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THE ACTION OF OUTER AUTOMORPHISMS ON BUNDLES OF CHIRAL BLOCKS

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Abstract

On the bundles of WZW chiral blocks over the moduli space of a punctured rational curve we construct isomorphisms that implement the action of outer automorphisms of the underlying affine Lie algebra. These bundle-isomorphisms respect the Knizhnik–Zamolodchikov connection and have finite order. When all primary fields are fixed points, the isomorphisms are endomorphisms; in this case, the bundle of chiral blocks is typically a reducible vector bundle. A conjecture for the trace of such endomorphisms is presented; the proposed relation generalizes the Verlinde formula. Our results have applications to conformal field theories based on non-simply connected groups and to the classification of boundary conditions in such theories.

1 Introduction and summary

1.1 Chiral blocks

Spaces of chiral blocks [1] are of considerable interest both in physics and in mathematics. In this paper we construct a natural class of linear maps on the spaces of chiral blocks of WZW conformal field theories and investigate their properties. Recall that a WZW theory is a two-dimensional conformal field theory which is characterized by an untwisted affine Lie algebra $\mathfrak{g}$ and a positive integer, the level.

The chiral block spaces of a rational conformal field theory are finite-dimensional complex vector spaces. In the WZW case, one associates to each complex curve C of genus g with m distinct smooth marked points and to each m-tuple $\vec{\Lambda}$ of integrable weights of $\mathfrak{g}$ at level $k^\vee$ a finite-dimensional vector space $\mathsf{B}_{\vec{\Lambda}}$ of chiral blocks. These spaces are the building blocks for the correlation functions of WZW theories both on surfaces with and without boundaries, and on orientable as well as unorientable surfaces (see e.g. [2]). Moreover, since WZW theories underly the construction of various other classes of two-dimensional conformal field theories, the spaces of WZW chiral blocks enter in the description of the correlation functions of those theories as well. They also form the space of physical states in certain three-dimensional topological field theories [3]. Finally they are of interest to algebraic geometers because via the Borel–Weil–Bott theorem they are closely related to spaces of holomorphic sections in line bundles over moduli spaces of flat connections (for a review, see e.g. [4]).

It is necessary (and rather instructive) not to restrict ones attention to the case of fixed insertion points, but rather to vary both the moduli of the curve C and the position of the marked points. In conformal field theory this accounts e.g. for the dependence of correlation functions on the positions of the fields.) This way one obtains a complex vector space $\mathsf{B}_{\vec{\Lambda}}$ over each point in the moduli space $\mathcal{M}_{g,m}$ of complex curves of genus g with m marked smooth points. It is known (see e.g. [5, 6]) that these vector spaces fit together into a vector bundle $\mathcal{B}_{\vec{\Lambda}}$ over $\mathcal{M}_{g,m}$, and that this vector bundle is endowed with a projectively flat connection, the Knizhnik–Zamolodchikov connection. (Sometimes the term 'chiral block' is reserved for sections in the bundle $\mathcal{B}_{\vec{\Lambda}}$ that are flat with respect to the Knizhnik–Zamolodchikov connection.) In this paper, we will restrict ourselves mainly to curves of genus $g=0$, but allow for arbitrary number $m\geq 2$ of marked points. For brevity, we will denote the moduli space of m distinct points on $\mathbb{P}^1$ simply by $\mathcal{M}_m$.

1.2 Automorphisms

It has been known for a long time that certain outer automorphisms of the affine Lie algebra $\mathfrak{g}$ that underlies a WZW theory play a crucial role in the construction of several classes of conformal field theories. The automorphisms in question are in one-to-one correspondence with the elements of the center $\mathcal{Z}$ of the relevant compact Lie group, i.e. of the real compact connected and simply connected Lie group G whose Lie algebra is the compact real form of the horizontal subalgebra $\bar{\mathfrak{g}}$ of the affine Lie algebra. These automorphisms underly the existence of non-trivial modular invariants, both of extension and of automorphism type, which describe the 'non-diagonal' WZW theories that are based [7] on non-simply connected quotients of the covering group G. (For the modular matrices of such theories see [8].) The same class of automorphisms also plays a crucial role in the construction of 'gauged' WZW models [9–11], i.e. coset conformal field theories [12].

Let us remark that the structures implied by such automorphisms have been generalized for arbitrary rational conformal field theories in the theory of so-called simple currents (for a review, see [14]).

It is also worth mentioning that in the representation theoretic approach to WZW chiral blocks [6, 13] one heavily relies on the loop construction of the affine algebra $\mathfrak{g}$. That description treats the simple roots of the horizontal subalgebra $\bar{\mathfrak{g}}$ and the additional simple root of $\mathfrak{g}$ on a rather different footing and accordingly does not reflect the full structure of $\mathfrak{g}$ in a symmetric way. In particular the symmetries of the Dynkin diagram of $\mathfrak{g}$ which are associated to the (classes of) outer automorphisms of $\mathfrak{g}$ that correspond to the center $\mathcal{Z}$ do not possess a particularly nice realization in the loop construction.

1.3 Main results

The basic purpose of this paper is to merge the two issues of chiral block spaces and outer automorphisms of affine Lie algebras. Thus our goal is to implement and describe the action of the outer automorphisms corresponding to the center $\mathcal{Z}$ of G on the spaces of chiral blocks of WZW theories. This constitutes in particular a necessary input for the definition of chiral blocks for the theories based on non-simply connected quotient groups of G. It also turns out to be an important ingredient in the description of the possible boundary conditions of WZW theories with non-diagonal partition function [15]. Moreover, our results are also relevant for the quest of establishing a Verlinde formula (in the sense of algebraic geometry) for moduli spaces of flat connections with non-simply connected structure group. (This might also pave the way to a rigorous proof of the Verlinde formula for more general conformal field theory models, such as coset conformal field theories.)

Let us now summarize informally the main results of this paper. Consider the direct sum of m copies of the coweight lattice of $\bar{\mathfrak{g}}$ and denote by $\Gamma_{\rm w}$ the subgroup consisting of those m-tuples $\vec{\bar\mu} = (\bar\mu_i)$ which sum to zero, $\sum_{i=1}^m \bar\mu_i = 0$. To each $\vec{\bar\mu} \in \Gamma_{\rm w}$ we associate a map $\vec\omega^\star$ on the set of integrable weights at level $k^\vee$ and, for each m-tuple $\vec\Lambda$ of integrable weights, a map

$$\vec\Theta^\star_{\vec{\bar\mu}}: \quad \mathcal{B}_{\vec\Lambda} \to \mathcal{B}_{\vec\omega^\star\vec\Lambda}$$

which is an isomorphism of the bundles $\mathcal{B}_{\vec\Lambda}$ of chiral blocks over the moduli space $\mathcal{M}_m$ and which projects to the identity on $\mathcal{M}_m$. We show that this map respects the Knizhnik–Zamolodchikov connections on $\mathcal{B}_{\vec\Lambda}$ and $\mathcal{B}_{\vec\omega^\star\vec\Lambda}$. Finally, we show that the map $\vec\Theta^\star_{\vec{\bar\mu}}$ is the identity map on $\mathcal{B}_{\vec\Lambda}$ whenever the collection $\vec{\bar\mu}$ of coweight lattice elements contains only elements of the coroot lattice of $\bar{\mathfrak{g}}$. We denote the sublattice of $\Gamma_{\rm w}$ that consists of such m-tuples $\vec{\bar\mu}$ by Γ. (With these properties the maps $\vec\Theta^\star_{\vec{\bar\mu}}$ constitute prototypical examples for so-called implementable automorphisms of the fusion rules. For the role of such automorphisms in the determination of possible boundary conditions for the conformal field theory, see [2, 16].)

Let us now discuss a few implications of the bundle-isomorphism $\vec\Theta^\star_{\vec{\bar\mu}}$. First, when $\vec\omega^\star\vec\Lambda \neq \vec\Lambda$, then $\vec\Theta^\star_{\vec{\bar\mu}}$ can be used to identify the two bundles $\mathcal{B}_{\vec\Lambda}$ and $\mathcal{B}_{\vec\omega^\star\vec\Lambda}$. On the other hand, suppose that the stabilizer $\Gamma_{\vec\Lambda}$, i.e. the subgroup of $\Gamma_{\rm w}$ that consists of all elements which satisfy $\vec\omega^\star\vec\Lambda = \vec\Lambda$, is strictly larger than Γ. In this case the finite abelian group $\Gamma_{\vec{\Lambda}}/\Gamma$ acts projectively by endomorphisms on each fiber. We then decompose each fiber into the direct sum of eigenspaces under this action. It is now crucial to note that the action of this finite group commutes with the Knizhnik–Zamolodchikov connection; this property implies that the eigenspaces fit together into sub-vector bundles of $\mathcal{B}_{\vec{\Lambda}}$. In other words, the bundle of chiral blocks is in this case a reducible vector bundle. Moreover, each of these subbundles comes equipped with its own Knizhnik–Zamolodchikov connection, which is just the restriction of the Knizhnik–Zamolodchikov connection of $\mathcal{B}_{\vec{\Lambda}}$ to the subbundle. This implies in particular that the chiral conformal field theory that underlies the WZW model with a modular invariant of extension type possesses a Knizhnik–Zamolodchikov connection as well.

This pattern – identification, respectively splitting of fixed points into eigenspaces – is strongly reminiscent of what has been found in coset conformal field theories [12]. It is the deeper reason why similar formulæ hold [8,12] for the so-called resolution of fixed points in the definition of coset conformal field theories and in simple current modular invariants.

The rank of the subbundles is the same for all values of the moduli. This rank is related via Fourier transformation over the subgroup $(\Gamma_{\vec{\Lambda}}/\Gamma)^{\circ}_{\epsilon}$ of regular elements of the finite abelian group $\Gamma_{\vec{\Lambda}}/\Gamma$ to the traces of $\vec{\Theta}^{\star}_{\vec{\bar{\mu}}}$ on any fiber. Thus we can conclude that these traces do not depend on the moduli either. We present a conjecture for a formula for these traces and support it by several consistency checks. We expect that several features of our construction can be generalized to curves of arbitrary genus. So far we do not know how to prove our trace formulæ. Some traces on spaces of chiral blocks on higher genus surfaces with only the vacuum module involved have, however, been computed [17] in some special cases, and in these cases they agree with our formula.

1.4 Organization of the paper

The rest of this paper is organized as follows. In section 2 we introduce (see equation (2.6)) a class of automorphisms of an untwisted affine Lie algebra $\mathfrak{g}$, which we call multi-shift automorphisms. These automorphisms can be extended uniquely to the semidirect sum of $\mathfrak{g}$ with the Virasoro algebra; they depend on a collection of m elements $\bar{\mu}_s$ of the coweight lattice, and also on a collection of pairwise distinct complex numbers z_s. We explain how such automorphisms are implemented on irreducible highest weight modules over the affine Lie algebra and determine their class modulo inner automorphisms.

In section 3 we introduce the algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$ of Lie algebra valued meromorphic functions over $\mathbb{P}^1$ with possible finite order poles at m prescribed different points, the so-called block algebra. The Lie algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$ can be regarded as a subalgebra of $\mathfrak{g}^m$, the direct sum of m copies of the affine Lie algebra $\mathfrak{g}$ with all centers identified. We observe that when the complex numbers z_s that enter in the definition of the multi-shift automorphisms are interpreted as the coordinates of points on $\mathbb{P}^1$, then a tensor product of the automorphisms of section 2 restricts to an automorphism of the block algebra, see formula (3.9). One should regard this latter automorphism as a global object from which the former automorphisms are obtained by local expansions. In section 4 we study the special case of multi-shift automorphisms that correspond to elements in the coroot lattice. We show that they are inner automorphisms with respect to the block algebra.

In section 5 we combine the implementation of these automorphisms on the modules of the affine Lie algebra at all insertion points, so as to obtain an implementation on the modules over the block algebra that arise from tensor products of the affine modules. We show that for fixed values of the insertion points these maps induce isomorphisms on the spaces of chiral blocks and prove that (upon suitably fixing the phases of the implementing maps) all these isomorphisms of chiral block spaces have finite order.

Up to this point the moduli of the problem, i.e. the positions of the punctures, are kept fixed in all our considerations. In section 6 we proceed to consider the dependence of our constructions on these moduli. Our guiding principle will be that the isomorphisms on the spaces of chiral blocks are compatible with the Knizhnik–Zamolodchikov connection. We show that the implementation on the modules can indeed be chosen in a such way that the Knizhnik–Zamolodchikov connection is preserved and that for each value of the moduli the implementation on the module provides the same projective representation of the group $\Gamma_{\rm w}$. We then show that for all values of the moduli the action of a multi-shift automorphism on the chiral blocks is the identity map when all coweights are actually coroots, so that in this case the implementation $\vec\Theta^\star_{\vec{\bar\mu}}$ of the automorphism on the *bundle* $\mathcal{B}$ of chiral blocks has finite order.

Finally, in section 7 we exploit these results to formulate a conjecture for a general formula (equation (7.4)) for the traces of the implementing maps $\vec\Theta^\star_{\vec{\bar\mu}}$. This conjecture suggests a generalization to curves of higher genus – see (7.7) – which is consistent with factorization. The expression (7.7) for the traces constitutes a generalization of the Verlinde formula. Surprisingly enough we also observe that in all examples that have been checked numerically, the traces turn out to be integers.

Details of various calculations are deferred to appendices.

2 Action of $\Gamma_{\rm w}$ on affine Lie algebras and their modules

We start our investigations by introducing a certain family of automorphisms of affine Lie algebras which we call multi-shift automorphisms. To determine the class of a multi-shift automorphism modulo inner automorphisms, we study its implementation on irreducible highest weight modules.

2.1 Multi-shift automorphisms of affine Lie algebras

In this subsection we introduce a distinguished class of automorphisms of untwisted affine Lie algebras $\mathfrak{g}$. Such a Lie algebra $\mathfrak{g}$ can be regarded as a centrally extended loop algebra, i.e. [1]

$$\mathfrak{g} := \bar{\mathfrak{g}} \otimes \mathbb{C}(\!(t)\!) \oplus \mathbb{C}K \,, \tag{2.1}$$

where $\bar{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra and t an indeterminate. We identify $\bar{\mathfrak{g}}$ with the zero mode subalgebra of $\mathfrak{g}$, i.e. $\bar{\mathfrak{g}} \ni \bar x \equiv \bar x {\otimes} t^0 \in \mathfrak{g}$, and refer to it as the *horizontal* subalgebra

[1] Often it is the subalgebra $\mathring{\mathfrak{g}} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ of $\mathfrak{g}$ that is generated when one allows only for Laurent polynomials rather than arbitrary Laurent series that one refers to as the affine Lie algebra. However, in conformal field theory one deals with $\mathfrak{g}$-representations for which every vector of the associated module (representation space) is annihilated by all but finitely many generators of the subalgebra $\mathring{\mathfrak{g}}_+$ of $\mathring{\mathfrak{g}}$ that corresponds to the positive roots. Any such representation of $\mathring{\mathfrak{g}}$ can be naturally promoted to a representation of the larger algebra $\mathfrak{g}$.

of $\mathfrak{g}$; the rank of $\bar{\mathfrak{g}}$ will be denoted by r. We may already remark at this point that our main interest in the multi-shift automorphisms of the affine Lie algebra $\mathfrak{g}$ stems from the fact that collections of m such automorphisms combine in a natural way to automorphisms of m copies of $\mathfrak{g}$ with identified centers. The latter automorphisms, in turn, restrict to automorphisms of an important class of infinite-dimensional subalgebras, the so-called block algebras $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$; those automorphisms of $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$ will be introduced in section 3. In the present section, however, we study the automorphisms of $\mathfrak{g}$ in their own right.

We will describe the automorphisms of the algebra $\mathfrak{g}$ by their action on the canonical central element K and on the elements $H^i{\otimes}f$ and $E^{\bar\alpha}{\otimes}f$ of $\mathfrak{g}$, where $f\in\mathbb{C}(\!(t)\!)$ is arbitrary and $\{H^i\,|\,i=1,2,...\,,r\}\cup\{E^{\bar\alpha}\,|\,\bar\alpha$ a $\bar{\mathfrak{g}}$-root$\}$ is a Cartan–Weyl basis of $\bar{\mathfrak{g}}$. (By choosing $f=t^n$ with $n\in\mathbb{Z}$ one would obtain a (topological) basis of $\mathfrak{g}$, but for the present purposes it is more convenient to write the formulæ for general $f\in\mathbb{C}(\!(t)\!)$.) In terms of these elements, the relations of the Lie algebra $\mathfrak{g}$ read $[K,\cdot\,]=0$ and

$$\begin{aligned} &[H^i{\otimes}f,H^j{\otimes}g]=G^{ij}\,\mathcal{R}es(\mathrm{d}f\,g)\,K\,,\\ &[H^i{\otimes}f,E^{\bar\alpha}{\otimes}g]=\bar\alpha^i\,E^{\bar\alpha}\otimes fg\,,\\ &[E^{\bar\alpha}{\otimes}f,E^{\bar\beta}{\otimes}g]=e_{\bar\alpha,\bar\beta}\,E^{\bar\alpha+\bar\beta}\otimes fg\qquad\text{for } \bar\alpha+\bar\beta \text{ a } \bar{\mathfrak{g}}\text{-root}\,,\\ &[E^{\bar\alpha}{\otimes}f,E^{-\bar\alpha}{\otimes}g]=\sum_{i=1}^{r}\bar\alpha_i\,H^i\otimes fg+\tfrac{2}{(\bar\alpha,\bar\alpha)}\,\mathcal{R}es(\mathrm{d}f\,g)\,K\,. \end{aligned} \tag{2.2}$$

Here $\mathrm{d}\equiv\partial/\partial t$, and the residue $\mathcal{R}es\equiv\mathcal{R}es_0$ of a formal Laurent series in t is defined by

$$\mathcal{R}es\big(\sum_{n=n_0}^{\infty}c_n\,t^n\big)=c_{-1}\,; \tag{2.3}$$

G^{ij} is the symmetrized Cartan matrix of the horizontal subalgebra $\bar{\mathfrak{g}}$, and $e_{\bar\alpha,\bar\beta}$ is the two-cocycle that furnishes structure constants of $\bar{\mathfrak{g}}$ via $[E^{\bar\alpha},E^{\bar\beta}]=e_{\bar\alpha,\bar\beta}E^{\bar\alpha+\bar\beta}$ when $\bar\alpha+\bar\beta$ is a $\bar{\mathfrak{g}}$-root.

The automorphisms to be constructed are characterized by a sequence of elements $\bar\mu_s$, for $s=1,2,...\,,m$ with m an integer $m\geq 2$, of the *coweight lattice* $\overset{\vee}{L}_{\rm w}$ of $\bar{\mathfrak{g}}$ (i.e. the lattice dual to the root lattice of $\bar{\mathfrak{g}}$) that add up to zero. The set

$$\Gamma_{\rm w}:=\{\vec{\bar\mu}\equiv(\bar\mu_1,\bar\mu_2,...\,,\bar\mu_m)\,|\,\bar\mu_s\in\overset{\vee}{L}_{\rm w}\text{ for all } s=1,2,...\,,m;\ \textstyle\sum_{s=1}^{m}\bar\mu_s=0\} \tag{2.4}$$

of such sequences forms a free abelian group with respect to elementwise addition. As an abstract group, we have the isomorphism

$$\Gamma_{\rm w}\cong(\overset{\vee}{L}_{\rm w})^{m-1}\,. \tag{2.5}$$

In addition, the automorphisms of our interest also depend on a sequence of pairwise distinct complex numbers z_s $(s=1,2,...\,,m)$, one of which is singled out. However, in contrast to the elements of $\Gamma_{\rm w}$, in this section we regard these numbers as fixed once and for all, and accordingly we do not refer to them in our notation for the automorphisms. Thus we will deal with precisely one automorphism of $\mathfrak{g}$ for each $\vec{\bar\mu}\in\Gamma_{\rm w}$; we denote this map by $\sigma_{\vec{\bar\mu}}$. Only in case we wish to stress the dependence of the automorphism on the label $s_\circ\in\{1,2,...\,,m\}$ that is singled out, we employ instead the notation $\sigma_{\vec{\bar\mu};s_\circ}$.

For any $\vec{\bar\mu}\in\Gamma_{\rm w}$, the automorphism $\sigma_{\vec{\bar\mu}}\equiv\sigma_{\vec{\bar\mu};s_\circ}$ is defined by

$$\begin{aligned}&\sigma_{\vec{\bar\mu}}(K):=K\,,\\&\sigma_{\vec{\bar\mu}}(H^i{\otimes}f):=H^i{\otimes}f+K\sum_{s=1}^m\bar\mu_s^i\,\mathcal{R}\mathrm{es}(\varphi_{s_\circ,s}\,f)\,,\\&\sigma_{\vec{\bar\mu}}(E^{\bar\beta}{\otimes}f):=E^{\bar\beta}{\otimes}f\cdot\prod_{s=1}^m(\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\beta)}\end{aligned}\tag{2.6}$$

with

$$\varphi_{s_\circ,s}(t):=\big(t+(z_{s_\circ}-z_s)\big)^{-1}\,.\tag{2.7}$$

Here for $q\in\mathbb{Q}$ we employ the notation f^q for the function with values

$$f^q(z)=(f(z))^q\tag{2.8}$$

for all $z\in\mathbb{C}$. In the sequel we will also use the short-hand

$$\Phi_{\bar\alpha}:=\prod_{s=1}^m(\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\alpha)}\tag{2.9}$$

for $\bar\alpha$ a $\bar{\mathfrak g}$-root.

It is readily checked that the mapping (2.6) indeed constitutes an automorphism of the affine Lie algebra $\mathfrak g$. Indeed, except for the bracket $[E^{\bar\alpha}{\otimes}f,E^{-\bar\alpha}{\otimes}g]$, all the relations (2.2) are trivially left invariant. As for the latter, invariance follows by the identity

$$\begin{aligned}\mathcal{R}\mathrm{es}\big(\mathrm{d}[f\cdot\Phi_{\bar\alpha}]\cdot g\cdot\Phi_{-\bar\alpha}\big)\,K\;&=\mathcal{R}\mathrm{es}\big(\mathrm{d}f\,g+fg\cdot\sum_{s=1}^m(\bar\mu_s,\bar\alpha)\,\varphi_{s_\circ,s}\big)\,K\\&=\mathcal{R}\mathrm{es}(\mathrm{d}f\,g)\,K+\sum_{i=1}^r\bar\alpha_i\,\sum_{s=1}^m\bar\mu_s^i\,\mathcal{R}\mathrm{es}(\varphi_{s_\circ,s}\,fg)\,K\,,\end{aligned}\tag{2.10}$$

which follows with the help of

$$\mathrm{d}\Phi_{\bar\alpha}=\Phi_{\bar\alpha}\cdot\sum_{s=1}^m(\bar\mu_s,\bar\alpha)\,\varphi_{s_\circ,s}\,.\tag{2.11}$$

Moreover, it follows immediately from the definition that

$$\sigma_{\vec{\bar\mu}'}\circ\sigma_{\vec{\bar\mu}}=\sigma_{\vec{\bar\mu}+\vec{\bar\mu}'}=\sigma_{\vec{\bar\mu}}\circ\sigma_{\vec{\bar\mu}'}\,.\tag{2.12}$$

Thus the automorphisms (2.6) respect the group law in $\Gamma_{\rm w}$, i.e. they furnish a representation of the abelian group $\Gamma_{\rm w}$ in $\mathrm{Aut}(\mathfrak g)$. This representation provides in fact a group *iso*morphism; accordingly we will henceforth identify the group of automorphisms $\sigma_{\vec{\bar\mu}}$ (2.6) with the group $\Gamma_{\rm w}$ (2.4).

As already mentioned, we refer to the automorphisms (2.6) as *multi-shift automorphisms* of the affine Lie algebra $\mathfrak g$. The origin of this terminology is that they are close relatives of the ordinary shift automorphisms $\sigma^{(0)}_{\bar\mu}$ of $\mathfrak{g}$ that appear in the literature, e.g. in [18–20]. In the conventional notation

$$H^i_n := H^i{\otimes}t^n\,, \qquad E^{\bar\alpha}_n := E^{\bar\alpha}{\otimes}t^n\,, \tag{2.13}$$

the action of such a 'single-shift' automorphism $\sigma^{(0)}_{\bar\mu}$ is given by

$$\begin{aligned} &\sigma^{(0)}_{\bar\mu}(K) = K\,,\\ &\sigma^{(0)}_{\bar\mu}(H^i_n) = H^i_n + \bar\mu^i\,\delta_{n,0}\,K\,,\\ &\sigma^{(0)}_{\bar\mu}(E^{\bar\alpha}_n) = E^{\bar\alpha}_{n+(\bar\mu,\bar\alpha)}\,, \end{aligned} \tag{2.14}$$

where $\bar\mu \in L^\vee_{\rm w}$. These ordinary shift automorphisms are recovered from the formula (2.6) as the special case where $\bar\mu_s = 0$ for $s \neq s_\circ$. In the case of (2.14), the coweight lattice element $\bar\mu$ is known as a *shift vector*; we will use this term also for the collection $\vec{\bar\mu}$ of coweight lattice elements that characterizes the multi-shift automorphisms (2.6).

Note that the ordinary shift automorphisms do not satisfy the constraint $\sum_{s=1}^m \bar\mu_s = 0$, i.e. do not belong to the class of automorphisms of $\mathfrak{g}$ that we consider here. In this context we should point out that when we dispense of the restriction $\sum_{s=1}^m \bar\mu_s = 0$, the formula (2.6) still provides us with an automorphism of $\mathfrak{g}$. The reason why we nevertheless impose that constraint will become clear in the next section, where we glue together several of the 'local' objects (2.6) to a 'global' object that only exists when the constraint is satisfied.

2.2 Implementation on modules

A natural question is to which class of outer modulo inner automorphisms a multi-shift automorphism $\sigma_{\vec{\bar\mu}}$ belongs. Before we can address this issue, we first have to study the implementation of $\sigma_{\vec{\bar\mu}}$ on irreducible highest weight modules.

We start by observing that for any automorphism σ of an affine Lie algebra $\mathfrak{g}$ and any irreducible highest weight module $(R,\mathcal{H})$ over $\mathfrak{g}$,

$$(\tilde R,\mathcal{H}) := (R\circ\sigma,\mathcal{H}) \tag{2.15}$$

furnishes again an irreducible highest weight module. Since this fact is interesting in itself, we pause to describe how it can be established. First, irreducibility of $(\tilde R,\mathcal{H})$ is immediate: if a subspace M of $\mathcal{H}$ is a non-trivial submodule under $\tilde R$, then it is a non-trivial submodule under R as well; but by the irreducibility of $\mathcal{H}$ such submodules do not exist. To check the highest weight property, it is sufficient to show that the module $(\tilde R,\mathcal{H})$ possesses at least one singular vector; this vector is necessarily unique (up to a scalar), since otherwise the module would not be irreducible. Now consider any Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$; then $\sigma(\mathfrak{b})$ is a Borel subalgebra again. Since any two Borel subalgebras are conjugated under an inner automorphism of $\mathfrak{g}$ [21], it follows that there exists an inner automorphism $\tilde\sigma$ of $\mathfrak{g}$ such that $\sigma(\mathfrak{b}) = \tilde\sigma(\mathfrak{b})$. This implies that the two automorphisms σ and $\tilde\sigma$ just differ by an automorphism that does not affect the triangular decomposition, i.e. by a so-called *diagram automorphism* ω (see e.g. [22]). Let us write

$$\tilde\sigma = \overrightarrow{\prod_i} \exp(\mathrm{ad}_{x_i})\,, \tag{2.16}$$

where a fixed ordering of the product is understood. We consider the vector

$$\tilde{v} := \overset{\leftarrow}{\prod_i} \exp(R(x_i))\, v\,, \tag{2.17}$$

where the opposite ordering of the product is understood and where v is the highest weight vector of $(R,\mathcal{H})$. (One can think of $\tilde{v}$ as a Bogoliubov transform of v.) One can check that

$$R(\sigma(x))\,\tilde{v} = R(\tilde{\sigma}\omega(x))\,\tilde{v} = \overset{\rightarrow}{\prod_i} \exp(R(x_i))R(\omega(x))\, v \tag{2.18}$$

(compare also the relation (5.25) below). This tells us that the highest weight properties of v imply analogous highest weight properties for $\tilde{v}$. This concludes the argument.

In addition, by similar arguments one can show that when σ is an inner automorphism, then the $\mathfrak{g}$-modules $(R,\mathcal{H})$ and $(\tilde{R},\mathcal{H})$ are isomorphic, and hence share the same highest weight. On the other hand, they do not necessarily share the same highest weight vector, of course. Thus in general the weight of the highest weight vector v of $(R,\mathcal{H})$ is different from its weight as a vector in $(\tilde{R},\mathcal{H})$; but the previous statement tells us that the difference must be an element of the root lattice of $\mathfrak{g}$.

We also note that any diagram automorphism ω restricts to an automorphism of the Cartan subalgebra $\mathfrak{g}_\circ$ of $\mathfrak{g}$ and hence by duality induces an automorphism $\omega^\star$ of the weight space of $\mathfrak{g}$, which constitutes a permutation $\Lambda \mapsto \omega^\star\Lambda$ of the finitely many integrable weights at fixed level. It follows that for each integrable $\mathfrak{g}$-weight Λ any multi-shift automorphism $\sigma_{\vec{\mu}}$ of $\mathfrak{g}$ induces a map

$$\Theta_{\vec{\mu}} \equiv \Theta_{\vec{\mu};s_\circ}: \quad \mathcal{H}_\Lambda \to \mathcal{H}_{\omega^\star\Lambda} \tag{2.19}$$

between the irreducible highest weight modules $\mathcal{H}_\Lambda$ and $\mathcal{H}_{\omega^\star\Lambda}$ over $\mathfrak{g}$ which obeys the twisted intertwining property

$$\Theta_{\vec{\mu}}\, x = \sigma_{\vec{\mu}}(x)\, \Theta_{\vec{\mu}} \tag{2.20}$$

for all $x \in \mathfrak{g}$; the map $\Theta_{\vec{\mu}}$ is characterized by this property up to a scalar factor. In the special case where $\sigma_{\vec{\mu}}$ is an inner automorphism of $\mathfrak{g}$ and hence of the form $\sigma_{\vec{\mu}} = \overset{\rightarrow}{\prod}_i \exp(\mathrm{ad}_{x_i})$, the map $\Theta_{\vec{\mu}}$ can be implemented by

$$\Theta_{\vec{\mu}} = \overset{\rightarrow}{\prod_i} \exp(x_i)\,, \tag{2.21}$$

where the right hand side is to be understood as an element of the universal enveloping algebra of $\mathfrak{g}$. Notice, though, that the implementation $\Theta_{\vec{\mu}}$ is only determined up to a phase. Indeed, owing to the fact that K is central so that $\exp(\mathrm{ad}_{\xi K}) = \mathrm{id}$, we can always modify $\Theta_{\vec{\mu}}$ by central terms,

$$\Theta_{\vec{\mu}} = \exp(f\,K) \overset{\rightarrow}{\prod_i} \exp(x_i)\,, \tag{2.22}$$

where f is an arbitrary element of $\mathcal{F}(\mathbb{P}^1_m)$. We will make use of this freedom later on. Moreover, notice that even if all automorphisms in $\sigma_{\vec{\mu}} = \prod_i \exp(\mathrm{ad}_{x_i})$ commute so that this product is well-defined without specifying the ordering, this does not imply that the same holds for the product that appears in the implementation (2.21). We will therefore choose some specific ordering of the insertion points and keep this ordering fixed in the sequel.

2.3 Outer automorphism classes

We are now in a position to determine in which outer automorphism class $[\sigma_{\vec{\mu}}]$ a multi-shift automorphism $\sigma_{\vec{\mu}}$ lies. To this end we take for the Laurent series f the constant function id and exploit the transformation property of the elements $H^i{\otimes}\mathrm{id}$. Because of the identity

$$\mathcal{R}es(\varphi_{s_\circ,s}\, t^n) = \begin{cases} 1 & \text{for } s = s_\circ,\ n = 0\,, \\ -(z_s - z_{s_\circ})^n & \text{for } s \neq s_\circ,\ n < 0\,, \\ 0 & \text{else} \end{cases} \tag{2.23}$$

for $n \in \mathbb{Z}$, we have in particular

$$\mathcal{R}es(\varphi_{s_\circ,s}) = \delta_{s,s_\circ}\,, \tag{2.24}$$

so that

$$\sigma_{\vec{\mu};s}(H^i{\otimes}\mathrm{id}) = H^i{\otimes}\mathrm{id} + \bar\mu^i_s\, K\,. \tag{2.25}$$

Expressed in terms of the notation (2.13), the transformation (2.25) reads

$$H^i_0 \;\mapsto\; H^i_0 + \bar\mu^i_s\, K\,. \tag{2.26}$$

When combined with the results of subsection 2.2 it follows that modulo the root lattice L of $\bar{\mathfrak{g}}$ the highest weight Λ of any irreducible highest weight module $(R,\mathcal{H})$ over $\mathfrak{g}$ and the highest weight $\tilde\Lambda$ of $(\tilde R,\mathcal{H}) \equiv (R\circ\sigma_{\vec{\mu};s},\mathcal{H})$ are related by

$$\bar{\tilde\Lambda} - \bar\Lambda = k\,\bar\mu_s \mod L\,, \tag{2.27}$$

where k is the eigenvalue of the central element K. Note that this is a statement about the *horizontal* parts of the highest weights; the transformation of the rest of the weight is then fixed by the fact that because of $\sigma_{\vec{\mu};s}(K) = K$ the level does not change, while the change in the grade is prescribed by the formula for $\sigma_{\vec{\mu};s}(L_0)$ that is given in (2.32) below. [2]

Now the quantity $\bar\lambda \bmod L$ is nothing but the *conjugacy class* of the $\bar{\mathfrak{g}}$-weight $\bar\lambda$. Taking also into account the relation $k = k^\vee\cdot(\bar\theta,\bar\theta)/2$ (with $\bar\theta$ the highest root of $\bar{\mathfrak{g}}$) between the eigenvalue k of K and the level $k^\vee$, we learn that the automorphism $\sigma_{\vec{\mu};s}$ leads to a change δ_s in the conjugacy class of level-1 modules that is given by $(\bar\theta,\bar\theta)/2$ times the shift vector $\bar\mu_s$, taken modulo the root lattice L, or what is the same,

$$\delta_s = \tfrac{(\bar\theta,\bar\theta)}{2}\left(\bar\mu_s \bmod L^\vee\right). \tag{2.28}$$

Furthermore, there is a natural one-to-one correspondence between conjugacy classes of $\bar{\mathfrak{g}}$-weights and classes of certain outer automorphisms of $\mathfrak{g}$. Namely, as abelian groups we have the isomorphisms

$$L_\mathrm{w}/L \cong L^\vee_\mathrm{w}/L^\vee \;\cong \mathcal{Z}(\mathfrak{g})\,, \tag{2.29}$$

where $\mathcal{Z}(\mathfrak{g})$ is the unique maximal abelian normal subgroup of the group $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ of outer automorphism classes of $\mathfrak{g}$. It follows in particular that $[\sigma_{\vec{\mu};s}]$ is the same outer automorphism class as the class $[\sigma^{(0)}_{\bar\mu_s}]$ of the ordinary shift automorphism $\sigma^{(0)}_{\bar\mu}$ (compare formula

[2] Of course the property that, modulo L, for fixed level there is a universal shift in the horizontal part of the highest weight is not shared by arbitrary automorphisms of $\mathfrak{g}$.

(2.14)) that is characterized by the same shift vector $\bar\mu = \bar\mu_s$. It is important to note that in general this way one cannot obtain *all* outer automorphism classes of $\mathfrak{g}$, but rather only those which belong to the subgroup $\mathcal{Z}(\mathfrak{g})$.

Let us also mention that the abelian subgroup $\mathcal{Z}(\mathfrak{g})$ of the group of diagram automorphisms is naturally isomorphic to the center of the compact, connected and simply connected real Lie group whose Lie algebra is the compact real form of $\bar{\mathfrak{g}}$. Furthermore, the elements of $\mathcal{Z}(\mathfrak{g})$ are in one-to-one correspondence with the simple currents [23, 24] of the WZW theory, which are those primary fields which correspond to the units of the fusion ring of the theory (see e.g. [14]).

2.4 Extension to the Virasoro algebra

Given a representation of the untwisted affine Lie algebra $\mathfrak{g}$, the affine Sugawara construction provides us with a representation of the Virasoro algebra $\mathcal{V}ir$. Recall that $\mathcal{V}ir$ is the Lie algebra with generators L_n for $n \in \mathbb{Z}$ and C and relations

$$[L_n, L_m] = (n-m)\, L_{n+m} + \tfrac{1}{24}\,(n^3-n)\,\delta_{n+m,0}\, C\,, \qquad [C, L_n] = 0\,. \tag{2.30}$$

The algebra $\mathcal{V}ir$ combines with $\mathfrak{g}$ as a semidirect sum $\mathcal{V}ir \oplus \mathfrak{g}$, with relations

$$[L_n, \bar x {\otimes} f] = -\bar x \otimes t^{n+1} \mathrm{d}f \tag{2.31}$$

for $n \in \mathbb{Z}$.

As we show in appendix A, the multi-shift automorphisms $\sigma_{\vec{\bar\mu};s_\circ}$ of $\mathfrak{g}$ can be extended in a unique manner to automorphisms of $\mathcal{V}ir \oplus \mathfrak{g}$. The action of such an automorphism on the generators of $\mathcal{V}ir$ reads $\sigma_{\vec{\bar\mu};s_\circ}(C) = C$ and

$$\sigma_{\vec{\bar\mu};s_\circ}(L_n) = L_n + \sum_{\ell\in\mathbb{Z}} \sum_s (\bar\mu_s, H_{n-\ell})\, \mathcal{R}es(t^\ell \varphi_{s_\circ,s}) + \tfrac12\, K \sum_{s,s'} (\bar\mu_s, \bar\mu_{s'})\, \mathcal{R}es(t^{n+1} \varphi_{s_\circ,s} \varphi_{s_\circ,s'}) \tag{2.32}$$

(since the extension is unique, we employ the same symbol for the automorphism of $\mathcal{V}ir \oplus \mathfrak{g}$ as for the automorphism of $\mathfrak{g}$). Note that by putting $\bar\mu_s = 0$ for $s \neq s_\circ$, this reduces to

$$\sigma^{(0)}_{\bar\mu}(L_n) = L_n + (\bar\mu, H_n) + \tfrac12\, K\,(\bar\mu, \bar\mu)\, \delta_{n,0}\,; \tag{2.33}$$

this is precisely the unique extension to the Virasoro algebra of the ordinary shift automorphism (2.14) of $\mathfrak{g}$.

For later reference we also mention that via the formula (2.20) and the affine Sugawara construction, the maps $\Theta_{\vec{\bar\mu}}\colon \mathcal{H}_\Lambda \to \mathcal{H}_{\omega^\star\Lambda}$ defined in (2.19) obey the twisted intertwining relations

$$\Theta_{\vec{\bar\mu};s_\circ} L_n = \sigma_{\vec{\bar\mu};s_\circ}(L_n)\, \Theta_{\vec{\bar\mu};s_\circ} \tag{2.34}$$

with regard to the Virasoro algebra.

3 Action of $\Gamma_{\rm W}$ on the block algebra

The automorphisms constructed in section 2 also provide us with corresponding automorphisms of direct sums of affine Lie algebras with identified centers. In this section we show that such an automorphism preserves a distinguished subalgebra of the latter algebra, namely the so-called block algebra which plays a crucial role in the description of chiral blocks.

3.1 The block algebra

In the previous section we have established that given a shift vector $\vec{\mu}\in\Gamma_{\rm w}$ together with a definite choice of distinguished label $s_\circ\in\{1,2,\ldots,m\}$, the prescription (2.6) provides us with an automorphism of the affine Lie algebra $\mathfrak{g}$. But the formula (2.6) of course makes sense for every choice $s_\circ=1,2,\ldots,m$ of this distinguished label, so that we are in fact even given a collection $\{\sigma_{\vec{\mu};s_\circ}\}$ of such automorphisms, one for each value of $s_\circ$, and all with the same shift vector $\vec{\mu}\in\Gamma_{\rm w}$. By a slight change of perspective we can rephrase this by saying that we are given an automorphism of the direct sum $\bigoplus_{s=1}^m\mathfrak{g}_s$ of m copies $\mathfrak{g}_s\cong\mathfrak{g}$, namely the one that on the sth summand $\mathfrak{g}_s$ acts as $\sigma_{\vec{\mu};s}$. Moreover, the so obtained automorphism restricts to an automorphism of the quotient

$$\mathfrak{g}^m := \big(\bigoplus_{s=1}^m \mathfrak{g}_s\big)\big/\mathcal{J}\,, \qquad \mathcal{J} := \langle K_s{-}K_{s'}\,|\, s,s'=1,2,...,m\rangle \tag{3.1}$$

of $\bigoplus_{s=1}^m\mathfrak{g}_s$ that is obtained by identifying the centers of the algebras $\mathfrak{g}_s$, because the ideal $\mathcal{J}$ is $\sigma_{\vec{\mu}}$-invariant.

While the specific form of the automorphisms is rather irrelevant for this simple observation, it will become crucial in the considerations that follow now, where we focus our attention to a different infinite-dimensional Lie algebra that can be embedded into (3.1). A convenient starting point for the description of this algebra is to recall that the automorphisms (2.6) also depend on a chosen sequence of m pairwise distinct numbers $z_s\in\mathbb{C}$. We now regard these numbers as the coordinates of m pairwise distinct points p_s on the complex projective line $\mathbb{P}^1$, to which we refer as the *insertion points* or *punctures*. More precisely, we regard the complex curve $\mathbb{P}^1$ as $\mathbb{C}\cup\{\infty\}$ and denote by z the standard global coordinate on $\mathbb{C}$; then we write

$$z_s = z(p_s)\,. \tag{3.2}$$

For the present purposes the points p_s are kept fixed; later on we will allow them to vary over the whole moduli space $\mathcal{M}_m$ of m pairwise distinct points on $\mathbb{P}^1$.

Given this collection of points, we can consider the space $\mathcal{F}(\mathbb{P}^1_m)$ of algebraic functions on the punctured Riemann sphere $\mathbb{P}^1_m\equiv\mathbb{P}^1\backslash\{p_1,p_2,\ldots,p_m\}$, i.e. of meromorphic functions on $\mathbb{P}^1$ whose poles are of finite order and lie at most at the punctures $\{p_1,p_2,\ldots,p_m\}$. The space $\mathcal{F}(\mathbb{P}^1_m)$ is an associative algebra, the product being given by pointwise multiplication. As a consequence, the tensor product

$$\bar{\mathfrak{g}}{\otimes}\mathcal{F} \equiv \bar{\mathfrak{g}}\otimes\mathcal{F}(\mathbb{P}^1_m) \tag{3.3}$$

of the horizontal subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ with the space $\mathcal{F}(\mathbb{P}^1_m)$ inherits a natural Lie algebra structure from the one of $\bar{\mathfrak{g}}$, with Lie bracket

$$[\bar{x}{\otimes}f,\bar{y}{\otimes}g] = [\bar{x},\bar{y}]\otimes fg \tag{3.4}$$

for $\bar{x},\bar{y}\in\bar{\mathfrak{g}}$ and $f,g\in\mathcal{F}(\mathbb{P}^1_m)$. In terms of the $\bar{\mathfrak{g}}$-generators H^i and $E^{\bar{\alpha}}$, this yields the same relations as in (2.2), but with the residue terms removed, and with f,g elements of $\mathcal{F}(\mathbb{P}^1_m)$ rather than formal power series.

The algebra $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$ provides a global realization of the symmetries of a WZW conformal field theory, which are locally realized through the affine Lie algebra $\mathfrak{g}$. As will be described in more detail later on, it constitutes an important ingredient in a representation theoretic description of the chiral blocks of WZW theories; for brevity, we will therefore refer to $\bar{\mathfrak{g}}\otimes\mathcal{F}$ as the *block algebra*. Clearly, the block algebra is spanned by the elements $H^i\otimes f$ and $E^{\bar\alpha}\otimes f$ with $i\in\{1,2,...,r\}$, $\bar\alpha$ a $\bar{\mathfrak{g}}$-root and $f\in\mathcal{F}(\mathbb{P}^1_m)$. (By allowing for arbitrary $f\in\mathcal{F}(\mathbb{P}^1_m)$ of course we do not obtain a basis of the block algebra; while a basis can easily be given, we will not need it here. Note that an automorphism of $\bar{\mathfrak{g}}\otimes\mathcal{F}$ is uniquely defined by its action on a basis, and hence a fortiori by its action on the elements $H^i\otimes f$ and $E^{\bar\beta}\otimes f$.)

Around each of the punctures p_s we can choose the local coordinate

$$\zeta_s = z - z_s\,. \tag{3.5}$$

By identifying these local coordinates with the indeterminate t of the loop construction of the affine Lie algebra $\mathfrak{g}_s\cong\mathfrak{g}$, one obtains an embedding of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$ in the direct sum of m copies of the loop algebra $\bar{\mathfrak{g}}_{\rm loop}\cong\bar{\mathfrak{g}}\otimes\mathbb{C}((t))$, and thereby also an embedding $\imath$ in the algebra $\mathfrak{g}^m$ that was introduced in (3.1). This is seen as follows. For any function $f\in\mathcal{F}(\mathbb{P}^1_m)$ we denote its expansion in local coordinates around the point p_s, considered as a Laurent series in the variable $t\equiv\zeta_s$, by $f_{|s}$. With this notation, the image of $\bar x\otimes f\in\bar{\mathfrak{g}}\otimes\mathcal{F}$ under the embedding $\imath$ is the sequence

$$\imath(\bar x\otimes f) = (\bar x\otimes f_{|1}, \bar x\otimes f_{|2}, ... , \bar x\otimes f_{|m})\,, \tag{3.6}$$

where $\bar x\otimes f_{|s}$ is regarded as an element of $\mathfrak{g}_s$. In short, the embedding in the sth summand is obtained by replacing the global function f by its local expansion $f_{|s}$ at the puncture p_s, or in other words, by its germ at p_s. Note that f is already determined completely by its germ at any single puncture. In particular, the block algebra is a proper subalgebra of $\mathfrak{g}^m$.

3.2 Automorphisms of the block algebra

Next we observe that also the functions $\varphi_{s_\circ,s}$ defined in (2.7) are the germs of globally defined functions in $\mathcal{F}(\mathbb{P}^1_m)$. Indeed, in view of (3.5) for each pair $s_\circ, s$ the function $\varphi_{s_\circ,s}$ (2.7) can be recognized as the local expansion

$$\varphi_{s_\circ,s}(\zeta_{s_\circ}) = (\zeta_{s_\circ}+z_{s_\circ}-z_s)^{-1} = \varphi^{(s)}_{|s_\circ}(\zeta_{s_\circ}) \tag{3.7}$$

at $p_{s_\circ}$ of the function $\varphi^{(s)}\in\mathcal{F}(\mathbb{P}^1_m)$ defined by

$$\varphi^{(s)}(z) := (z-z_s)^{-1}\,. \tag{3.8}$$

At this point the meaning of the requirement $\sum_{s=1}^m\bar\mu_s=0$ becomes clear: it ensures that the function $\prod_s(\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}$ possesses poles only at the punctures p_s and hence lies in $\mathcal{F}(\mathbb{P}^1_m)$. It therefore follows in particular that the functions $f\cdot\prod_s(\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\beta)}$ that appear in the automorphism (2.6) of $\mathfrak{g}$ are the local germs at $p_{s_\circ}$ of globally defined functions $f\cdot\prod_s(\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}$. Of course, each of the germs individually already contains all information on the global function; this is reflected by the fact that for all choices of $s_\circ$ the Laurent series $(\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\beta)}$ all involve one and the same shift vector $\vec\mu$.

From these observations we finally deduce that to any shift vector $\vec\mu\in\Gamma_{\rm w}$ we can associate a linear map of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$ which acts as

$$\vec\sigma_{\vec\mu}(H^i\otimes f) = H^i\otimes f\,, \qquad \vec\sigma_{\vec\mu}(E^{\bar\beta}\otimes f) = E^{\bar\beta}\otimes f\cdot\prod_{s=1}^m(\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}\,. \tag{3.9}$$

It is readily checked that this map constitutes an automorphism of $\bar{\mathfrak{g}}\otimes\mathcal{F}$. Note that on purpose here we employ a similar symbol $\vec{\sigma}_{\vec{\mu}}$ as for the automorphisms $\sigma_{\vec{\mu}}$ of $\mathfrak{g}$ in (2.6); indeed these maps should be regarded as the global and local realizations, respectively, of one and the same basic structure. To be precise, the local expansions of the automorphism (3.9) of the block algebra reproduce the automorphisms (2.6) of $\mathfrak{g}_s$ only up to the central terms. The latter are needed in order to really have an automorphism of $\mathfrak{g}_s$ rather than only an automorphism of the corresponding loop algebra; however, because of the identification of the centers of the subalgebras $\mathfrak{g}_s$ in $\mathfrak{g}^m$ (3.1), upon summing over all insertion points these terms cancel owing to the residue theorem:

$$\begin{aligned}\vec{\sigma}_{\vec{\mu}}(H^i\otimes f) \;&\hat{=}\; \vec{\sigma}_{\vec{\mu}}\big(\textstyle\sum_{s=1}^m H^i_{(s)}\otimes f_{|s}\big) = \displaystyle\sum_{s=1}^m \sigma_{\vec{\mu};s}(H^i\otimes f_{|s}) \;\hat{=}\; H^i\otimes f + K \sum_{s,s'=1}^m \bar{\mu}^i_{s'}\, \mathcal{R}es(\varphi_{s,s'}\, f_{|s}) \\ &= H^i\otimes f + K \sum_{s'=1}^m \bar{\mu}^i_{s'} \cdot \sum_{s=1}^m \mathcal{R}es\big((\varphi^{(s')} f)_{|s}\big) = H^i\otimes f\,.\end{aligned} \tag{3.10}$$

As the reader may already have noticed, for any $\bar{\mathfrak{g}}$-root $\bar{\alpha}$ not only the poles, but also the zeroes of the meromorphic function $\prod_s(\varphi^{(s)})^{-(\bar{\mu}_s,\bar{\alpha})}$ occur at the punctures p_s. In other words, these functions belong to the subset

$$\mathcal{F}^*(\mathbb{P}^1_m) := \{f\in\mathcal{F}(\mathbb{P}^1_m)\,|\,f^{-1}\in\mathcal{F}(\mathbb{P}^1_m)\} \tag{3.11}$$

of $\mathcal{F}(\mathbb{P}^1_m)$, where in accordance with the prescription (2.8) the symbol f^{-1} stands for the function that has values inverse to those of f, i.e. $f^{-1}(p)=(f(p))^{-1}$ for all $p\in\mathbb{P}^1$. The elements of this subset $\mathcal{F}^*(\mathbb{P}^1_m)\subset\mathcal{F}(\mathbb{P}^1_m)$, which is in fact a subalgebra, are called the invertible elements or *units* of $\mathcal{F}(\mathbb{P}^1_m)$.

At this point it is appropriate to mention that the point at infinity of $\mathbb{P}^1\cong\mathbb{C}\cup\{\infty\}$ is by no means distinguished geometrically, but acquires its special role only through the choice of coordinates. Choosing a different quasi-global coordinate $\tilde{z}$ on $\mathbb{P}^1$ one would assign the coordinate value ∞ to a different geometrical point. Accordingly we should also allow for $z=\infty$ as the value of z at one of the insertion points, for definiteness say $z_m=\infty$. As it turns out, in this situation the discussion above indeed goes through, but with some modifications.

An obvious modification consists in the fact that in place of $\zeta_s=z-z_s$ (3.5) a good local coordinate at $z_m=\infty$ is provided by $\zeta_m=z^{-1}$. Accordingly we must e.g. replace the formula (2.7) for $\varphi_{s_\circ,s}$ by

$$\varphi_{s_\circ,s}(t) := \left\{ \begin{array}{ll} (t+z_{s_\circ}-z_s)^{-1} & \text{for } z_{s_\circ}\neq\infty\,, \\ (t^{-1}-z_s)^{-1} & \text{for } z_{s_\circ}=\infty\,. \end{array}\right. \tag{3.12}$$

However, this is not the only change that needs to be made. One finds that in fact in the formula (2.6) for the automorphisms of $\mathfrak{g}$ one must now only include the contributions from $s\in\{1,2,...\,,m-1\}$, but not from $s=m$, i.e. one now has

$$\sigma_{\vec{\mu};s_\circ}(H^i\otimes f) := H^i\otimes f + K\sum_{s=1}^{m-1}\bar{\mu}^i_s\,\mathcal{R}es(\varphi_{s_\circ,s}\,f)\,, \qquad \sigma_{\vec{\mu};s_\circ}(E^{\bar{\beta}}\otimes f) := E^{\bar{\beta}}\otimes f\cdot\Phi_{\bar{\beta}}\,. \tag{3.13}$$

where the definition (2.9) of $\Phi_{\bar\alpha}$ is to be replaced by

$$\Phi_{\bar\alpha} := \prod_{s=1}^{m-1} (\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\alpha)} \,. \tag{3.14}$$

Moreover, the formula (3.13) applies only for $s_\circ \in \{1,2,...,m{-}1\}$, while for $s_\circ = m$ it gets replaced by

$$\sigma_{\vec\mu;m}(H^i{\otimes}f) := H^i{\otimes}f - K \sum_{s=1}^{m-1} \bar\mu_s^i\, \mathcal{R}es(t^{-2}\varphi_{s_\circ,s}\, f)\,, \qquad \sigma_{\vec\mu;m}(E^{\bar\beta}{\otimes}f) := E^{\bar\beta}{\otimes}f \cdot \Phi_{\bar\beta}\,. \tag{3.15}$$

Similarly, instead by (3.9), the associated automorphism of the block algebra $\bar{\mathfrak g}{\otimes}\mathcal F$ is now given by

$$\vec\sigma_{\vec\mu}(H^i{\otimes}f) = H^i{\otimes}f\,, \qquad \vec\sigma_{\vec\mu}(E^{\bar\beta}{\otimes}f) = E^{\bar\beta}{\otimes}f \cdot \prod_{s=1}^{m-1} (\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}\,. \tag{3.16}$$

After these replacements all calculations go through as before. In particular, still the functions $\prod_{s=1}^{m-1}(\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\beta)}$ are the local germs of the globally defined function $\prod_{s=1}^{m-1}(\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}$ at $p_{s_\circ}$ for *all* $s_\circ \in \{1,2,...,m\}$. Also, that function is still an element of $\mathcal F^*(\mathbb P^1_m)$. (On the other hand, of course now one can no longer invoke the identity $\sum_{s=1}^m \bar\mu_s = 0$ to exclude poles at infinity; but this is also no longer needed, since ∞ is a puncture. Explicitly, now the divisor of the function $\prod_s(\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}$ consists of the points p_s, $s=1,2,...,m{-}1$, with finite value of z_s, which have multiplicity $-(\bar\mu_s,\bar\beta)$, and in addition of ∞ which has multiplicity $\sum_{s=1}^{m-1}(\bar\mu_s,\bar\beta) = -(\bar\mu_m,\bar\beta)$.) Briefly, the function $\prod_{s=1}^{m-1}(\varphi^{(s)})^{-(\bar\mu_s,\bar\beta)}$ has already by itself a pole respectively zero of the correct order at ∞.

Let us also note that at $z_m = \infty$ the formula (2.25) for $\sigma_{\vec\mu;s_\circ}(H^i{\otimes}\mathrm{id})$ still applies. Namely, in this case we have

$$\sigma_{\vec\mu;m}(H^i{\otimes}\mathrm{id}) = H^i{\otimes}\mathrm{id} - K \sum_{s=1}^{m-1} \bar\mu_s^i\, \mathcal{R}es(t^{-2}\,(t^{-1}-z_s)^{-1})\,. \tag{3.17}$$

Using the fact that $\mathcal{R}es(t^{-2}(t^{-1}-z_s)^{-1}) = 1$ independently of s together with the condition that the shift vectors add up to zero, this is nothing but

$$\sigma_{\vec\mu;m}(H^i{\otimes}\mathrm{id}) = H^i{\otimes}\mathrm{id} + \bar\mu_m^i\, K\,. \tag{3.18}$$

It follows in particular that the statement that the outer automorphism class $[\sigma_{\vec\mu;s}]$ of $\sigma_{\vec\mu;s}$ can be read off the coweight lattice element $\bar\mu_s$ is still true for $s=m$ with $z_m=\infty$. We also mention that in the case $z_m = \infty$ the residue theorem still ensures the cancellation of central terms as in formula (3.10). Namely, while the summation over s' in (3.10) is now only from 1 to $m{-}1$, the summation over s still ranges from 1 to m; the desired result then follows after taking into account that $\mathcal{R}es\big((\varphi^{(s')}f)_{|m}\big) = \mathcal{R}es_\infty(\varphi^{(s')}f) = -\mathcal{R}es_0(t^{-2}\varphi^{(s')}f)$.

Finally we remark that the Sugawara construction provides us at each of the insertion points p_s, with a Virasoro algebra $\mathcal{V}ir \equiv \mathcal{V}ir_s$. We write $L_n^{(s)}$ for the generators of the Virasoro algebra $\mathcal{V}ir_s$ associated to $\mathfrak g_s$. Also note that in the formula (2.32) for the action of a multi-shift automorphism on $\mathcal{V}ir$ the sums over s and s' extend over all punctures except, when present, the one at ∞.

4 Inner automorphisms

In subsection 2.2 we have seen how an automorphism of the affine Lie algebra $\mathfrak{g}$ can be implemented on $\mathfrak{g}$-modules. We can now implement the tensor product of m multi-shift automorphisms on the tensor product of m $\mathfrak{g}$-modules. We will see that this leads to a well-defined map on the spaces of chiral blocks because such a tensor product of multi-shift automorphisms leaves the block algebra $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$ invariant. One convenient property that one would like to achieve for the maps on the chiral blocks is that they are of finite order. As it turns out, to this end we must find a suitable description for those multi-shift automorphisms which are inner automorphisms of $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$.

The subset

$$\Gamma := \{\vec{\bar\mu} \equiv (\bar\mu_1,\bar\mu_2,\ldots,\bar\mu_m) \,|\, \bar\mu_s \in L^{\vee} \;\text{ for all } s=1,2,\ldots,m;\; \textstyle\sum_{s=1}^m \bar\mu_s = 0\} \tag{4.1}$$

of those elements of $\Gamma_{\rm w}$ for which the allowed range of the shift vectors $\bar\mu_s$ is restricted to the co*root* lattice $L^{\vee} \subseteq L^{\vee}_{\rm w}$ of $\bar{\mathfrak{g}}$ clearly is a subgroup of $\Gamma_{\rm w}$. In this section we show that the automorphisms (2.6) of $\mathfrak{g}$ and (3.9) of $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$ associated to shift vectors that lie in Γ are *inner* automorphisms of the block algebra $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$, i.e. automorphisms that can be written as $\prod_i \exp(\mathrm{ad}_{x_i})$ with suitable elements x_i of $\mathfrak{g}$ respectively $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$;[3] a fortiori they are then also inner automorphisms of the ambient algebra $\mathfrak{g}^m \supset \bar{\mathfrak{g}}{\otimes}\mathcal{F}$.

For establishing this result we need to introduce some special inner automorphisms of the block algebra $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$, and for these we need some particular elements of $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$. For convenience, and without loss of generality, in this section on we assume that 0 and ∞ are among the insertion points p_s, and number the latter in such a way that $z_m = \infty$; this has the technical advantage that for each insertion point p_s with $s \in \{1,2,\ldots,m{-}1\}$[4] the function $\varphi^{(s)}$ defined by (3.8) already lies in the subalgebra $\mathcal{F}^*(\mathbb{P}^1_m)$ of units of $\mathcal{F}(\mathbb{P}^1_m)$. The relevant special elements of $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$ are then of the form

$$X_{\bar\alpha,f} := E^{\bar\alpha}{\otimes}f + E^{-\bar\alpha}{\otimes}f^{-1}\,, \tag{4.2}$$

where $E^{\bar\alpha} \in \bar{\mathfrak{g}}$ is a step operator of the horizontal subalgebra $\bar{\mathfrak{g}}$ and $f \in \mathcal{F}^*(\mathbb{P}^1_m)$ is a unit in $\mathcal{F}(\mathbb{P}^1_m)$.

Furthermore, for any two elements $X, Y \in \bar{\mathfrak{g}}{\otimes}\mathcal{F}$ and any complex number ξ, we denote by $\mathrm{Ad}_{\xi;X,Y}$ the inner automorphism

$$\mathrm{Ad}_{\xi;X,Y} := \exp(\mathrm{ad}_{-\xi Y}) \circ \exp(\mathrm{ad}_{\xi X}) \tag{4.3}$$

of the block algebra. We are interested in automorphisms of the form $\mathrm{Ad}_{\mathrm{i}\pi/2;X_{\bar\alpha,f_1},X_{\bar\alpha,f_2}}$ with an arbitrary positive $\bar{\mathfrak{g}}$-root $\bar\alpha$ and some suitable functions $f_1, f_2 \in \mathcal{F}^*(\mathbb{P}^1_m)$. By direct computation we find that

$$\mathrm{Ad}_{\mathrm{i}\pi/2;X_{\bar\alpha,f_1},X_{\bar\alpha,f_2}}(H^i{\otimes}f) = H^i{\otimes}f \tag{4.4}$$

for all $i = 1,2,\ldots,r$ and

$$\mathrm{Ad}_{\mathrm{i}\pi/2;X_{\bar\alpha,f_1},X_{\bar\alpha,f_2}}(E^{\bar\beta}{\otimes}f) = E^{\bar\beta} \otimes (f_1/f_2)^{-(\bar\alpha^\vee,\bar\beta)}\, f \tag{4.5}$$

[3] In the present section we only show that $\vec{\bar\mu} \in \Gamma$ is sufficient for having an inner automorphism. But as we shall see later on, this is also necessary.

[4] Recall that there is no function $\varphi^{(s)}$ associated to ∞.

for all $\bar{\mathfrak{g}}$-roots $\bar\beta$; here $\bar\alpha^\vee = 2\bar\alpha/(\bar\alpha,\bar\alpha) \in L^\vee$ is the coroot associated to the $\bar{\mathfrak{g}}$-root $\bar\alpha$. (Though straightforward, this calculation is somewhat lengthy; some relevant formulæ are collected in appendix B.)

The results (4.4) and (4.5) imply in particular that any two automorphisms of this form commute. Also, only the quotient f_1/f_2 appears in (4.5), so that without loss of generality we can put $f_2 = \mathrm{id}$. It follows in particular that when we compose inner automorphisms of the form $\mathrm{Ad}_{\mathrm{i}\pi/2;X_{\bar\alpha,f_1},X_{\bar\alpha,f_2}}$ with the special choice $f_1 = \varphi^{(s)}$ as defined by (3.8) for some $s \in \{1,2,\dots,m{-}1\}$ and $f_2 = \mathrm{id}$ and with arbitrary $\bar{\mathfrak{g}}$-roots $\bar\alpha$, then we arrive at automorphisms of the form studied previously. Indeed,

$$\prod_{s=1}^{m-1}\prod_{i_s=1}^{\ell_s} \mathrm{Ad}_{\mathrm{i}\pi/2;X_{\bar\alpha_{i_s},\varphi^{(s)}},X_{\bar\alpha_{i_s},1}} = \vec\sigma_{\vec{\bar\mu}}\,, \tag{4.6}$$

where $\sigma_{\vec{\bar\mu}}$ is the multi-shift automorphism of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$ that acts as in (3.16), with

$$\vec{\bar\mu} = (\bar\mu_1,\bar\mu_2,\dots,\bar\mu_m)\,, \qquad \bar\mu_s = \sum_{i_s=1}^{\ell_s} \bar\alpha_{i_s}^\vee \in L^\vee \quad \text{for } s = 1,2,\dots,m{-}1 \tag{4.7}$$

(and where the ordering in the product is irrelevant). In short, for any choice of the $\bar{\mathfrak{g}}$-roots $\bar\alpha_{i_s}$ we obtain an inner automorphism of the form (3.16) with $\bar\mu_s \in L^\vee$. Moreover, by appropriately choosing these roots we can obtain *every* automorphism $\sigma_{\vec{\bar\mu}}$ (3.16) for which $\vec{\bar\mu} \in \Gamma$. It follows in particular that any automorphism $\vec\sigma_{\vec{\bar\mu}}$ with $\vec{\bar\mu} \in \Gamma$ is an *inner* automorphism of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$, as claimed.

When we replace the functions f etc. that appeared in the considerations above, which are elements of $\mathcal{F}(\mathbb{P}^1_m)$, by functions that have the same mapping prescription but are regarded as Laurent series in the relevant local coordinate ζ_s, respectively as formal Laurent series in the variable t, then we can immediately repeat all steps in the derivation of the formulæ (4.4) and (4.5); thereby for each summand $\mathfrak{g}_s$, $s = 1,2,\dots,m{-}1$, of the algebra $\bigoplus_{s=1}^m \mathfrak{g}_s \supset \mathfrak{g}^m$ we construct a certain class of commuting inner automorphisms. The action of these automorphisms differs from the one of the automorphisms of the block algebra precisely in that there arise additional terms that are proportional to the central element $K \equiv K_s$ of $\mathfrak{g}_s$. We find that (4.4) gets modified to (for some details see appendix B)

$$\mathrm{Ad}_{\mathrm{i}\pi/2;X_{\bar\alpha,f_1},X_{\bar\alpha,f_2}}(H^i\otimes f) = H^i\otimes f - (\bar\alpha^\vee)^i K\left(\mathcal{R}es(f_1^{-1}\mathrm{d}f_1\, f) - \mathcal{R}es(f_2^{-1}\mathrm{d}f_2\, f)\right), \tag{4.8}$$

while the formula (4.5) applies without any change also to the present situation.

We now investigate the case where these Laurent series are precisely the local expansions of the functions $\varphi^{(s)}$ studied previously. Thus for any $s, s_\circ = 1,2,\dots,m{-}1$ we have to deal with the inner automorphism $\mathrm{Ad}_{\mathrm{i}\pi/2;x_s,y_s}$ of $\mathfrak{g}_s$ with

$$x_s = E^{\bar\alpha}\otimes\varphi_{s_\circ,s} + E^{-\bar\alpha}\otimes(\varphi_{s_\circ,s})^{-1} \qquad \text{and} \qquad y_s = (E^{\bar\alpha} + E^{-\bar\alpha})\otimes\mathrm{id}\,, \tag{4.9}$$

where $\varphi_{s_\circ,s}(t)$ is the local expansion (2.7) at $p_{s_\circ}$ of the function $\varphi^{(s)}$ as defined by (3.8). By inserting these functions in the general formulæ obtained above, we learn that the automorphism $\mathrm{Ad}_{\mathrm{i}\pi/2;x_s,y_s}$ acts as

$$\begin{aligned} \mathrm{Ad}_{\mathrm{i}\pi/2;x_s,y_s}(H^i\otimes f_{|s_\circ}) &= H^i\otimes f_{|s_\circ} + (\bar\alpha^\vee)^i K\,\mathcal{R}es(\varphi_{s_\circ,s} f_{|s_\circ})\,,\\ \mathrm{Ad}_{\mathrm{i}\pi/2;x_s,y_s}(E^{\bar\beta}\otimes f_{|s_\circ}) &= E^{\bar\beta}\otimes f_{|s_\circ}(\varphi_{s_\circ,s})^{-(\bar\alpha^\vee,\bar\beta)}\,. \end{aligned} \tag{4.10}$$

Next we take the product of such automorphisms that is analogous to the product in (4.6). Thereby for each $s_\circ = 1, 2, ..., m{-}1$ we arrive at automorphisms $\sigma_{\vec\mu;s_\circ}$ of $\mathfrak{g}_{s_\circ}$ which act as $\sigma_{\vec\mu;s_\circ}(K) = K$ and

$$\begin{aligned} \sigma_{\vec\mu;s_\circ}(H^i{\otimes}f_{|s_\circ}) &= H^i\otimes f_{|s_\circ} + K\, \textstyle\sum_{s=1}^{m-1} \bar\mu^i_s\, \mathcal{R}es(\varphi_{s_\circ,s}\, f_{|s_\circ})\,, \\ \sigma_{\vec\mu;s_\circ}(E^{\bar\beta}{\otimes}f_{|s_\circ}) &= E^{\bar\beta}\otimes f_{|s_\circ}\cdot \textstyle\prod_{s=1}^{m-1}(\varphi_{s_\circ,s})^{-(\bar\mu_s,\bar\beta)}\,. \end{aligned} \tag{4.11}$$

For $s_\circ = m$ an analogous result holds, with the formula in the first line replaced by

$$\sigma_{\vec\mu;m}(H^i{\otimes}f_{|m}) = H^i\otimes f_{|m} - K \sum_{s=1}^{m-1} \bar\mu^i_s\, \mathcal{R}es(t^{-2}\varphi_{m,s}\, f_{|m}) \tag{4.12}$$

(and with $\varphi_{m,s}$ as given in (3.12)). Thus again we have succeeded in constructing all those multi-shift automorphisms $\sigma_{\vec\mu}$ – this time the automorphisms (2.6) of the affine Lie algebra $\mathfrak{g}$ – for which the shift vector $\vec\mu$ lies in the subgroup Γ of $\Gamma_{\rm w}$, and hence can conclude that all such automorphisms are *inner* automorphisms of $\mathfrak{g}$.

Of course, the function $f_{|s_\circ}(\varphi_{s_\circ,s})^{-(\bar\alpha^\vee,\bar\beta)} = f_{|s_\circ}(\varphi^{(s)}_{|s_\circ})^{-(\bar\alpha^\vee,\bar\beta)}$ is nothing but the local expansion of the function $f(\varphi^{(s)})^{-(\bar\alpha^\vee,\bar\beta)}$ at p_s. Hence in particular upon summation over s the central terms in the transformation (4.10) of $H^i{\otimes}f$ cancel owing to the residue theorem. Thus the automorphisms (4.11) are the local realizations of the automorphism (4.6) of the block algebra.

One may also analyze the analogous inner multi-shift automorphisms of the Virasoro algebra. This is briefly mentioned at the end of appendix B.

5 Implementation on chiral blocks

5.1 Implementation on tensor products

The next step is now to implement a tensor product of multi-shift automorphisms on a tensor product of modules of the affine Lie algebra. This gives rise to a projective action of the group $\Gamma_{\rm w}$ on this tensor product and, of course, also to a dual action on the algebraic dual of the tensor product. As we will explain, the space B of chiral blocks can be identified with a subspace in this algebraic dual. We will see that the action of $\Gamma_{\rm w}$ can be restricted to B and that this action has finite order on the subspace B.

Let us first recall the existence of the maps $\Theta_{\vec\mu} \equiv \Theta_{\vec\mu;s_\circ}$ (2.19). They act on the $s_\circ$th factor of the tensor product

$$\vec{\mathcal{H}} \equiv \mathcal{H}_{\vec\Lambda} = \mathcal{H}_{\Lambda_1}\otimes\mathcal{H}_{\Lambda_2}\otimes\cdots\otimes\mathcal{H}_{\Lambda_m} \tag{5.1}$$

of irreducible highest weight modules over $\mathfrak{g}$, i.e. $\Theta_{\vec\mu}$: $\mathcal{H}_{\Lambda_{s_\circ}} \to \mathcal{H}_{\omega^\star\Lambda_{s_\circ}}$, with $\omega^\star$ the permutation of integrable weights that was described before (2.19). The tensor product

$$\vec\Theta_{\vec\mu} := \Theta_{\vec\mu;1}\otimes\Theta_{\vec\mu;2}\otimes\cdots\otimes\Theta_{\vec\mu;m} \tag{5.2}$$

of these maps for all $s_\circ = 1, 2, ..., m$ provides us with an analogous map between tensor products,

$$\vec\Theta_{\vec\mu}:\quad \mathcal{H}_{\vec\Lambda} \to \mathcal{H}_{\vec\omega^\star\vec\Lambda}\,, \tag{5.3}$$

where $\vec{\omega}^\star\vec{\Lambda} = (\omega^\star\Lambda_1, \omega^\star\Lambda_2, \dots, \omega^\star\Lambda_m)$. Now note that this map is defined for any $\vec{\bar\mu} \in \Gamma_{\rm w}$. Moreover,

$$\vec{\bar\mu} \;\mapsto\; \vec\Theta_{\vec{\bar\mu}} \tag{5.4}$$

constitutes a projective representation of $\Gamma_{\rm w}$, i.e. up to possibly a $U(1)$-valued cocycle ϵ, the group law of $\Gamma_{\rm w}$ is also realized by the action of the maps $\vec\Theta_{\vec{\bar\mu}}$ on the modules,

$$\vec\Theta_{\vec{\bar\mu}_1}\,\vec\Theta_{\vec{\bar\mu}_2} = \epsilon(\vec{\bar\mu}_1,\vec{\bar\mu}_2)\,\vec\Theta_{\vec{\bar\mu}_1+\vec{\bar\mu}_2}\,; \tag{5.5}$$

moreover, the cocycle on $\Gamma_{\rm w}$ is uniquely determined by a cocycle on the finite group $\Gamma_{\rm w}/\Gamma$. This result can be established as follows. First we realize that the highest weight vector of the $\mathfrak{g}$-module $\mathcal{H}_\Lambda$ can be required to be normalized and then is unique up to a phase. Accordingly the maps $\Theta_{\vec{\bar\mu};s_\circ}$ are unique up to a phase, too. Our aim is to show that we can choose this phase as a function of $\vec{\bar\mu}$ in a suitable manner. To this end we proceed in several steps.

We first define the maps $\vec\Theta_{\vec{\bar\mu}}$ for elements $\vec{\bar\mu} = \vec{\bar\gamma} \in \Gamma$ of the subgroup Γ (4.1). This group is a finitely generated free abelian group, so there is a finite set $\{\vec{\bar\gamma}_{(j)}\}$ of independent generators. For instance one can choose these generators to be all of the form

$$\bar\gamma_{(j);s_1} = \bar\beta^\vee\,, \qquad \bar\gamma_{(j);s_2} = -\bar\beta^\vee\,, \qquad \bar\gamma_{(j);s} = 0 \quad \text{for } \; s \neq s_1, s_2 \tag{5.6}$$

with suitable choices of $s_1, s_2 \in \{1,2,\dots,m\}$ with $s_1 < s_2$ and simple $\bar{\mathfrak{g}}$-roots $\bar\beta$. Now for each $\vec{\bar\gamma}_{(j)}$ we fix the freedom in the definition of the implementing maps by explicitly prescribing a preferred implementation $\vec\Theta_{\vec{\bar\gamma}_{(j)}}$ through elements of the block algebra. Concretely, in the case of generators of the form (5.6) we set

$$\begin{aligned}\vec\Theta_{\vec{\bar\gamma}_{(j)}} :=\; & {\rm id}\otimes\cdots\otimes{\rm id}\otimes\exp\big(-\tfrac{{\rm i}\pi}{2}X_{\bar\beta,1}\big)\exp\big(\tfrac{{\rm i}\pi}{2}X_{\bar\beta,\varphi_{s_\circ,s_1}}\big)\otimes{\rm id}\otimes \\ & \cdots\otimes{\rm id}\otimes\exp\big(-\tfrac{{\rm i}\pi}{2}X_{\bar\beta,1}\big)\exp\big(\tfrac{{\rm i}\pi}{2}X_{\bar\beta,\varphi_{s_\circ,s_2}}\big)\otimes{\rm id}\otimes\cdots\otimes{\rm id}\,,\end{aligned} \tag{5.7}$$

where the notation $X_{\bar\alpha,f}$ is as introduced in (4.2) and where the non-trivial maps occur at the s_1th and s_2th factor of the tensor product. A priori it might be expected that the maps $\vec\Theta_{\vec{\bar\gamma}_{(j)}}$ for different values of j do not commute, but rather satisfy

$$\vec\Theta_{\vec{\bar\gamma}_{(j)}}\,\vec\Theta_{\vec{\bar\gamma}_{(k)}} = \eta(\vec{\bar\gamma}_{(j)},\vec{\bar\gamma}_{(k)})\,\vec\Theta_{\vec{\bar\gamma}_{(j)}}\,\vec\Theta_{\vec{\bar\gamma}_{(k)}}\,. \tag{5.8}$$

But as a matter of fact, when there are non-trivial chiral blocks in $\vec{\mathcal{H}}^\star$ (which is the case we will be most interested in), then the numbers η introduced this way are equal to one. To see this, we note that the fact that the implementing maps (5.7) are realized through elements of the block algebra implies (compare the remarks around (5.25) below) that each of the $\vec\Theta_{\vec{\bar\gamma}_{(j)}}$ acts as the identity on chiral blocks. The result then follows by acting with both sides of (5.8) (or more precisely, with their implementation on the blocks, as defined in formula (5.29) below) on a chiral block.

We now write any arbitrary $\vec{\bar\gamma} \in \Gamma$ uniquely as a linear combination $\vec{\bar\gamma} = \sum_j n_j\,\vec{\bar\gamma}_{(j)}$ and define the image of $\vec{\bar\gamma}$ under the map (5.4) as

$$\vec\Theta_{\vec{\bar\gamma}} := \prod_j (\vec\Theta_{\vec{\bar\gamma}_{(j)}})^{n_j}\,. \tag{5.9}$$

By the result just obtained, the order of the factors in this product does not matter. Also, by the same argument as before one concludes that

$$\vec{\Theta}_{\vec{\gamma}_1}\,\vec{\Theta}_{\vec{\gamma}_2} = \vec{\Theta}_{\vec{\gamma}_1+\vec{\gamma}_2} = \vec{\Theta}_{\vec{\gamma}_2}\,\vec{\Theta}_{\vec{\gamma}_1} \qquad \text{for all } \ \vec{\gamma}_1,\vec{\gamma}_2\in\Gamma\,. \tag{5.10}$$

Further, suppose that we have made a choice also for the implementations $\vec{\Theta}_{\vec{\mu}}$ for all other $\vec{\mu}\in\Gamma_{\rm w}$ (which we do not yet specify, because the actual choice is not relevant for the argument). Then again by the same reasoning as before, i.e. by considering the induced maps on the blocks, one deduces that (5.10) in fact generalizes to

$$\vec{\Theta}_{\vec{\gamma}}\,\vec{\Theta}_{\vec{\mu}} = \vec{\Theta}_{\vec{\mu}}\,\vec{\Theta}_{\vec{\gamma}} \qquad \text{for all } \ \vec{\gamma}\in\Gamma,\ \vec{\mu}\in\Gamma_{\rm w}\,. \tag{5.11}$$

Next we choose some set ${\rm M}=\{\vec{\kappa}_{(i)}\}$ of coset representatives for $\Gamma_{\rm w}/\Gamma$, which is a finite abelian group. For each of the $\vec{\kappa}_{(i)}$ we make some choice of the implementing map $\vec{\Theta}_{\vec{\kappa}_{(i)}}$, with the arbitrary phase for the moment still left unspecified. Having made these choices, every $\vec{\mu}\in\Gamma_{\rm w}$ can be uniquely written as $\vec{\mu}=\vec{\gamma}(\vec{\mu})+\vec{\kappa}_{(i(\vec{\mu}))}$ with $\vec{\gamma}(\vec{\mu})\in\Gamma$ and $\vec{\kappa}_{(i(\vec{\mu}))}\in{\rm M}$; we then define

$$\vec{\Theta}_{\vec{\mu}} := \vec{\Theta}_{\vec{\gamma}(\vec{\mu})}\,\vec{\Theta}_{\vec{\kappa}_{(i(\vec{\mu}))}} \tag{5.12}$$

(where the order of the factors is again irrelevant). Note that this way we have also defined all maps $\vec{\Theta}_{\vec{\kappa}_{(i)}+\vec{\kappa}_{(j)}}$, i.e. in particular also for those cases where $\vec{\kappa}_{(i)}+\vec{\kappa}_{(j)}\not\in{\rm M}$. Therefore we can define phases $\epsilon(\vec{\kappa}_{(i)},\vec{\kappa}_{(j)})$ by

$$\vec{\Theta}_{\vec{\kappa}_{(i)}}\,\vec{\Theta}_{\vec{\kappa}_{(j)}} = \epsilon(\vec{\kappa}_{(i)},\vec{\kappa}_{(j)})\,\vec{\Theta}_{\vec{\kappa}_{(i)}+\vec{\kappa}_{(j)}}\,. \tag{5.13}$$

Finally, combining formula (5.13) with the result (5.11), we learn that for arbitrary $\vec{\mu},\vec{\nu}\in\Gamma_{\rm w}$ we have

$$\vec{\Theta}_{\vec{\mu}}\,\vec{\Theta}_{\vec{\nu}} = \vec{\Theta}_{\vec{\gamma}(\vec{\mu})}\,\vec{\Theta}_{\vec{\kappa}_{(i(\vec{\mu}))}}\,\vec{\Theta}_{\vec{\gamma}(\vec{\nu})}\,\vec{\Theta}_{\vec{\kappa}_{(i(\vec{\nu}))}} = \epsilon(\vec{\kappa}_{(i(\vec{\mu}))},\vec{\kappa}_{(i(\vec{\nu}))})\,\vec{\Theta}_{\vec{\gamma}(\vec{\mu})+\vec{\gamma}(\vec{\nu})}\,\vec{\Theta}_{\vec{\kappa}_{(i(\vec{\mu}))}+\vec{\kappa}_{(i(\vec{\nu}))}}\,. \tag{5.14}$$

When $\vec{\kappa}_{(i(\vec{\mu}))}+\vec{\kappa}_{(i(\vec{\nu}))}\in{\rm M}$, this yields immediately

$$\vec{\Theta}_{\vec{\mu}}\,\vec{\Theta}_{\vec{\nu}} = \epsilon(\vec{\kappa}_{(i(\vec{\mu}))},\vec{\kappa}_{(i(\vec{\nu}))})\,\vec{\Theta}_{\vec{\mu}+\vec{\nu}}\,, \tag{5.15}$$

while for $\vec{\kappa}_{(i(\vec{\mu}))}+\vec{\kappa}_{(i(\vec{\nu}))}\not\in{\rm M}$ the same result is obtained after envoking the definition (5.12) and the identity (5.10).

We thus conclude that, as claimed, the group law of $\Gamma_{\rm w}$ is realized by the maps $\vec{\Theta}_{\vec{\mu}}$ up to a cocycle ϵ, and furthermore this cocycle is induced from the cocycle on $\Gamma_{\rm w}/\Gamma$ that was introduced in formula (5.13). For later reference we also note that (compare e.g. [25]) the center $\mathcal{Z}(\mathbb{C}_\epsilon(\Gamma_{\rm w}/\Gamma))$ of the twisted group algebra $\mathbb{C}_\epsilon(\Gamma_{\rm w}/\Gamma)$ is the ordinary group algebra $\mathcal{Z}(\mathbb{C}_\epsilon(\Gamma_{\rm w}/\Gamma))=\mathbb{C}((\Gamma_{\rm w}/\Gamma)^\circ_\epsilon)$ of the subgroup

$$(\Gamma_{\rm w}/\Gamma)^\circ_\epsilon := \{[\vec{\kappa}_{(i)}]\in\Gamma_{\rm w}/\Gamma \mid \epsilon(\vec{\kappa}_{(i)},\vec{\kappa}_{(j)})=\epsilon(\vec{\kappa}_{(j)},\vec{\kappa}_{(i)}) \text{ for all } [\vec{\kappa}_{(j)}]\in\Gamma_{\rm w}/\Gamma\} \tag{5.16}$$

of so-called regular elements of $\Gamma_{\rm w}/\Gamma$.

At this point it is appropriate to recall that so far we have left the phase choices for the maps $\vec{\Theta}_{\vec{\kappa}_{(i)}}$ undetermined. Changing these phases will change the cocycle ϵ by a coboundary.

Thus by adjusting these phases we can achieve to obtain some preferred representative cocycle in the cohomology class of ϵ. From the cohomological properties of finite abelian groups (see e.g. [26]) it follows in particular that this way we can achieve the property that all the numbers $\epsilon(\vec{\vec{\kappa}}_{(i)},\vec{\vec{\kappa}}_{(j)})$ are roots of unity. In the sequel we will often assume that such a phase choice has been made.

Taken together, these results will allow us to implement the multi-shift automorphisms also on chiral blocks in such a way that we even obtain a projective action of the group $\Gamma_{\rm w}$ on the space of chiral blocks. To make the implementation of $\Gamma_{\rm w}$ on blocks explicit, we need an appropriate description of the chiral blocks in terms of the action of the block algebra.

5.2 Chiral blocks from co-invariants

In a representation theoretic approach, the chiral blocks of a conformal field theory are constructed with the help of *co-invariants* $\lfloor\vec{\mathcal{H}}\rfloor_{\mathfrak{B}}$ of tensor products $\vec{\mathcal{H}}\equiv\mathcal{H}_{\vec{\Lambda}}$ (5.1) of irreducible modules over the chiral algebra with respect to a suitable block algebra $\mathfrak{B}$ (see e.g. [6, 2]). This statement involves two new ingredients that need to be explained. First, in general, by a co-invariant of a module V over some Lie algebra $\mathfrak{h}$ one means the quotient vector space

$$\lfloor V\rfloor_{\mathfrak{h}} := V\,/\,\mathsf{U}^+(\mathfrak{h})V\,, \tag{5.17}$$

where $\mathsf{U}^+(\mathfrak{h})=\mathfrak{h}\mathsf{U}(\mathfrak{h})$ with $\mathsf{U}(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$. When the $\mathfrak{h}$-module V is fully reducible, then (5.17) is just the submodule of $\mathfrak{h}$-singlets in V, but generically it is a genuine quotient which cannot be identified with a subspace of V. And second, the action of the block algebra $\mathfrak{B}=\bar{\mathfrak{g}}\otimes\mathcal{F}$ on the tensor product vector space $\vec{\mathcal{H}}$ is defined by its expansions in local coordinates, i.e. for $X=\bar{x}\otimes f$ and $v=v_1\otimes v_2\otimes\cdots\otimes v_m$ one has

$$X\,v := \sum_{s=1}^{m} v_1\otimes v_2\otimes\cdots\otimes v_{s-1}\otimes(\bar{x}\otimes f_{|s})v_s\otimes v_{s+1}\otimes\cdots\otimes v_m\,, \tag{5.18}$$

where $\bar{x}\otimes f_{|s}$ is regarded as an element of the loop algebra $\bar{\mathfrak{g}}_{\rm loop}$ and hence of $\mathfrak{g}$, or more precisely, as the representation matrix of that element of $\mathfrak{g}$ in the $\mathfrak{g}$-representation R_{Λ_s}. (That this yields a $\bar{\mathfrak{g}}\otimes\mathcal{F}$-representation follows by

$$\begin{aligned}\big((\bar{x}\otimes f)(\bar{y}\otimes g)-(\bar{y}\otimes g)(\bar{x}\otimes f)\big)(v_1\otimes v_2\otimes\cdots\otimes v_m) &= \sum_{s=1}^{m} v_1\otimes v_2\otimes\cdots\otimes[\bar{x}\otimes f_{|s},\bar{y}\otimes g_{|s}]v_s\otimes\cdots\otimes v_m\\ &= \sum_{s=1}^{m} v_1\otimes v_2\otimes\cdots\otimes([\bar{x},\bar{y}]\otimes(fg)_{|s})v_s\otimes\cdots\otimes v_m\\ &\qquad + K\,\kappa(\bar{x},\bar{y})\,\Big(\sum_{s=1}^{m}\mathcal{R}es_{p_s}(\mathrm{d}f\,g)\Big)\,v_1\otimes v_2\otimes\cdots\otimes v_m\,,\end{aligned} \tag{5.19}$$

where in the first equality one uses the fact that terms acting on different tensor factors cancel and in the second equality the bracket relations of $\mathfrak{g}$ are inserted. The terms in (5.19) that involve the central element K cancel as a consequence of the residue formula, while the other terms add up to $[\bar{x}\otimes f,\bar{y}\otimes g]\,v$, where the Lie bracket is the one of $\bar{\mathfrak{g}}\otimes\mathcal{F}$ as defined in (3.4).)

The finite-dimensional vector spaces $\lfloor\vec{\mathcal{H}}\rfloor_{\mathfrak{B}}$ of co-invariants play the role of the dual spaces $\mathsf{B}^\star$ of the chiral block spaces B, i.e.

$$\mathsf{B}_{\vec\Lambda} = \big(\lfloor\mathcal{H}_{\vec\Lambda}\rfloor_{\mathfrak{B}}\big)^\star . \tag{5.20}$$

By duality, the chiral blocks can then also be regarded as the *invariants* (or in other words, singlet submodules) in the algebraic dual $\vec{\mathcal{H}}^\star$ of $\vec{\mathcal{H}}$, i.e.

$$\mathsf{B}_{\vec\Lambda} = (\mathcal{H}^\star_{\vec\Lambda})^{\mathfrak{B}} \,; \tag{5.21}$$

in this description, the blocks are linear forms β on $\vec{\mathcal{H}}$ with the property

$$\langle\beta\,,Xv\rangle \equiv \beta(Xv) = 0 \tag{5.22}$$

for all $X\in\bar{\mathfrak{g}}\otimes\mathcal{F}$ and all $v\in\vec{\mathcal{H}}$.

5.3 Isomorphisms of chiral blocks

In (5.17) we followed the habit of suppressing the symbol R for the representation by which $\mathfrak{h}$ acts on the vector space V, e.g. $\lfloor V\rfloor_{\mathfrak{h}}$ is a shorthand for $\lfloor V\rfloor_{R(\mathfrak{h})}$. This cannot cause any confusion as long as we only deal with a single $\mathfrak{h}$-module (R,V) which is based on the vector space V. On the other hand, as is easily checked, given any automorphism σ of $\mathfrak{h}$, together with (R,V) also $(\tilde R,V):=(R\circ\sigma,V)$ furnishes a module over $\mathfrak{h}$. Accordingly we are then dealing with two different actions of $\mathfrak{h}$ on V, and hence with two different spaces (5.17) of co-invariants. It is then natural to try to associate to any automorphism σ of $\mathfrak{h}$ a corresponding mapping of the respective spaces of co-invariants. This is easily achieved once we are given a linear map $\Theta_\sigma\colon V\to V$ with the property that $\Theta_\sigma\,R(x)\,v=\tilde R(x)\,\Theta_\sigma\,v$ for all $x\in\mathfrak{h}$ and all $v\in V$, or in short (suppressing again the symbol R),

$$\Theta_\sigma\,x = \sigma(x)\,\Theta_\sigma \tag{5.23}$$

for all $x\in\mathfrak{h}$. Namely, we observe that via the prescription $\sigma(\mathbf{1}):=\mathbf{1}$ and linearity, the automorphism σ of $\mathfrak{h}$ extends to an automorphism of the enveloping algebra $\mathsf{U}(\mathfrak{h})$ that respects the filtration, hence also to an automorphism of $\mathsf{U}^+(\mathfrak{h})$. Because of (5.23) the prescription

$$\lfloor V\rfloor_{R(\mathfrak{h})} \ni [v] \;\mapsto\; [\Theta_\sigma v] \in \lfloor V\rfloor_{\tilde R(\mathfrak{h})} \tag{5.24}$$

then supplies us with a well-defined mapping from $\lfloor V\rfloor_{R(\mathfrak{h})}$ to $\lfloor V\rfloor_{\tilde R(\mathfrak{h})}$, and by construction this is in fact an isomorphism of vector spaces.

Now it is a general fact that *inner* automorphisms act trivially on co-invariants. Namely, an inner automorphism σ of $\mathfrak{h}$ can by definition be written as a product of finitely many automorphisms $\sigma_i=\exp(\mathrm{ad}_{x_i})$ for some elements $x_i\in\mathfrak{h}$. Moreover, for every $y\in\mathfrak{h}$ we have (compare the formulæ (2.18) and (2.21)) [5]

$$R\circ\sigma_i(y) = R(\exp(\mathrm{ad}_{x_i})(y)) = \exp(R(x_i))\,R(y)\,\exp(-R(x_i))\,. \tag{5.25}$$

[5] This may be checked by replacing x_i by ξx_i and comparing both sides order by order in the dummy variable ξ. In the special case that $\mathfrak{h}$ integrates to a group, the formula is an immediate consequence of the fact that $\exp(\mathrm{ad}_x)(y)=\gamma y\gamma^{-1}$, where γ is a group element such that $\gamma=\exp(x)$.

By expanding the exponentials it then follows in particular that

$$\tilde{R}(y)\, v = R\circ\sigma(y)\, v = R(y)\, v \mod \mathsf{U}^+(\mathfrak{h})V \tag{5.26}$$

for all $v\in V$ and all $y\in\mathfrak{h}$. By the definition of co-invariants, this means that the modules (R,V) and $(\tilde{R},V)$ possess the same co-invariants. By duality, this also means that the dual spaces to these modules possess the same invariants.

When applied to the situation of our interest, these general observations tell us that as far as the study of chiral blocks is concerned we need to regard the automorphisms $\vec\sigma_{\vec\mu}$ of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$ that we defined in (3.16) above only modulo inner automorphisms. Accordingly we should determine the outer automorphism class to which a multi-shift automorphism $\vec\sigma_{\vec\mu}$ of $\bar{\mathfrak{g}}\otimes\mathcal{F}$ belongs. We have already addressed this question in section 2 for the case of multi-shift automorphisms of the affine Lie algebra $\mathfrak{g}$. Now we study the same issue for the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$. We first note that the result of section 2 implies that whenever at least one of the vectors $\bar\mu_s$ is not a coroot, then $\sigma_{\vec\mu;s}$ is an *outer* automorphism of the corresponding affine Lie algebra $\mathfrak{g}_s$; it follows that in this case it is also an outer automorphism of $\mathfrak{g}^m$, and thereby a fortiori an outer automorphism of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$.

In other words, the subgroup Γ (4.1) already exhausts the set of those shift vectors $\vec\mu\in\Gamma_{\rm w}$ for which $\sigma_{\vec\mu}$ is an inner automorphism of $\bar{\mathfrak{g}}\otimes\mathcal{F}$. Thus the group $\Gamma_{\rm out}$ of outer modulo inner multi-shift automorphisms of $\bar{\mathfrak{g}}\otimes\mathcal{F}$ is precisely the factor group

$$\Gamma_{\rm out} = \Gamma_{\rm w}/\Gamma\,. \tag{5.27}$$

According to the isomorphism (2.29) between $L^\vee_{\rm w}/L^\vee$ and the unique maximal abelian normal subgroup $\mathcal{Z}(\mathfrak{g})$ of $\mathrm{Out}(\mathfrak{g})=\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$, we thus have

$$\Gamma_{\rm out} \cong (\mathcal{Z}(\mathfrak{g}))^{m-1}\,. \tag{5.28}$$

In particular, $\Gamma_{\rm out}$ is a *finite* group of order $\mathrm{ord}(\Gamma_{\rm out})=|L_{\rm w}/L|^{m-1}$. For future reference we mention that a set of distinguished representatives of the elements of $\mathrm{Out}(\mathfrak{g})$ is provided by the diagram automorphisms $\omega_{\vec\mu;s}$ of $\mathfrak{g}$ (see subsection 2.2); thus elements of $\Gamma_{\rm out}$ may be regarded as collections of suitable diagram automorphisms whose product is the identity. We also recall that when implementing the group $\Gamma_{\rm w}$ through the maps $\vec\Theta_{\vec\mu}$ which satisfy (5.5), we are effectively dealing with a two-cocycle on the finite abelian group $\Gamma_{\rm w}/\Gamma=\Gamma_{\rm out}$.

Explicitly, for every $\vec\mu\in\Gamma_{\rm w}$ we have a map $\vec\Theta^\star_{\vec\mu}$ from the chiral blocks $\mathsf{B}_{\vec\Lambda}=(\mathcal{H}^\star_{\vec\Lambda})^{\mathfrak{B}}$ to $\mathcal{H}^\star_{\vec\omega^\star\vec\Lambda}$ which acts as

$$\langle\vec\Theta^\star_{\vec\mu}(\beta)\,,v\rangle := \langle\beta\,,\vec\Theta_{\vec\mu}^{-1}v\rangle\,. \tag{5.29}$$

The previous results imply, first, that the image of this map is in $\mathsf{B}_{\vec\omega^\star\vec\Lambda}$, i.e. that for any block β, $\vec\Theta^\star_{\vec\mu}(\beta)$ is again a chiral block, and second, that this map depends on $\vec\mu$ only via its class $[\vec\mu]$ in $\Gamma_{\rm out}$. In short, for each $[\vec\mu]\in\Gamma_{\rm out}$ we have constructed an isomorphism

$$\vec\Theta^\star_{\vec\mu}\equiv\vec\Theta^\star_{[\vec\mu]}:\qquad \mathsf{B}_{\vec\Lambda}\cong\mathsf{B}_{\vec\omega^\star\vec\Lambda}\,. \tag{5.30}$$

(Analogously, we also have an isomorphism between the respective dual spaces, $\lfloor\mathcal{H}_{\vec\Lambda}\rfloor_{\bar{\mathfrak{g}}\otimes\mathcal{F}}\cong\lfloor\mathcal{H}_{\vec\omega^\star\vec\Lambda}\rfloor_{\bar{\mathfrak{g}}\otimes\mathcal{F}}$.) Note that it already follows from the Verlinde formula and elementary properties of simple currents that the spaces $\mathsf{B}_{\vec\Lambda}$ and $\mathsf{B}_{\vec\omega^\star\vec\Lambda}$ have the same dimension; being finite-dimensional, it is then trivial that they are isomorphic as vector spaces. The virtue of the result (5.30) is, however, that it provides us with a canonical realization of this isomorphism. As we will soon see, this realization possesses the additional non-trivial property to be compatible with a variation of the moduli in the problem, i.e. with different choices of the insertion points. Moreover, knowing the isomorphism (5.30) one can also study the problem of *fixed point resolution*, which arises whenever the collection $\vec{\Lambda}$ of $\mathfrak{g}$-weights is left invariant by $\vec{\omega}^\star$; this issue will be addressed in section 7 below.

Finally, according to the remarks at the end of subsection 5.1, we can (and do) employ the freedom in defining the maps $\Theta_{\vec{\mu}}$ so as to achieve the property that all cocycle factors in the projective representation of $\Gamma_{\rm w}/\Gamma$ are roots of unity. Together with the fact that $\Gamma_{\rm w}/\Gamma$ is a finite group, it follows that the maps $\vec{\Theta}^\star_{\vec{\mu}}$ all have finite order.

6 Bundles of blocks

In all the considerations above we have regarded the punctures p_s as held fixed, i.e. we have analyzed block algebras and chiral blocks at a single point of the moduli space $\mathcal{M} \equiv \mathcal{M}_m$ of m-punctured projective curves. We now address the issues that arise when the insertion points are allowed to vary over the whole moduli space $\mathcal{M}$. Recall from the discussion in subsection 5.1 that in the implementation $\vec{\Theta}_{\vec{\mu}}$ of an automorphism of $\mathfrak{g}^m$ a phase is still undetermined. Clearly, if we choose this phase at random for each point in the moduli space, we cannot expect to obtain quantities on the bundle of chiral blocks that vary smoothly with the moduli. On the other hand, the bundle of chiral blocks carries a natural projectively flat connection, the Knizhnik–Zamolodchikov connection [27–29, 6]. Our rationale will therefore be to implement the maps $\sigma_{\vec{\mu}}(\vec{z})$ for different values of the moduli $\vec{z}$ in such a way that they preserve the Knizhnik–Zamolodchikov connection.

6.1 Connections and bundles over the moduli space

The moduli space $\mathcal{M}$ is $(\mathbb{P}^1)^m$ minus the union of diagonals, i.e.

$$\mathcal{M} = \{(p_1,p_2,\dots,p_m) \,|\, p_s \in \mathbb{P}^1,\ p_s \neq p_{s'} \text{ for } s \neq s'\} . \tag{6.1}$$

When considered as depending on the values of the punctures, the spaces B of chiral blocks combine to a vector bundle $\mathcal{B}$ over $\mathcal{M}$ [6]. Before we study that bundle, we introduce a few other bundles which in this context are of interest as well.

We first consider the trivial bundle over $\mathcal{M}$ with fiber given by the infinite-dimensional Lie algebra $\mathfrak{g}^m$ defined in (3.1),

$$\mathfrak{g}^m \times \mathcal{M} \to \mathcal{M} . \tag{6.2}$$

This bundle carries a fiberwise action of the group $\Gamma_{\rm w}$, i.e. an action by automorphisms of the total space that cover the identity on the base space $\mathcal{M}$. For every $\vec{\mu} \in \Gamma_{\rm w}$ we write $\sigma_{\vec{\mu}}(\vec{z})$ for the map on the fiber over the point $\vec{p} \in \mathcal{M}$ with coordinates $\vec{z} \equiv (z_1,z_2,\dots,z_m)$. On the bundle (6.2) we have a flat connection D which acts on smooth sections $g(\vec{z})$ of (6.2) as

$$D_s g(\vec{z}) := \partial_s g(\vec{z}) + [L^{(s)}_{-1}, g(\vec{z})] \tag{6.3}$$

for $s = 1,2,\dots,m$, where $\partial_s \equiv \partial/\partial z_s$ and $L^{(s)}_{-1} \in \mathcal{V}\!ir_s$ is a generator of the Virasoro algebra associated to $\mathfrak{g}_s$.

Moreover, the bundle (6.2) possesses a *subbundle of block algebras*, which is defined by taking for each $\vec{p}\in\mathcal{M}$ the corresponding block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F} = \bar{\mathfrak{g}}\otimes\mathcal{F}(\mathbb{P}^1\backslash\{p_1,p_2,...,p_m\})\subset\mathfrak{g}^m$. The smooth sections of this bundle will be denoted by $X(\vec{z})$. It is important to realize that while the block algebras at different points $\vec{p}$ are isomorphic as abstract Lie algebras, they are not naturally isomorphic, and the subbundle is not necessarily a trivial bundle. Still, by construction, the automorphisms $\sigma_{\vec{\mu}}$ restrict to automorphisms on the subbundle of block algebras.

A crucial property of the connection (6.3) is that it preserves the subbundle of block algebras. Explicitly, this can be seen as follows. The algebra $\mathcal{F}(\mathbb{P}^1_m)$ is generated algebraically by the constant function $\varphi^{(0)}:=\mathrm{id}$ and the functions $\varphi^{(s)}$ with $s=1,2,...,m$, where $\varphi^{(s)}(z)=(z-z_s)^{-1}$.[6] By induction with respect to the 'length' ℓ of $f=\varphi^{(i_1)}\varphi^{(i_2)}\cdots\varphi^{(i_\ell)}$ and using the Leibniz rule, it follows that this property is established as soon as it is shown to hold for each of the functions $\varphi^{(s)}$. Thus we consider the element $X=\bar{x}\otimes\varphi^{(s)}$ of the block algebra. For $s=0$ we trivially have $\partial_{s'}X=0$ and $[L_{-1}^{(s')},X]=0$ for all $s'=1,2,...,m$. For $s\in\{1,2,...,m\}$ and $s'\neq s$ both $\partial_{s'}X$ and $[L_{-1}^{(s')},X]$ have a component only in $\mathfrak{g}_{s'}$, namely

$$[L_{-1}^{(s')},X]_{|s'} = \bar{x}\otimes(t+z_{s'}-z_s)^{-2} = -\partial_{s'}X_{|s'} \tag{6.4}$$

so that again $DX=0$. Finally, for $s\in\{1,2,...,m\}$ and $s'=s$, the commutator $[L_{-1}^{(s')},X]$ is as in (6.4), but now for $\partial_{s'}X$ the component in $\mathfrak{g}_{s'}$ vanishes while there are additional contributions in all $\mathfrak{g}_{s''}$ with $s''\neq s$, namely $\partial_{s'}X_{|s''}=\bar{x}\otimes(t+z_{s''}-z_s)^{-2}$. Thus in this case $D_{s'}X$ is not zero; however, it is still an element of the block algebra $\bar{\mathfrak{g}}\otimes\mathcal{F}$, namely $D_{s'}X=\bar{x}\otimes(z-z_s)^{-2}$, since we encountered precisely the local expansions of this element of $\bar{\mathfrak{g}}\otimes\mathcal{F}$.

Next we consider the trivial bundle over $\mathcal{M}$ with fiber given by the tensor product $\vec{\mathcal{H}}\equiv\mathcal{H}_{\vec{\Lambda}}$ (5.1) of irreducible highest weight modules $\mathcal{H}_{\Lambda_s}$ over the affine Lie algebra $\mathfrak{g}$,

$$\vec{\mathcal{H}}\times\mathcal{M}\to\mathcal{M}\,. \tag{6.5}$$

Again this bundle is endowed with a flat connection ∇, which acts on smooth sections $v(\vec{z})$ of (6.5) as

$$\nabla_{\!s}v(\vec{z}) := \big(\partial_s + L_{-1}^{(s)}\big)v(\vec{z}) \tag{6.6}$$

for $s=1,2,...,m$; we call this connection the *Knizhnik–Zamolodchikov connection*. And again on the bundle (6.5) there is a fiberwise action of a certain discrete group. This group is the (proper) subgroup $\Gamma_{\rm fix}$ of $\Gamma_{\rm w}$ that consists of all those elements $\vec{\mu}$ of $\Gamma_{\rm w}$ for which all associated maps $\Theta_{\vec{\mu};s}$ of the tensor factors (see (2.19)) of $\vec{\mathcal{H}}$ are *endo*morphisms. This group is given by

$$\Gamma_{\rm fix} = \Gamma_{\vec{\Lambda}} := \{\vec{\mu}\in\Gamma_{\rm w}\,|\,\omega^\star_{\vec{\mu};s}(\Lambda_s)=\Lambda_s \text{ for all } s=1,2,...,m\}\,. \tag{6.7}$$

Note that by the results of subsection 5.3, the definition of $\Gamma_{\vec{\Lambda}}$ depends on the automorphisms $\vec{\sigma}_{\vec{\mu}}$ only through the class $[\vec{\mu}]$ in $\Gamma_{\rm out}$ (5.27), or what is the same, through the associated diagram automorphisms $\vec{\omega}_{\vec{\mu}}$. We call $\Gamma_{\vec{\Lambda}}$ the *stabilizer subgroup* of $\Gamma_{\rm w}$ associated to the weights $\Lambda_1,...,\Lambda_m$. For every $\vec{\mu}\in\Gamma_{\vec{\Lambda}}$ we write $\vec{\Theta}_{\vec{\mu}}(\vec{z})$ for the map on the fiber over $\vec{p}\in\mathcal{M}$ in the bundle (6.5).

[6] When the puncture p_m is at $z_m=\infty$, for $s=m$ we rather have $\varphi^{(s)}(z)=z$. The corresponding changes in the arguments below are obvious, and we refrain from writing them down explicitly.

6.2 Moduli dependence of the twisted intertwiners

To determine how the implementation of the automorphism depends on the moduli, we first study how the multi-shift automorphism $\vec{\sigma}_{\vec{\mu}}$ changes when the punctures are varied (while the shift vector $\vec{\bar{\mu}}$ is kept fixed). We denote by $\vec{p}^{[0]}$ a reference point on $\mathcal{M}$ and study how the automorphism $\vec{\sigma}_{\vec{\mu}} \equiv \vec{\sigma}_{\vec{\mu}}(\vec{z})$ differs from $\vec{\sigma}_{\vec{\mu}}(\vec{z}^{[0]})$, or in other words, what the automorphism $\vec{\delta}_{\vec{\mu}}$ defined by

$$\vec{\sigma}_{\vec{\mu}}(\vec{z}) = \vec{\delta}_{\vec{\mu}}(\vec{z};\vec{z}^{[0]}) \circ \vec{\sigma}_{\vec{\mu}}(\vec{z}^{[0]}) \tag{6.8}$$

looks like. Note that we have to interpret $\vec{\delta}_{\vec{\mu}}$ as an automorphism of the algebra $\mathfrak{g}^m$; since the block algebra, regarded as a subalgebra of $\mathfrak{g}^m$, varies with the moduli, it does not make sense to look for an automorphism of the block algebra.

For every $s_\circ = 1, 2, \ldots, m$ the explicit form of the map $\delta_{\vec{\mu}} \equiv \delta_{\vec{\mu};s_\circ}$ follows directly from the formula (2.6) for $\sigma_{\vec{\mu}} \equiv \sigma_{\vec{\mu};s_\circ}$. We get $\delta_{\vec{\mu}}(K) = K$ and

$$\begin{aligned} \delta_{\vec{\mu}}(H^i \otimes f) &= H^i \otimes f + K \sum_{s=1}^{m} \bar{\mu}_s^i \, \mathcal{R}es\big((\varphi_{s_\circ,s} - \varphi_{s_\circ,s}^{[0]})\, f\big)\,, \\ \delta_{\vec{\mu}}(E^{\bar{\beta}} \otimes f) &= E^{\bar{\beta}} \otimes f \cdot \prod_{s=1}^{m} (\varphi_{s_\circ,s}/\varphi_{s_\circ,s}^{[0]})^{-(\bar{\mu}_s,\bar{\beta})} \end{aligned} \tag{6.9}$$

with $\varphi_{s_\circ,s}$ as defined in (2.7) [7] and $\varphi_{s_\circ,s}^{[0]}(t) := (t + z_{s_\circ}^{[0]} - z_s^{[0]})^{-1}$.

For simplicity, let us for the moment restrict our attention to the particular case where $z_s^{[0]} = z_s$ except for some fixed value $u \in \{1, 2, \ldots, m\}$. Then $\vec{\delta}_{\vec{\mu}}$ acts on $\mathfrak{g}^m$ as $\vec{\delta}_{\vec{\mu}}(K) = K$ and

$$\begin{aligned} \vec{\delta}_{\vec{\mu}}(H^i \otimes f) &= \begin{cases} H^i \otimes f + K\, \bar{\mu}_u^i \, \mathcal{R}es\big((\varphi_{s_\circ,u} - \varphi_{s_\circ,u}^{[0]})\, f\big) & \text{for } s_\circ \neq u\,, \\ H^i \otimes f + K \sum\limits_{\substack{s=1 \\ s\neq u}}^{m} \bar{\mu}_s^i \, \mathcal{R}es\big((\varphi_{u,s} - \varphi_{u,s}^{[0]})\, f\big) & \text{for } s_\circ = u\,, \end{cases} \\ \vec{\delta}_{\vec{\mu}}(E^{\bar{\beta}} \otimes f) &= \begin{cases} E^{\bar{\beta}} \otimes f \cdot (\varphi_{s_\circ,u}/\varphi_{s_\circ,u}^{[0]})^{-(\bar{\mu}_u,\bar{\beta})} & \text{for } s_\circ \neq u\,, \\ E^{\bar{\beta}} \otimes f \cdot \prod\limits_{\substack{s=1 \\ s\neq u}}^{m} (\varphi_{u,s}/\varphi_{u,s}^{[0]})^{-(\bar{\mu}_s,\bar{\beta})} & \text{for } s_\circ = u\,. \end{cases} \end{aligned} \tag{6.10}$$

Next we observe that a variation of the moduli does not change the class of the automorphism $\sigma_{\vec{\mu}}$ modulo inner automorphisms. As a consequence, $\vec{\delta}_{\vec{\mu}}$ is in fact an inner automorphism of $\mathfrak{g}^m$. As can be verified by direct calculation (see appendix C), we have in fact

$$\vec{\delta}_{\vec{\mu}} = \begin{cases} \exp(\mathrm{ad}_{(\bar{\mu}_u,H)\otimes g_{s_\circ,u}}) & \text{for } s_\circ \neq u\,, \\ \exp(\mathrm{ad}_{\sum_{s\neq u}(\bar{\mu}_s,H)\otimes \tilde{g}_s}) & \text{for } s_\circ = u\,, \end{cases} \tag{6.11}$$

where

$$g_{s_\circ,u}(t) := \ln \frac{t + z_{s_0} - z_u}{t + z_{s_0} - z_u^{[0]}} \qquad \text{and} \qquad \tilde{g}_s(t) := \ln \frac{t + z_u - z_s}{t + z_u^{[0]} - z_s}\,. \tag{6.12}$$

[7] Here and below we suppress again the obvious modifications that arise when the puncture p_m is at $z_m = \infty$.

(Here we choose some definite branch of the logarithm. It is readily checked that all the results below do not depend on this choice. For $z_u \neq z_u^{[0]}$, $g_{s_\circ,u}$ can be considered as an element of $\mathbb{C}(\!(t)\!)$, for which the constant part is defined only modulo $2\pi\mathrm{i}\mathbb{Z}$.)

Now recall from section 5 that associated to the automorphism $\sigma_{\vec\mu}$ of $\mathfrak{g}^m$ there comes the map $\Theta_{\vec\mu}$ that for each tensor factor of $\vec{\mathcal{H}}$ is defined as in (2.19). This map is present for any $\vec z \in \mathcal{M}$; moreover, according to the relation (6.8) the maps $\Theta_{\vec\mu}(\vec z)$ and $\Theta_{\vec\mu}(\vec z^{[0]})$ are related by an analogous map that is associated to the inner automorphism $\vec\delta_{\vec\mu}$. Using the results (2.22) and (6.11) we learn that explicitly we have

$$\Theta_{\vec\mu;s_\circ}(\vec z) = \exp\big((\bar\mu_u,H)\otimes g_{s_\circ,u} + K\tilde h_{s_\circ}\big) \circ \Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) \qquad \text{for} \ \ s_\circ \neq u\,, \tag{6.13}$$

respectively

$$\Theta_{\vec\mu;s_\circ}(\vec z) = \exp\big(\sum_{\substack{s=1\\ s\neq u}}^{m} (\bar\mu_s,H)\otimes \tilde g_s + K\tilde h_u\big) \circ \Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) \qquad \text{for} \ \ s_\circ = u\,. \tag{6.14}$$

At this stage the functions $\tilde h_{s_\circ}$, which are introduced here to take care of the fact that the implementation is only determined up to a phase, can still be chosen arbitrarily; we will make use of this freedom later on.

Combining these results with the twisted intertwining relations (2.34), we can show that the commutator $[\nabla_u, \vec\Theta_{\vec\mu}(\vec z)]$ has the local expansions

$$[\nabla_u, \Theta_{\vec\mu;s_\circ}(\vec z)]\Big|_{z_u=z_u^{[0]}} = \Big(-(\bar\mu_u,H)\otimes\varphi^{[0]}_{s_\circ,u} + K\,\hat g_{s_\circ}\Big)\,\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})\,, \tag{6.15}$$

where

$$\hat g_{s_\circ} = \frac{\partial}{\partial z_u}\,\tilde h_u\Big|_{z_u=z_u^{[0]}} - \sum_{\substack{s=1\\ s\neq u}}^{m} \frac{(\bar\mu_s,\bar\mu_u)}{z_u^{[0]}-z_s^{[0]}} \qquad \text{for} \ \ s_\circ = u\,, \tag{6.16}$$

while $\hat g_{s_\circ} = \frac{\partial}{\partial z_u}\,\tilde h_{s_\circ}\Big|_{z_u=z_u^{[0]}}$ for $s_\circ \neq u$. (The derivation of (6.15) is explained in some detail in appendix C.) Collecting these local expansions, we learn that up to central terms we simply have

$$[\nabla_u, \Theta_{\vec\mu}(\vec z^{[0]})] = Y_{\vec\mu;u}\,\Theta_{\vec\mu}(\vec z^{[0]})\,, \tag{6.17}$$

where

$$Y_{\vec\mu;u} := -\,(\bar\mu_u,H)\otimes(z-z_u^{[0]})^{-1} \tag{6.18}$$

is an element of the block algebra. Concerning the central terms, we claim that we can employ the freedom that is present in the choice of the functions $\tilde h_{s_\circ}$ so as to achieve

$$\hat g_{s_\circ} \equiv 0 \qquad \text{for all} \ \ \vec p \in \mathcal{M}\,. \tag{6.19}$$

Clearly, this is possible at the point $\vec p = \vec p^{[0]}$, by just choosing [8]

$$\tilde h_{s_\circ} \equiv 0 \quad \text{for} \;\; s_\circ \neq u \qquad \text{and} \qquad \tilde h_u = \sum_{\substack{s=1\\ s\neq u}}^m (\bar\mu_s, \bar\mu_u) \, \ln \frac{z_u - z_s}{z_u^{[0]} - z_s^{[0]}} \,. \tag{6.20}$$

To discuss how the situation looks like globally, we first have to generalize the formula (6.15) to the case where all punctures may vary. By analogous considerations as above one can check that the generalization of (6.13) and (6.14) reads

$$\Theta_{\vec\mu;s_\circ}(\vec z) = \exp\big(\sum_{s=1}^m (\bar\mu_s, H) \otimes g_{s_\circ,s} + K h_{s_\circ} \big) \circ \Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) \tag{6.21}$$

for all $s_\circ$, where

$$g_{s_\circ,s}(t) := \ln \frac{t + z_{s_0} - z_s}{t + z_{s_0}^{[0]} - z_s^{[0]}} \,, \tag{6.22}$$

which for all $s_\circ$ and all s is a sensible Laurent series. (Also note that the notation $g_{s_\circ,u}$ is in agreement with the definition (6.12), and $g_{u,s} = \tilde g_s$; the functions $h_{s_\circ}$ are still to be determined.) The same arguments as in appendix C then lead to a formula analogous to (6.15),

$$[\nabla_u, \Theta_{\vec\mu;s_\circ}(\vec z)]\big|_{\vec z = \vec z^{[0]}} = \big(-(\bar\mu_u, H) \otimes \varphi^{[0]}_{s_\circ,u} + K \, \hat g_{s_\circ} \big) \, \Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) \,, \tag{6.23}$$

with

$$\hat g_{s_\circ} = \frac{\partial}{\partial z_u} \, h_{s_\circ}\big|_{\vec z = \vec z^{[0]}} - \sum_{\substack{s=1\\ s\neq u}}^m \frac{(\bar\mu_s,\bar\mu_u)}{z_u^{[0]} - z_s^{[0]}} \tag{6.24}$$

for all $s_\circ$. Our aim is now to choose the functions $h_{s_\circ}$ is such a way that in the tensor product $\vec\Theta_{\vec\mu}$ the terms proportional to K in the exponent cancel so that we are again left with an element of the block algebra. Thus we need again $\sum_{s_\circ} \hat g_{s_\circ} \equiv 0$, i.e.

$$\sum_{s_\circ=1}^m \frac{\partial}{\partial z_u} \, h_{s_\circ} = \sum_{\substack{s=1\\ s\neq u}}^m \frac{(\bar\mu_s,\bar\mu_u)}{z_u - z_s} \tag{6.25}$$

(note that the appearance of the summation is in agreement with the fact that $\mathfrak g^m$ is obtained from a direct sum of affine Lie algebras by identifying the centers). The choice

$$h_{s_\circ}(\vec z) = \tfrac12 \sum_{\substack{s=1\\ s\neq s_\circ}}^m (\bar\mu_{s_\circ}, \bar\mu_s) \, \ln \frac{z_{s_\circ} - z_s}{z_{s_\circ}^{[0]} - z_s^{[0]}} \tag{6.26}$$

[8] Here the freedom in the choice of the branch of the logarithm is not completely irrelevant. But since the level is always integral while the inner product $(\bar\mu_s, \bar\mu_u)$ is rational, the twisted intertwiner $\Theta_{\vec\mu;s_\circ}$ is determined only up to multiplication by a root of unity, and the presence of this root of unity does not destroy the important property of $\Theta_{\vec\mu;s_\circ}$ to have finite order. Moreover, the precise value of the order really matters only in the case of fixed points, and closer inspection shows that in that case the number $(\bar\mu_s, \bar\mu_u)$ is in fact always an integer so that the order is not changed at all.

for all $s_\circ = 1, 2, \ldots, m$ indeed satisfies this requirement. Note that the requirement (6.25) determines $h_{s_\circ}$ only up to a $\vec{z}$-independent constant; the choice made here will be convenient later on.[9] We conclude that we can choose the phase of $\vec{\Theta}_{\vec{\mu}}(\vec{z})$ in such a manner that the Knizhnik–Zamolodchikov connection is preserved at every point of the moduli space.

6.3 Transformation properties of flat sections

We are now in a position to investigate the bundle of chiral blocks and the transformation properties of those sections in the bundle which are flat with respect to the Knizhnik–Zamolodchikov connection. To this end we consider the trivial bundle

$$\vec{\mathcal{H}}^\star \times \mathcal{M} \to \mathcal{M}\,, \tag{6.27}$$

where the fiber $\vec{\mathcal{H}}^\star$ is the algebraic dual of $\vec{\mathcal{H}}$, and in this bundle the *subbundle $\mathcal{B}$ of chiral blocks* which are the singlets under the (dual) action of the block algebra. This implies that the smooth sections $\beta(\vec{z})$ of $\mathcal{B}$ obey

$$\langle \beta(\vec{z})\,, X(\vec{z}) v(\vec{z}) \rangle = 0 \tag{6.28}$$

for all sections $X(\vec{z})$ of the bundle of block algebras and all sections $v(\vec{z})$ of the trivial bundle (6.5). This bundle $\mathcal{B}$ is of finite rank [6]. We endow the trivial bundle (6.27) with the connection that is dual to the connection (6.6); it restricts to a connection on the subbundle $\mathcal{B}$. We will use the term Knizhnik–Zamolodchikov connection also for both the connection on the dual bundle and for its restriction to $\mathcal{B}$. (Note, however, that a frequent convention in the literature is to reserve the term Knizhnik–Zamolodchikov connection only for the connection on $\mathcal{B}$.)

An action of the group $\Gamma_{\rm fix}$ on $\mathcal{B}$ can be defined by

$$\langle \vec{\Theta}^\star_{\vec{\mu}}(\beta(\vec{z}))\,, v(\vec{z}) \rangle := \langle \beta(\vec{z})\,, \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle \tag{6.29}$$

for all sections $v(\vec{z})$ (recall that $\vec{\Theta}^\star_{\vec{\mu}}(\beta)$ is again a chiral block). We have

$$\begin{aligned} \langle \nabla_s (\vec{\Theta}^\star_{\vec{\mu}} \beta(\vec{z}))\,, v(\vec{z}) \rangle &= \partial_s \langle \vec{\Theta}^\star_{\vec{\mu}} \beta(\vec{z})\,, v(\vec{z}) \rangle - \langle \vec{\Theta}^\star_{\vec{\mu}} \beta(\vec{z})\,, \nabla_s v(\vec{z}) \rangle \\ &= \partial_s \langle \beta(\vec{z})\,, \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle - \langle \beta(\vec{z})\,, \vec{\Theta}^{-1}_{\vec{\mu}} \nabla_s v(\vec{z}) \rangle\,. \end{aligned} \tag{6.30}$$

Using the result (6.17), this can also be written as

$$\begin{aligned} \langle \nabla_s (\vec{\Theta}^\star_{\vec{\mu}} \beta(\vec{z}))\,, v(\vec{z}) \rangle &= \partial_s \langle \beta(\vec{z})\,, \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle - \langle \beta(\vec{z})\,, \nabla_s \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle + \langle \beta(\vec{z})\,, Y_{-\vec{\mu};s}\, \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle \\ &= \partial_s \langle \beta(\vec{z})\,, \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle - \langle \beta(\vec{z})\,, \nabla_s \vec{\Theta}^{-1}_{\vec{\mu}} v(\vec{z}) \rangle\,, \end{aligned} \tag{6.31}$$

where in the second line we used the fact that according to the definition (6.18), $Y_{-\vec{\mu};s}$ is an element of the block algebra. Note that $\vec{\Theta}_{\vec{\mu}}(\vec{z})$ depends smoothly on $\vec{z}$, so that $\vec{\Theta}^{-1}_{\vec{\mu}}(\vec{z}) v(\vec{z})$ is a smooth section as well. Let now $\beta(\vec{z})$ be a *flat* section in $\mathcal{B}$,[10] i.e. $\nabla\beta = 0$, or more explicitly,

$$0 = \langle \nabla_s \beta(\vec{z})\,, v(\vec{z}) \rangle = \partial_s \langle \beta(\vec{z})\,, v(\vec{z}) \rangle - \langle \beta(\vec{z})\,, \nabla_s v(\vec{z}) \rangle \tag{6.32}$$

[9] Also note that in the general case considered here there is of course less freedom in the choice of these functions than we had for the analogous functions $\tilde{h}_{s_\circ}$ in the special case treated before. In particular, the simple choice made in (6.20) is no longer available.

[10] Sometimes in the literature the term 'chiral block' is reserved for such flat sections.

for all smooth sections $v(\vec z)$ and all $s=1,2,\dots,m$. Applying this formula to the smooth section $\vec\Theta_{\vec\mu}^{-1}(\vec z)v(\vec z)$, we see from (6.31) that

$$\langle\nabla_s(\vec\Theta^\star_{\vec\mu}(\vec z)\beta(\vec z))\,,v(\vec z)\rangle = 0\,. \tag{6.33}$$

This means that together with $\beta(\vec z)$ also the section $\vec\Theta^\star_{\vec\mu}(\vec z)\beta(\vec z)$ is flat.

6.4 Verification of the projective action of $\Gamma_{\rm w}$

In subsection 5.1 we have seen that for every given value $\vec z$ of the moduli the maps $\vec\Theta_{\vec\mu}(\vec z)$ that implement the automorphisms $\vec\sigma_{\vec\mu}$ on the $\mathfrak g^m$-modules can be chosen such that they respect the group law of $\Gamma_{\rm w}$ up to a two-cocycle ϵ. Suppose we have made this choice for some point $\vec z^{[0]}\in\mathcal M$. To extend $\vec\Theta_{\vec\mu}$ to other values of the moduli, we have imposed the requirement that the extension should be compatible with the Knizhnik–Zamolodchikov connection. This requirement can only be satisfied when the implementation $\vec\Theta_{\vec\mu}(\vec z)$ is chosen in a suitable manner. We will now show that the specific implementation $\vec\Theta_{\vec\mu}(\vec z)$ that we have already chosen above for each $\vec\mu\in\Gamma_{\rm w}$ still respects the group law of $\Gamma_{\rm w}$ at *every* point in $\mathcal M$ up to the *same* cocycle ϵ.

To start, we recall the formula (6.21), i.e. $\Theta_{\vec\mu;s_\circ}(\vec z)=A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]})\,\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})$, with

$$A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]}) = \exp\big(\sum_{s=1}^m(\bar\mu_s,H)\otimes g_{s_\circ,s}+Kh_{s_\circ}\big)\,; \tag{6.34}$$

here $g_{s_\circ,s}(t;\vec z)$ and $h_{s_\circ}(\vec z;\vec\mu)$ are defined by (6.22) and (6.26), respectively. It follows that

$$\begin{aligned}\epsilon(\vec\mu,\vec\nu)\,\Theta_{\vec\mu+\vec\nu;s_\circ}(\vec z) &= A_{\vec\mu+\vec\nu;s_\circ}(\vec z;\vec z^{[0]})\circ\Theta_{\vec\mu+\vec\nu;s_\circ}(\vec z^{[0]})\\ &= A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]})\circ\exp\big(\sum_{s=1}^m(\bar\nu_s,H)g_{s_\circ,s}+K\,[h_{s_\circ}(\vec z;\vec\mu{+}\vec\nu)-h_{s_\circ}(\vec z;\vec\mu)]\big)\\ &\qquad\circ\,\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})\,\Theta_{\vec\nu;s_\circ}(\vec z^{[0]})\,.\end{aligned} \tag{6.35}$$

By commuting the exponential through $\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})$ we arrive at a similar exponential, but with $(\sigma^{[0]}_{\vec\mu})^{-1}$ applied to the argument. With the help of the identity (D.14) we then get explicitly

$$\epsilon(\vec\mu,\vec\nu)\,\Theta_{\vec\mu+\vec\nu;s_\circ}(\vec z) = A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]})\circ\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})\circ\exp(\tilde Y_{s_\circ})\circ\Theta_{\vec\nu;s_\circ}(\vec z^{[0]}) \tag{6.36}$$

with

$$\begin{aligned}\tilde Y_{s_\circ} &:= \sum_s(\bar\nu_s,H)\,g_{s_\circ,s}+K\Big(-\sum_{s,s'}(\bar\mu_{s'},\bar\nu_s)\,\mathcal{R}\mathrm{es}(\varphi_{s_\circ,s'}g_{s_\circ,s})\\ &\qquad+\tfrac12\sum_{\substack{s\\ s\ne s_\circ}}\ln\frac{z_{s_\circ}-z_s}{z^{[0]}_{s_\circ}-z^{[0]}_s}\big[(\bar\mu_{s_\circ}{+}\bar\nu_{s_\circ},\bar\mu_s{+}\bar\nu_s)-(\bar\mu_{s_\circ},\bar\mu_s)\big]\Big)\\ &= \sum_s(\bar\nu_s,H)\,g_{s_\circ,s}+\tfrac12\,K\sum_{\substack{s\\ s\ne s_\circ}}\ln\frac{z_{s_\circ}-z_s}{z^{[0]}_{s_\circ}-z^{[0]}_s}\,(\bar\nu_{s_\circ},\bar\nu_s)\\ &\qquad+\tfrac12\,K\sum_{\substack{s\\ s\ne s_\circ}}\ln\frac{z_{s_\circ}-z_s}{z^{[0]}_{s_\circ}-z^{[0]}_s}\,\big[(\bar\nu_{s_\circ},\bar\mu_s)-(\bar\mu_{s_\circ},\bar\nu_s)\big]\,.\end{aligned} \tag{6.37}$$

Upon exponentiation, the terms in the first line of the last expression just yield the correct factor $A_{\vec\nu;s_\circ}(\vec z;\vec z^{[0]})$, while the rest of the terms amount to an additional phase. Now this is the situation at the puncture $s_\circ$; taking into account that the centers of the affine algebras $\mathfrak{g}_s$ are identified in $\mathfrak{g}^m$, we finally have to add up the prefactors of K for all insertion points $s_\circ = 1,2,\ldots,m$. Doing so, the two sums in the prefactor cancel each other,

$$\sum_{\substack{s_\circ,s\\ s\neq s_\circ}} \ln \frac{z_{s_\circ}-z_s}{z^{[0]}_{s_\circ}-z^{[0]}_s} \left[(\bar\mu_{s_\circ},\bar\nu_s) - (\bar\nu_{s_\circ},\bar\mu_s) \right] = 0\,. \tag{6.38}$$

As a consequence, we have

$$\begin{aligned} \epsilon(\vec\mu,\vec\nu)\,\vec\Theta_{\vec\mu+\vec\nu}(\vec z) &= A_{\vec\mu+\vec\nu}(\vec z;\vec z^{[0]})\,\vec\Theta_{\vec\mu+\vec\nu}(\vec z^{[0]}) \\ &= A_{\vec\mu}(\vec z;\vec z^{[0]})\vec\Theta_{\vec\mu}(\vec z^{[0]}) \circ A_{\vec\nu}(\vec z;\vec z^{[0]})\vec\Theta_{\vec\nu}(\vec z^{[0]}) = \vec\Theta_{\vec\mu}(\vec z)\circ\vec\Theta_{\vec\nu}(\vec z)\,. \end{aligned} \tag{6.39}$$

We conclude that, as claimed, with our chosen implementation the group law of $\Gamma_{\rm w}$ is respected at every point of the moduli space $\mathcal{M}$ up to a $\vec z$-independent cocycle. (Note that it is the representative cocycle ϵ itself, and not just its cohomology class, that is independent of the moduli.)

Moreover, using the relation

$$\begin{aligned} &\exp\left(\tfrac{\mathrm{i}\pi}{2}\,(E^{\bar\alpha_s}{\otimes}\varphi_{s_\circ,s} + E^{-\bar\alpha_s}{\otimes}\varphi^{-1}_{s_\circ,s})\right) \\ &\quad = \exp\left(-\tfrac12\,H^{\bar\alpha_s}{\otimes}g_{s_\circ,s}\right)\,\exp\left(\tfrac{\mathrm{i}\pi}{2}\,(E^{\bar\alpha_s}{\otimes}\varphi^{[0]}_{s_\circ,s} + E^{-\bar\alpha_s}{\otimes}(\varphi^{[0]}_{s_\circ,s})^{-1})\right)\,\exp\left(\tfrac12\,H^{\bar\alpha_s}{\otimes}g_{s_\circ,s}\right), \end{aligned} \tag{6.40}$$

one can show that the preferred implementation of inner multi-shift automorphisms introduced in subsection 5.1 (see (5.7)) obeys

$$\Theta_{\vec\mu;s_\circ}(\vec z) = A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]})\,\Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) \tag{6.41}$$

precisely as in (6.21), which implies that at every point in the moduli space this implementation is compatible with the Knizhnik–Zamolodchikov connection. (For details, we refer to appendix D.) We conclude in particular that in each fiber of the bundle $\mathcal{B}$ the map $\vec\Theta^\star_{\vec\mu}$ has finite order.

7 Fixed point resolution

7.1 Fixed points

In the previous section we have constructed a projective action of the group $\Gamma_{\rm w}$ on the tensor product $\vec{\mathcal{H}}$ in such a way that for each $\vec\mu\in\Gamma$ the twisted intertwiner $\vec\Theta_{\vec\mu}$ is represented by the product of exponentials of elements in the block algebra $\bar{\mathfrak{g}}{\otimes}\mathcal{F}\equiv\bar{\mathfrak{g}}{\otimes}\mathcal{F}(\vec z^{[0]})$. As seen in subsection 5.3 this has in particular the consequence that in all fibers over the moduli space $\mathcal{M}_m$ the induced maps on the space $\mathsf{B}\equiv\mathsf{B}(\vec z)$ of chiral blocks have finite order and realize, modulo the fixed cocycle ϵ, the group law of the finite abelian group $\Gamma_{\rm w}/\Gamma$.

A particularly interesting situation arises when the automorphism $\sigma_{\vec\mu}$ is not inner, but still does not change the isomorphism class of a module $\mathcal{H}_\Lambda$. More precisely, given an m-tuple $\vec\Lambda$ of integrable weights of the affine Lie algebra $\mathfrak{g}$, we associate to it the subgroup of $\Gamma_{\rm w}$ that leaves each of the irreducible highest weight modules $\mathcal{H}_{\Lambda_s}$ invariant up to isomorphism. This subgroup is precisely the stabilizer $\Gamma_{\vec\Lambda}$ of $\vec\Lambda$ as defined in (6.7). For every m-tuple $\vec\Lambda$ the stabilizer $\Gamma_{\vec\Lambda}$ definitely contains Γ as a subgroup. If it is larger than Γ, then it also contains elements which do not necessarily act as a multiple of the identity on the blocks. In this case we call the m-tuple $\vec\Lambda$ of $\mathfrak{g}$-weights a *fixed point*.

Now the cocycle $\epsilon\equiv\epsilon_{\vec\Lambda}$ on $\Gamma_{\rm w}/\Gamma$ induces a cocycle on the subgroup $\Gamma_{\vec\Lambda}/\Gamma$, which we denote by the same symbol. Further, when applied to fixed points, the results of subsection 5.3 tell us that each fiber of the vector bundle $\mathcal{B}$ of chiral blocks can be split into finitely many subspaces B^ψ that are invariant under the projective action of the group $\Gamma_{\vec\Lambda}$, or rather, of the quotient by its subgroup Γ. These invariant subspaces, whose dimensions may be larger than one, are in correspondence with the irreducible representations of the twisted group algebra $\mathbb{C}_\epsilon(\Gamma_{\vec\Lambda}/\Gamma)$ of the finite group $\Gamma_{\vec\Lambda}/\Gamma$. Thus [11] the label ψ of the invariant subspace B^ψ can be taken to be a character of the center $\mathcal{Z}(\mathbb{C}_\epsilon(\Gamma_{\vec\Lambda}/\Gamma))$ of $\mathbb{C}_\epsilon(\Gamma_{\vec\Lambda}/\Gamma)$, which in turn (compare the remarks around formula (5.16)) is the group algebra $\mathcal{Z}(\mathbb{C}_\epsilon(\Gamma_{\vec\Lambda}/\Gamma))=\mathbb{C}((\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon)$ of the subgroup $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$ of regular elements of $\Gamma_{\vec\Lambda}/\Gamma$; thus in short, $\psi\in((\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon)^\star$. Now the Knizhnik–Zamolodchikov connection is preserved under the map $\vec\Theta^\star_{\vec\mu}$; therefore it restricts to the various invariant subspaces and they fit together into sub-vector bundles of $\mathcal{B}$. In particular, the dimensions of the invariant subspaces B^ψ do not depend on the moduli. Hence as soon as $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$ is non-trivial, the bundle $\mathcal{B}$ of chiral blocks is reducible (as a vector bundle). To actually establish this fact, we have to show that the ranks of at least two subbundles $\mathcal{B}^\psi$ are non-zero, which in fact follows from the conjectured formula for the rank to be discussed below. We refer to the decomposition of the bundle $\mathcal{B}$ into the subbundles of invariant subspaces under $\vec\Theta_{\vec\mu}$ as *fixed point resolution*. Our results imply that the Knizhnik–Zamolodchikov connection consistently restricts to these subbundles; thus even after fixed point resolution we are still given a Knizhnik–Zamolodchikov connection. The present situation should be compared with the situation in coset conformal field theories, where it is known [12] that the characters, i.e. the zero-point blocks on the torus, of fixed points decompose in a similar way under the action of an outer automorphism. This structural analogy should also explain the striking similarities in the modular matrices for extension modular invariants [8] and coset conformal field theories [12].

We pause for a side remark. One might be tempted to speculate that the converse is also true, i.e. that the bundle $\mathcal{B}$ splits into a direct sum of subbundles only if fixed points are involved. This, however, does not seem to be true, as the following example at higher genus shows. The bundle of zero-point blocks on the torus is given by the characters of the theory, and the representation of the mapping class group on this bundle is just given by the usual modular group representation on the characters. This bundle definitely does not involve fixed points. On the other hand it is known that the representation of the modular group is reducible if the theory contains non-trivial simple currents. (One well known example for this phenomenon is the fact that in superconformal field theories the character-valued indices of fields in the Ramond sector span a closed subspace under modular transformations.)

[11] See e.g. [25] for the representation theory of twisted group algebras.

7.2 Trace formulæ

We have seen that the dimensions of the invariant subspaces B^ψ do not depend on the moduli. In this subsection we present a conjecture for a general formula for these dimensions. Via Fourier transformation over the finite abelian group $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$ of regular elements of $\Gamma_{\vec\Lambda}/\Gamma$, these dimensions are related to the traces of the implementing maps $\vec\Theta^\star_{\vec{\bar\mu}}$ on the fibers. More precisely, we need to choose a representative $\vec\mu \in \Gamma_{\vec\Lambda}$ for every element $\vec\omega$ of $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$; the result does not depend on the choice of representative modulo Γ, because the map $\vec\Theta^\star_{\vec{\bar\mu}}$ depends on $\vec\mu \in \Gamma_{\vec\Lambda}$ only through its class $\vec\omega$. Now recall that $\omega_{\vec\mu;s}$ denotes the diagram automorphism that is in the same class as $\sigma_{\vec\mu;s}$. We have also seen that the latter is in the same class as the ordinary single-shift automorphism $\sigma^{(0)}_{\bar\mu_s}$; accordingly, instead of $\omega_{\vec\mu;s}$ we may also use the notation $\omega_{\bar\mu_s}$ or, for brevity, ω_s. In this notation, the condition that $\sum_{s=1}^m \bar\mu_s = 0$ tells us that we have

$$\prod_{s=1}^m \omega_s = \mathrm{id}\,. \tag{7.1}$$

Moreover, as diagram automorphisms are in one-to-one correspondence with classes of outer automorphisms, we identify the m-tuple $\vec\omega$ of outer automorphisms with the corresponding element of the group $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$.

Also recall that the implementing maps $\vec\Theta^\star_{\vec{\bar\mu}}$ were defined only up to a phase. The invariant contents of the conjecture we are going to spell out consists of a formula for the dimensions of the invariant subspaces B^ψ. However, it will be convenient not to write down directly this formula for the dimension of B^ψ, but to present instead the trace

$$\mathrm{T}_{\vec\Lambda;\vec\omega} \equiv \mathrm{T}_{\vec\Lambda;\vec\sigma_{\vec\mu}} := \mathrm{tr}_{\mathcal{B}_{\vec\Lambda}}\, \vec\Theta^\star_{\vec{\bar\mu}} \tag{7.2}$$

for a given choice of implementation $\vec\Theta^\star_{\vec{\bar\mu}}$. Of course, unlike the dimensions of the invariant subspaces themselves, these numbers do depend on the chosen implementation. For a definite choice of this phase, the relation between the dimensions $\dim \mathsf{B}^\psi$ and traces reads

$$\dim \mathsf{B}^\psi = |(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon|^{-1} \sum_{\vec\omega\in(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon} \psi^*(\vec\omega)\, \mathrm{T}_{\vec\Lambda;\vec\omega}\,. \tag{7.3}$$

The facts that the right hand side has an independent meaning and that the traces on the left hand side depend on the implementation are reconciled by the following observation. First, we do require that the maps $\vec\Theta^\star_{\vec{\bar\mu}}$ realize the group structure of Γ_w projectively and that they are the identity for all $\vec\mu \in \Gamma$. This already restricts the choice of possible scalar factors, but still leaves some indeterminacy. Indeed, the remaining freedom consists of modifying the implementing maps by phases that furnish a character φ of the group $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$. In short, once we fix the choice of implementation in such a way that our requirements are satisfied, we can label the eigenspaces by characters ψ of $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$, but when making an allowed change in the implementation, the labelling of the eigenspaces is by different characters $\psi' = \psi\varphi$ of $(\Gamma_{\vec\Lambda}/\Gamma)^\circ_\epsilon$.

Having explained the dependence of the traces (7.2) on the choice of implementation, we are now in a position to present our conjecture for these numbers. The results of [8] for the S-matrix of integer spin simple current modular invariants suggest, when combined with the ordinary Verlinde formula, that there exists an allowed implementation for which the traces in the case $m=3$ are given by

$$\mathrm{T}_{\vec{\Lambda};\vec{\omega}} = \sum_{\Lambda'} \frac{S^{\omega_1}_{\Lambda_1,\Lambda'} S^{\omega_2}_{\Lambda_2,\Lambda'} S^{\omega_3}_{\Lambda_3,\Lambda'}}{S_{\Omega,\Lambda'}} \,. \tag{7.4}$$

Here the summation extends over all integrable $\mathfrak{g}$-weights at level $k^\vee$ which are fixed under each of the permutations $\omega_s^\star$ for $s=1,2,3$; Ω is the label for the vacuum primary field, while $S_{\Lambda,\Lambda'}$ denotes the entries of the modular S-matrix of the WZW theory based on $\mathfrak{g}$, which is given by the Kac–Peterson formula [30]

$$S_{\Lambda,\Lambda'} = \mathcal{N} \sum_{\bar w\in\overline{W}} \mathrm{sign}(\bar w)\, \exp\big[-\tfrac{2\pi\mathrm{i}}{k^\vee+g^\vee}\, (\bar w(\bar\Lambda+\bar\rho), \bar\Lambda'+\bar\rho)\big] \,. \tag{7.5}$$

And finally, $S^\omega_{\Lambda,\Lambda'}$ are the entries of the modular S-matrix for some other WZW theory. Namely, to each pair consisting of an affine Lie algebra $\mathfrak{g}$ and a diagram automorphism ω of $\mathfrak{g}$ one can associate another affine Lie algebra $\mathfrak{g}^\omega$, the so-called orbit Lie algebra; S^ω is just given by the Kac–Peterson formula for $\mathfrak{g}^\omega$. (For more details, in particular on how the fixed point weights Λ_s are to be interpreted as weights of $\mathfrak{g}^\omega_s$, see [22, 31]. For convenience, we have also listed in table 1 the orbit Lie algebras [12] of untwisted affine Lie algebras with respect to all relevant diagram automorphisms.)

Table 1: *Orbit Lie algebras of untwisted affine Lie algebras* [22].

$\mathfrak{g}$	ω	N	$\mathfrak{g}^\omega$
$A^{(1)}_{n-1}$	ω_n	n	$\{0\}$
$A^{(1)}_{n-1}$	$(\omega_n)^{n/N}$	$N<n$	$A^{(1)}_{(n/N)-1}$
$B^{(1)}_n$	ω	2	$\tilde B^{(2)}_{n-1}$
$C^{(1)}_2$	ω	2	$A^{(2)}_1$
$C^{(1)}_{2n}$	ω	2	$\tilde B^{(2)}_n$
$C^{(1)}_{2n+1}$	ω	2	$C^{(1)}_n$
$D^{(1)}_n$	$\omega_{\rm v}$	2	$C^{(1)}_{n-2}$
$D^{(1)}_{2n}$	$\omega_{\rm s}$	2	$B^{(1)}_n$
$D^{(1)}_{2n+1}$	$\omega_{\rm s}$	4	$C^{(1)}_{n-1}$
$E^{(1)}_6$	ω	3	$G^{(1)}_2$
$E^{(1)}_7$	ω	2	$F^{(1)}_4$

[12] $\tilde B^{(2)}_n$ stands for the unique series of twisted affine Lie algebras whose characters furnish a module of the modular group and which have simple roots of three different lengths.

It is worth mentioning that in a first step, formula (7.3) should be regarded as an identity that holds in each fiber separately. But since the dimension $\dim \mathsf{B}^{\psi}$ is independent of the choice of the fiber, by inverse Fourier transformation one concludes that the traces also do not depend on the point in moduli space over which the trace is taken. Accordingly, there is no moduli dependence in our conjecture (7.4).

It is a remarkable empirical observation that, in all cases that have been checked numerically, the expression (7.4) gives an integral (though not necessarily non-negative) result. [13] Notice that this property is much stronger than the obvious requirement that the dimensions $\dim \mathsf{B}^{\psi}$ of the invariant subspaces must be integral. (Also, when the constraint (7.1) is not satisfied, then the expression on the right hand side of (7.4) is zero.)

We conjecture that the formula (7.4) indeed holds true and, moreover, that it directly generalizes to more complicated cases. To prepare the ground for this generalization, we write (7.4) in the equivalent form

$$\mathrm{T}_{\vec{\Lambda};\vec{\omega}} = \sum_{\Lambda'} \frac{S^{\omega_1}_{\Lambda_1,\Lambda'}}{S_{\Omega,\Lambda'}} \cdot \frac{S^{\omega_2}_{\Lambda_2,\Lambda'}}{S_{\Omega,\Lambda'}} \cdot \frac{S^{\omega_3}_{\Lambda_3,\Lambda'}}{S_{\Omega,\Lambda'}} \cdot |S_{\Omega,\Lambda'}|^2 \,. \tag{7.6}$$

The generalization to all $m \geq 3$ then simply consists in replacing the product $\prod_{s=1}^{3} S^{\omega_s}_{\Lambda_s,\Lambda'}/S_{\Omega,\Lambda'}$ by the analogous product $\prod_{s=1}^{m} S^{\omega_s}_{\Lambda_s,\Lambda'}/S_{\Omega,\Lambda'}$.

The last factor on the right hand side of the formula (7.6) should find its heuristic interpretation as a Weyl integration volume at genus zero (compare e.g. the derivation of the Verlinde formula in the framework of Chern–Simons theory [32]). Accordingly, for a surface of genus g we can speculate that the exponent gets replaced by the Euler number $\chi = 2-2g$, leading to an expression of the form

$$\sum_{\Lambda'} |S_{\Omega,\Lambda'}|^{2-2g} \prod_{s=1}^{m} \frac{S^{\omega_s}_{\Lambda_s,\Lambda'}}{S_{\Omega,\Lambda'}} \,, \tag{7.7}$$

which is consistent with factorization.

Unfortunately, a complete proof of (7.4) is not known at present and remains a challenge for future work. Note that such a proof would in particular include a proof of the ordinary Verlinde formula for WZW theories, namely for the special case when $\omega_s = \mathrm{id}$ for all s.

[13] The relevant calculations have been performed with the program `kac` which has been written by A.N. Schellekens and is available at `http://norma.nikhef.nl/~t58/kac.html`.
Helpful discussions with Bert Schellekens are gratefully acknowledged.

A The Virasoro algebra

In this appendix we collect some information about multi-shift automorphisms of the Virasoro algebra and its semidirect sum with the affine Lie algebra $\mathfrak{g}$.

We first show that for an arbitrary automorphism σ of an untwisted affine Lie algebra $\mathfrak{g}$, the extension to the semi-direct sum with the Virasoro algebra is unique, if it exists. To see this, consider two maps σ_i: $\mathcal{V}ir \to \mathcal{V}ir \oplus \mathfrak{g}$, $i=1,2$, such that both (σ,σ_1) and (σ,σ_2) furnish an automorphism of the semi-direct sum $\mathcal{V}ir \oplus \mathfrak{g}$. Then we have

$$[\sigma_1(L_n),\sigma(x_m)] = -m\,\sigma(x_{m+n}) = [\sigma_2(L_n),\sigma(x_m)]\,, \tag{A.1}$$

from which we learn that for every n the combination $\sigma_1(L_n)-\sigma_2(L_n)$ commutes with all of $\mathfrak{g}$, so that $\sigma_1(L_n)-\sigma_2(L_n)=\xi_n K+\eta_n C$ with $\xi_n,\eta_n\in\mathbb{C}$ for $n\in\mathbb{Z}$. But this in turn implies that $[\sigma_1(L_n),\sigma_1(L_m)]=[\sigma_2(L_n),\sigma_2(L_m)]$, or more explicitly

$$(n-m)\,\sigma_1(L_{n+m}) + \tfrac{1}{24}\,(n^3-n)\,\delta_{n+m,0}\,C = (n-m)\,\sigma_2(L_{n+m}) + \tfrac{1}{24}\,(n^3-n)\,\delta_{n+m,0}\,C\,. \tag{A.2}$$

This finally implies that $\xi_n=0=\eta_n$ for all n, and hence $\sigma_1=\sigma_2$ as claimed.

Next we check that the prescription (2.32) indeed provides us with an extension of the automorphism $\sigma_{\vec{\bar\mu};s_\circ}$ of $\mathfrak{g}$ as defined in (2.6). We first verify that the relation (2.31) is preserved when $\bar x=H^i$. We have

$$\begin{aligned}[\sigma_{\vec{\bar\mu};s_\circ}(L_n),\sigma_{\vec{\bar\mu};s_\circ}(H^i{\otimes}f)] &= [L_n+\textstyle\sum_{\ell\in\mathbb{Z}}\sum_s(\bar\mu_s,H_{n-\ell}){\otimes}\mathcal{R}es(t^\ell\varphi_{s_\circ,s}),H^i{\otimes}f] \\ &= -H^i\otimes t^{n+1}\mathrm{d}f + K\sum_{\ell\in\mathbb{Z}}\sum_s \bar\mu^i_s\,\mathcal{R}es(t^\ell\varphi_{s_\circ,s})\,\mathcal{R}es(\mathrm{d}t^{n-\ell}\,f)\,.\end{aligned} \tag{A.3}$$

Integrating by parts within the second residue and using the identity

$$\mathcal{R}es(fg) = \sum_{\ell\in\mathbb{Z}}\mathcal{R}es(t^{\ell-1}f)\,\mathcal{R}es(t^{-\ell}g) \qquad \text{for} \quad f,g\in\mathbb{C}((t)) \tag{A.4}$$

(which follows immediately by substituting f and g by their Laurent expansions), this reduces to

$$\begin{aligned}[\sigma_{\vec{\bar\mu};s_\circ}(L_n),\sigma_{\vec{\bar\mu};s_\circ}(H^i{\otimes}f)] &= -H^i\otimes t^{n+1}\mathrm{d}f - K\textstyle\sum_s\bar\mu^i_s\,\mathcal{R}es(t^{n+1}\varphi_{s_\circ,s}\,\mathrm{d}f) \\ &= -\sigma_{\vec{\bar\mu};s_\circ}(H^i{\otimes}t^{n+1}\mathrm{d}f) = \sigma_{\vec{\bar\mu};s_\circ}\big([L_n,H^i{\otimes}f]\big)\,,\end{aligned} \tag{A.5}$$

which is the desired result. Similarly, with the help of the identities (2.11) and (2.23) we calculate

$$\begin{aligned}[\sigma_{\vec{\bar\mu};s_\circ}(L_n),\sigma_{\vec{\bar\mu};s_\circ}(E^{\bar\alpha}{\otimes}f)] &= [L_n+\textstyle\sum_{\ell\in\mathbb{Z}}\sum_s(\bar\mu_s,H_{n-\ell})\,\mathcal{R}es(t^\ell\varphi_{s_\circ,s}),E^{\bar\alpha}{\otimes}f\Phi_{\bar\alpha}] \\ &= -E^{\bar\alpha}\otimes t^{n+1}\mathrm{d}(f\Phi_{\bar\alpha}) + \sum_{\ell\in\mathbb{Z}}\sum_s(\bar\mu_s,\bar\alpha)\,E^{\bar\alpha}\otimes t^{n-\ell}f\Phi_{\bar\alpha}\,\mathcal{R}es(t^\ell\varphi_{s_\circ,s}) \\ &= -E^{\bar\alpha}\otimes t^{n+1}\mathrm{d}(f\Phi_{\bar\alpha}) + E^{\bar\alpha}\otimes f\Phi_{\bar\alpha}\sum_s(\bar\mu_s,\bar\alpha)\varphi_{s_\circ,s}\,t^{n+1} \\ &= -E^{\bar\alpha}\otimes t^{n+1}\mathrm{d}(f\Phi_{\bar\alpha}) + E^{\bar\alpha}\otimes t^{n+1}f\,\mathrm{d}\Phi_{\bar\alpha} = -E^{\bar\alpha}\otimes t^{n+1}\,(\mathrm{d}f)\,\Phi_{\bar\alpha} \\ &= -\sigma_{\vec{\bar\mu};s_\circ}(E^{\bar\alpha}{\otimes}t^{n+1}\mathrm{d}f) = \sigma_{\vec{\bar\mu};s_\circ}\big([L_n,E^{\bar\alpha}{\otimes}f]\big)\,.\end{aligned} \tag{A.6}$$

Finally we check that we are indeed dealing with an automorphism of $\mathcal{V}ir_{s_\circ}$. We obtain

$$\begin{aligned}
[\sigma_{\vec{\bar\mu};s_\circ}(L_n),\sigma_{\vec{\bar\mu};s_\circ}(L_m)] &= [L_n + \textstyle\sum_{\ell\in\mathbb{Z}}\sum_s(\bar\mu_s,H_{n-\ell}){\otimes}\mathcal{R}\mathrm{es}(t^\ell\varphi_{s_\circ,s}),\\
&\qquad\qquad L_m + \textstyle\sum_{\ell'\in\mathbb{Z}}\sum_{s'}(\bar\mu_{s'},H_{m-\ell'}){\otimes}\mathcal{R}\mathrm{es}(t^{\ell'}\varphi_{s_\circ,s'})]\\
&= (n-m)\,L_{n+m} + \frac{1}{24}\,(n^3-n)\,\delta_{n+m,0}\,C\\
&\quad + \sum_{\ell'\in\mathbb{Z}}\sum_{s'}(\bar\mu_{s'},H_{m-\ell'+n})\otimes\mathcal{R}\mathrm{es}(t^{\ell'}\varphi_{s_\circ,s'})\,[-(m-\ell')]\\
&\quad - \sum_{\ell\in\mathbb{Z}}\sum_{s}(\bar\mu_{s},H_{n-\ell+m})\otimes\mathcal{R}\mathrm{es}(t^{\ell}\varphi_{s_\circ,s})\,[-(n-\ell)]\\
&\quad + K\sum_{\ell,\ell'\in\mathbb{Z}}\sum_{s,s'}(\bar\mu_s,\bar\mu_{s'})\,(n{-}\ell)\delta_{n-\ell+m-\ell',0}\,\mathcal{R}\mathrm{es}(t^\ell\varphi_{s_\circ,s})\,\mathcal{R}\mathrm{es}(t^{\ell'}\varphi_{s_\circ,s'})\,.
\end{aligned} \tag{A.7}$$

The terms in the second and third line combine to $(n-m)\sum_{\ell\in\mathbb{Z}}\sum_s(\bar\mu_s,H_{n+m-\ell}){\otimes}\mathcal{R}\mathrm{es}(t^\ell\varphi_{s_\circ,s})$. The central term in the fourth line can be rewritten as

$$\begin{aligned}
K\sum_{\ell,\ell'\in\mathbb{Z}}&\sum_{s,s'}(\bar\mu_s,\bar\mu_{s'})\,\big(\tfrac{n-m}{2}+\tfrac{\ell'-\ell}{2}\big)\,\delta_{n+m-\ell-\ell',0}\,\mathcal{R}\mathrm{es}(t^\ell\varphi_{s_\circ,s})\,\mathcal{R}\mathrm{es}(t^{\ell'}\varphi_{s_\circ,s'})\\
&= \tfrac{n-m}{2}\,K\sum_{s,s'}(\bar\mu_s,\bar\mu_{s'})\,\sum_{\ell\in\mathbb{Z}}\mathcal{R}\mathrm{es}(t^\ell\varphi_{s_\circ,s})\,\mathcal{R}\mathrm{es}(t^{n+m-\ell}\varphi_{s_\circ,s'})\\
&= \tfrac{n-m}{2}\,K\sum_{s,s'}(\bar\mu_s,\bar\mu_{s'})\,\sum_{\ell\in\mathbb{Z}}\mathcal{R}\mathrm{es}(t^{n+m+1}\varphi_{s_\circ,s}\varphi_{s_\circ,s'})\,,
\end{aligned} \tag{A.8}$$

where in the first step we used the fact that except for the explicit factor $(\ell'-\ell)/2$ the expression is symmetric in ℓ and ℓ', and in the second step we employed again the identity (2.23). Collecting all terms, we see that indeed

$$[\sigma_{\vec{\bar\mu};s_\circ}(L_n),\sigma_{\vec{\bar\mu};s_\circ}(L_m)] = \sigma_{\vec{\bar\mu};s_\circ}\big([L_n,L_m]\big) \tag{A.9}$$

as required.

B Inner multi-shift automorphisms

Here we collect some details about the derivation of some of the results stated in section 4. Let us consider the map ad_X for the element

$$X = E^{\bar\alpha}{\otimes}f_+ + E^{-\bar\alpha}{\otimes}f_- \tag{B.1}$$

of the block algebra $\bar{\mathfrak{g}}{\otimes}\mathcal{F}$. We have

$$\begin{aligned}
\mathrm{ad}_X(H^i{\otimes}f) &= -\bar\alpha^i\,(E^{\bar\alpha}{\otimes}f_+f - E^{-\bar\alpha}{\otimes}f_-f)\,,\\
\mathrm{ad}_X^2(H^i{\otimes}f) &= 2\bar\alpha^i\,(\bar\alpha^\vee,H)\otimes f_+f_-f\,,\\
\mathrm{ad}_X^3(H^i{\otimes}f) &= -4\bar\alpha^i\,(E^{\bar\alpha}{\otimes}f_+^2f_-f - E^{-\bar\alpha}{\otimes}f_+f_-^2f)
\end{aligned} \tag{B.2}$$

etc., and hence

$$\begin{aligned}\exp(\mathrm{ad}_{\xi X})(H^i{\otimes}f) = H^i{\otimes}f - \tfrac{1}{2}\,\bar\alpha^i\,(E^{\bar\alpha}{\otimes}f_+ - E^{-\bar\alpha}{\otimes}f_-)\,\sinh(2\xi\sqrt{f_+f_-})\,(f_+f_-)^{-1/2}\,f \\ + \tfrac{1}{2}\,\bar\alpha^i\,(\bar\alpha^\vee,H)\otimes[\cosh(2\xi\sqrt{f_+f_-})-1]\,f\,.\end{aligned} \tag{B.3}$$

In the special case where $f_- = (f_+)^{-1}$ (recall that this restricts $f_+ \in \mathcal{F}(\mathbb{P}^1_m)$ to lie in the subalgebra $\mathcal{F}^*(\mathbb{P}^1_m)$), this reduces to

$$\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X})(H^i{\otimes}f) = (\bar w_{\bar\alpha}(H))^i \otimes f\,, \tag{B.4}$$

where $\bar w_{\bar\alpha}(H) = H - (\bar\alpha^\vee,H)\bar\alpha$ is the image of H under the Weyl reflection $\bar w_{\bar\alpha}$. Note that this result does not depend on the function f_+ at all; as a consequence, by making use of $\bar w_{\bar\alpha}^2 = \mathrm{id}$ one immediately deduces the relation (4.4).

When applying ad_X to $E^{\bar\beta}{\otimes}f \in \bar{\mathfrak{g}}{\otimes}\mathcal{F}$, we distinguish between several cases. First assume that $\bar\beta = \pm\bar\alpha$. Then

$$\begin{aligned}\exp(\mathrm{ad}_{\xi X})(E^{\pm\bar\alpha}{\otimes}f) = E^{\pm\bar\alpha}{\otimes}f \mp \tfrac{1}{2}\,(\bar\alpha^\vee,H)\otimes\sinh(2\xi\sqrt{f_+f_-})\,(f_+f_-)^{-1/2}\,f_\mp f \\ \pm \tfrac{1}{2}\,(E^{\bar\alpha}{\otimes}f_+ - E^{-\bar\alpha}{\otimes}f_-)\,[\cosh(2\xi\sqrt{f_+f_-})-1]\,(f_\pm)^{-1}f\,.\end{aligned} \tag{B.5}$$

Similarly, for $\bar\beta \neq \pm\bar\alpha$ we find the following. When neither $\bar\beta+\bar\alpha$ nor $\bar\beta-\bar\alpha$ is a $\bar{\mathfrak{g}}$-root, we simply have $\mathrm{ad}_X(E^{\bar\beta}{\otimes}f)=0$. When either $\bar\beta+\bar\alpha$ or $\bar\beta-\bar\alpha$ (but not both) is a $\bar{\mathfrak{g}}$-root, while $\bar\beta\pm 2\bar\alpha$ is not a $\bar{\mathfrak{g}}$-root, then we obtain

$$\begin{aligned}\exp(\mathrm{ad}_{\xi X})(E^{\bar\beta}{\otimes}f) = {}& E^{\bar\beta}\otimes\cosh(\xi\sqrt{\eta f_+f_-})\,f \\ & + e_{\pm\bar\alpha,\bar\beta}\,E^{\bar\beta\pm\bar\alpha}\otimes\sinh(\xi\sqrt{\eta f_+f_-})\,(\eta f_+f_-)^{-1/2}\,f_\pm f\,,\end{aligned} \tag{B.6}$$

where $\eta \equiv \eta_{\pm\bar\alpha} := e_{\pm\bar\alpha,\bar\beta}e_{\mp\bar\alpha,\bar\beta\pm\bar\alpha}$, with $e_{\bar\alpha,\bar\beta}$ structure constants of the horizontal subalgebra $\bar{\mathfrak{g}}$. Actually, from the fact that the $\bar\alpha$-string through $\bar\beta$ has only two elements, it follows that $e_{\pm\bar\alpha,\bar\beta}$ and $e_{\mp\bar\alpha,\bar\beta\pm\bar\alpha}$ can only take the values ±1, and that $\bar\beta$ and $\bar\beta\pm\bar\alpha$ have the same length, so that the general identity $(\bar\beta\pm\bar\alpha,\bar\beta\pm\bar\alpha)/(\bar\beta,\bar\beta) = e_{\pm\bar\alpha,\bar\beta}/e_{\mp\bar\alpha,\bar\beta\pm\bar\alpha}$ tells us that $\eta=1$. When the $\bar\alpha$-string through $\bar\beta$ has more elements, then the calculations become still a bit more lengthy. We refrain from describing all different possibilities, because the calculations remain straightforward. As an illustration, let $\bar\beta+\bar\alpha$ and $\bar\beta+2\bar\alpha$, but neither $\bar\beta-\bar\alpha$ nor $\bar\beta+3\bar\alpha$ be $\bar{\mathfrak{g}}$-roots; then we have

$$\begin{aligned}\mathrm{ad}_X(E^{\bar\beta}{\otimes}f) &= e_{\bar\alpha,\bar\beta}\,E^{\bar\alpha+\bar\beta}\otimes f_+f\,, \\ \mathrm{ad}_X^2(E^{\bar\beta}{\otimes}f) &= e_{\bar\alpha,\bar\beta}\,(e_{\bar\alpha,\bar\alpha+\bar\beta}\,E^{2\bar\alpha+\bar\beta}\otimes f_+ + e_{-\bar\alpha,\bar\alpha+\bar\beta}\,E^{\bar\beta}\otimes f_-)\,f_+f\,, \\ \mathrm{ad}_X^3(E^{\bar\beta}{\otimes}f) &= \eta' e_{\bar\alpha,\bar\beta}\,E^{\bar\alpha+\bar\beta}\otimes f_+^2f_-f\end{aligned} \tag{B.7}$$

with $\eta' := e_{\bar\alpha,\bar\alpha+\bar\beta}e_{-\bar\alpha,2\bar\alpha+\bar\beta} + e_{-\bar\alpha,\bar\alpha+\bar\beta}e_{\bar\alpha,\bar\beta}$, leading to

$$\begin{aligned}\exp(\mathrm{ad}_{\xi X})(E^{\bar\beta}{\otimes}f) = {}& E^{\bar\beta}{\otimes}f + e_{\bar\alpha,\bar\beta}\,E^{\bar\alpha+\bar\beta}\otimes\sinh(\xi\sqrt{\eta' f_+f_-})\,(\eta' f_+f_-)^{-1/2}\,f_+f \\ & + e_{\bar\alpha,\bar\beta}\,(e_{\bar\alpha,\bar\alpha+\bar\beta}\,E^{2\bar\alpha+\bar\beta}\otimes f_+ + e_{-\bar\alpha,\bar\alpha+\bar\beta}\,E^{\bar\beta}\otimes f_-) \\ & \qquad [\cosh(\xi\sqrt{\eta' f_+f_-})-1]\,(\eta' f_+f_-)^{-1}\,f_+f\,.\end{aligned} \tag{B.8}$$

In the special case $f_+f_-=1$ from (B.5) we get

$$\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X})(E^{\pm\bar\alpha}\otimes f) = E^{\pm\bar\alpha}\otimes f \mp (E^{\bar\alpha}\otimes f_+ - E^{-\bar\alpha}\otimes f_-)\,(f_\pm)^{-1} f = E^{\mp\bar\alpha}\otimes f_\mp^2 f \tag{B.9}$$

for $\bar\beta=\pm\bar\alpha$. When in addition $Y\in\bar{\mathfrak{g}}\otimes\mathcal{F}$ is of the same form as X, but with $f_\pm$ replaced by $g_\pm\in\mathcal{F}^*(\mathbb{P}^1_m)$, then it follows from (B.9) that

$$\mathrm{Ad}_{\mathrm{i}\pi/2;X,Y}(E^{\pm\bar\alpha}\otimes f) = E^{\pm\bar\alpha}\otimes(f_+)^{\mp2}(g_+)^{\pm2} f\,. \tag{B.10}$$

Analogously, the special case $f_+f_-=1$ of (B.6) yields

$$\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X})(E^{\bar\beta}\otimes f) = \mathrm{i}\,e_{\pm\bar\alpha,\bar\beta}\,E^{\bar\beta\pm\bar\alpha}\otimes f_\pm f\,. \tag{B.11}$$

Similarly, from (B.8) one obtains (using also the fact that $\bar\beta$ and $\bar\beta+2\bar\alpha$ must be long roots while $\bar\alpha$ and $\bar\beta+\bar\alpha$ must be short roots, which implies the identities

$$\begin{aligned}
& e_{\bar\alpha,\bar\alpha+\bar\beta}e_{-\bar\alpha,2\bar\alpha+\bar\beta} = (e_{\bar\alpha,\bar\alpha+\bar\beta})^2\cdot\frac{(\bar\alpha+\bar\beta,\bar\alpha+\bar\beta)}{(2\bar\alpha+\bar\beta,2\bar\alpha+\bar\beta)} = 2^2\cdot\tfrac12 = 2\,,\\
& e_{\bar\alpha,\bar\beta}e_{-\bar\alpha,\bar\alpha+\bar\beta} = (e_{\bar\alpha,\bar\beta})^2\cdot\frac{(\bar\beta,\bar\beta)}{(\bar\alpha+\bar\beta,\bar\alpha+\bar\beta)} = 1^2\cdot 2 = 2\,,\\
& |e_{\bar\alpha,\bar\beta}e_{\bar\alpha,\bar\alpha+\bar\beta}| = 2\,,
\end{aligned} \tag{B.12}$$

so that in particular $\eta'=4$),

$$\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X})(E^{\bar\beta}\otimes f) = \pm\,E^{2\bar\alpha+\bar\beta}\otimes f_+^2 f\,. \tag{B.13}$$

Taking again also Y of the form described above, the relations (B.11) and (B.13) lead to

$$\mathrm{Ad}_{\mathrm{i}\pi/2;X,Y}(E^{\bar\beta}\otimes f) = E^{\bar\beta}\otimes f_\pm g_\mp f \qquad\text{and}\qquad \mathrm{Ad}_{\mathrm{i}\pi/2;X,Y}(E^{\bar\beta}\otimes f) = E^{\bar\beta}\otimes(f_+g_-)^2 f\,, \tag{B.14}$$

respectively.

For the remaining cases, the calculations are completely parallel and lead to results analogous to (B.14), with $f_\pm g_\mp$ raised to the power $-(\bar\alpha^\vee,\bar\beta)$. This finally leads to the formula (4.5) of the main text.

Next we comment on the analogous calculations for the case where the functions f are Laurent series (in the local coordinate ζ_s respectively the formal variable t). First we compute that, for $f_+f_-=1$, the second formula in (B.2) changes to

$$\mathrm{ad}_X^2(H^i\otimes f) = 2\bar\alpha^i\,(\bar\alpha^\vee,H)\otimes f_+f_-f - 2\bar\alpha^i K\,\frac{2}{(\bar\alpha,\bar\alpha)}\,\mathcal{R}es(f_+f\,\mathrm{d}f_-)\,, \tag{B.15}$$

so that (B.3) acquires an additional contribution proportional to the central element,

$$-\tfrac12\,(\bar\alpha^\vee)^i K\,\mathcal{R}es(f_+f\mathrm{d}f_-)\,(\cosh(2\xi)-1))\,, \tag{B.16}$$

which when specialized to $\xi=\mathrm{i}\pi/2$ becomes $(\bar\alpha^\vee)^iK\,\mathcal{R}es(f_+f\mathrm{d}f_-)$. Performing the same calculation also with the analogous element Y in place of X, we then arrive at the formula (4.8).

In the calculation of $\exp(\mathrm{ad}_{\xi X})(E^{\bar\beta}\otimes f)$, a change occurs only for $\bar\beta=\pm\bar\alpha$; in this case we get

$$\begin{aligned}
\mathrm{ad}_X(E^{\pm\bar\alpha}\otimes f) &= \mp(\bar\alpha^\vee,H)\otimes f_\mp f - K\,\tfrac{2}{(\bar\alpha,\bar\alpha)}\,\mathcal{R}\mathrm{es}(f_\mp \mathrm{d}f)\,,\\
\mathrm{ad}_X^2(E^{\pm\bar\alpha}\otimes f) &= \pm 2\,(E^{\bar\alpha}\otimes f_+ - E^{-\bar\alpha}\otimes f_-)\,f_\mp f\,,\\
\mathrm{ad}_X^3(E^{\pm\bar\alpha}\otimes f) &= \mp 4\,(\bar\alpha^\vee,H)\otimes f_\mp f - 4K\,\tfrac{2}{(\bar\alpha,\bar\alpha)}\,\mathcal{R}\mathrm{es}(f_\mp \mathrm{d}f)\,,
\end{aligned} \tag{B.17}$$

etc. The residue terms add up to $-(\bar\alpha,\bar\alpha)^{-1}\,K\,\mathcal{R}\mathrm{es}(f_\mp\mathrm{d}f)\,\sinh(2\xi)$, which vanishes for $\xi=\mathrm{i}\pi/2$, while the other terms reproduce (B.6).

Finally let us compute the action of ad_X on the Virasoro generators L_n. By (2.31) we find, for $f_+f_-=1$,

$$\begin{aligned}
\mathrm{ad}_X(L_n) &= E^{\bar\alpha}\otimes t^{n+1}\mathrm{d}f_+ + E^{-\bar\alpha}\otimes t^{n+1}\mathrm{d}f_-\,,\\
\mathrm{ad}_X^2(L_n) &= -2\,(\bar\alpha^\vee,H^i)\otimes f_-\mathrm{d}f_+\,t^{n+1} + 2\,\tfrac{2}{(\bar\alpha,\bar\alpha)}K\,\mathcal{R}\mathrm{es}(t^{n+1}\mathrm{d}f_+\mathrm{d}f_-)\,,\\
\mathrm{ad}_X^3(L_n) &= 4\,(E^{\bar\alpha}\otimes t^{n+1}\mathrm{d}f_+ + E^{-\bar\alpha}\otimes t^{n+1}\mathrm{d}f_-)\,,
\end{aligned} \tag{B.18}$$

and hence

$$\begin{aligned}
\exp(\mathrm{ad}_{\xi X})(L_n) = L_n &+ \tfrac12\,\sinh(2\xi)\,(E^{\bar\alpha}\otimes t^{n+1}\mathrm{d}f_+ + E^{-\bar\alpha}\otimes t^{n+1}\mathrm{d}f_-)\\
&- \tfrac12\,[\cosh(2\xi)-1]\,\big((\bar\alpha^\vee,H)\otimes f_-\mathrm{d}f_+\,t^{n+1} - \tfrac{2}{(\bar\alpha,\bar\alpha)}K\,\mathcal{R}\mathrm{es}(t^{n+1}\mathrm{d}f_+\mathrm{d}f_-)\big)\,.
\end{aligned} \tag{B.19}$$

Specializing to $\xi=\mathrm{i}\pi/2$, we have

$$\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X})(L_n) = L_n + (\bar\alpha^\vee,H)\otimes f_-\mathrm{d}f_+\,t^{n+1} - \tfrac{2}{(\bar\alpha,\bar\alpha)}K\,\mathcal{R}\mathrm{es}(t^{n+1}\mathrm{d}f_+\mathrm{d}f_-)\,. \tag{B.20}$$

C On the moduli dependence of inner automorphisms

Our first aim in this appendix is to check the formula (6.11), which asserts that the automorphism $\delta_{\vec\mu}$ acting as in (6.10) can be represented as $\delta_{\vec\mu}=\exp(\mathrm{ad}_{(\bar\mu_u,H)\otimes g_{s_\circ,u}})$ for $s_\circ\neq u$ and as $\delta_{\vec\mu}=\exp(\mathrm{ad}_{\sum_{s\neq u}(\bar\mu_s,H)\otimes\tilde g_s})$ for $s_\circ=u$, with $g_{s_\circ,u}$ and $\tilde g_s$ as defined in (6.12). First, definitely

$$\exp(\mathrm{ad}_{(\bar\mu_u,H)\otimes g_{s_\circ,u}})(K) = K = \exp(\mathrm{ad}_{(\bar\mu_s,H)\otimes\tilde g_s})(K)\,. \tag{C.1}$$

Second, using the relations $\mathrm{d}g_{s_\circ,u}=\varphi_{s_\circ,u}-\varphi^{[0]}_{s_\circ,u}$ and

$$[(\bar\mu_u,H)\otimes g_{s_\circ,u},H^i\otimes f] = \bar\mu^i_u\,\mathcal{R}\mathrm{es}(\mathrm{d}g_{s_\circ}\,f)\,K \tag{C.2}$$

together with $[K,\cdot\,]=0$, we have

$$\exp(\mathrm{ad}_{(\bar\mu_u,H)\otimes g_{s_\circ,u}})(H^i\otimes f) = H^i\otimes f + K\,\bar\mu^i_u\,\mathcal{R}\mathrm{es}\big((\varphi_{s_\circ,u}-\varphi^{[0]}_{s_\circ,u})\,f\big)\,, \tag{C.3}$$

and similarly

$$\exp(\mathrm{ad}_{(\bar\mu_s,H)\otimes\tilde g_s})(H^i\otimes f) = H^i\otimes f + K\,\bar\mu^i_s\,\mathcal{R}\mathrm{es}\big((\varphi_{u,s}-\varphi^{[0]}_{u,s})\,f\big)\,. \tag{C.4}$$

And finally, by exponentiating the identity $[(\bar\mu_u,H)\otimes g_{s_\circ,u},E^{\bar\beta}\otimes f]=(\bar\mu_u,\bar\beta)E^{\bar\beta}\otimes g_{s_\circ,u}f$ we obtain

$$\exp(\mathrm{ad}_{(\bar\mu_u,H)\otimes g_{s_\circ,u}})(E^{\bar\beta}\otimes f) = E^{\bar\beta}\otimes\exp((\bar\mu_u,\bar\beta)\,g_{s_\circ,u})\,f = E^{\bar\beta}\otimes f\cdot\Big(\frac{t+z_{s_0}-z_u}{t+z_{s_0}-z_u^{[0]}}\Big)^{(\bar\mu_u,\bar\beta)}\,, \qquad (C.5)$$

and analogously

$$\exp(\mathrm{ad}_{(\bar\mu_s,H)\otimes\tilde g_s})(E^{\bar\beta}\otimes f) = E^{\bar\beta}\otimes f\cdot\Big(\frac{t+z_u-z_s}{t+z_u^{[0]}-z_s}\Big)^{(\bar\mu_s,\bar\beta)}\,. \qquad (C.6)$$

Putting these results together we see that $\delta_{\vec\mu}$ is indeed given by (6.11).

Next we derive the formula (6.15) for the commutator $[\nabla_u,\Theta_{\vec\mu;s_\circ}]$. Recall that $\nabla_u=\partial/\partial z_u+L^{(u)}_{-1}$. Thus we first study the effect of differentiating with respect to z_u. Employing the identities

$$\frac{\partial}{\partial z_u}\,g_{s_\circ} = -\varphi_{s_\circ,u} \quad\text{for}\;\; s_\circ\neq u \qquad\text{and}\qquad \frac{\partial}{\partial z_u}\,\tilde g_s\Big|_{z_u=z_u^{[0]}} = \varphi_{u,s} \quad\text{for}\;\; s\neq u\,, \qquad (C.7)$$

we derive

$$\begin{aligned}\frac{\partial}{\partial z_u}\,\Theta_{\vec\mu;s_\circ}(\vec z)\Big|_{z_u=z_u^{[0]}} &= \Big(-(\bar\mu_u,H)\,\varphi^{[0]}_{s_\circ,u}+K\,\frac{\partial}{\partial z_u}\,\tilde h_{s_\circ}\Big|_{z_u=z_u^{[0]}}\Big)\,\Theta_{\vec\mu;s_\circ}(\vec z^{\,[0]}) \quad\text{for}\;\; s_\circ\neq u\,,\\ \frac{\partial}{\partial z_u}\,\Theta_{\vec\mu;s_\circ}(\vec z)\Big|_{z_u=z_u^{[0]}} &= \Big(\sum_{\substack{s=1\\ s\neq u}}^m(\bar\mu_s,H)\,\varphi^{[0]}_{u,s}+K\,\frac{\partial}{\partial z_u}\,\tilde h_u\Big|_{z_u=z_u^{[0]}}\Big)\,\Theta_{\vec\mu;s_\circ}(\vec z^{\,[0]}) \quad\text{for}\;\; s_\circ=u\,.\end{aligned} \qquad (C.8)$$

Concerning the commutation with $L^{(u)}_{-1}$, we use $\Theta_{\vec\mu;s}(\vec z)L^{(s)}_{-1}=\sigma_{\vec\mu;s}(L^{(s)}_{-1})\,\Theta_{\vec\mu;s}(\vec z)$ which follows by (2.34). Also, according to (2.32) we have explicitly

$$\sigma_{\vec\mu;u}(L^{(u)}_{-1}) = L^{(u)}_{-1}+\sum_{\ell\in\mathbb Z}\sum_{s=1}^m(\bar\mu_s,H_{-1-\ell})\,\mathcal{R}es(t^\ell\varphi_{u,s})+\tfrac12\,K\sum_{s,s'=1}^m(\bar\mu_s,\bar\mu_{s'})\,\mathcal{R}es(\varphi_{u,s}\varphi_{u,s'})\,. \qquad (C.9)$$

The two terms of the right hand side can be rewritten with the help of

$$\sum_{\ell\in\mathbb Z}H^i_{-1-\ell}\,\mathcal{R}es(t^\ell\varphi_{u,s}) \equiv \sum_{\ell\in\mathbb Z}H^i\otimes t^{-1-\ell}\,\mathcal{R}es(t^\ell\varphi_{u,s}) = \sum_{\ell\in\mathbb Z}H^i\otimes(\varphi_{u,s})_{-1-\ell}\,t^{-1-\ell} = H^i\otimes\varphi_{u,s} \qquad (C.10)$$

and

$$\sum_{s,s'=1}^m(\bar\mu_s,\bar\mu_{s'})\,\mathcal{R}es(\varphi_{u,s}\varphi_{u,s'}) = 2\sum_{\substack{s=1\\ s\neq u}}^m\frac{(\bar\mu_s,\bar\mu_u)}{z_u-z_s} \qquad (C.11)$$

so as to obtain

$$[\Theta_{\vec\mu;u},L^{(u)}_{-1}] = \big(\sigma_{\vec\mu;u}(L^{(u)}_{-1})-L^{(u)}_{-1}\big)\,\Theta_{\vec\mu;u} = \Big(\sum_{s=1}^m(\bar\mu_s,H)\otimes\varphi_{u,s}+K\sum_{\substack{s=1\\ s\neq u}}^m\frac{(\bar\mu_s,\bar\mu_u)}{z_u-z_s}\Big)\,\Theta_{\vec\mu;u}\,. \qquad (C.12)$$

Combining this with (C.8), we finally get

$$[\nabla_u,\Theta_{\vec\mu;s_\circ}(\vec z)] = -(\bar\mu_u,H)\otimes\varphi^{[0]}_{s_\circ,u}\,\Theta_{\vec\mu;s_\circ}(\vec z^{\,[0]})+K\,\frac{\partial}{\partial z_u}\,\tilde h_{s_\circ}\Big|_{z_u=z_u^{[0]}}\,\Theta_{\vec\mu;s_\circ}(\vec z^{\,[0]}) \quad\text{for}\;\; s_\circ\neq u\,, \qquad (C.13)$$

while for $s_\circ=u$ several terms cancel, leading to

$$[\nabla_u,\Theta_{\vec\mu;u}(\vec z)]\Big|_{z_u=z_u^{[0]}} = -(\bar\mu_u,H)\otimes t^{-1}\,\Theta_{\vec\mu;u}(\vec z^{\,[0]})+K\,\Big(\frac{\partial}{\partial z_u}\,\tilde h_u\Big|_{z_u=z_u^{[0]}}-\sum_{\substack{s=1\\ s\neq u}}^m\frac{(\bar\mu_s,\bar\mu_u)}{z_u-z_s}\Big)\,\Theta_{\vec\mu;u}(\vec z^{\,[0]})\,. \qquad (C.14)$$

This is indeed equivalent to the formula (6.15).

D On the implementation of inner automorphisms

In this appendix we present the calculations which show that the specific implementation of inner multi-shift automorphisms that was described in (5.7) satisfies the relation (6.41) and explain why this implies that the implementation can be consistently chosen at all points of the moduli space $\mathcal{M}_m$.

We consider first the case of a single coroot $\bar\alpha_s^\vee$ for some fixed s, i.e. $\bar\mu_s = \bar\alpha_s^\vee$ while $\bar\mu_{s'} = 0$ for $s' \neq s$. We write

$$X_{s_\circ,s} \equiv X_{s_\circ,s;\alpha_s}(\vec z) := E^{\bar\alpha_s} \otimes \varphi_{s_\circ,s} + E^{-\bar\alpha_s} \otimes \varphi_{s_\circ,s}^{-1} \tag{D.1}$$

and

$$X^{[0]}_{s_\circ,s} := X_{s_\circ,s;\alpha_s}(\vec z^{\,[0]}) = E^{\bar\alpha_s} \otimes \varphi^{[0]}_{s_\circ,s} + E^{-\bar\alpha_s} \otimes (\varphi^{[0]}_{s_\circ,s})^{-1} \,. \tag{D.2}$$

We claim that $X_{s_\circ,s}$ and $X^{[0]}_{s_\circ,s}$ are related by (6.40), which in terms of the present notations reads

$$\exp(\tfrac12\, \mathrm{i}\pi\, X_{s_\circ,s}) = U_{s_\circ,s}(\vec z)\, \exp(\tfrac12\, \mathrm{i}\pi\, X^{[0]}_{s_\circ,s})\, U^{-1}_{s_\circ,s}(\vec z)\,, \tag{D.3}$$

where

$$U_{s_\circ,s}(\vec z) \equiv U_{s_\circ,s;\bar\alpha_s^\vee}(\vec z) := \exp\big(-\tfrac12\, H^{\bar\alpha_s} \otimes g_{s_\circ,s}(\vec z)\big) \tag{D.4}$$

with $g_{s_\circ,s}$ as defined in (6.22) and $H^{\bar\alpha_s} \equiv (\bar\alpha_s^\vee, H)$. To prove (D.3), we show that both sides satisfy the same differential equation; the relation then holds because owing to $U_{s_\circ,s}(\vec z^{\,[0]}) = \mathbf{1}$ both sides are identical at $\vec z = \vec z^{\,[0]}$. To obtain the derivative of the left hand side of (D.3) we first compute

$$\begin{aligned} \frac{\partial}{\partial z_u}\, X_{s_\circ,s} &= (\delta_{u,s} - \delta_{u,s_\circ})\, (E^{\bar\alpha_s}\otimes\varphi^2_{s_\circ,s} + E^{-\bar\alpha_s}\otimes\mathrm{id}) \\ &= -\tfrac12\, (\delta_{u,s} - \delta_{u,s_\circ})\, \mathrm{ad}_{X_{s_\circ,s}}(H^{\bar\alpha_s} \otimes \varphi_{s_\circ,s}) \equiv \tfrac12\, (\delta_{u,s} - \delta_{u,s_\circ})\, \mathrm{ad}_{H^{\bar\alpha_s}\otimes\varphi_{s_\circ,s}}(X_{s_\circ,s})\,, \end{aligned} \tag{D.5}$$

where in the transition to the second line we used the formula (B.2). It follows immediately that

$$\frac{\partial}{\partial z_u}\, \exp\big(\tfrac{\mathrm{i}\pi}{2}\, X_{s_\circ,s}\big) = \tfrac12\, (\delta_{u,s} - \delta_{u,s_\circ})\, \big(H^{\bar\alpha_s}\otimes\varphi_{s_\circ,s}\, \exp\big(\tfrac{\mathrm{i}\pi}{2}\, X_{s_\circ,s}\big) - \exp\big(\tfrac{\mathrm{i}\pi}{2}\, X_{s_\circ,s}\big)\, H^{\bar\alpha_s}\otimes\varphi_{s_\circ,s}\big)\,. \tag{D.6}$$

As for the right hand side of (D.3), we have

$$\begin{aligned} \frac{\partial U_{s_\circ,s}(\vec z)}{\partial z_u} &= -\tfrac12\, U_{s_\circ,s}(\vec z)\cdot H^{\bar\alpha_s} \otimes \frac{\partial}{\partial z_u}\, g_{s_\circ,s} \\ &= -\tfrac12\, U_{s_\circ,s}(\vec z)\cdot H^{\bar\alpha_s} \otimes \varphi_{s_\circ,s}\, (\delta_{u,s_\circ} - \delta_{u,s}) \equiv \tfrac12\, (\delta_{u,s} - \delta_{u,s_\circ})\, H^{\bar\alpha_s} \otimes \varphi_{s_\circ,s} \cdot U_{s_\circ,s}(\vec z) \end{aligned} \tag{D.7}$$

and

$$\frac{\partial}{\partial z_u}\, U^{-1}_{s_\circ,s}(\vec z) = -U^{-1}_{s_\circ,s}(\vec z)\, \frac{\partial U_{s_\circ,s}(\vec z)}{\partial z_u}\, U^{-1}_{s_\circ,s}(\vec z) = -\tfrac12\, (\delta_{u,s} - \delta_{u,s_\circ})\, U^{-1}_{s_\circ,s}(\vec z)\cdot H^{\bar\alpha_s} \otimes \varphi_{s_\circ,s}\,; \tag{D.8}$$

therefore the product $U_{s_\circ,s}(\vec z)\, \exp(\tfrac12\, \mathrm{i}\pi\, X^{[0]}_{s_\circ,s})\, U^{-1}_{s_\circ,s}(\vec z)$ indeed satisfies the same differential equation as $\exp(\tfrac12\, \mathrm{i}\pi\, X_{s_\circ,s})$ in (D.6).

Next we note that according to the result (B.4) and (B.16), we also have

$$\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X_{s_\circ,s}})(H^{\bar\beta}\otimes f) = \bar w_{\bar\alpha}(H^{\bar\beta}) \otimes f - (\bar\alpha^\vee, \bar\beta^\vee)\, K\, \mathcal{R}es\big(\varphi^{-1}_{s_\circ,s} \mathrm{d}\varphi_{s_\circ,s}\, f\big)\,. \tag{D.9}$$

Thus in particular

$$\begin{aligned}\exp(\mathrm{ad}_{(\mathrm{i}\pi/2)X_{s_\circ,s}})(H^{\bar\alpha_s}\otimes f) &= -H^{\bar\alpha_s}\otimes f - (\bar\alpha_s^\vee,\bar\alpha_s^\vee)\,K\,\mathcal{R}es\big(\varphi_{s_\circ,s}^{-1}\mathrm{d}\varphi_{s_\circ,s}\,f\big)\,,\\ \exp(\mathrm{ad}_{-(\mathrm{i}\pi/2)Y_s})(H^{\bar\alpha_s}\otimes f) &= -H^{\bar\alpha_s}\otimes f\,,\end{aligned} \tag{D.10}$$

where we introduced

$$Y_s := (E^{\bar\alpha_s}+E^{-\bar\alpha_s})\otimes\mathrm{id}\,. \tag{D.11}$$

In other words, $\exp(-\frac12\,\mathrm{i}\pi\,Y_s)\,H^{\bar\alpha_s}\otimes f = -H^{\bar\alpha_s}\otimes f\,\exp(-\frac12\,\mathrm{i}\pi\,Y_s)$, which implies in particular that

$$\exp(-\tfrac12\,\mathrm{i}\pi\,Y_s)\,U_{s_\circ,s}^{-1}(\vec z) = U_{s_\circ,s}(\vec z)\,\exp(-\tfrac12\,\mathrm{i}\pi\,Y_s)\,, \tag{D.12}$$

while

$$\exp\big(\tfrac{\mathrm{i}\pi}{2}\,X_{s_\circ,s}\big)\,U_{s_\circ,s}(\vec z) = U_{s_\circ,s}^{-1}(\vec z)\,\exp\big(\tfrac{\mathrm{i}\pi}{2}\,X_{s_\circ,s}\big)\cdot\exp\big(-\tfrac{2}{(\bar\alpha_s,\bar\alpha_s)}\,K\,\mathcal{R}es(\varphi_{s_\circ,s}^{-1}\mathrm{d}\varphi_{s_\circ,s}\,g_{s_\circ,s})\big)\,, \tag{D.13}$$

which by $\mathrm{d}\varphi_{s_\circ,s} = -\varphi_{s_\circ,s}^2$ and

$$\mathcal{R}es(\varphi_{s_\circ,s'}g_{s_\circ,s}) = \begin{cases}\delta_{s_\circ,s'}\,\ln\frac{z_{s_\circ}-z_s}{z_{s_\circ}^{[0]}-z_s^{[0]}} & \text{for } s\neq s_\circ\,,\\ 0 & \text{for } s=s_\circ\end{cases} \tag{D.14}$$

reduces to

$$\exp(\tfrac12\,\mathrm{i}\pi\,X_{s_\circ,s})\,U_{s_\circ,s}(\vec z) = U_{s_\circ,s}^{-1}(\vec z)\,\exp(\tfrac12\,\mathrm{i}\pi\,X_{s_\circ,s})\,. \tag{D.15}$$

Now according to our prescription (5.7), at each individual point $\vec z$ in the moduli space the inner automorphism $\mathrm{Ad}_{\mathrm{i}\pi/2;X_{s_\circ,s},Y_s}$ of the block algebra $\bar{\mathfrak g}\otimes\mathcal F$ is implemented by

$$\tilde\Theta_{\bar\alpha_s^\vee;s_\circ}(\vec z) = \exp(-\tfrac12\,\mathrm{i}\pi\,Y_s)\,\exp(\tfrac12\,\mathrm{i}\pi\,X_{s_\circ,s})\,. \tag{D.16}$$

Combining the results (D.3), (D.12) and (D.15), we learn that this twisted intertwiner satisfies

$$\begin{aligned}\tilde\Theta_{\bar\alpha_s^\vee;s_\circ}(\vec z^{[0]}) &= \exp(-\tfrac12\,\mathrm{i}\pi\,Y_s)\,\exp(\tfrac12\,\mathrm{i}\pi\,X_{s_\circ,s}^{[0]})\\ &= \exp(-\tfrac12\,\mathrm{i}\pi\,Y_s)\,U_{s_\circ,s}^{-1}(\vec z)\,\exp(\tfrac12\,\mathrm{i}\pi\,X_{s_\circ,s})\,U_{s_\circ,s}(\vec z)\\ &= U_{s_\circ,s}^2(\vec z)\,\exp(-\tfrac12\,\mathrm{i}\pi\,Y_s)\,\exp(\tfrac12\,\mathrm{i}\pi\,X_{s_\circ,s}) = U_{s_\circ,s}^2(\vec z)\,\tilde\Theta_{\bar\alpha_s^\vee;s_\circ}(\vec z)\,.\end{aligned} \tag{D.17}$$

We now return to the general case where the components $\vec{\vec\mu}$ are arbitrary elements of the coroot lattice. We note that because of

$$\mathcal{R}es\big(\mathrm{d}g_{s_\circ,s}\,g_{s_\circ,s'}\big) = 0 \tag{D.18}$$

for all values of s,s' any two operators of the form (D.4) commute (when taken at the same point $\vec z$ in the moduli space $\mathcal M_m$ of course). Moreover, according to (4.8), up to central terms $U_{s_\circ,s}(\vec z)$ also commutes with the twisted intertwiner $\tilde\Theta_{\bar\beta_{s'}^\vee;s_\circ}(\vec z)$ for $s'\neq s$ and arbitrary coroots $\bar\beta^\vee\equiv\bar\beta_{s'}^\vee$; more precisely, using the identity (D.14) we find

$$\begin{aligned}\tilde\Theta_{\bar\beta_{s'}^\vee;s_\circ}(\vec z)\,U_{s_\circ,s}(\vec z) &= U_{s_\circ,s}(\vec z)\,\tilde\Theta_{\bar\beta_{s'}^\vee;s_\circ}(\vec z)\,\cdot\exp\Big[-\tfrac12\,(\bar\beta_{s'}^\vee,(\bar\alpha_s^\vee))\,K\,\mathcal{R}es(\varphi_{s_\circ,s'}\,g_{s_\circ,s})\Big]\\ &= U_{s_\circ,s}(\vec z)\,\tilde\Theta_{\bar\beta_{s'}^\vee;s_\circ}(\vec z)\,\cdot\exp\Big[-\tfrac12\,(\bar\beta_{s'}^\vee,(\bar\alpha_s^\vee))\,\delta_{s_\circ,s'}\,K\,\ln\tfrac{z_{s_\circ}-z_s}{z_{s_\circ}^{[0]}-z_s^{[0]}}\Big]\,.\end{aligned} \tag{D.19}$$

The final expression will involve the square of the exponential with the central element, because it is $U^2_{s_\circ,s}(\vec{z})$ that we commute to the left. Moreover, taking into account the chosen ordering of the factors in $\tilde\Theta_{\vec\mu;s_\circ}$, such a term will appear if and only if $s > s_\circ$. Thus defining $\tilde\Theta_{\vec\mu;s_\circ}$ according to the prescription (5.7), i.e. as

$$\tilde\Theta_{\vec\mu;s_\circ}(\vec z) := \prod_s^{\rightarrow} \Big(\prod_{i_s}^{\rightarrow} \exp(-\tfrac12\, \mathrm{i}\pi\, Y_{\alpha_{i_s};s_\circ})\, \exp(\tfrac12\, \mathrm{i}\pi\, X_{\alpha_{i_s};s,s_\circ}) \Big) \tag{D.20}$$

when $\bar\mu_s = \sum_{i_s} \bar\alpha^\vee_{i_s}$ for $s = 1,2,\dots,m{-}1$, the equation (D.17) generalizes to

$$\tilde\Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) = U^2_{s_\circ;\vec\mu}(\vec z)\, \tilde\Theta_{\vec\mu;s_\circ}(\vec z) \cdot \exp\Big[-\sum_{\substack{s=1\\ s>s_\circ}}^{m-1} (\bar\mu_{s_\circ},\bar\mu_s)\, K\, \ln \frac{z_{s_\circ}-z_s}{z^{[0]}_{s_\circ}-z^{[0]}_s} \Big] \tag{D.21}$$

with

$$U_{s_\circ;\vec\mu}(\vec z) := \exp\Big(-\sum_{s=1}^{m-1} \tfrac12\, (\bar\mu_s, H) \otimes g_{s_\circ,s}(\vec z) \Big)\,. \tag{D.22}$$

Using the invariance of the exponent under exchange of s with $s_\circ$, (D.21) may also be rewritten as

$$\tilde\Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) = U^2_{s_\circ;\vec\mu}(\vec z)\, \tilde\Theta_{\vec\mu;s_\circ}(\vec z) \cdot \exp\Big[-\tfrac12 \sum_{\substack{s=1\\ s\neq s_\circ}}^{m-1} (\bar\mu_{s_\circ},\bar\mu_s)\, K\, \ln \frac{z_{s_\circ}-z_s}{z^{[0]}_{s_\circ}-z^{[0]}_s} \Big]\,. \tag{D.23}$$

We are now finally in a position to make contact to the relation (6.21) that corresponds to the compatibility with the Knizhnik–Zamolodchikov connection. First, comparison of our results with the definition (6.34) [14] (see also (6.22) and (6.26)), tells us that we may rewrite (D.23) in the form of (6.41), i.e.

$$\tilde\Theta_{\vec\mu;s_\circ}(\vec z) = A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]})\, \tilde\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})\,. \tag{D.24}$$

Moreover, we already know that the implementation is consistent at the specific point $\vec z^{[0]}$, i.e. we have $\tilde\Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) = \Theta_{\vec\mu;s_\circ}(\vec z^{[0]})$. Together with the relation (6.21) between $\Theta_{\vec\mu;s_\circ}(\vec z)$ and $\Theta_{\vec\mu;s_\circ}(\vec z^{[0]})$ the result (D.24) therefore implies that in fact

$$\tilde\Theta_{\vec\mu;s_\circ}(\vec z) = A_{\vec\mu;s_\circ}(\vec z;\vec z^{[0]})\, \Theta_{\vec\mu;s_\circ}(\vec z^{[0]}) = \Theta_{\vec\mu;s_\circ}(\vec z) \tag{D.25}$$

at every point $\vec z$ in the moduli space $\mathcal{M}$ and hence that, as claimed, the implementation is consistent on all of $\mathcal{M}$.

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[14] The summation over s in (6.34) now extends only up to $m{-}1$, owing to our decision to keep one puncture at $z_m = \infty$ when dealing with inner multi-shift automorphisms.

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