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# Explicit solution of the quantum three-body Calogero-Sutherland model

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**Abstract**

The class of quantum integrable systems associated with root systems was introduced in [OP 1977] as a generalization of the Calogero–Sutherland systems [Ca 1971, Su 1972]. In a recent note [Pe 1998], it was proved that for such systems with a potential $v(q) = \kappa(\kappa - 1)\sin^{-2} q$, the series in the product of two wave functions is the $\kappa$-deformation of the Clebsch–Gordan series. This yields recursion relations for the wave functions of those systems and, related to them, for generalized zonal spherical functions on symmetric spaces.
In the present note, this approach is used to compute the explicit expressions for the three-body Calogero–Sutherland wave functions, which are the Jack polynomials. We conjecture that similar results are also valid for the more general two-parameters deformation ($(q, t)$-deformation) introduced by Macdonald [Ma 1988].

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# 1 Introduction

The class of quantum integrable systems associated with root systems was introduced in [OP 1977] (see also [OP 1978]) as generalization of the Calogero–Sutherland systems [Ca 1971, Su 1972]. Such systems depend on one real parameter $\kappa$ (for the type $A_n$, $D_n$ and $E_6, E_7, E_8$), on two parameters (for the type $B_n, C_n, F_4$ and $G_2$) and on three parameters for the type $BC_n$. These parameters are related to the coupling constants of the quantum system.

For special values of this parameter $\kappa$, the wave functions correspond to the characters of the compact simple Lie groups ($\kappa = 1$) [We 1925/26], or to zonal spherical functions on symmetric spaces ($\kappa = 1/2, 2, 4$) [Ha 1958, He 1978]. At arbitrary values of $\kappa$, they provide an interpolation between these objects.

This class has many remarkable properties. Let us only mention that the wave functions of such systems are a natural generalization of special functions (hypergeometric functions) to the case of several variables. The history of the problem and some results can be found in [OP 1983]. It was recently shown in [Pe 1998], that the product of two wave functions is a finite linear combination of analogous functions, namely of functions that appear in the corresponding Clebsch–Gordan series. In other words, this deformation ($\kappa$-deformation) does not change the Clebsch–Gordan series. For rank 1, one obtains the well-known cases of the Legendre, Gegenbauer and Jacobi polynomials and the limiting cases of the Laguerre and Hermite polynomials (see for example [Vi 1968]). Some other cases were also considered in [He 1955, Ja 1970, Ko 1974, Ja 1975, Se 1977, Vr 1976, Vr 1984, Ma 1988, La 1989] and [St 1989]. In this note we use this property in order to obtain the explicit expressions for the Jack polynomials[3] of type $A_2$ which give the solution of the three-body Calogero–Sutherland model. For special values of $\kappa = 1/2, 2, 4$ we obtain the explicit expressions for zonal polynomials of type $A_2$.

We conjecture that these results remain valid for the Macdonald polynomials of type $A_2$ [Ma 1988] and this will be the subject of a separate communication [PRZ 1998].

# 2 General description

The systems under consideration are described by the Hamiltonian (for more details see [OP 1983]):

$$H = \frac{1}{2}\, p^2 + U(q), \qquad p^2 = (p, p) = \sum_{j=1}^{l} p_j^2, \tag{2.1}$$

[3] We use the name of Jack polynomials, although, strictly speaking, they are slightly different from those introduced by Jack [Ja 1970]. Another possible denomination is generalized Gegenbauer polynomials [Ja 1975].

where $p = (p_1, ..., p_l)$, $p_j = -i \frac{\partial}{\partial q_j}$, is a momentum vector operator, and $q = (q_1, ..., q_l)$ is a coordinate vector in the $l$-dimensional vector space $V \sim \mathbb{R}^l$ with standard scalar product $(\alpha, q)$. They are a generalization of the Calogero–Sutherland systems [Ca 1971, Su 1972] for which $\{\alpha\} = \{e_i - e_j\}$, $\{e_j\}$ being a standard basis in $V$. The potential $U(q)$ is constructed by means of a certain system of vectors $R^+ = \{\alpha\}$ in $V$ (the so-called root system):

$$U = \sum_{\alpha \in R^+} g_\alpha^2 \, v(q_\alpha), \qquad q_\alpha = (\alpha, q), \qquad g_\alpha^2 = \kappa_\alpha(\kappa_\alpha - 1). \tag{2.2}$$

The constants satisfy the condition $g_\alpha = g_\beta$, if $(\alpha, \alpha) = (\beta, \beta)$. Such systems are completely integrable for five types of potential [OP 1983]. In this note we consider only the $A_2$ case with potential $v(q) = \sin^{-2} q$.

# 3 The Clebsch–Gordan series

In this section, we recall the main results of [Pe 1998] and specialize them to the $A_2$ case. The Schrödinger equation for this quantum system with $v(q) = \sin^{-2} q$ has the form

$$H \, \Psi^\kappa = E(\kappa) \, \Psi^\kappa; \qquad H = -\,\Delta_2 + U(q_1, q_2, q_3), \qquad \Delta_2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial q_j^2} \tag{3.1}$$

with

$$U(q_1, q_2, q_3) = \kappa(\kappa - 1) \left( \sin^{-2}(q_1 - q_2) + \sin^{-2}(q_2 - q_3) + \sin^{-2}(q_3 - q_1) \right). \tag{3.2}$$

The ground state wave function and its energy are

$$\Psi_0^\kappa(q) = \left( \prod_{j<k}^{3} \sin(q_j - q_k) \right)^\kappa, \qquad E_0(\kappa) = 8\kappa^2. \tag{3.3}$$

Substituting $\Psi_\lambda^\kappa = \Phi_\lambda^\kappa \, \Psi_0^\kappa$ in (3.1) we obtain

$$-\,\Delta^\kappa \, \Phi_\lambda^\kappa = \varepsilon_\lambda(\kappa) \, \Phi_\lambda^\kappa, \qquad \Delta^\kappa = \Delta_2 + \Delta_1^\kappa, \qquad \varepsilon_\lambda(\kappa) = E_\lambda(\kappa) - E_0(\kappa). \tag{3.4}$$

Here the operator $\Delta_1^\kappa$ takes the form

$$\Delta_1^\kappa = \kappa \sum_{j<k}^{3} \cot(q_j - q_k) \left( \frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_k} \right). \tag{3.5}$$

It is easy to see that $\Delta^\kappa$ leaves invariant the set of symmetric polynomials in variables $\exp(2iq_j)$. Such polynomials $m_\lambda$ are labelled by an $SU(3)$ highest weight $\lambda = m\lambda_1 + n\lambda_2$, with $m, n$ non-negative integers and $\lambda_{1,2}$ the two fundamental weights. In general,

$$\Phi_\lambda^\kappa = \sum_{P^+ \ni \mu \leq \lambda} C_\lambda^\mu(\kappa) \, m_\mu, \qquad m_\mu = \sum_{\lambda' \in W \cdot \mu} e^{2i(q, \lambda')}, \tag{3.6}$$

where $P^+$ denotes the cone of dominant weights, $W$ the Weyl group, and $C^\mu_\lambda(\kappa)$ are some constants, taking care of the wave function normalization.

The most remarkable result of [Pe 1998] is that the product of two wave functions is a finite sum of wave functions (a sort of the $\kappa$-deformed Clebsch–Gordan series)

$$\Phi^\kappa_\mu \, \Phi^\kappa_\lambda = \sum_{\nu \in D_\mu(\lambda)} C^\nu_{\mu\lambda}(\kappa) \, \Phi^\kappa_\nu. \tag{3.7}$$

In this equation, $D_\mu(\lambda) = (D_\mu + \lambda) \cap P^+$, where $D_\mu$ is the weight diagram of the representation with highest weight $\mu$.

Since $\Phi^\kappa_\mu$ are symmetric functions of $\exp(2iq_j)$, it is convenient to work with a new set of variables

$$\begin{aligned} &z_1 = e^{2iq_1} + e^{2iq_2} + e^{2iq_3}, \qquad z_2 = e^{2i(q_1+q_2)} + e^{2i(q_2+q_3)} + e^{2i(q_3+q_1)}, \\ &z_3 = e^{2i(q_1+q_2+q_3)}. \end{aligned} \tag{3.8}$$

As we are in the centre-of-mass frame ($\sum_i p_i = 0$), the wave functions depend on two variables only, which we choose to be $z_1$ and $z_2$ (it is consistent to set $z_3 = 1$). In these variables, and up to a normalisation factor, $\Delta^\kappa$ reads ($\partial_i = \partial/\partial z_i$):

$$\Delta^\kappa = (z_1^2 - 3z_2)\,\partial_1^2 + (z_2^2 - 3z_1)\,\partial_2^2 + (z_1 z_2 - 9)\,\partial_1\partial_2 + (3\kappa + 1)\,(z_1\partial_1 + z_2\partial_2). \tag{3.9}$$

Its eigenvalues are

$$\varepsilon_{m,n}(\kappa) = m^2 + n^2 + mn + 3\kappa(m+n). \tag{3.10}$$

For the rest of this note, we will use a different normalization for the polynomials $\Phi^\kappa_\lambda$ and denote them by $P^\kappa_{m,n}$. In [Pe 1998] their general structure was outlined

$$P^\kappa_{m,n}(z_1, z_2) = \sum_{p,q} C^{p,q}_{m,n}(\kappa)\, z_1^p\, z_2^q = z_1^m\, z_2^n + \text{lower terms}, \tag{3.11}$$

with $p + q \le m + n$ and $p - q \equiv m - n (\mathrm{mod}\ 3)$. The first polynomials are easy to find:

$$P^\kappa_{0,0} = 1, \qquad P^\kappa_{1,0} = z_1, \qquad P^\kappa_{0,1} = z_2. \tag{3.12}$$

In [Pe 1998] simple instances of (3.7) for $P^\kappa_\lambda = P^\kappa_{1,0}$ or $P^\kappa_{0,1}$ were given

$$z_1\, P^\kappa_{m,n} = P^\kappa_{m+1,n} + a_{m,n}(\kappa)\, P^\kappa_{m,n-1} + c_m(\kappa)\, P^\kappa_{m-1,n+1}, \tag{3.13}$$

$$z_2\, P^\kappa_{m,n} = P^\kappa_{m,n+1} + \tilde{a}_{m,n}(\kappa)\, P^\kappa_{m-1,n} + c_n(\kappa)\, P^\kappa_{m+1,n-1}, \tag{3.14}$$

where

$$a_{m,n}(\kappa) = \tilde{a}_{n,m}(\kappa) = c_n(\kappa)\, c_{m+n+\kappa}(\kappa), \tag{3.15}$$

$$c_m(\kappa) = \frac{e(m)}{e(\kappa+m)}, \qquad e(m) = \frac{m}{m-1+\kappa}. \tag{3.16}$$

In the next section, we will build the polynomials with the help of these recursion relations.

## 4 Results

As a first step towards the complete solution, it is instructive to compute the simpler $P^\kappa_{m,0}$ polynomials, which were considered first by Jack [Ja 1970] (see also [Pr 1984]). Combining the recursion relations (3.13) and (3.14), we get

$$P^\kappa_{m+1,0} = z_1\, P^\kappa_{m,0} - c_m\, z_2\, P^\kappa_{m-1,0} + d_m\, P^\kappa_{m-2,0}, \tag{4.1}$$

where $d_m = c_m\, c_{m-1}\, c_{m-1+\kappa}$ (for brevity we drop the $\kappa$ dependence in $c_m$). From the general structure (3.11) of $P^\kappa_{m,n}$, it is natural to decompose $P^\kappa_{m,0}$ as

$$P^\kappa_{m,0} = \sum_{l=0}^{[\frac{m}{3}]} z_1^{m-3l}\, Q_l^{\kappa,m}(y), \qquad y = \frac{z_2}{z_1^2}, \tag{4.2}$$

$Q_l^{\kappa,m}(y)$ being a polynomial in $y$. Then the recursion relation (4.1) implies that these $Q_l^{\kappa,m}$ satisfy

$$\begin{aligned} Q_0^{\kappa,m+1}(y) &= Q_0^{\kappa,m}(y) - c_m\, y\, Q_0^{\kappa,m-1}(y), & (4.3)\\ Q_l^{\kappa,m+1}(y) &= Q_l^{\kappa,m}(y) - c_m\, y\, Q_l^{\kappa,m-1}(y) + d_m\, Q_{l-1}^{\kappa,m-1}(y). & (4.4) \end{aligned}$$

The first relation involves only $Q_l^{\kappa,m}$ with $l = 0$ and can be readily solved with the help of the Gegenbauer polynomials $C^\kappa_m(t)$ as

$$\begin{aligned} Q_0^{\kappa,m}(y) &= \sum_{i=0}^{[\frac{m}{2}]} \frac{(-1)^i}{i!}\, \frac{m!}{(m-2i)!}\, \frac{\Gamma(\kappa+m-i)}{\Gamma(\kappa+m)}\, y^i \\ &= \frac{y^{m/2}}{e(m+1)_{-m}}\, C^\kappa_m\left(\frac{1}{2\sqrt{y}}\right), \end{aligned} \tag{4.5}$$

where $1/e(x+1)_{-i}$ denotes the product[4] $e(x)\, e(x-1) \cdots e(x-i+1)$. For higher $l$, we try the following ansatz

$$Q_l^{\kappa,m}(y) = \alpha_l^m\, Q_0^{\kappa+l,m-3l}(y), \tag{4.6}$$

which solves (4.4), provided that the constants $\alpha_l^m$ are

$$\alpha_l^m = \frac{m!}{l!\,(m-3l)!}\, \frac{\Gamma(\kappa+m-2l)}{\Gamma(\kappa+m)}. \tag{4.7}$$

Therefore, we conclude that the polynomials $P^\kappa_{m,0}(z_1, z_2)$ are just some particular linear combinations of the one-variable Gegenbauer polynomials. One obtains the other set of polynomials $P^\kappa_{0,n}$ using the relation $P^\kappa_{0,n}(z_1, z_2) = P^\kappa_{n,0}(z_2, z_1)$.

[4] Similarly, for positive $i$, $e(x)_i = e(x)\, e(x+1) \cdots e(x+i-1)$. This is a functional generalization of the Pochhammer symbol $(x)_i = \Gamma(x+i)/\Gamma(x), i \in \mathbb{Z}$ used later in the text.

The recursion relation (4.1) is also very useful to derive a generating function for the $P^\kappa_{m,0}$ polynomials. Indeed, plugging in (4.1) the following function

$$G^\kappa_0(u) = \sum_{m=0}^{\infty} e(m+1)_{-m}\, u^m\, P^\kappa_{m,0}, \tag{4.8}$$

we obtain the first-order differential equation, easily solved by

$$G^\kappa_0(u) = (1 - z_1 u + z_2 u^2 - u^3)^{-\kappa}. \tag{4.9}$$

This generating function is perfectly suited to prove some basic properties of these polynomials, such as

$$\partial_1 P^\kappa_{m,0} = m\, P^{\kappa+1}_{m-1,0}, \qquad \partial_2 P^\kappa_{m,0} = -\frac{m(m-1)}{\kappa+m-1}\, P^{\kappa+1}_{m-2,0}. \tag{4.10}$$

We will build the general polynomials with the help of the $P^\kappa_{m,0}$, using the property

$$P^\kappa_{m,0}\, P^\kappa_{0,n} = \sum_{i=0}^{\min(m,n)} \gamma^i_{m,n}\, P^\kappa_{m-i,n-i}. \tag{4.11}$$

This is a consequence of Eq. (3.7), with the notable difference that the sum on the right-hand side is over a restricted domain (actually, it parallels exactly the $SU(3)$ Clebsch–Gordan decomposition).

For the proof, we proceed by iteration, assuming that (4.11) is valid up to $(m,n)$. Then, with repeated use of (3.13) and (3.14), we get

$$P^\kappa_{m,0}\, P^\kappa_{0,n+1} = \sum_{i=0}^{\min(m,n+1)} \gamma^i_{m,n+1}\, P^\kappa_{m-i,n+1-i} + c_n\, \delta^i_{m,n+1}\, P^\kappa_{m+1-i,n-1-i}, \tag{4.12}$$

where we defined

$$\begin{aligned}
\gamma^i_{m,n+1} &= \gamma^i_{m,n} + \tilde{a}_{m-i+1,n-i+1}\, \gamma^{i-1}_{m,n} - c_n\, c_{m-i+1}\, \gamma^{i-1}_{m,n-1}, &(4.13)\\
\delta^i_{m,n+1} &= c_n^{-1}\, c_{n-i}\, \gamma^i_{m,n} - \gamma^i_{m,n-1} - a_{m-i+1,n-i}\, \gamma^{i-1}_{m,n-1} \\
&\quad + c_{n-1}\, c_{\kappa+n-1}\, \gamma^{i-1}_{m,n-2}. &(4.14)
\end{aligned}$$

From the polynomials normalisation, we already know that $\gamma^0_{m,n} = 1$, and after a straigthforward computation, the solution to (4.13) is found to be

$$\gamma^i_{m,n} = \frac{e(2\kappa+m+n+1-i)_{-i}}{e(1)_i\, e(m+1)_{-i}\, e(n+1)_{-i}}, \tag{4.15}$$

which implies that $\delta^i_{m,n+1} = 0$ in (4.14).

The constructive aspect of this formula lies in its inverted form.

**Theorem 1**. The Jack polynomials $P^{\kappa}_{m,n}$ of type $A_2$ are given by the formula

$$P^{\kappa}_{m,n} = \sum_{i=0}^{\min(m,n)} \beta^{i}_{m,n}\, P^{\kappa}_{m-i,0}\, P^{\kappa}_{0,n-i}, \tag{4.16}$$

where the constants are

$$\beta^{i}_{m,n} = \frac{(-1)^i}{i!\,(\kappa+1)_{-i}}\, \frac{3\kappa+m+n-2i}{3\kappa+m+n-i}\, \frac{(\kappa+m)_{-i}\,(\kappa+n)_{-i}\,(2\kappa+m+n)_{-i}}{(m+1)_{-i}\,(n+1)_{-i}\,(3\kappa+m+n)_{-i}}. \tag{4.17}$$

Note that the $\beta^{i}_{m,n}$ are obtained using the relation

$$\beta^{i}_{m,n} = -\sum_{j=0}^{i-1} \beta^{j}_{m,n}\, \gamma^{i-j}_{m-j,n-j}. \tag{4.18}$$

From this theorem, we see that the construction of a general polynomial $P^{\kappa}_{m,n}$ is similar to the construction of $SU(3)$ representations from tensor products of the two fundamental representations.

Likewise, one can explicitly study other types of decompositions, such as

$$P^{\kappa}_{m,0}\, P^{\kappa}_{n,0} = \sum_{i=0}^{\min(m,n)} \tilde{\gamma}^{i}_{m,n}\, P^{\kappa}_{m+n-2i,i}, \tag{4.19}$$

with the coefficients

$$\tilde{\gamma}^{i}_{m,n} = \frac{e(\kappa+m+n+1-i)_{-i}}{e(1)_i\, e(m+1)_{-i}\, e(n+1)_{-i}}. \tag{4.20}$$

The proof is essentially the same as for (4.11). Here again, the summation on the right-hand side is on a restricted domain, compared to (3.7).

**Theorem 2**. There is another formula for polynomials $P^{\kappa}_{m,n}$ at $m \geq n$:

$$\tilde{\gamma}^{n}_{m+n,n}\, P^{\kappa}_{m,n} = \sum_{i=0}^{n} \tilde{\beta}^{i}_{mn}\, P^{\kappa}_{m+n+i,0}\, P^{\kappa}_{n-i,0}, \tag{4.21}$$

where

$$\tilde{\beta}^{i}_{m,n} = \frac{(-1)^i}{i!\,(\kappa+1)_{-i}}\, \frac{m+2i}{m}\, \frac{(\kappa+m+n)_i}{(m+n+1)_i}\, \frac{(m)_i}{(\kappa+m+1)_i}\, \frac{(\kappa+n)_{-i}}{(n+1)_{-i}}. \tag{4.22}$$

This theorem is a simple consequence of (4.19), and the coefficients $\tilde{\beta}^{i}_{m,n}$ are found using

$$\tilde{\beta}^{i}_{m,n} = -\left(\tilde{\gamma}^{n-i}_{m+n+i,n-i}\right)^{-1} \sum_{j=0}^{i-1} \tilde{\beta}^{j}_{m,n}\, \tilde{\gamma}^{n-i}_{m+n+j,n-j}. \tag{4.23}$$

As a by-product of (4.16), specializing it to the case $\kappa = 1$, where $P^{\kappa}_{m,n}$ are nothing but the $SU(3)$ characters, we get

$$P^1_{m,n} = P^1_{m,0}\, P^1_{0,n} - P^1_{m-1,0}\, P^1_{0,n-1}. \tag{4.24}$$

From this we easily deduce the generating function for $SU(3)$ characters (see e.g. [PS 1978])

$$G^1(u,v) = \sum_{m,n=0}^{\infty} u^m\, v^n\, P^1_{m,n} = \frac{1-uv}{(1-z_1\, u + z_2\, u^2 - u^3)\,(1-z_2\, v + z_1\, v^2 - v^3)}. \tag{4.25}$$

# 5 Conclusion

In this letter we have solved the quantum three-body Calogero–Sutherland model exactly. The wave functions are known to be Jack polynomials, and our construction gives explicit expansion of them. They appear to be constructed with Gegenbauer polynomials.

Since the wave functions correspond, for special values of $\kappa$, to zonal spherical polynomials, we have obtained, as a by-product, explicit expression for zonal spherical functions of the symmetric spaces $SU(3)/SO(3)$ ($\kappa = 1/2$), $SU(3)\times SU(3)/SU(3)$ ($\kappa = 1$), $SU(6)/Sp(3)$ ($\kappa = 2$), and $E_{6(-78)}/F_4$ ($\kappa = 4$).

Due to the algebraic framework, many aspects of this work can be applied to the $N$-body model, for instance Eq. (4.11) is easy to generalize to the $SU(N)$ case.

Let us also remark that preliminary investigations indicate that relations similar to (3.13) hold in the case of the Macdonald polynomials.

# Appendix: Explicit expressions for $P^{\kappa}_{mn}$ with $m+n \leq 4$

In addition to those already given in the main text, we list here the first few polynomials $P^{\kappa}_{m,n}$ with $m+n \leq 4$:

$$\begin{aligned}
P^{\kappa}_{2,0} &= z_1^2 - \frac{2}{\kappa+1} z_2 \\
P^{\kappa}_{1,1} &= z_1 z_2 - \frac{3}{2\kappa+1} \\
P^{\kappa}_{3,0} &= z_1^3 - \frac{6}{\kappa+2} z_1 z_2 + \frac{6}{(\kappa+1)(\kappa+2)} \\
P^{\kappa}_{2,1} &= z_1^2 z_2 - \frac{2}{\kappa+1} z_2^2 - \frac{3\kappa+1}{(\kappa+1)^2} z_1 \\
P^{\kappa}_{4,0} &= z_1^4 - \frac{12}{\kappa+3} z_1^2 z_2 + \frac{12}{(\kappa+2)(\kappa+3)} z_2^2 + \frac{24}{(\kappa+2)(\kappa+3)} z_1 \\
P^{\kappa}_{3,1} &= z_1^3 z_2 - \frac{6}{\kappa+2} z_1 z_2^2 - \frac{3(3\kappa+2)}{(\kappa+2)(2\kappa+3)} z_1^2 + \frac{30}{(\kappa+2)(2\kappa+3)} z_2
\end{aligned}$$

$$P^{\kappa}_{2,2} \;=\; z_1^2 z_2^2 - \frac{2}{\kappa+1}(z_1^3+z_2^3) - \frac{12(\kappa-1)}{(\kappa+1)(2\kappa+3)} z_1 z_2 + \frac{9(\kappa-1)}{(\kappa+1)^2(2\kappa+3)}$$


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