

EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH
CERN - SL DIVISION

CERN-SL-98-015 AP
CLIC Note 362

ANALYTICAL TREATMENT OF SINGLE BUNCH TRANSVERSE
DYNAMICS IN
LINACS WITH WAKEFIELDS

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In this paper we present an analytical treatment of the equation of motion of single bunch particles traveling in the linac of a linear collider, in the presence of wakefields. Using a somewhat simplified model, the equation is solved for a Gaussian distribution of charge, a linear variation of the wakefield along the bunch, a smooth focusing and in the absence of acceleration. The specificity of the method consists of preventing the appearance of artificial secular terms and keeping at any stage the intrinsic tune-shift that characterizes the problem and stabilizes the motion. Hence, a first order perturbation becomes sufficient provided that a non-standard perturbation expansion, specially developed for this analysis, be however applied. Solutions for the particle off-sets within the bunch are obtained and their contributions to the effective emittance calculated. The results explain the observed features of beam breakup and BNS damping, and reproduce the bunch dipole-oscillations visible on the animated graphics after simulations with the code MUSTAFA. This treatment provides in addition a closed expression for the tune shift along the bunch and confirms the existence of an optimum BNS damping setting which differs from autophasing in a single bunch mode.

Geneva, Switzerland
30.04.98

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1 Introduction

For some time, we have been involved in numerical simulations of the beam dynamics of both a single bunch and a train of bunches transported in a linear accelerator (linac) where the wakefields can be strong [1]. We therefore thought that it would be very useful to complement this work by an analytical treatment of the problem and to have whenever possible explicit solutions of the beam motion for our better understanding of the physics as well as a direct comparison with the tracking results of our computer code MUSTAFA [2]. In this report, we deal with the analysis of the single bunch transverse dynamics.

This implies analytically solving the equations of motion for individual particles distributed in a single bunch, travelling along a linac with unavoidable wakefields. Since we are mostly interested to understand the effects on the vertical emittance which eventually determines the luminosity in linear colliders, we shall focus our analysis on the equation of the vertical motion that describes the interaction of the bunch tail with the vertical wakefield generated by the head. It is well known from previous theoretical studies ([3] to [6]) that this may lead to beam breakup instability; and a powerful technique for alleviating the detrimental effects on the emittance, termed BNS damping according to the names of the authors, has been proposed [7] and further analysed ([8] and [9]).

Our present interest mostly concerns the rise of resonant instabilities due to the presence of real secular terms, the importance of the related driving terms, the self-induced stabilization mechanism associated with possible focusing-frequency detuning within the bunch, the various BNS damping regimes with in particular the so-called "autophasing" [10] that corresponds to a constant dipole-mode of betatron oscillation of the whole single-bunch, and the proof of existence of an optimum BNS damping adjustment that minimizes the emittance growth and is different from the autophasing mentioned.

Solving analytically the equation of motion almost unavoidably implies the introduction of some simplification assumptions, that we can try to remove in subsequent developments, but which have to be carefully selected in order to keep the characteristics of the motion which drive the phenomena that we want to study. With this in mind, we consulted the literature on the subject and convinced ourselves of the usefulness of the present work which has the specificity to prevent the appearance of artificial secular terms during the treatment and to keep at any stage the intrinsic tune-shift that conditions the whole problem. In order to illustrate this point, we would like to mention two particular approaches which have been used in attempts to describe and estimate the vertical emittance dilution. The first method is based on a standard perturbation expansion ([5],[6] and [11]) for small energy deviations and transverse wakefields. In spite of its interest, this approach has the intrinsic weakness that, at each order of the perturbation, the driving term naturally contains the solution of the previous order which oscillates with exactly the same betatron frequency as the one present in the homogeneous part of the equation of motion. This corresponds to artificial secular terms which are generated at each order, cancel each other in most cases provided the expansion goes to infinity and the convergence is alright, but may give a questionable result when the expansion is truncated. The second approach we like to quote is the one adopted in [12] and [13], based on the Laplace transform of the equation of motion. Although this method reproduces some main features of previous studies ([14] and [15]), solving the equation in the Laplace variables requires simplifications like flat bunch-charge distribution. In addition, the authors neglect the second order term in the equation of motion for the slowly varying amplitude. Disregarding the second derivative could possibly be at the origin of an uncorrect or incomplete description of the resonances associated with a driven oscillation and its absence could seemingly have consequences on the subsequent analysis.

Starting with the general equation of motion for particles distributed in a single bunch, we introduce a simplified model in order to avoid cumbersome mathematics in the first place (Section 2). The method used is based on resolving the variable into one which only depends on s (coherent oscillation) and one which depends on s and z (incoherent motion), if s is the position along the linac and z the coordinate inside the bunch. The form of the obtained equations (Section 3) naturally leads to the autophasing condition quoted above. Solving these equations requires a perturbative treatment that we generalized into an original partial perturbation expansion (Appendix) in order to keep the detuning property and avoid the rise of a secular term, as mentioned already. This makes it possible to write the solution up to the first order of the perturbation, which is here the product of the transverse wakefield by the off-set of the slices of charge, and to give an interesting polynomial expression of the tune-shift with z . Finally, the contribution of this perturbation to the effective emittance increase can be calculated (Section 4). The results of the whole analysis can then be compared with numerical simulations done with MUSTAFA for the same model. They firstly give a clear illustration of the different BNS damping regimes and nicely confirm the existence of an optimum that minimizes the single-bunch emittance growth. Secondly they reproduce remarkably well the vertical off-sets put in evidence by animated graphics after tracking with the CLIC parameters [2]. The analysis explains the beam behaviour, such as the large tail oscillations in the absence of damping, the stability and detuning whether some BNS damping is applied or the bunch is injected with a slope, the specific autophasing condition and the growing number of nodes and oscillations of the transverse positions along the bunch.

2 Equation of motion in the transverse plane and its solution

Since in most linear colliders a flat beam design (low vertical to horizontal beam size ratio) is used, emittance blow up due to wakefields is most critical in the vertical plane and leads to serious reductions in luminosity at interaction when no correction is applied. The equation of motion in the vertical plane using a weak focusing model and in presence of transverse wakefields is [5]

$$\frac{\partial^2 x(s, z)}{\partial s^2} + \frac{\gamma'(s, z)}{\gamma(s, z)} \frac{\partial x(s, z)}{\partial s} + q^2 [1 + \Delta k(s, z)] (x(s, z) - x_Q(s)) = \frac{C(1 - \delta(s, z))}{\gamma_r(s)} \int_0^z \rho(z^*) w^T(z - z^*) [x(s, z^*) - x_A(s)] dz^* \quad (1)$$

Note that total derivatives w.r.t. s are denoted by $'$. The quantities contained in the above basic equation have the following meanings:

- $x(s, z)[m]$: The transverse vertical displacement of a "slice of charge" as function of the position z inside the bunch and the distance s that has been passed by the whole bunch inside the linac of the collider.
- $\gamma(s, z)$: The energy Lorentz factor within the bunch, as a function of s and z in general. The reference energy factor $\gamma_r(s)$ is taken at the head of the bunch and is thus only a function of s . The s dependence is linear and given by

$$\gamma_r(s) = \gamma_0 + gs \quad (2)$$

where γ_0 represents the value at injection ($s = 0$) and g the Lorentz factor gain by acceleration per unit length.

- $q[m^{-1}]$: The weak focusing tune of the lattice. For a regular FODO lattice q is equal to the inverse average of the β function of this lattice

$$q = \frac{1}{\bar{\beta}} \quad (3)$$

where [5]

$$\bar{\beta} = \frac{L_{cell}}{2} \left(\cot \frac{\mu_{cell}}{2} + \frac{2}{3} \tan \frac{\mu_{cell}}{2} \right) \quad (4)$$

- $\Delta k(s, z)$: Focusing force (in addition to the magnetic lattice focusing) that depends on the position z inside the bunch as well as on the distance s inside the linac. This force can origin from an energy spread along the bunch caused by the acceleration sine wave and longitudinal wakefields or by a correction scheme containing RF quadrupoles. This additional force leads to a detuning across the bunch and can stabilize the resonant excitation of the transverse motion due to transverse wakefields [7].
- $C[\frac{As}{V}]$: A constant (independent of s and z) being defined as

$$C = 4\pi \epsilon_0 r_e N \quad (5)$$

where ϵ_0 is the dielectric constant, r_e the classical electron radius and N the number of charged particles in the electron or positron bunch.

- $\delta(s, z)$: The energy spread across the bunch as function of the bunch coordinate z , and of s in the presence of acceleration.
- $\rho(z)[m^{-1}]$: Line density charge distribution of the bunch. The distribution is supposed to be cut at the head and tail of the bunch defined by $z = 0$ and $z = l_b$ (bunchlength) respectively.
- $w^T(z)[\frac{V}{Cbmz}]$: The transverse wakefield. At the head of the bunch ($z = 0$) w^T is zero.
- $x_Q(s)[m]$: Missalignments of quadrupoles as function of s .
- $x_A(s)[m]$: Missalignments of accelerating cavities as function of s .

Taking into account the fact that the accelerating cavity missalignments do only depend on s , we may extract the function $x_A(s)$ from the integral expression in Eq. (2) and rewrite this equation as an inhomogeneous problem:

$$\begin{aligned} \frac{\partial^2 x(s, z)}{\partial s^2} + \frac{\gamma'(s, z)}{\gamma(s, z)} \frac{\partial x}{\partial s} + q^2[1 + \Delta k(s, z)]x(s, z) = & \quad (6) \\ \frac{C(1 - \delta(s, z))}{\gamma_r(s)} \int_0^z \rho(z^*) w^T(z - z^*) x(s, z^*) dz^* + & \\ \frac{C(1 - \delta(s, z))}{\gamma_r(s)} x_A(s) \int_0^z \rho(z^*) w^T(z - z^*) dz^* - q^2[1 + \Delta k(s, z)]x_Q(s) & \end{aligned}$$

The above equation (7) is a second order, linear, partial integro-differential equation with nonconstant coefficients. Assuming that the bunch is injected without initial vertical displacement $x(s = 0, z) = 0$ it is clear that the inhomogeneous part of the above equation (part not containing $x(s, z)$) will drive noncoherent (depending on the coordinate z) bunch oscillations that lead to emittance blowup within the linac.

In the subsequent sections, we shall analytically approximate and investigate the solutions of this equation in order to gain some understanding of the underlying complex dynamics of this multi-dimensional system. The method for searching for a solution has obviously to deal with resonance and secular terms (the word *secular* here means *growing with the distance s* and comes from the astro-physics mainly). It must allow to either distinguish between real secular terms due to the presence of an actual resonance and artificial ones which may result from an iterative resolution associated with an unavoidable truncation, or simply prevent altogether the generation of any artificial secular term. We shall see that it is indeed possible in our case to develop a special perurbation method which follows the second track.

2.1 Introduction of a simplified model equation

In order to help the search for an actual solution of the above equation we use the following simplifications:

1. Instead of solving the inhomogeneous equation (7) that contains in principle random functions for the cavity and quadrupole missalignments, we solve the homogeneous problem with initial conditions:

$$x(0, z) = \alpha_0 + \alpha_1 z \quad (7)$$

$$\frac{dx}{ds}(0, z) = 0 \quad (8)$$

where α_0 is of the order of the RMS value of the missalignments and $\alpha_1 z$ takes care of the fact that - since the inhomogeneous part of Eq. (7) depends also on z - a slope along the bunch will be created. The detailed values for α_0 and α_1 have to be chosen in order to best represent a given set of misalignments.

2. The acceleration within one linac sector is neglected and $\gamma(s)$ is replaced by the initial value γ_0 of this sector. This simplification is expected to result in a pessimistic representation of the dynamics since the adiabatic damping of the transverse amplitude proportional to $1/\sqrt{\gamma(s)}$ will not appear in the description of the model.
3. The wakefield within one bunch is assumed to be linear in z and described by

$$w^T(z) = W_0 \frac{z}{l_b} \quad (9)$$

hence, W_0 is the value of w^T at the tail of the bunch where z is equal to the bunch length l_b . This represents an excellent approximation for short bunches like in the case of CLIC [1].

4. The line density charge distribution ρ , taken as a Gaussian, has been modeled by a polynomial distribution in the limits of $[-2\sigma, +2\sigma]$. This has been realized by a decomposition of the function

$$y = e^{-\frac{x^2}{2}} \quad (10)$$

into Chebyshev polynomials in $x = [-2,2]$ ([16]) with an accuracy better than 4% of the maximum. This leads to the following expansion:

$$e^{-\frac{1}{2}x^2} \approx 1 - \frac{41}{100}x^2 + \frac{1}{20}x^4 \quad (11)$$

Fig.1 shows a superposition of the Gaussian and its polynomial representation. The Gaussian is represented by a full line and the polynomial approximation by points.

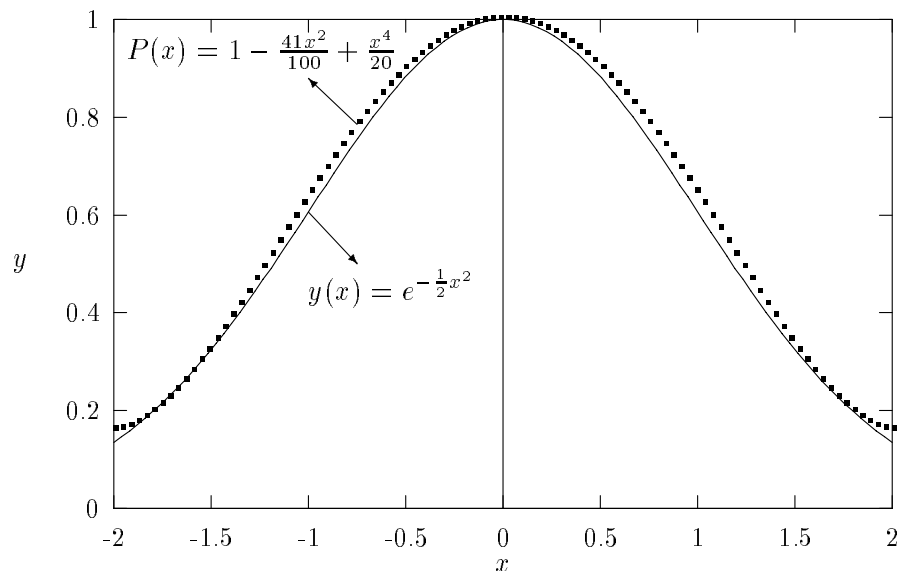


Fig. 1: Comparison between Gaussian and polynomial approximation

5. Since the wakefield is considered as a perturbation to the unperturbed betatron motion and $\delta(z) = \frac{\Delta p}{p_0}(z) \ll 1$, we neglect the term proportional to $\delta \cdot w$ on the right hand side of Eq. (7).

After all these simplifications of the model, the problem treated can be summarized by the actual equation to be solved and which now reads as

$$\frac{\partial^2 x(s, z)}{\partial s^2} + q^2[1 + \Delta k(z)]x(s, z) = \frac{CW_0}{\gamma_0 l_b} \int_0^z \rho(z^*)(z - z^*)x(s, z^*)dz^* \quad (12)$$

and is subject to the initial conditions (7) and (8). If we define $z = 0$ as the head of the truncated bunch at $-2\sigma_z$ and $z = l_b$ as the tail of the same bunch truncated at $+2\sigma_z$ we obtain from Eq. (11) the following expression for the line density of charge:

$$\rho(z) = \frac{75}{46l_b} \left[1 - \frac{41}{100} \left(\frac{4z}{l_b} - 2 \right)^2 + \frac{1}{20} \left(\frac{4z}{l_b} - 2 \right)^4 \right] \quad (13)$$

where the factor $\frac{75}{46}$ results from the normalization condition for ρ :

$$\int_{-2}^2 \rho(z)dz = 1 \quad (14)$$

3 Separation of variables and perturbation theory

This section deals with the analytical approximation to the solution of Eq. (12). This equation is a partial integro differential equation of second order for a single function $x(s, z)$. Frequently equations of this kind can be treated by choosing a separation ansatz for the independent variables and this is treated in the following subsection. Proceeding in this way we naturally arrive at the autophasing condition meaning that one particular non zero solution of the equation becomes only dependent on s but not on z . The equation depending on s and z obtained from the separation of variables is then solved by use of a special perturbation approach called **partial perturbation expansion**. This method as well as its practical application to our problem are also presented in a subsection. Finally the zero and first order perturbation equations are solved in closed form.

3.1 Separation of the independent variables

We split the actual solution into a coherent part (only depending on s but not on the bunch coordinate z) and an incoherent part which depends on both variables. hence

$$x(s, z) = X(s) + y(s, z) \quad (15)$$

Inserting this ansatz into the original equation results in the system

$$\begin{aligned} \frac{d^2 X}{ds^2} + q^2 X &= -\frac{\partial^2 y}{\partial s^2} - q^2 y - q^2 \Delta k(z) X(s) - q^2 \Delta k(z) y + \\ \frac{CW_0}{\gamma_0 l_b} X(s) \int_0^z \rho(z^*) (z - z^*) dz^* &+ \frac{CW_0}{\gamma_0 l_b} \int_0^z \rho(z^*) (z - z^*) y(s, z^*) dz^* = \kappa \end{aligned} \quad (16)$$

Since the left hand side depends only on s while the right hand side depends on s and z , this equation can only hold if both sides are equal to a separation constant κ . If we want to identify X with the unperturbed betatron motion this separation constant has to be equal to zero. We therefore obtain:

$$\begin{aligned} \frac{d^2 X}{ds^2} + q^2 X &= 0 \quad (17) \\ \frac{\partial^2 y}{\partial s^2} + q^2 [1 + \Delta k(z)] y &= \\ X(s) \left[-q^2 \Delta k(z) + \frac{CW_0}{\gamma_0 l_b} \int_0^z \rho(z^*) (z - z^*) dz^* \right] &+ \frac{CW_0}{\gamma_0 l_b} \int_0^z \rho(z^*) (z - z^*) y(s, z^*) dz^* \end{aligned} \quad (18)$$

The coherent equation (17) is given by the unperturbed betatron equation and its solution according to the initial condition (7) is

$$X(s) = \alpha_0 \cos qs = \alpha_0 \cos \frac{s}{\beta} \quad (19)$$

Considering the case of no z dependent focusing across the bunch ($\Delta k(z) = 0$), we are faced by a resonant situation meaning that the frequency q of the unperturbed betatron motion appears on the right hand side and generates a secular solution in s . This is related to the familiar head to tail instability of a single bunch traveling through a structure with wakefields. In order to get rid of the resonance excitation term it is necessary to introduce a spread in tune across the bunch [10] as

$$\Delta k(z)_{AUTO} = \frac{CW_0}{\gamma_0 l_b q^2} \int_0^z \rho(z^*) (z - z^*) dz^* \quad (20)$$

In this paper we do not specify the mechanism creating the detuning (RF quads or energy spread). However, in order to study the bunch dynamics also in the case of no correction or only a partial correction we define the actual detuning as

$$\Delta k(z) = \lambda \Delta k(z)_{AUTO} = \lambda \frac{CW_0}{\gamma_0 l_b q^2} \int_0^z \rho(z^*) (z - z^*) dz^* \quad (21)$$

where $\lambda = 0$ corresponds to no correction while $\lambda = 1$ describes the autophasing condition (resonance suppressed). With this definition (21) the tune spread $\Delta k(z)$ is linked to the wakefield and is directly proportional to the integral of the right hand side in Eq. (20). It means that the detuning function is supposed to be exactly the same as the wakefield integral function (instead of a linear or quadratic polynomial for instance) and that both functions simultaneously tend to zero when W_0 vanishes. Then (18) becomes

$$\begin{aligned} \frac{\partial^2 y}{\partial s^2} + q^2 \left[1 + \frac{\lambda CW_0}{\gamma_0 l_b q^2} \int_0^z \rho(z^*) (z - z^*) dz^* \right] y = \\ (1 - \lambda) \frac{CW_0}{\gamma_0 l_b} X(s) \int_0^z \rho(z^*) (z - z^*) dz^* + \frac{CW_0}{\gamma_0 l_b} \int_0^z \rho(z^*) (z - z^*) y(s, z^*) dz^* \end{aligned} \quad (22)$$

and is subject to the initial conditions

$$y(0, z) = \alpha_1 z \quad (23)$$

$$\frac{\partial y}{\partial s}(0, z) = 0 \quad (24)$$

From the above equation of motion we see that even canceling exactly the resonant excitation term (first term on the r.h.s of Eq. (18)), we remain with two contributions depending on the wakefield. However, in this autophasing condition ($\lambda = 1$) there exists a singular solution $y = 0$ and thus $x = X(s)$ being completely coherent and independent of the wakefield ([8] and [10]). The problem is that this special solution **does only fit** with the initial conditions (7) and (8) if the bunch is injected without initial slope in the bunch coordinate z ($\alpha_1 = 0$). In a realistic case however, although at injection the bunch is almost located parallel to the central axes, a slope in z will be immediately generated by the quadrupole and cavity misalignments present all along the linac. This fact in our model is precisely represented by a non zero initial displacement and slope in z as has been discussed in the previous section. From these arguments it becomes clear that in the generic case two terms depending on the transverse wakefield continue to influence the solution for all values of λ .

- The betatron frequency q is shifted depending on the bunch coordinate z and is given by

$$\bar{q}(z) = q \sqrt{1 + \frac{\lambda CW_0}{\gamma_0 l_b q^2} \int_0^z \rho(z^*) (z - z^*) dz^*} \quad (25)$$

The effect of this term is twofold:

1. Firstly, the resonance created by the term proportional to $X(s)$ of Eq. (18) is detuned because the frequency of $X(s)$ is q while the eigenfrequency of the oscillator is given by $\bar{q}(z)$. Hence the resonant solution at $\lambda = 0$ is transferred to a beating solution for $\lambda > 0$. It is obvious that the effect of detuning increases from the head to the tail of the bunch. At the head of the bunch the detuning vanishes but also the resonant excitation term becomes zero at this position and no resonant effect is visible.

2. The z dependent detuning across the bunch will clearly **decohere** the motion of the bunch. This effect disappears for the resonant case $\lambda = 0$ and increases with λ .

While the effect of detuning with increasing λ will **lower** the emittance blow-up of the bunch traveling along the linac the effect of decoherence of the bunch due to detuning will **increase** the emittance. So we are dealing with two counteracting effects and it is not obvious that the autophasing condition will always result in the strongest possible emittance reduction.

- The integral equation part

$$\frac{CW_0}{\gamma_0 l_b} \int_0^z \rho(z^*) (z - z^*) y(s, z^*) dz^* \quad (26)$$

which is small w.r.t. the excitation term proportional to $X(s)$ in Eq. (18) (as will be shown in the following section) can be thought as an additional excitation term to the oscillatory motion of the bunch. However, since this term contains the solution inside the integral, unlike to the $X(s)$ dependent term, this contribution has a **continuous** z dependent frequency spectrum. However its frequency dependence with z differs from the z dependent frequency term on the left hand side. Hence it is to expect that this contribution does not generate a new resonant component of the solution but rather distort "weakly" the solution determined by the rest of the equation.

3.2 BNS Damping and Autophasing Concepts

After the introduction of a **BNS-damping parameter** λ , that characterizes the amount and distribution of detuning (z -dependent, additional focusing), a certain number of comments can be done in order to better specify the meaning of such a focusing correction:

- In a real situation, the equation (21) with $\lambda = 1$ can never be exactly satisfied, since the gradient correction (done through either an energy-spread or micro-wave quadrupoles) has a z -dependence which follows the sine-wave form of the acceleration voltage and the longitudinal wakefield function $w^L(z)$. In such a situation, the actual correction $\Delta k(s, z)$ applied can formally be written as follows:

$$\begin{aligned} \Delta k(s, z) &= A(s) \sin [2\pi f_{RF}/c + \phi_k] + B(s) w_L(z) \cong \\ &\cong \Delta k(z)_{AUTO} = \frac{CW_0}{\gamma_0 l_b q^2} \int_0^z \rho(z^*) (z - z^*) dz^* \end{aligned} \quad (27)$$

This means that the function Δk is never formally identical to the transverse wake field integral of Eq. (21), such as the difference $\Delta k(z) - \Delta k(z)_{AUTO}$ cannot exactly be equal to zero over the total length of the bunch. This is illustrated in Fig.2 where the wake integral is compensated by a sine-wave correction within $8\sigma_z$ ($\sigma_z =$ the r.m.s. bunch length) of a bunch with gaussian charge distribution. In this arbitrary example, the compensation is only fully effective around the bunch center ($\pm\sigma_z$). Note that in addition such a local compensation is not possible in the presence of strong focusing [11], since the wakfield has a constant sign but the quadrupole gradients alternate.

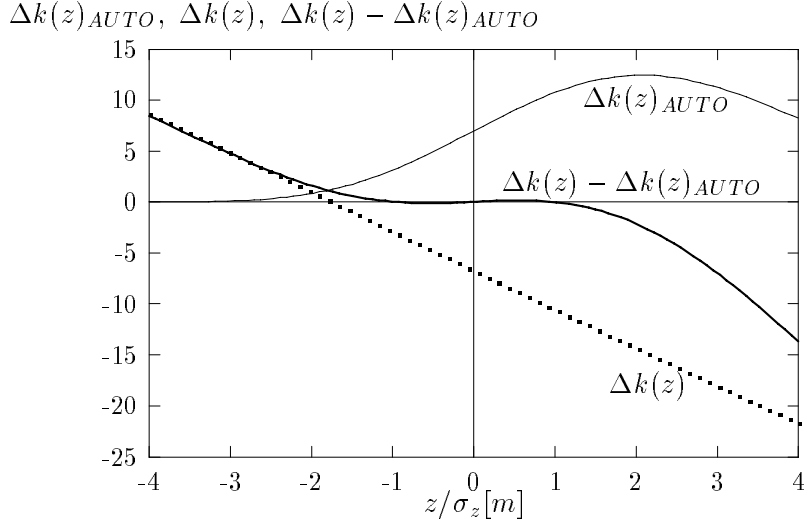


Fig.2 Compensation of the wakefield integral by a sine-wave focusing

- When only a fraction of the wakefield integral is compensated, according to the definition of λ ($\lambda \neq 0, \lambda < 1$),

$$\Delta k(z) = \lambda \cdot \Delta k(z)_{AUTO} \quad (28)$$

the motion satisfies the equation (22). Since the wakefield driving term of the right hand side, proportional to the coherent amplitude $X(s)$ oscillating with q , is not entirely cancelled anymore, there is, on the one hand, a possibility of resonance. On the other hand, looking at the coefficient in front of $y(s, z)$ of the left hand side, there is an additional detuning proportional to λ that shifts the initial tune q out of the resonance.

- Considering next that the damping parameter λ increases from 0 to 1, it is interesting to describe the physics of the effects which take place and result from the equation (22).

— $\lambda = 0$. There is no direct detuning of the focusing (q^2), but a **resonance** due to the driving term containing $X(s)$. In other words, the coherent part of the motion drives the incoherent part to instability. As indicated above, it can however be noted that in Eq. (22) the second driving term proportional to the product $W_0 y$, much smaller than the first one, may generate a weak detuning effect when the amplitude x is already large under the impact of the betatron oscillation $X(s)$.

— $0 < \lambda < 1$. Detuning simultaneously moves the motion away from the resonance associated with $\lambda = 0$ and creates a decoherence of the motion within the bunch. These are the two **counteracting effects** mentioned before; firstly the resonance behaviour is transformed into a beating or **stabilization**, which provokes a strong **reduction** of the rapid emittance growth, and secondly the decoherence generates more and more amplitude wiggles along z which tend to **increase** the emittance. We consequently look for a **minimum of the emittance growth** at some $\lambda > 0$ but $\neq 1$. This minimum is expected to be a rather flat function of λ .

In this whole interval of the parameter λ and even when $\lambda > 1$, the physics described exactly corresponds to what is called **BNS damping** ([7]), since there is indeed a stabilization associated with detuning.

— $\lambda = 1$. This corresponds to the **autophasing** condition ([10]) already quoted before. It is just an undistinguished point in the emittance growth curve as a function of λ , a point which has no particular meaning and does not correspond in general to a minimum blow-up of a single bunch. It can only be associated to the unrealistic situation where a "flat" bunch, injected with an off-set ($\alpha_0 \neq 0$ and $\alpha_1 = 0$) in a linac with perfectly aligned components, follows coherent betatron oscillations during its whole travelling (no amplitude wiggles in the bunch and no emittance growth); indeed $y(s, z) \equiv 0$ is solution of Eq. (22) in this very special case.

3.3 Perturbative Treatment of the Solution

In this section we analytically approximate the solution of the model equation (18). Unfortunately the integral equation part (26) prevents a closed form solution of the equation, at least to our knowledge. A Laplace transform applied to the equation like in Ref. [12] does not work since we consider a nonconstant $\rho(z)$ as well as linear growing wakefield and therefore a variable coefficient in front of the unknown $y(s, z)$ prevents a successful use of this method. However, perturbative expansions can still lead to an analytic approximation of the solution of Eq. (18). As mentioned before, we consider the wakefield as a perturbation to the unperturbed betatron motion. Hence we first reformulate our problem such as to obtain a variable being exactly zero as W_0 vanishes. Inspecting the initial conditions (23) and (24) it can be seen that y only becomes zero at $W_0 = 0$ if the initial slope across the bunch vanishes, i.e. $\alpha_1 = 0$ (given the link between the tune spread and the wakefield assume in Eq. (21)). In order to justify a perturbation method we therefore split y as

$$y(s, z) = \alpha_1 z \cos qs + v(s, z) \quad ; \quad v(0, z) = 0, \quad \frac{\partial}{\partial s} v(0, z) = 0 \quad (29)$$

In addition we introduce the dimensionless bunch coordinate ζ defined by

$$\zeta = \frac{z}{l_b} \quad (30)$$

Then

$$y(s, \zeta) = \alpha_1 l_b \zeta \cos qs + v(s, \zeta) \quad (31)$$

and the new equation for v becomes

$$\begin{aligned} \frac{\partial^2 v}{\partial s^2} + \bar{q}^2(\zeta)v &= \frac{CW_0}{\gamma_0 l_b} \cos qs \left[[\alpha_0(1 - \lambda) - \alpha_1 l_b \zeta \lambda] \int_0^\zeta l_b^2 \rho(l_b \zeta) (\zeta - \zeta^*) d\zeta^* + \right. \\ &\left. \alpha_1 \int_0^\zeta l_b^3 \zeta^* \rho(l_b \zeta) (\zeta - \zeta^*) d\zeta^* \right] + \frac{CW_0 \epsilon}{\gamma_0 l_b} \int_0^\zeta l_b^2 \rho(l_b \zeta^*) (\zeta - \zeta^*) v(s, \zeta^*) d\zeta^* \end{aligned} \quad (32)$$

where

$$\bar{q}^2(\zeta) = q^2 \left[1 + \lambda \epsilon \frac{CW_0}{\gamma_0 l_b q^2} \int_0^\zeta l_b^2 \rho(l_b \zeta^*) (\zeta - \zeta^*) d\zeta^* \right] \quad (33)$$

The integral in the last equation can be evaluated using the polynomial representation of ρ to give

$$\bar{q}^2(\zeta) = q^2 \left[1 + \epsilon \frac{\lambda CW_0}{\gamma_0 q^2} \left(\frac{16}{23} \zeta^6 - \frac{48}{23} \zeta^5 + \frac{79}{46} \zeta^4 + \frac{1}{23} \zeta^3 + \frac{3}{23} \zeta^2 + 0 \right) \right] \quad (34)$$

In Fig.3 we show $\bar{q}(\zeta)$ using the above formula for typical CLIC main linac parameters.

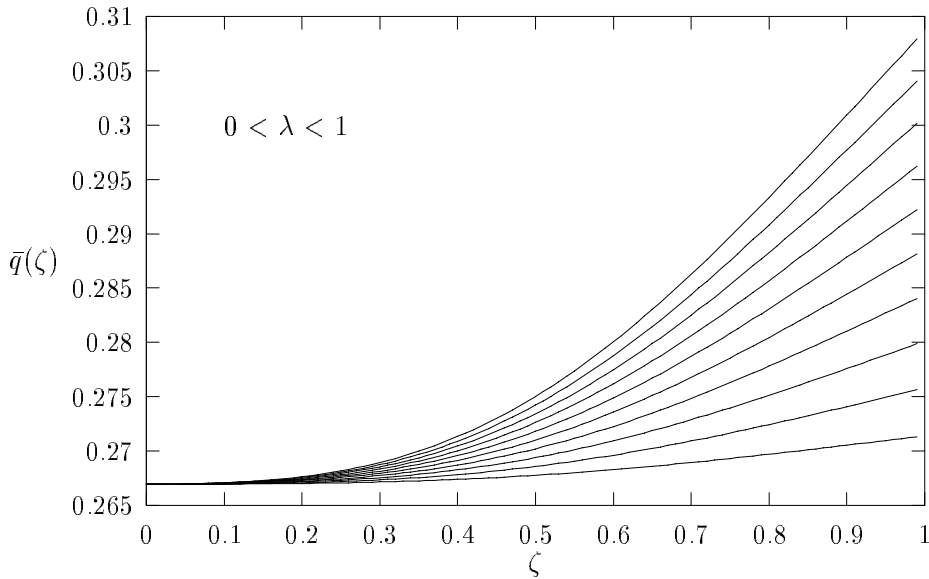


Fig. 3 Tune variation across a single bunch for CLIC parameters and different detuning strengths λ

The formal perturbation parameter ϵ has been introduced in order to indicate that products of W_0 and $v(s, \zeta)$ are considered as a perturbation to the rest of the equation. It will be set to unity at the end of the computations.

3.3.1 The Partial Expansion Method

In principle one can apply a direct perturbation ansatz for the solution of Eq. (32) of the form:

$$v(s, \zeta) \approx \sum_{n=0}^{\infty} \epsilon^n v^{(n)}(s, \zeta) \quad (35)$$

However, if we do so the lowest order contribution $v^{(0)}(s, \zeta)$ satisfies Eq. (32) for $\epsilon = 0$ and this is a driven linear oscillator equation with a driving term having the same frequency as the undriven oscillator. Hence a secular term will arise meaning that at least to this order v is unbounded. This result is in contradiction with the fact that the additional focusing taken proportional to the wakefield detunes the resonant effect generated by the right hand side of (32) and we expect a bounded solution for $\lambda > 0$. In order to avoid this effect and to keep the detuning property through all orders in the perturbation we use the method of **partial perturbation expansion** worked out by the authors. It consists in replacing ϵ in the detuning part (33) of Eq. (32) by a general parameter E and to expand only w.r.to the perturbation parameter ϵ multiplying the integral term on the right hand side of this equation. At the end of the computations E is reset to ϵ to result in

$$v(s, \zeta) \approx \sum_{n=0}^N v^{(n)}(s, \zeta; \epsilon) \epsilon^n \quad (36)$$

A detailed treatment of this method together with a mathematical justification of its validity is presented in the Appendix. Following this method we obtain the following equations for the

contributions $v^{(0)}(s, \zeta; E)$ and $v^{(1)}(s, \zeta; E)$:

$$\frac{\partial^2 v^{(0)}}{\partial s^2} + \bar{q}^2(\zeta; E)v^{(0)} = \frac{CW_0}{\gamma_0 l_b} \cos qs \left[[\alpha_0(1 - \lambda) - \alpha_1 l_b \zeta \lambda] \int_0^\zeta l_b^2 \rho(l_b \zeta^*) (\zeta - \zeta^*) d\zeta^* + \alpha_1 \int_0^\zeta l_b^3 \zeta^* \rho(l_b \zeta) (\zeta - \zeta^*) d\zeta^* \right] \quad (37)$$

$$\frac{\partial^2 v^{(1)}}{\partial s^2} + \bar{q}^2(\zeta; E)v^{(1)} = \frac{CW_0}{\gamma_0 l_b} \int_0^\zeta l_b^2 \rho(l_b \zeta^*) (\zeta - \zeta^*) v^{(0)}(s, \zeta^*; E) d\zeta^* \quad (38)$$

As it can be verified no secular term will appear in the solutions of Eq. (37) and (38) since the exciter frequencies in the driving terms are different from $\bar{q}(\zeta)$ which appears in the homogeneous parts of the perturbation equations.

3.3.2 Zero and first order result

The perturbation equations (37) and (38) are linear driven-oscillator equations of the type

$$u'' + q^2 u = F(s) \quad (39)$$

and are solved by the Greens function formula:

$$u(s) = \frac{\sin qs}{q} \int_0^s \cos qs^* F(s^*) ds^* - \frac{\cos qs}{q} \int_0^s \sin qs^* F(s^*) ds^* \quad (40)$$

For $v^{(0)}$ using this method we get after setting $E = \epsilon = 1$:

$$v^{(0)}(s, \zeta) = \left[\alpha_0 \frac{1 - \lambda}{\lambda} + \left(-\zeta + \frac{\zeta}{35\lambda} \frac{800\zeta^4 - 2240\zeta^3 + 1659\zeta^2 + 35\zeta + 70}{32\zeta^4 - 96\zeta^3 + 79\zeta^2 + 2\zeta + 6} \right) l_b \alpha_1 \right] \times [\cos qs - \cos(\bar{q}(\zeta)s)] \quad (41)$$

Using once more a Chebyshev expansion it can be seen that

$$\zeta \frac{800\zeta^4 - 2240\zeta^3 + 1659\zeta^2 + 35\zeta + 70}{32\zeta^4 - 96\zeta^3 + 79\zeta^2 + 2\zeta + 6} \approx 15\zeta \quad (42)$$

within $0 < \zeta < 1$ and thus a simplified expression for $v^{(0)}$ of the form

$$v^{(0)}(s, \zeta) = \left[\alpha_0 \frac{\lambda - 1}{\lambda} + \left(-\zeta + \frac{3\zeta}{7\lambda} \right) \alpha_1 l_b \right] [\cos qs - \cos(\bar{q}(\zeta)s)] = A(\zeta) [\cos qs - \cos(\bar{q}(\zeta)s)] \quad (43)$$

is obtained. In addition the ζ dependent tune given by Eq. (34) is approximated by a quadratic polynomial as

$$\bar{q}(\zeta) \approx q + \frac{\lambda CW_0}{4\gamma_0 q} \zeta^2 = q + a\lambda \zeta^2 \quad (44)$$

This approximation has been chosen such that \bar{q} agrees with the full expression (34) at $\zeta = 0$ and $\zeta = 1$.

From Eq. (38) and by once more applying the Greens function formula (40) $v^{(1)}(s, \zeta)$ reads as

$$v^{(1)}(s, \zeta) = \frac{CW_0}{\gamma_0 l_b} \left[\frac{\sin(\bar{q}(\zeta)s)}{\bar{q}(\zeta)} \int_0^s \int_0^\zeta K(\zeta, \zeta^*) \cos(\bar{q}(\zeta)s) [\cos qs^* - \cos(\bar{q}(\zeta^*)s^*)] d\zeta^* ds^* - \frac{\cos(\bar{q}(\zeta)s)}{\bar{q}(\zeta)} \int_0^s \int_0^\zeta K(\zeta, \zeta^*) \sin(\bar{q}(\zeta)s) [\sin qs^* - \cos(\bar{q}(\zeta^*)s^*)] d\zeta^* ds^* \right] \quad (45)$$

with

$$K(\zeta, \zeta^*) = l_b^2 \rho(l_b \zeta^*)(\zeta - \zeta^*) A(\zeta^*) \quad (46)$$

Doing first the integration w.r.t s and keeping only the low frequency terms to this perturbation order gives

$$\begin{aligned} v^{(1)}(s, \zeta) = & \frac{CW_0}{\gamma_0 l_b} \frac{\sin(\bar{q}(\zeta)s)}{2\bar{q}(\zeta)} \left[\frac{\sin[(q - \bar{q}(\zeta))s]}{q - \bar{q}(\zeta)} \int_0^\zeta K(\zeta, \zeta^*) d\zeta^* - \right. \\ & \left. \int_0^\zeta K(\zeta, \zeta^*) \frac{\sin[(\bar{q}(\zeta) - \bar{q}(\zeta^*))s]}{\bar{q}(\zeta) - \bar{q}(\zeta^*)} d\zeta^* \right] - \\ & \frac{CW_0}{\gamma_0 l_b} \frac{\cos(\bar{q}(\zeta)s)}{2\bar{q}(\zeta)} \left[\frac{\cos[(q - \bar{q}(\zeta))s] - 1}{q - \bar{q}(\zeta)} \int_0^\zeta K(\zeta, \zeta^*) d\zeta^* - \right. \\ & \left. \int_0^\zeta K(\zeta, \zeta^*) \frac{\cos[(\bar{q}(\zeta) - \bar{q}(\zeta^*))s] - 1}{\bar{q}(\zeta) - \bar{q}(\zeta^*)} d\zeta^* \right] \end{aligned} \quad (47)$$

Computing the integral over $K(\zeta, \zeta^*)$ in the last equation and expressing the tune differences as functions of ζ and ζ^* we finally arrive at

$$\begin{aligned} v^{(1)}(s, \zeta) = & \frac{CW_0}{\gamma_0 l_b} \frac{\sin(\bar{q}(\zeta)s)}{2\bar{q}(\zeta)} \left[\frac{\sin(\lambda a \zeta^2 s)}{\lambda a \zeta^2} \left(\frac{(1 - \lambda)\alpha_0 l_b}{2\lambda} \zeta^2 - \left(1 - \frac{3}{7\lambda}\right) \frac{\alpha_1 l_b^2}{5} \zeta^3 \right) - \right. \\ & \left. \int_0^\zeta K_R(\zeta^*) \frac{\sin(\lambda a(\zeta^2 - \zeta^{*2})s)}{\lambda a(\zeta + \zeta^*)} d\zeta^* \right] - \\ & \frac{CW_0}{\gamma_0 l_b} \frac{\cos(\bar{q}(\zeta)s)}{2\bar{q}(\zeta)} \left[-\frac{\cos(\lambda a \zeta^2 s) - 1}{\lambda a \zeta^2} \left(\frac{(1 - \lambda)\alpha_0 l_b}{2\lambda} \zeta^2 - \left(1 - \frac{3}{7\lambda}\right) \frac{\alpha_1 l_b^2}{5} \zeta^3 \right) - \right. \\ & \left. \int_0^\zeta K_R(\zeta^*) \frac{\cos(\lambda a(\zeta^2 - \zeta^{*2})s) - 1}{\lambda a(\zeta + \zeta^*)} d\zeta^* \right] \end{aligned} \quad (48)$$

The function K_R is defined as

$$K_R(\zeta^*) = \frac{K(\zeta, \zeta^*)}{\zeta - \zeta^*} = P_3(\zeta^*) \left[\left(1 - \frac{1}{\lambda}\right) \alpha_0 l_b + \zeta^* \left(1 - \frac{3}{7\lambda}\right) \alpha_1 l_b^2 \right] \quad (49)$$

with

$$P_3(\zeta^*) = -\frac{6}{23} 80 \zeta^{*4} - 160 \zeta^{*3} + 79 \zeta^{*2} + \zeta^* + 1 \quad (50)$$

Unfortunately no closed form representation of the two integrals

$$I_1(\zeta) = \frac{1}{a\lambda} \int_0^\zeta K_R(\zeta^*) \frac{\sin(\lambda a(\zeta^{*2} - \zeta^2)s)}{\zeta + \zeta^*} d\zeta^* = \frac{1}{\lambda} \int_0^\zeta F_1(\zeta, \zeta^*) d\zeta^* \quad (51)$$

$$I_2(\zeta) = \frac{1}{a\lambda} \int_0^\zeta K_R(\zeta^*) \frac{\cos(\lambda a(\zeta^{*2} - \zeta^2)s) - 1}{\zeta + \zeta^*} d\zeta^* = \frac{1}{\lambda} \int_0^\zeta F_2(\zeta, \zeta^*) d\zeta^* \quad (52)$$

could be found. For this reason I_1 and I_2 have been approximated by a Simpsons rule over the integration limits in ζ^* with the three samples $F(\zeta, 0)$, $F(\zeta, \frac{\zeta}{2})$ and $F(\zeta, \zeta) = 0$. Hence,

$$I_{1,2} \approx \frac{\zeta}{6\lambda} [F_{1,2}(\zeta, 0) + 4F_{1,2}(\zeta, \frac{\zeta}{2})] \quad (53)$$

This last approximation is good for sufficiently small values of s where the integrand is a monotonic function in $0 < \zeta^* < 1$ and is also good for large s where the integrand is rapidly oscillating and the integral tends to average out these oscillations. However, there might exist a transition region between the two regimes where the Simpsons approximation becomes weak and more samples would be required.

4 Evaluation of the results

Having to our disposal the analytic solution of Eq. (12) in the form (note that the formal perturbation parameters ϵ and E have been set to unity)

$$\begin{aligned}
 x(s, \zeta) &= (\alpha_0 + \alpha_1 \zeta l_b) \cos qs + \left[\alpha_0 \frac{1 - \lambda}{\lambda} + \alpha_1 l_b \zeta \left(-1 + \frac{3}{7\lambda} \right) \right] [\cos(\bar{q}(\zeta)s) - \cos qs] + \\
 v^{(1)}(\zeta, s) & \\
 \zeta &= \frac{z}{l_b}
 \end{aligned} \tag{54}$$

where $v^{(1)}$ is given by Eq. (48), we want to study the behaviour of the solution when applied to the CLIC main linac. For this purpose we use the following parameters:

Parameter	Value	Unit
α_0	$-10 < \alpha_0 < 10$	μm
α_1	$-0.5 < \alpha_1 < 0.5$	-
γ_0	18000	-
W_0	$7 \cdot 10^{17}$	$\frac{V}{Asm^2}$
$q = \frac{1}{\beta_u}$	0.2669	$\frac{1}{m}$
$l_b (4\sigma_z)$	200	μm

Table 1: CLIC main linac parameters

The initial displacement α_0 has been set to the RMS misalignments of the quadrupoles and cavities of $-10\mu m < \alpha_0 < +10\mu m$ while for the initial slope α_1 in ζ values out of the interval $-0.5 < \alpha_1 < 0.5$ are used. The energy Lorentz factor γ_0 corresponds to the injection energy of $9GeV$ and the transverse dipole wakefield at the tail of the bunch ($z = 4\sigma_z$) has been taken from our simulation program MUSTAFA ([2]). In the two following subsections we show the behaviour of the solution we obtained, compare our result to some tracking examples and derive a closed expression for the asymptotic emittance blow-up as a function of all the parameters.

4.1 Behaviour of the solution

The following series of figures shows the typical behaviour of the solution $x(s, \zeta)$ as the bunch proceeds through the CLIC main linac. In all the subsequent figures $X(s)$, the coherent betatron part of x , has been suppressed and only

$$y(s, \zeta) = \alpha_1 \zeta l_b \cos qs + v(s, \zeta) \tag{55}$$

is plotted. The first two cases show the solution for $\lambda = 0$, i.e. no focusing force additional to the normal magnetic focusing has been applied. Hence, we are in a resonant situation and we observe increasing oscillation amplitudes as s increases. In Fig.4 we represent the solution for $\alpha_1 = 0$ at a distance of $80m$ after injection and the same situation at $520m$ in Fig.5. While the full line relates to the analytic result the points are showing the result obtained by the tracking with the MUSTAFA code [2]. It is obvious that the agreement between the analysis and the simulation is very good. At the given distance of $80m$ downstream the linac we see a coherent oscillation growing in amplitude as we proceed from the head towards the tail of the bunch. At $520m$ - due to the resonant behaviour of this example the amplitude of oscillation has grown considerably but we also observe a certain detuning across the bunch that is due to the integral equation part

in Eq. (32) and is described by the first order perturbation contribution $v^{(1)}(s, \zeta)$. Hence this term is able in principle to stabilize the motion although this only takes place at comparably large amplitudes.

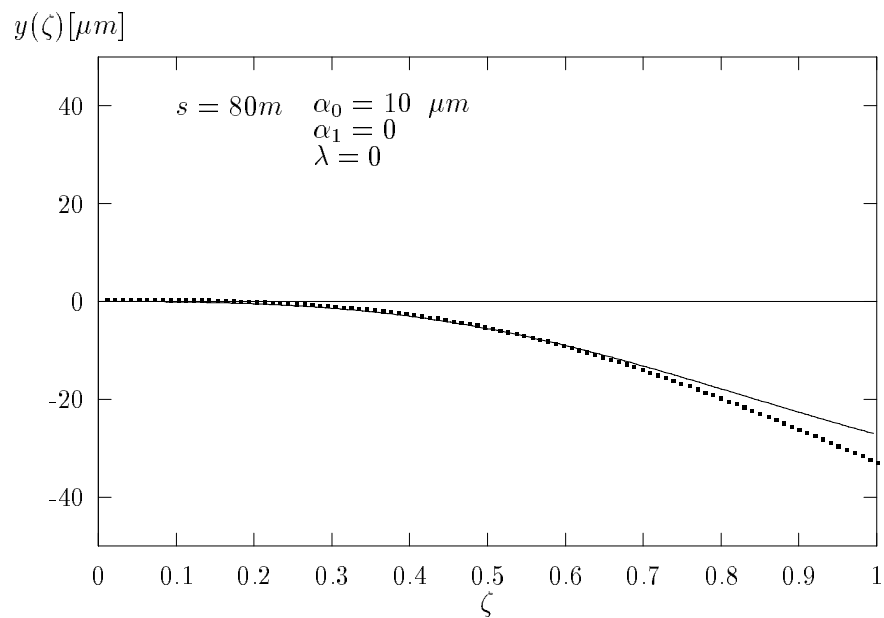


Fig. 4 Resonant solution in CLIC at $s = 80m$

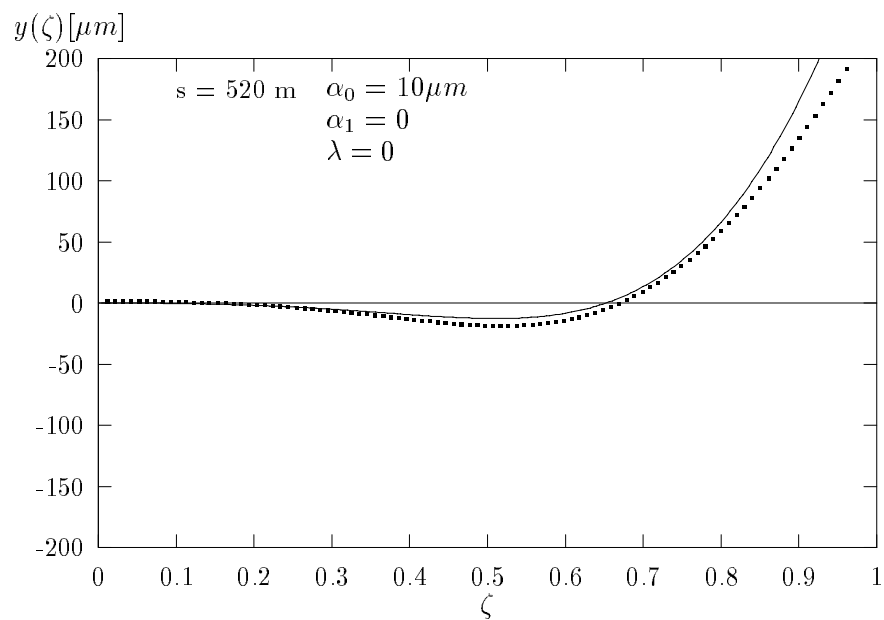


Fig. 5 Resonant solution in CLIC at $s = 520 m$

In the next figure (Fig.6) we use $\alpha_0 = -10\mu m$ and $\alpha_1 = 0.5$ and include an additional focusing of a strength equal to 70% of the one representing the autophasing condition ($\lambda = 0.7$). The

distance downstream the linac $s = 520$ m has been kept from the previous example.

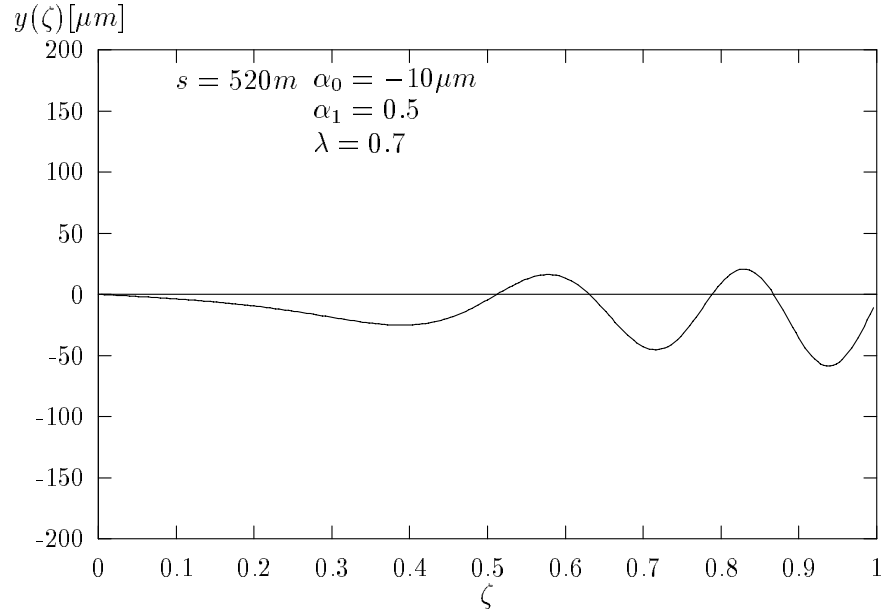


Fig. 6 Detuned solution in CLIC with $\lambda = 0.7$

In Fig.7 we reproduce the same situation but for $\lambda = 1$, meaning that we exactly fulfill the autophasing condition.

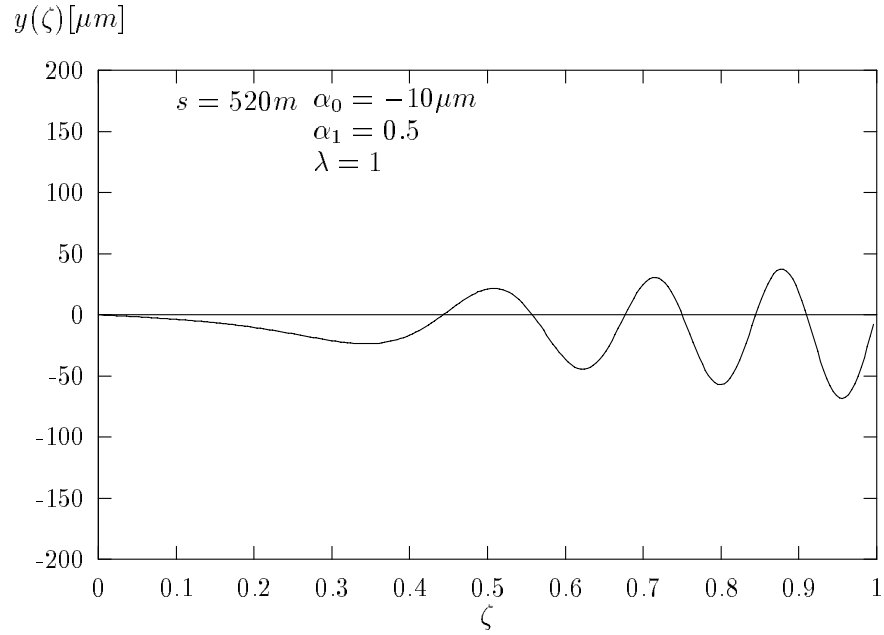


Fig. 7 Autophasing solution in CLIC

The rising decoherence along the bunch can be well observed in the following two figures (Figs.8 - 9), which show y for $s = 1000$ m and $s = 2000$ m.

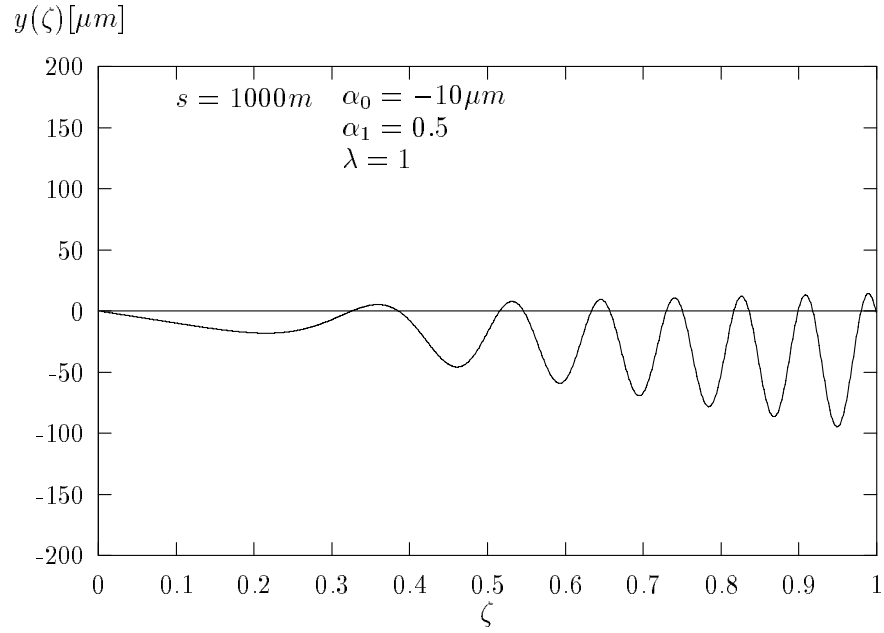


Fig. 8 Autophasing solution at $s = 1000 m$

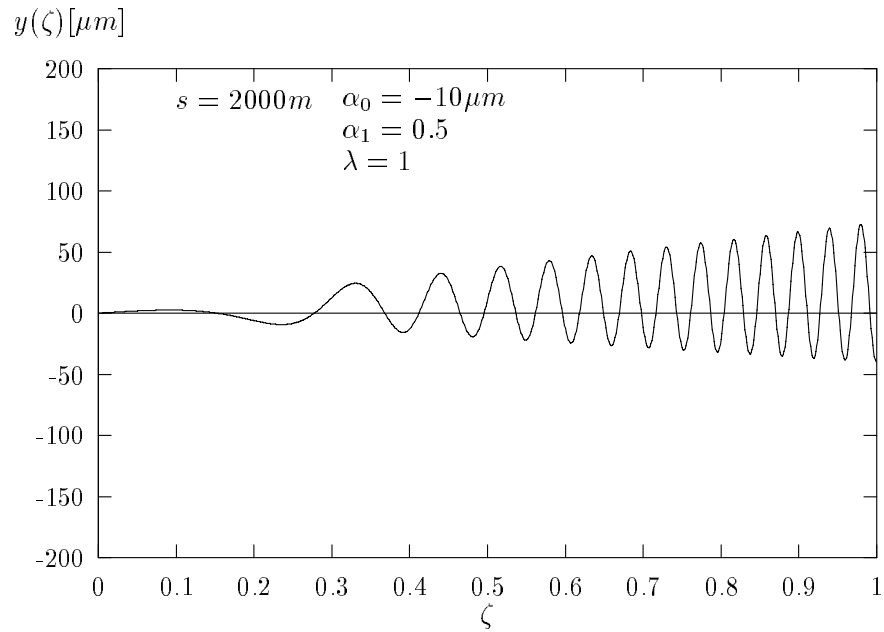


Fig. 9 Autophasing solution at $s = 2000m$

It is interesting to compare the cases of $\lambda = 0.7$ and $\lambda = 1$ at the same distance $s = 520 m$ and the same initial conditions (Figs.6 - 7): We see that in the autophasing case the oscillation contains two more nodes compared to the case $\lambda = 0.7$, i.e. there is more incoherence visible in the autophasing condition. Hence we suspect (as has been discussed in section 3.1) that the minimum emittance blow up in this case will be closer to $\lambda = 0.7$ than to $\lambda = 1$. Qualitatively this is linked to the fact that the amplitudes of the oscillations within the bunch are comparable but slightly larger in the case $\lambda = 1$ and that the sum of the square amplitudes is likely larger in

the presence of more nodes or oscillation waves. This shift of the minimum emittance from $\lambda = 1$ to lower values of λ will be shown and explained in the following section. However, this effect is depending on the actual value of α_1 , i.e. on the initial slope along the bunch. For $\alpha_1 = 0$ and $\lambda = 1$ the solution $y = 0$ of Eq. (22) is compatible with the initial conditions (23) and (24) and hence is an exact solution of our equation. Then of course no emittance blow-up is present.

4.2 Emittance Dilution

Since a low emittance of the beam at the interaction point is needed to provide high luminosity collisions it is interesting to study the emittance dilution due to wakefields in the accelerating structures of the collider. If we consider the emittance increase due to transverse wakes in a single bunch, the total normalised emittance at the end of the main linac is given by

$$\gamma\epsilon_{tot} = \gamma\epsilon_{inj} + \Delta(\gamma\epsilon_y) \quad (56)$$

where

$$\Delta(\gamma\epsilon_y) = l_b\gamma_0 \int_0^1 \rho(\zeta) \left[qy^2(s, \zeta) + \frac{1}{q} \left(\frac{\partial y}{\partial s}(s, \zeta) \right)^2 \right] d\zeta \quad (57)$$

and y is given by (55). Instead of v we will use $v^{(0)}(s, \zeta)$ as given in Eq. (41) since it is believed to give the strongest contribution. Although y has then a relatively simple form, the integral in (57) becomes non elementary, leading to a very complex expression with trigonometric and Fresnel functions and extends over several pages. Hence, for the general case we decided to replace the analytic expression for the integral by curves obtained from a numerical procedure using Romberg integration [17].

4.2.1 Dilution in the absence of external detuning

At first, we consider a case with no external detuning (neglecting the detuning due to longitudinal wakefields) across the bunch ($\lambda = 0$). In such an example, the emittance is unboundedly growing since the tail of the bunch is in resonance with the head. The figure below illustrates this resonant situation for initial conditions which either include or discard a slope with z along the bunch. In both cases, the emittance grows quadratically to infinity as expected from a linear increase of the amplitude y or x .

In this special example a simple analytic expression for $\Delta(\gamma\epsilon_y)$ can also be obtained. In order to derive it we first take the limit of y as λ tends to zero. Applying the rule of De L'Hopital to Eq. (41) and using Eq. (55) we get

$$\lim_{\lambda \rightarrow 0} y(s, \zeta) = \alpha_1 l_b \zeta \cos qs + \frac{CW_0}{q\gamma_0} \left(\frac{1}{4}\alpha_0 \zeta^2 + \frac{3}{28}l_b \alpha_1 \zeta^3 \right) s \sin qs \quad (58)$$

Equation 58 explicitly shows what is called a *secular term* associated to a situation of resonance. The amplitude, solution of the equation of motion, is given by a product of the form $s \sin qs$ and therefore linearly rises with the distance s along the linac.

Inserting this limit value for y into (57) and performing the integration over ζ - containing just polynomials in this case - we find (when only the dominant secular terms, proportional to s^2 for the emittance, are retained):

$$\Delta(\gamma\epsilon_y) = \frac{C^2\alpha_0 W_0^2}{q\gamma_0} \left[\frac{205}{18032}\alpha_1 l_b + \frac{341}{19320}\alpha_0 \right] s^2 \quad (59)$$

This formula represents very well the quadratic behaviour of the curves seen in Fig.10 (note that for instance in the case $\alpha_1 = 0.2$ the emittance at 800 m is four times the one at 400 m).

$$\gamma_0 \Delta \epsilon_y [10^7] \text{ m}$$

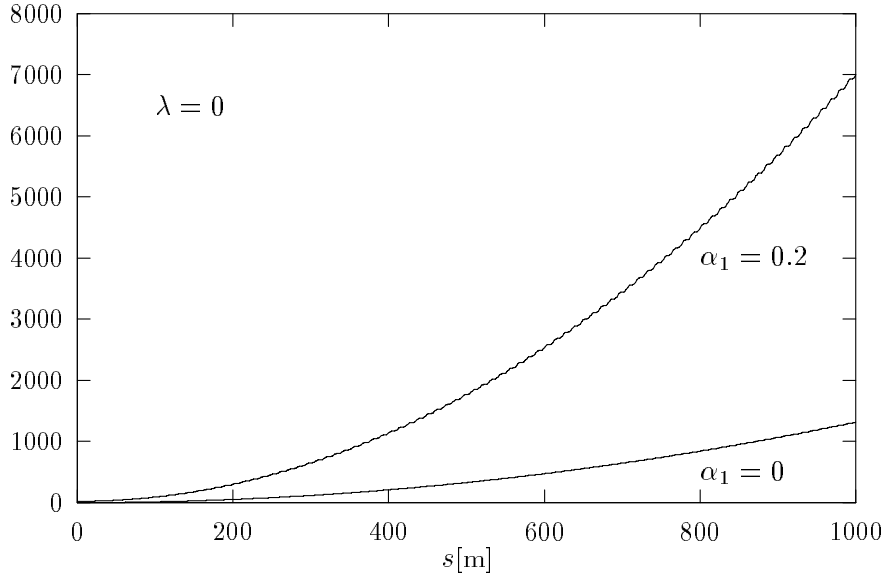


Fig. 10 Emittance increase as function of s in the resonant case with $\alpha_1 = 0$ and $\alpha_1 = 0.2$

4.2.2 Dilution in the presence of BNS damping

Next, some BNS damping is supposed to be applied $\lambda \neq 0$ and the effect of the corresponding detuning on the emittance is presented. We choose $\alpha_0 = 10 \mu\text{m}$, $\alpha_1 = 0$ as before and $\lambda = 0.7$. Comparing to the first example of the preceding section, we now are off the resonance because of the detuning present across the bunch. Therefore, after some initial growth, the emittance tends to a constant value and remains much smaller than in the resonant case. The two lines in Fig.11 correspond to the evaluation of the analytic formula (57), and the tracking results obtained with MUSTAFA [2], respectively. In the numerical simulations, RF quadrupoles have been added to the normal magnetic focusing lattice, the strengths of which have been set to 70 % (according to $\lambda = 0.7$) of the strengths required for minimizing the emittance growth after 1000m.

Note that in the tracking code we are not able to exactly reproduce the situation of the analytic model since the focusing function produced by the RF quadrupoles follows essentially the RF sine wave while in the analysis the focusing function reproduces exactly the wakefield integral,

$$\int_0^z \rho(z^*) w(z - z^*) dz^* \quad (60)$$

and these two functions are never identical, as underlined in section 3.1. However, the agreement - specially of the asymptotic value of the emittance for large s - is good.

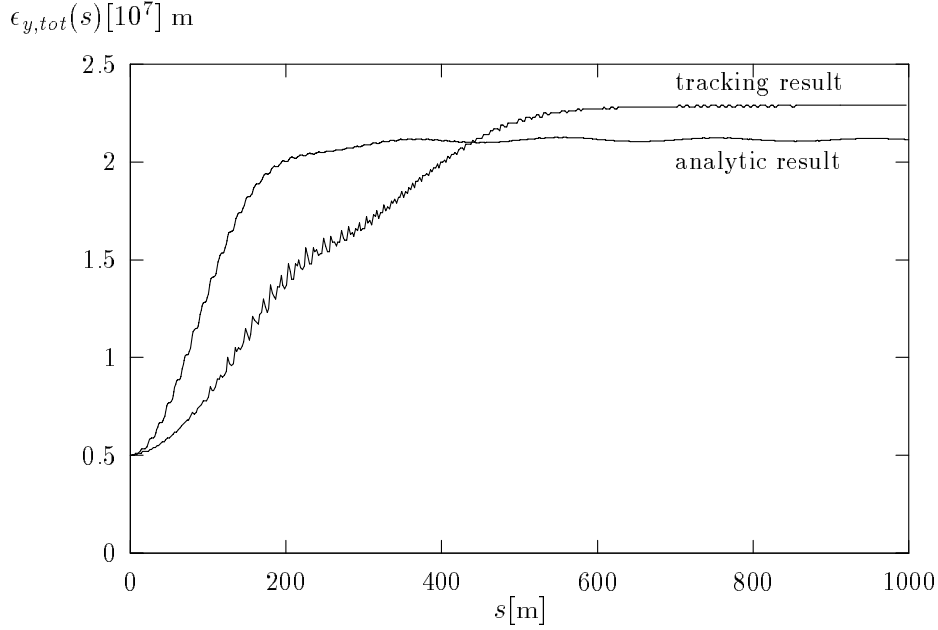


Fig. 11 Total emittance as function of s when $\alpha_0 = 10\mu m$, $\alpha_1 = 0$ and $\lambda = 0.7$

Due to the near resonant situation for $\lambda > 0$ the emittance first starts to grow but finally becomes stabilised due to the increasing decoherence of the bunch caused by the detuning at nearly constant amplitude (see Figs.7-9). For large values of s more oscillations appear as we scan through the bunch from head to tail and they all increase in frequency as the bunch advances along the linac. Hence the projected area of all the parts of the bunch tends to a constant as s increases if the amplitudes remains bounded. However, the amplitudes **are bounded** since every slice of the bunch, actually submitted to detuning, oscillates off the resonance with a beating amplitude defined by its proper distance in frequency relative to the resonant case $\lambda = 0$.

We now compute the limiting upper bound value of the emittance as s tends to infinity and $\lambda > 0$. This is achieved by considering the fact that the integrand in (57) contains infinitely fast oscillating terms in ζ as $s \rightarrow \infty$ because of the term $\cos \bar{q}(\zeta)s$ that occurs in y and $\frac{\partial y}{\partial s}$, $\bar{q}(\zeta)$ being a monotonic function. As s tends to infinity these terms will not influence the integral value since they are averaged out by quadrature. Hence they are replaced by zero before integration. In this case the integrand becomes polynomial in ζ and the integral can be carried out analytically leading to the expression

$$\begin{aligned}
\lim_{s \rightarrow \infty} (\gamma_0 \epsilon_y) &= \gamma_0 \epsilon_{inj} + \left[\frac{4329}{39445} \gamma_0 q \alpha_1^2 l_b^2 + \frac{6}{7} \gamma_0 q \alpha_0 \alpha_1 l_b + 2 \gamma_0 q \alpha_0^2 \right] \lambda^{-2} + \\
&\left[\left(-\frac{1443}{5635} \gamma_0 q + \frac{1023}{157780} \frac{C W_0}{q} \right) l_b^2 \alpha_1^2 + \left(-\frac{13}{7} \gamma_0 q + \frac{957}{22540} \frac{C W_0}{q} \right) l_b \alpha_0 \alpha_1 + \right. \\
&\left. \left(-4 \gamma_0 q + \frac{481}{6440} \frac{C W_0}{q} \right) \alpha_0^2 \right] \lambda^{-1} + \\
&\left(\frac{481}{1610} \gamma_0 q - \frac{341}{11270} \frac{C W_0}{q} + \frac{5295}{11107712} \frac{C^2 W_0^2}{\gamma_0 q^3} \right) l_b^2 \alpha_1^2 + \\
&\left(\gamma_0 q - \frac{319}{2254} \frac{C W_0}{q} + \frac{205}{72128} \frac{C^2 W_0^2}{\gamma_0 q^3} \right) l_b \alpha_0 \alpha_1 + \left(2 \gamma_0 q - \frac{481}{3220} \frac{C W_0}{q} + \frac{341}{77280} \frac{C^2 W_0^2}{\gamma_0 q^3} \right) \alpha_0^2 \\
&+ \left[\left(\frac{341}{9660} \frac{C W_0}{q} - \frac{1765}{793408} \frac{C^2 W_0^2}{\gamma_0 q^3} \right) l_b^2 \alpha_1^2 + \left(\frac{319}{3220} \frac{C W_0}{q} - \frac{1025}{108192} \frac{C^2 W_0^2}{\gamma_0 q^3} \right) l_b \alpha_0 \alpha_1 + \right. \\
&\left. \left(\frac{481}{6440} \frac{C W_0}{q} - \frac{341}{38640} \frac{C^2 W_0^2}{\gamma_0 q^3} \right) \alpha_0^2 \right] \lambda + \\
&\left[\frac{1765}{680064} \frac{C^2 W_0^2 \alpha_1^2 l_b^2}{\gamma_0 q^3} + \frac{205}{30912} \frac{C^2 W_0^2 \alpha_0 \alpha_1 l_b}{\gamma_0 q^3} + \frac{341}{77280} \frac{C^2 W_0^2 \alpha_0^2}{\gamma_0 q^3} \right] \lambda^2 \tag{61}
\end{aligned}$$

Using once again the parameters of the CLIC main linac as given in Table 1, $\alpha_0 = 10 \mu m$, $\alpha_1 = 0$ and a vertical injected emittance of $0.5 \cdot 10^{-7}$, this formula gives the asymptotic value of $2.29 \cdot 10^{-7} mrad$ which is in perfect agreement with the value $2.28 \cdot 10^{-7}$ found by Romberg integration at $s = 1000$ m (see also Fig.11).

At first view it seems striking that the limit value for the emittance according to (61) is different from zero if the wakefield W_0 is zero even if α_1 , the initial slope, is set to zero. However, this comes from the fact that even infinitesimally small wakefields will drive a certain emittance growth after some (infinitely long) distance down the linac and this growth will eventually reach an asymptotic value. In order to know if the asymptotic value given in the above equation is relevant for a given linac length we need information about the distance at which the asymptotic value is nearly reached. Such an estimate can be made by taking into account the fact that the solution $v^{(0)}(s, \zeta)$ contains two close frequencies q and \bar{q} in the form

$$v^{(0)} = A(\zeta) [\cos qs - \cos \bar{q}(\zeta)s] \tag{62}$$

Hence the minimum distance after which the asymptotic behaviour can be observed is given by the beating period between these two cosine functions. If we compute this beating at the center of the bunch where the charge density is maximum we find

$$s_{crit} = \frac{2\pi}{\bar{q}(\frac{1}{2}) - q} = \frac{32\pi \gamma_0 q}{\lambda C W_0} \tag{63}$$

Using as before the CLIC parameters we find

$$s_{crit} \approx 800 m \tag{64}$$

It may also look surprising that, the amplitudes being bounded, the emittance tends to an asymptotic value as s increases to infinity (infinitely long linac), according to Eq. (61), i.e. the normalised emittance does not grow any more when s is large enough. However, this can easily be understood. Indeed, in the simplified model of Section 2.1, the random misalignments of the

components have been represented by appropriate initial conditions which include a slope in z along the bunch. Strictly speaking, this is equivalent to a single misalignment of any component (e.g. a quadrupole) at an arbitrary position s in the linac (which obviously introduces a kick and an off-set downstream, but also a z -slope, because of the unavoidable chromaticity) and the preceding conclusion strictly applies. In a real linac however, there is a large succession of such misalignments, the number of which increases with the linac length. If they are random, their effects on the transverse oscillation amplitude will combine quadratically and the emittance is expected to monotonously grow with the total number of the misaligned components in the linac, hence with the linac length. The emittance will then result from the addition of contributions of the type of Eq. 61, but remain bounded (though arbitrarily large) since the number of components is always finite in a given linac.

4.2.3 Variation of the Emittance Dilution with the BNS Damping Parameter λ

We are eventually interested in the asymptotic value of emittance growth as a function of the additional focusing given by λ . According to the results derived in the previous sections, $\lambda = 0$ corresponds to a resonance and the asymptotic emittance tends to infinity as seen from (61). When λ increases above zero the emittance is expected to decrease significantly. However, for large values of λ the terms proportional to λ and λ^2 become dominant and finally will raise the emittance. Hence we find a certain value of λ for which the emittance growth will reach a minimum. The following figures 12-14 demonstrate this effect. Besides using the CLIC parameters from Table 1 as before we select $\alpha_0 = -10\mu m$ and various values $\alpha_1 = 0.5, 0.25$ and 0 respectively.

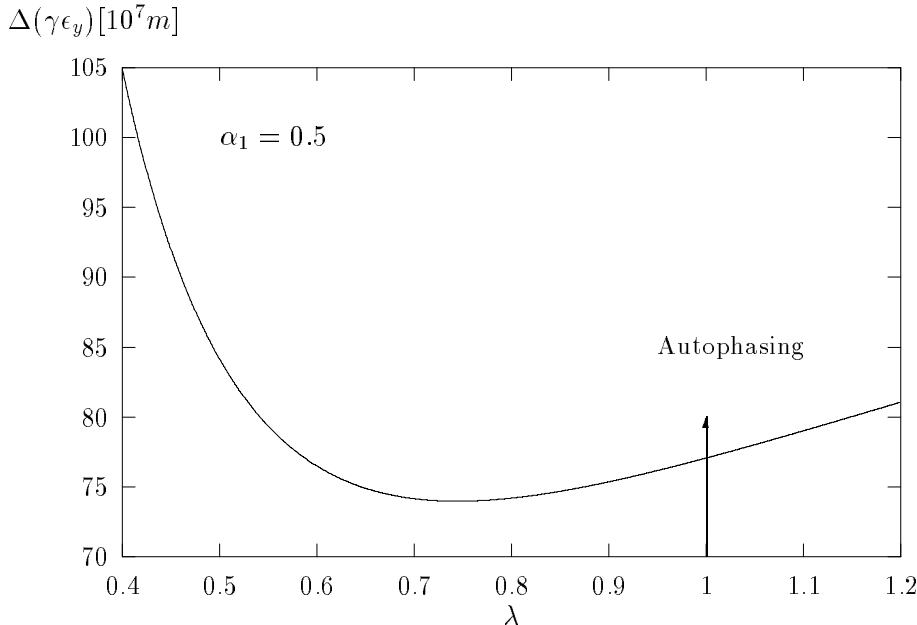


Fig. 12 Asymptotic emittance growth as function of λ when $\alpha_1 = 0.5$

Fig.12 for $\alpha_1 = 0.5$ confirms the conjecture made in section 3.1 stating that **the autophasing case in general is not leading to the minimum single bunch emittance**. Instead the minimum is shifted to a lower value of λ where the two effects, increasing decoherence of the bunch with λ (leading to emittance increase) and increasing distance from the resonance with λ (leading to emittance decrease) are best balanced. The actual minimum of emittance in this case appears at a value of λ about 75% of the one corresponding to the autophasing condition.

In the next figure $\alpha_1 = 0.25$ and the minimum emittance is shifted from 0.75 to 0.83. The reason for this change is that we are closer to the case $\alpha_1 = 0$ where $v(s, \zeta)$ becomes exactly zero according to the initial conditions $v(0, \zeta) = \frac{\partial v}{\partial s}(0, \zeta) = 0$.

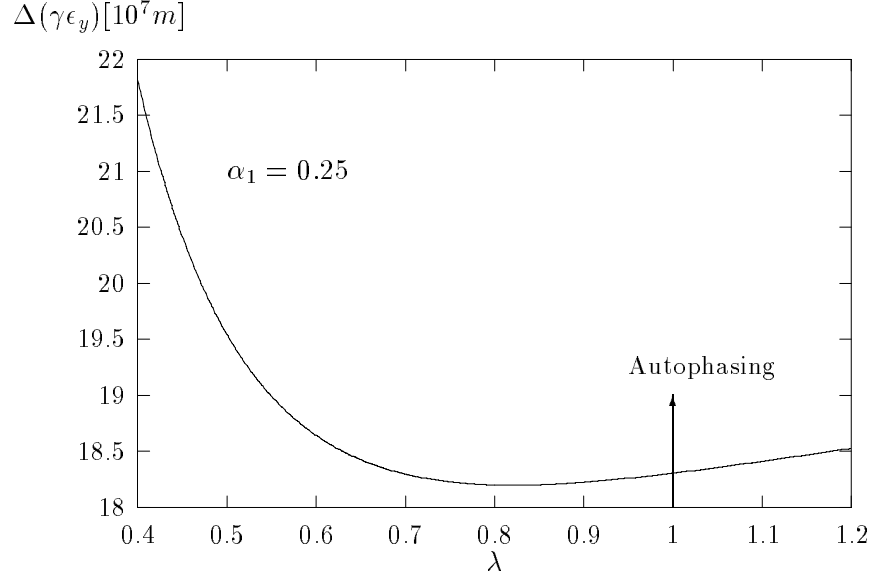


Fig. 13 Asymptotic emittance growth as function of λ when $\alpha_1 = 0.25$

Finally in the last figure we show the case $\alpha_1 = 0$ (no initial slope across the bunch) and indeed the minimum emittance ($\Delta\gamma_0\epsilon_y = 0$!) occurs for the exact autophasing condition.

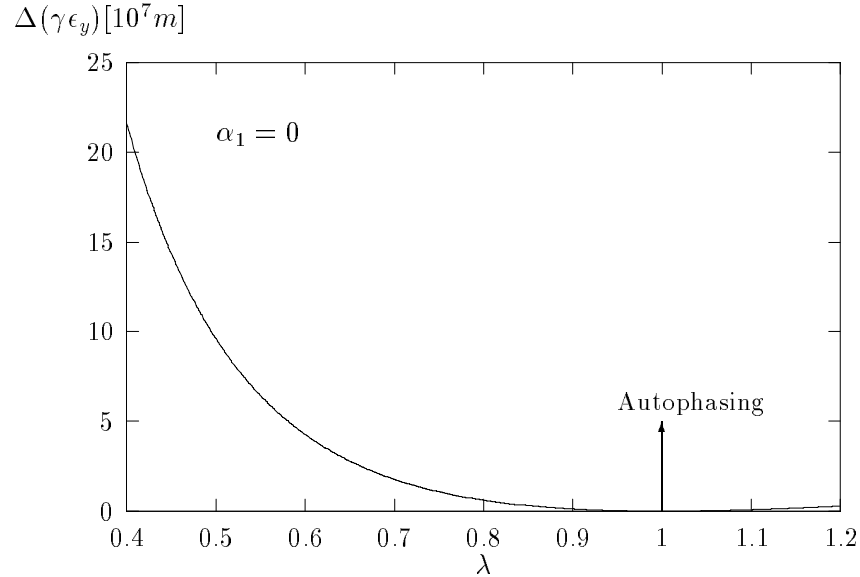


Fig. 14 Asymptotic emittance growth as function of λ when $\alpha_1 = 0$

However, in the actual linac, quadrupole and cavity misalignments are unavoidable and therefore a slope in z along the bunch is always present, since generated by these misalignments over the first few tens of meters of travelling in the linac. This precisely corresponds to simultaneously use $\alpha_0 \neq 0$ as well as $\alpha_1 \neq 0$ in our model and, in this sense, the last figure (Fig.14) is neither realistic nor specially meaningful in practice.

5 Conclusions

Starting from the general equation of the single bunch motion in the transverse plane, some simplifications are introduced in the model before searching for a solution. The most important one is that, instead of including the effects of random cavity and quadrupole misalignments, we solve the problem with initial conditions which correspond to an injection offset α_0 of the order of the r.m.s. misalignment as well as to a slope $\alpha_1 z$ along the bunch representing the unavoidable presence of such a slope when the bunch traverses misaligned components. In addition, the linac is supposed to be divided in sectors and the effect of the acceleration within each sector is not included, which gives a pessimistic representation of the reality. The transverse wakefield variation in the bunch is taken linear and the gaussian charge density is well approximated by Chebyshev polynomials.

In the process of deriving a solution, the vertical displacement $x(s)$ of a slice of charge is resolved in two variables, one termed $X(s)$ which only depends on the distance s and another $y(s, z)$ which depends on s as well as the position z in the bunch. The second variable y is in turn split in two parts, the first one satisfying the initial slope-condition $\alpha_1 z$ while the second called $v(s, z)$ is chosen in such a way as to become zero when both the wakefields and the initial slope vanish. A perturbation treatment is then applied to the equation for v , considering the wakefield as the perturbation to the betatron motion. However, a particular partial perturbation expansion worked out by the authors (see Appendix) is used in order to keep the detuning property through all orders and hence prevent the generation of artificial resonant terms. With this method, zero and first order solutions have been derived as a function of a parameter λ which represents the fraction of the wakefield driving term that is damped by focusing variation across the bunch, using either microwave quadrupoles or relative momentum modulation. Autophasing corresponds to $\lambda = 1$, for which the driving term is exactly compensated when $\alpha_1 = 0$, since the damping function in our model is assumed to be the same as and proportional to the wakefield perturbation integral. At $\lambda = 0$, no damping is applied but a small detuning still appears and slowly stabilizes the beam because of the presence, to the first order of the perturbation, of an integrand proportional to v that shifts the frequency of the motion at large s . Various levels of BNS damping correspond to values of λ between 0 and 1.

The behaviour of the analytical solution is evaluated with the CLIC parameters and compared with the simulations done with the code MUSTAFA. The agreement between the two is shown to be good in the case $\alpha_1 = 0$ and $\lambda = 0$, i.e. in the presence of the small natural detuning mentioned above. Examples with $\alpha_1 = 0.5$ and different damping fraction λ exhibit the rise of oscillations along the bunch due to detuning, with a number of nodes or oscillation periods that increases with the distance s but an amplitude that remains about the same. It is recognized that the emittance growth rises with the number of nodes and therefore with the distance s , in these examples.

The emittance dilution is estimated by keeping the zero order term which is believed to give the strongest contribution. A closed expression is proposed in the limit of absence of damping ($\lambda = 0$) and confirms the expected quadratic increase of the emittance in such a resonant situation. In the presence of a significant detuning fraction ($\lambda = 0.7$), the emittance first grows rapidly with s and then reaches an asymptotic value when the stabilizing effect takes place. Here again, analytical and numerical results are in good agreement.

For intermediate values $0 < \lambda < 1$, it is possible to find an analytical expression for the asymptotic value of the emittance blow-up as s tends to infinity, by averaging the fast oscillating terms before performing the quadrature. It is then very interesting to observe the variation of the asymptotic emittance with the damping fraction λ . At $\lambda = 0$, there is a resonance, the expression diverges and the emittance tends to infinity. As λ increases, the emittance is expected to decrease significantly because of the detuning and compensation effect. However, for large λ -values, the resonance is cancelled but the emittance rises again since detuning induces more and more nodes or oscillations within the bunch. This behaviour of the effective emittance is directly visible in the formula by the simultaneous presence of terms proportional to λ^{-1} and λ^{-2} responsible for the divergence when $\lambda \rightarrow 0$, as well as linear and quadratic terms with λ which explain the subsequent rise when λ is near or larger than 1. Hence, there exists a given λ -value for which the emittance blow-up is minimum and this confirms the observation made by numerical simulations. This minimum only lays at $\lambda = 1$ when $\alpha_1 = 0$, but occurs at some intermediate value $0 < \lambda < 1$ when α_1 is included. In other words, when $\alpha_1 \neq 0$, i.e. in the presence of linac-component misalignments, the conjecture that autophasing is not in general leading to the minimum single-bunch emittance is corroborated by the theory.

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A The partial perturbation expansion method

In this paper a non standard perturbation technique has been applied to preserve the detuning properties of the dynamical system which describes a charged bunch traveling in a linac with wakefields. In order to justify this method we investigate it for a general case of a linear operator and a linear or nonlinear perturbing function. Since in this work we only proceed to first order in the perturbation, also the theoretical treatment will be restricted to the same order although we will arrive at a conjecture covering arbitrary orders in the perturbation expansion.

We consider a problem of the kind:

$$L_{00}(X) + \epsilon L_{01}(X) = \epsilon P(X) \quad (65)$$

where L_{00} and L_{01} are linear algebraic or differential operators (e.g. $\frac{\partial^2}{\partial t^2} + \omega^2$) and $P(X)$ is a linear or nonlinear operator. As can be seen the linear part itself consists of an unperturbed part L_{00} and a weak contribution ϵL_{01} . We require that L_{00} and L_{01} are invertable and that the linearized problem

$$L_{00}(X) + \epsilon L_{01}(X) = 0 \quad (66)$$

allows a closed form solution for X . We now compare the classical straightforward perturbation expansion of this problem to the partial expansion method used in this paper.

- The classical (total) expansion method to first order:

We rewrite (67) as

$$L_{00}(X) = \epsilon [P(X) - L_{01}(X)] \quad (67)$$

and let

$$X = X_0 + \epsilon X_1 + O(\epsilon^2) \quad (68)$$

To zero and first order in ϵ we then obtain

$$L_{00}(X_0) = 0 \Rightarrow X_0 = L_{00}^{-1}(0) \quad (69)$$

$$L_{00}(X_1) = P(X_0) - L_{01}(X_0) \Rightarrow X_1 = L_{00}^{-1}[P(X_0) - L_{01}(X_0)] \quad (70)$$

and thus

$$X = L_{00}^{-1}(0) + \epsilon L_{00}^{-1}[P(L_{00}^{-1}(0)) - L_{01}L_{00}^{-1}(0)] + O(\epsilon^2) \quad (71)$$

- The partial expansion method:

In this case the entire linear operator serves as unperturbed part of the equation and a perturbation expansion is only applied to $\epsilon P(X)$. In order to provide this we replace ϵ in the linear operator part by a general parameter E which after performing the expansion on the remaining ϵ will be reset to ϵ . Hence,

$$L_{00}(X) + EL_{01}(X) = \epsilon P(X) \quad (72)$$

Expanding w.r.t. ϵ we are lead to

$$X = X_0(E) + \epsilon X_1(E) + O(\epsilon^2) \quad (73)$$

and the equations for $X_0(E)$ and $X_1(E)$ are

$$L_{00}(X_0) + EL_{01}(X_0) = 0 \quad (74)$$

$$L_{00}(X_1) + EL_{01}(X_1) = P(X_0) \quad (75)$$

Since we are interested only in first order solutions we expand X_0 and X_1 in this system w.r.t E

$$X_0 = X_{00} + EX_{01} \quad (76)$$

$$X_1 = X_{10} + EX_{11} \quad (77)$$

and by comparing like powers in E we find to first order in E :

$$X_0(E) = L_{00}^{-1}(0) - EL_{00}^{-1}L_{01}L_{00}^{-1}(0) + O(E^2) \quad (78)$$

$$X_1 = L_{00}^{-1}P(L_{00}^{-1}(0)) + EX_{11} + O(E^2) \quad (79)$$

Then the solution X according to (73) has the form

$$X = L_{00}^{-1}(0) - EL_{00}^{-1}L_{01}L_{00}^{-1}(0) + \epsilon[L_{00}^{-1}P(L_{00}^{-1}(0)) + EX_{11}] + O(\epsilon^2) \quad (80)$$

If we replace $E = \epsilon$ in Eq. (80) and expand once more to first order in ϵ then X becomes

$$X = L_{00}^{-1}(0) + \epsilon[L_{00}^{-1}P(L_{00}^{-1}(0)) - L_{00}^{-1}L_{01}L_{00}^{-1}(0)] + O(\epsilon^2) \quad (81)$$

which agrees with the result found from the classical expansion method (71). This proof has been extended to second order with the equivalent result. Therefore we make the conjecture

Conjecture: Partial perturbation expansion applied to Eq. (67) to order $N > 0$ leads to an expression

$$\sum_{n=0}^N x_n(E)\epsilon^n \quad (82)$$

which, for $E = \epsilon$ agrees up to order N with the total expansion of Eq. (67) w.r.t. ϵ .

It is clear however that solving the equations (74) and (75) $X_0(E)$ as well as $X_1(E)$ contain all powers in E and thus the final result $x(\epsilon)$ will be generally represented by a non polynomial function in the perturbation parameter.

As an instructive example we use the algebraic equation:

$$x - 1 + \epsilon x + \epsilon x^2 = 0 \quad (83)$$

One of the two exact solutions is given by

$$x_{EX} = \frac{-1 - \epsilon + \sqrt{1 + 6\epsilon + \epsilon^2}}{2\epsilon} \quad (84)$$

In order to approximate this solution we use the classical and partial perturbation expansion methods and compare the obtained results.

- The classical expansion method:

We rewrite the equation as

$$x - 1 = -\epsilon(x + x^2) \quad (85)$$

and expand up to third order

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + O(\epsilon^4) \quad (86)$$

Inserting this expansion into (85) and comparing like powers in ϵ we obtain the following linear recursive system for the $x_0 \cdots x_3$:

$$x_0 - 1 = 0 \quad (87)$$

$$x_1 = -x_0 - x_0^2 \quad (88)$$

$$x_2 = -x_1 - 2x_0x_1 \quad (89)$$

$$x_3 = -x_2 - 2x_0x_2 - x_1^2 \quad (90)$$

which results in

$$x_{tot}(\epsilon) = 1 - 2\epsilon + 6\epsilon^2 - 22\epsilon^3 + O(\epsilon^4) \quad (91)$$

- The partial expansion method:

In this case the equation (83) is rewritten as

$$x - 1 + Ex = -\epsilon x^2 \quad (92)$$

where ϵ in the linear part has been replaced by a general parameter E . Now we expand only ("partially") w.r.t remaining ϵ term and using

$$x(E; \epsilon) = x_0(E) + \epsilon x_1(E) + \epsilon^2 x_2(E) + \epsilon^3 x_3(E) + O(\epsilon^4) \quad (93)$$

we get the system

$$x_0 + Ex_0 = 1 \quad (94)$$

$$x_1 + Ex_1 = -x_0^2 \quad (95)$$

$$x_2 + Ex_2 = -2x_0x_1 \quad (96)$$

$$x_3 + Ex_3 = -2x_0x_2 - x_1^2 \quad (97)$$

and thus

$$x(E; \epsilon) = \frac{1}{1+E} - \frac{\epsilon}{(1+E)^3} + \frac{2\epsilon^2}{(1+E)^5} - \frac{5\epsilon^3}{(1+E)^7} + O(\epsilon^4) \quad (98)$$

or - after resubstituting $E = \epsilon$ -

$$x_{part}(\epsilon) = \frac{1}{1+\epsilon} - \frac{\epsilon}{(1+\epsilon)^3} + \frac{2\epsilon^2}{(1+\epsilon)^5} - \frac{5\epsilon^3}{(1+\epsilon)^7} \quad (99)$$

Expanding the latter result up to fifth order in ϵ leads

$$x_{part}(\epsilon) = 1 - 2\epsilon + 6\epsilon^2 - 22\epsilon^3 + 76\epsilon^4 - 226\epsilon^5 + O(\epsilon^6) \quad (100)$$

and indeed this expansion agrees with the total expansion of x up to third order in ϵ .

In the following table we compare the result of perturbation orders 0 to 3 coming from the two perturbation methods to the exact solution of the example equation (83) for $\epsilon = 0.1$.

Order N	$x_{tot}(0.1)$	$x_{part}(0.1)$
0	1.00000	0.90909
1	0.80000	0.83396
2	0.86000	0.84638
3	0.83800	0.84381

Comparison between the total and the partial expansion method

The exact solution follows from Eq. (84) and to five digits is

$$x_{EX}(0.1) = 0.84429 \quad (101)$$

This table indicates that the partial expansion method at least for the specified value of ϵ is more accurate than the total expansion method. In order to be more complete we finally compare the two approximation results to the exact solution in the interval

$$\frac{\sqrt{8} - 2}{6 - 2\sqrt{8}} \approx -0.17 < \epsilon < 1 \quad (102)$$

The left end of the interval is related to the bifurcation of the real solutions of Eq. (83) to complex ones. As can be seen in the Fig.15 there is a very good agreement between the partial expansion method and the exact solution over all the considered interval while the total expansion method only agrees well with the exact solution in a small interval around $\epsilon = 0$. This is not surprising if we consider the fact that the total expansion in ϵ represents the exact Taylor series representation of the exact solution at $\epsilon = 0$. From the famous criterion of Cauchy-Weierstrass it is known that the convergence radius of such a series is given by the closest distance from the expansion point to a complex (or real) singularity or branching point. Hence this case the radius of convergence is given by

$$\rho = \left| \frac{\sqrt{8} - 2}{6 - 2\sqrt{8}} \right| \approx 0.17 \quad (103)$$

and it is evident that above $\epsilon = 0.2$ the total expansion method diverges completely from the exact solution. However this limitation does not apply to the partial expansion result because it is not a polynomial expansion.

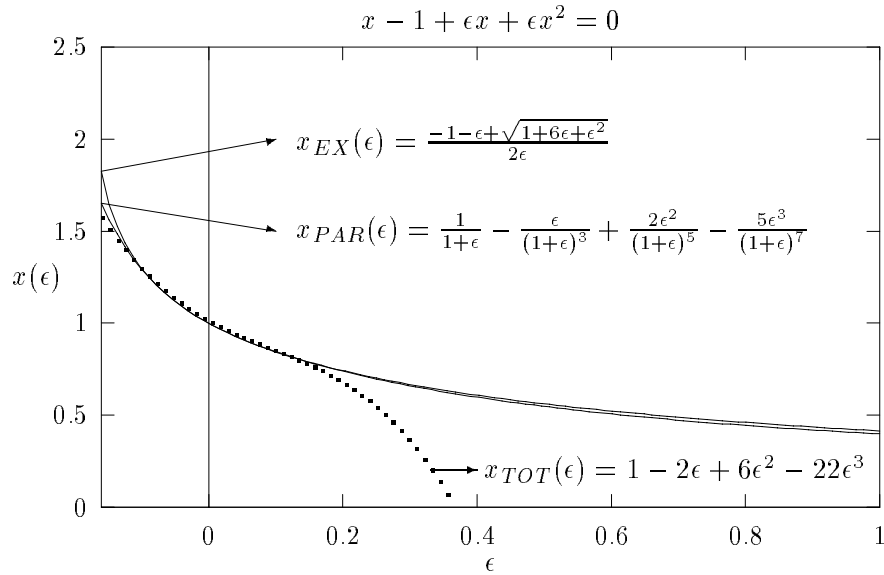


Fig. 15 Comparison of the exact solution with perturbation expansions

It is also interesting to mention that the partial expansion method in the present example gives the correct asymptotic behaviour as $\epsilon \rightarrow \infty$.