the numerical coefficient r (which ultimately relates these cutoff parameters to underlying physical *mass scales*) is defined as

$$r \equiv \pi\, (X_\delta)^{-2/\delta} \qquad (A.15)$$

where X_δ is defined in (3.10). As we discussed in Sect. 3, such a precise value of X_δ or r cannot be deduced purely on the basis of such a non-renormalizable theory alone, and instead requires outside information. However, in Appendix B we shall show that this is the correct relative normalization factor that relates the cutoff parameters $r\Lambda^{-2}$ and $r\mu_0^{-2}$ in (A.14) to the underlying physical mass scales Λ and μ_0 used in Sect. 3.

Looking at (A.13), we see that the limit of the usual four-dimensional theory without Kaluza-Klein modes can be obtained by setting $\vartheta_3 = 1$. This is equivalent to setting $R \to 0$, which essentially makes all Kaluza-Klein modes infinitely massive. For all values of δ, this then produces the expected result

$$\Pi(0) = \frac{g^2}{6\pi^2} \ln \frac{\Lambda}{\mu_0} = \frac{g^2 b}{8\pi^2} \ln \frac{\Lambda}{\mu_0} \qquad (A.16)$$

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where we have identified $b = 4/3$ as the beta-function coefficient of our single Dirac fermion.

Let us now generalize this result to the present case of the full MSSM. As we discussed in Sect. 3, not all of the MSSM states have Kaluza-Klein excitations. Indeed, while the zero-mode states with $n_i = 0$ correspond to the full MSSM spectrum (for which the corresponding beta-function coefficients are denoted b_i), only some of these fields will have Kaluza-Klein excitations. The beta-function coefficients $\tilde{b}_i$ corresponding to these Kaluza-Klein modes at each non-zero mass level $\{n_i\}$ are given in (3.9). Thus, generalizing (A.13) to the case of the full MSSM, we find that

$$
\begin{aligned}
\Pi(0) \;&=\; \frac{g_i^2 b_i}{8\pi^2} \ln \frac{\Lambda}{\mu_0} \;+\; \frac{g_i^2 \tilde{b}_i}{16\pi^2} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{ \left[\vartheta_3 \left(\frac{it}{\pi R^2} \right) \right]^\delta - 1 \right\} \\
&=\; \frac{g_i^2 (b_i - \tilde{b}_i)}{8\pi^2} \ln \frac{\Lambda}{\mu_0} \;+\; \frac{g_i^2 \tilde{b}_i}{16\pi^2} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{ \vartheta_3 \left(\frac{it}{\pi R^2} \right) \right\}^\delta \;. \qquad \text{(A.17)}
\end{aligned}
$$

In the first line, we have explicitly separated the zero-mode contributions (which yield the first term) from the higher-mode contributions (which yield the second term). Since the ϑ_3 functions implicitly include the contributions from the zero-modes, we have explicitly subtracted these contributions from the integrand of the second term by writing $\vartheta_3^\delta - 1$. Thus, passing to the second line of (A.17), we see that the second term represents the contributions from the complete Kaluza-Klein towers that would have existed if the zero-mode states had matched the excited states in our theory, while the first term represents the compensating adjustment that arises because the zero-modes and excited states are actually different. Taken together, (A.17) then implies that

$$
\alpha_i^{-1}(\Lambda) \;=\; \alpha_i^{-1}(\mu_0) \;-\; \frac{b_i - \tilde{b}_i}{2\pi} \ln \frac{\Lambda}{\mu_0} \;-\; \frac{\tilde{b}_i}{4\pi} \int_{r\Lambda^{-2}}^{r\mu_0^{-2}} \frac{dt}{t} \left\{ \vartheta_3 \left(\frac{it}{\pi R^2} \right) \right\}^\delta \;. \qquad \text{(A.18)}
$$

The result (A.18) gives the exact running of the MSSM gauge couplings in the presence of an infinite tower of Kaluza-Klein states associated with δ extra dimensions compactified on circles of radius R. Indeed, the effect of the Kaluza-Klein modes is completely incorporated within the ϑ_3 function. Note that this result is true for *any* mass scales Λ and μ_0 — in particular, we need not identify μ_0 with R^{-1}.

However, we expect that it is a valid description of the physics to treat the appropriate subset of the MSSM as being in D flat dimensions for energy scales much larger than R^{-1}. We shall now demonstrate in what sense this is true.

Let us suppose, for the moment, that μ_0 and Λ are both much larger than R^{-1}. (Of course, strictly speaking, we will ultimately want to identify μ_0 with R^{-1}, but we will assume $\mu_0 \gg R^{-1}$ for now and defer a discussion of the errors this introduces until later.) In this case, we have $t/R^2 \ll 1$, and we can approximate the ϑ_3 function using (A.11), obtaining

$$
\vartheta_3 \left(\frac{it}{\pi R^2} \right) \;\approx\; R\sqrt{\frac{\pi}{t}} \;. \qquad \text{(A.19)}
$$

Inserting this approximation back into (A.18) and evaluating the integral, we then obtain

$$\alpha_i^{-1}(\Lambda) \;=\; \alpha_i^{-1}(\mu_0) \;-\; \frac{b_i - \tilde{b}_i}{2\pi} \ln\frac{\Lambda}{\mu_0} \;-\; \frac{\tilde{b}_i X_\delta}{2\pi\delta}\, R^\delta \left(\Lambda^\delta - \mu_0^\delta\right)\,. \qquad (A.20)$$

If we now identify R^{-1} with μ_0, we find

$$\alpha_i^{-1}(\Lambda) \;=\; \alpha_i^{-1}(\mu_0) \;-\; \frac{b_i - \tilde{b}_i}{2\pi} \ln\frac{\Lambda}{\mu_0} \;-\; \frac{\tilde{b}_i X_\delta}{2\pi\delta} \left[\left(\frac{\Lambda}{\mu_0}\right)^\delta - 1\right]\,, \qquad (A.21)$$

in agreement with (3.12). Thus, for sufficiently high energy scales, we see that our explicit Kaluza-Klein calculation reproduces the gauge coupling relations used in Sect. 3.

Finally, we must discuss the validity of the approximation $\mu_0, \Lambda \gg R^{-1}$ that was used in obtaining (A.19), especially in light of the fact that we ultimately wish to identify $\mu_0 = R^{-1}$ and $\Lambda = M'_{\rm GUT}$. To what extent does this disturb the validity of the above calculation? Due to the difficulty of analytically integrating the ϑ-function, this question is best answered numerically. However, it turns out that one cannot discern any difference between Fig. 1 (in which the exact results (A.18) are plotted) and the approximate results based on (3.12). Thus, we conclude that the assumption of D flat spacetime dimensions for the non-chiral sector of the MSSM at energy scales above μ_0 provides an excellent approximation to the full Kaluza-Klein theory.

Appendix B

In this Appendix, we shall show that the truncated Kaluza-Klein theory is also an excellent approximation to the full Kaluza-Klein theory for the purposes of calculating the scale-dependence of the gauge couplings and Yukawa couplings. This will ultimately enable us to calculate the exact value of X_δ given in (3.10).

To do this, let us for the moment consider a simple five-dimensional $U(1)$ gauge theory along with an arbitrary spectrum of zero-mode ground states with beta-function coefficient b_1 as well as an arbitrary tower of Kaluza-Klein excitations with beta-function coefficient $\tilde{b}_1$. If the extra dimension has radius R, then this Kaluza-Klein tower of excited states will consist of two Kaluza-Klein states of mass $\approx \mu_0 \equiv R^{-1}$, two additional Kaluza-Klein states of mass $\approx 2\mu_0$, two more of mass $\approx 3\mu_0$, and so forth. There are two Kaluza-Klein states at each mass level because there are two possible directions for the Kaluza-Klein momentum in the fifth direction.

Let us now consider the running of our $U(1)$ gauge coupling in the presence of this Kaluza-Klein spectrum. We shall first ignore the effects of the zero-modes (which by themselves always give logarithmic running), and concentrate solely on the contributions of the non-zero Kaluza-Klein excitations. Rather than consider

all of these Kaluza-Klein states running in the loops at once (as in Appendix A), our fundamental idea here will be to introduce these states only at their thresholds, two each at every mass threshold $m_n = n\mu_0$. If our Kaluza-Klein tower has beta-function coefficient $\tilde{b}_1$, then each time we cross a threshold the effective beta-function coefficient increases by $2\tilde{b}_1$ because each threshold produces two extra massive copies of the same Kaluza-Klein states. Thus, after n thresholds (i.e., for energy scales $n\mu_0 \leq \mu \leq (n+1)\mu_0$), our beta-function coefficient has grown to $b_1 + 2n\tilde{b}_1$. Adding these incremental contributions together (and restoring the contribution from the zero-modes), we thus find the result

$$\alpha_1^{-1}(\mu) \;=\; \alpha_1^{-1}(M_Z) \;-\; \frac{b_1}{2\pi}\ln\frac{\mu}{M_Z} \;-\; \frac{\tilde{b}_1}{2\pi}\left(2n\ln\frac{\mu}{\mu_0} - 2\ln n!\right) \;. \tag{B.1}$$

In other words, for any value of μ, our gauge couplings will be given by (B.1) where we identify $n \equiv [\mu/\mu_0]$ where $[r]$ signifies the greatest integer not exceeding r.

Since we are interested in physics only below the scale $M'_{\rm GUT}$, in this approach we are free to disregard Kaluza-Klein states of masses exceeding $M'_{\rm GUT}$. Thus, at every step in the evolution of our $U(1)$ gauge coupling, we have a *completely renormalizable field theory*. Indeed, the only difference relative to the usual MSSM is a finite set of extra states whose masses are regularly spaced in multiples of μ_0. Note that we have chosen to limit our attention to an *abelian* gauge group. This is because our Kaluza-Klein states will necessarily include massive copies of our low-energy gauge bosons, and it is immediately clear that abelian massive gauge bosons are consistent with renormalizability. However, it turns out that such a truncated Kaluza-Klein theory is renormalizable even in the case of non-abelian gauge groups; this will be discussed in Appendix C.

Given that this theory is completely renormalizable, there are no ambiguities regarding the interpretation of cutoffs and mass scales. Indeed, (B.1) may be regarded as a true renormalization group equation. We can therefore compare this prediction with that based on our non-renormalizable higher-dimensional theory in order to resolve any numerical cutoff or mass-scale ambiguities.

To do this, let us consider the prediction (3.13) from our non-renormalizable field theory, restricted to the $U(1)$ case. For simplicity, we may formally take a derivative of (3.13) to write

$$\frac{d}{d\ln\mu}\alpha_1^{-1}(\mu) \;=\; -\frac{b_1 - \tilde{b}_1}{2\pi} \;-\; \frac{\tilde{b}_1 X_\delta}{2\pi}\left(\frac{\mu}{\mu_0}\right)^\delta \tag{B.2}$$

where we have rewritten $\Lambda \to \mu$ for notational convenience. Indeed, we may regard (B.2) as providing a *definition* for X_δ. Note that this result should hold, in principle, for *any* value of μ, no matter how large. Let us now compare this differential equation with the prediction of our above discrete approach for $\delta = 1$. It is clear that our

discrete approach gives the corresponding RGE

$$\frac{d}{d \ln \mu} \alpha_1^{-1}(\mu) \;=\; -\frac{b_1 - \tilde{b}_1}{2\pi} - \frac{\tilde{b}_1}{\pi} \left[\frac{\mu}{\mu_0} \right] \;. \tag{B.3}$$

Once again, this result should hold for any μ, no matter how large. As $\mu/\mu_0 \to \infty$, we can approximate $[\mu/\mu_0] \approx \mu/\mu_0$ in (B.3). We thus immediately find that $X_1 = 2$.

We may also check this result numerically. Taking $X_1 = 2$, it is straightforward to verify that that (B.1) and (3.13) give closely matching curves, even for relatively small values of μ. In Fig. 16, we show an extreme case: we take $\mu_0 = 10^5$ GeV, and compare the discrete result (B.1) against the *full* analytical result (A.18). Moreover, for this figure we have artificially inflated the value of $\tilde{b}_1$ (taking $\tilde{b}_1 = b_1 = 33/5$ rather than its true value $\tilde{b}_1 = 3/5$) in order to magnify the differences between the two curves and render these differences visible in Fig. 16. Even with this magnification, we see that the agreement is excellent. Thus, we see that we are free to interpret Λ as the physical mass scale provided we take $X_1 = 2$.
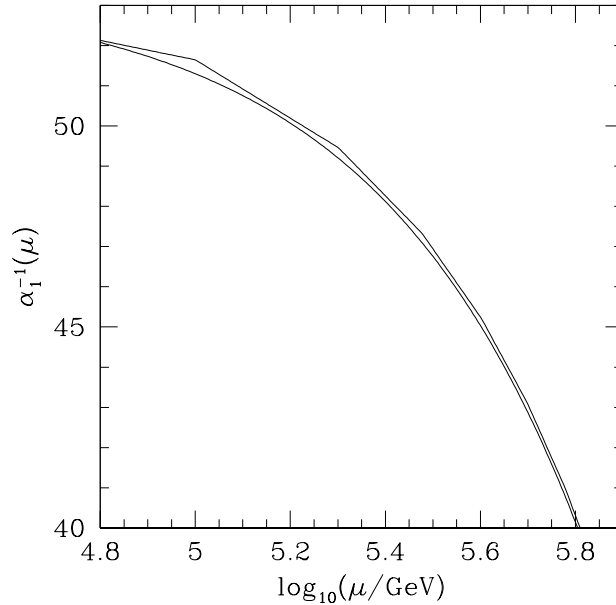

Figure 16: Comparison between the renormalizable discrete threshold approach (upper curve) and the full non-renormalizable approach (lower curve). We have taken $\mu_0 = 10^5$ GeV and $\delta = 1$, and for the clarity of this figure we have artificially *magnified* the difference between the two curves by a factor of 11 (see text). We see that the agreement between the two curves remains excellent, even in this extreme case.

Finally, let us consider the situation in higher dimensions. Once again, our discrete

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threshold approach yields the RGE

$$\frac{d}{d \ln \mu} \alpha_1^{-1}(\mu) \;=\; -\frac{b_1 - \tilde{b}_1}{2\pi} \;-\; \frac{\tilde{b}_1}{2\pi} \, N(\mu, \mu_0) \tag{B.4}$$

where $N(\mu, \mu_0)$ is the number of Kaluza-Klein states with masses less than μ. Note that by definition, $N(\mu, \mu_0)$ is the number of solutions to the equation

$$\sum_{i=1}^{\delta} n_i^2 \;\leq\; \left(\frac{\mu}{\mu_0}\right)^2 \;, \qquad n_i \in \mathbb{Z} \;. \tag{B.5}$$

For large μ/μ_0, this is well-approximated as the volume of a δ-dimensional sphere of radius μ/μ_0:

$$N(\mu, \mu_0) \;=\; \frac{\pi^{\delta/2}}{\Gamma(1 + \delta/2)} \left(\frac{\mu}{\mu_0}\right)^{\delta} \;. \tag{B.6}$$

Comparing with (B.2) then yields the result for X_δ quoted in (3.10). Thus, once again, we find that we may interpret the cutoff Λ in Sect. 3 as a physical mass scale provided we take the appropriate value for X_δ. Although our analysis in this section is restricted to the case of a $U(1)$ gauge group, these values for X_δ are universal for all gauge groups because they reflect nothing more than the universal enhancement factors due to the appearance of Kaluza-Klein states and/or extra spacetime dimensions. Indeed, as we shall demonstrate in Appendix C, the truncated Kaluza-Klein theory is renormalizable even in the non-abelian case.

Finally, let us briefly discuss the significance of the fact that we can model the scale-dependence (or cutoff-dependence) of the gauge couplings as resulting from an effective renormalizable theory.* In general, since we are evolving the physics from the infrared to the ultraviolet within the context of a non-renormalizable field theory, there is always the danger that there will exist additional relevant operators at higher scales whose effects we are not including. In general, this should limit the validity of our approach. However, for the purposes of examining the evolution of gauge and Yukawa couplings (which are ultimately wavefunction renormalization calculations), such operators will have no effect. This then explains why the truncated Kaluza-Klein theory succeeds so well in modelling the evolution of these couplings, which has been our main focus in this paper. Nevertheless, it is possible that there will exist physical processes for which such operators will play an important role, and for which a renormalizable truncated Kaluza-Klein theory will not be appropriate.

Appendix C

In this Appendix, we shall provide a technical background discussion concerning the renormalizability and gauge invariance of truncated Kaluza-Klein theories.

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*We thank R. Rattazzi for discussions on this point.

Specifically, we shall demonstrate the renormalizability of the non-abelian truncated Kaluza-Klein theory on the $\mathbb{Z}_2$ orbifold discussed in Sect. 2 by demonstrating the close analogy between this theory and the ordinary Higgs mechanism of four-dimensional gauge theories (which we know preserves renormalizability).

For simplicity, we shall restrict ourselves to the pure gauge part, and begin the discussion by considering abelian gauge fields. For an abelian gauge theory in five dimensions, the pure gauge Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab} \qquad (\text{C.1})$$

where $a, b = 1, ..., 5$ and where $F_{ab} \equiv \partial_a A_b - \partial_b A_a$. Let us now compactify the fifth dimension on a circle of radius R and rescale our Kaluza-Klein modes:

$$(A_\mu^{(n)}, A_5^{(n)}) \to \sqrt{2} \, (A_\mu^{(n)}, A_5^{(n)}) \ . \qquad (\text{C.2})$$

We then obtain

$$\mathcal{L} = -\frac{1}{4} \sum_{n=0}^{\infty} F_{\mu\nu}^{(n)2} + \frac{1}{4} \sum_{n=1}^{\infty} (\partial_\mu A_5^{(n)} + \frac{n}{R} A_\mu^{(n)})^2 \ . \qquad (\text{C.3})$$

Note that the resulting Lagrangian is gauge-invariant for any $A_\mu^{(n)}$, with non-linear gauge transformations

$$\begin{aligned} A_\mu^{(n)} &\to A_\mu^{(n)} + \partial_\mu \theta^{(n)} \\ A_5^{(n)} &\to A_5^{(n)} - \frac{n}{R} \theta^{(n)} \end{aligned} \qquad (\text{C.4})$$

where $\theta^{(n)}$ is the gauge transformation parameter. It is straightforward to compare this result with that of the usual $U(1)$ abelian Higgs model. If we define the massive gauge field

$$B_\mu^{(n)} = A_\mu^{(n)} + \frac{R}{n} \partial_\mu A_5^{(n)} \ , \qquad (\text{C.5})$$

we see that $B_\mu^{(n)}$ has mass n/R. We thus construct the analogy with the abelian Higgs model by associating

$$e \iff n \ , \qquad v \iff 1/R \qquad (\text{C.6})$$

where e is the electric charge of the abelian gauge field in the abelian Higgs model and where v is the VEV of the Higgs field. The Nambu-Goldstone boson of the spontaneously broken $U(1)$ associated with $A_\mu^{(n)}$ is therefore $A_5^{(n)}$.

In five dimensions, Lorentz invariance allows us to add a gauge-fixing term, and the full pure gauge Lagrangian reads

$$\mathcal{L} = -\frac{1}{4} F_{ab}^2 - \frac{\lambda}{2} (\partial^a A_a)^2 \ . \qquad (\text{C.7})$$

Note that when reduced to four dimensions, the gauge-fixing term becomes

$$-\frac{\lambda}{2}(\partial^a A_a)^2 \;=\; -\frac{\lambda}{2}\sum_{n=0}^{\infty}\left[\partial^\mu A_\mu^{(n)} - \frac{n}{R}A_5^{(n)}\right]^2 \;.\qquad\text{(C.8)}$$

Thus, for $\lambda = 1$, we see that we obtain a four-dimensional 't Hooft renormalizable gauge. Hence the dimensional reduction of the Lagrangian in (C.7) gives the four-dimensional result

$$
\begin{aligned}
\mathcal{L} \;=\; &-\frac{1}{4}\sum_{n=0}^{\infty}F_{\mu\nu}^{(n)2} - \frac{1}{2}\sum_{n=0}^{\infty}\left[(\partial^\mu A_\mu^{(n)})^2 - \frac{n^2}{R^2}(A_\mu^{(n)})^2\right]\\
&+\frac{1}{2}\sum_{n=1}^{\infty}\left[(\partial_\mu A_5^{(n)})^2 - \frac{n^2}{R^2}(A_5^{(n)})^2\right]\;.
\end{aligned}
\qquad\text{(C.9)}
$$

For $n \geq 1$, this Lagrangian then leads to the propagators $\Delta_{\mu\nu}^{(n)}(k)$ and $\Delta^{(n)}(k)$ for the gauge fields and Nambu-Goldstone bosons respectively, where

$$
\begin{aligned}
\Delta_{\mu\nu}^{(n)}(k) \;&=\; \frac{-ig_{\mu\nu}}{k^2 - n^2/R^2 + i\epsilon}\\
\Delta^{(n)}(k) \;&=\; \frac{i}{k^2 - n^2/R^2 + i\epsilon}\;.
\end{aligned}
\qquad\text{(C.10)}
$$

These propagators are well-behaved as $k \to \infty$. Thus, we conclude that the whole theory is renormalizable for any (finite) number of Kaluza-Klein states. Of course, the theory becomes non-renormalizable if we consider the full, infinite tower of Kaluza-Klein states.

Having explicitly demonstrated that the truncated abelian Kaluza-Klein theory is renormalizable, we can now easily repeat the above steps for the non-abelian case. Every step carries through as before. Thus, we conclude that even the *non-abelian* truncated Kaluza-Klein theory is renormalizable. For completeness, we give the non-linear realization of the gauge symmetry for the massive Kaluza-Klein levels:

$$
\begin{aligned}
\delta A_\mu^{a(n)} \;&=\; \partial_\mu\theta^{a(n)} - \frac{1}{2}f^{abc}\sum_m\left[A_\mu^{b(n-m)} + A_\mu^{b(n+m)}\right]\theta^{c(m)}\\
\delta A_5^{a(n)} \;&=\; -\frac{n}{R}\theta^{a(n)} - \frac{1}{2}f^{abc}\sum_m\left[A_5^{b(n-m)} + A_5^{b(n+m)}\right]\theta^{c(m)}
\end{aligned}
\qquad\text{(C.11)}
$$

where f^{abc} are the structure constants of the non-abelian gauge group and where $\theta^{a(n)}$ are the gauge transformation parameters.

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