The Influence of Higher Fock States in Light-Cone Gauge Theories

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In the light-cone Fock state expansion of gauge theories, the influence of non-valence states may be significant in precision non-perturbative calculations. In two-dimensional gauge theories, it is shown how these states modify the behaviour of the light-cone wavefunction in significant ways relative to endemic choices of variational ansatz. Similar effects in four-dimensional gauge theories are briefly discussed.

I. INTRODUCTION

Light-cone quantisation is believed to be an especially efficient hamiltonian method for dealing with many-body relativistic field theory [1]. One of the tenets of this method is that the valence approximation to a system of light-cone partons provides a much better approximation than the analogous Tamm-Dancoff truncation in other quantisation schemes. This chiefly follows from the peculiar inverse relation between light-cone momentum and free energy:

$$k^- = \frac{m^2 + \mathbf{k}^2}{2k^+} \ . \qquad (1.1)$$

Here, $k^\pm = (k^0 \pm k^3)/\sqrt{2}$ are the light-cone momentum and energy, $\mathbf{k} = (k^1, k^2)$ the transverse momenta, and m the mass of a parton. Since $k^+ > 0$ and is conserved, to create more partons costs significant energy because the new particles carry smaller k^+ than their parent(s). Moreover, one might naively expect wavefunctions to vanish at small k^+ to ensure finite energy. In this report it will be shown how non-valence states nevertheless modify the behaviour of light-cone wavefunctions, *including the valence sector*, when one or more parton momenta are small. Rather than wavefunctions simply vanishing at the boundaries of phase space, the requirements of finite energy in general lead to cancelations between the valence and non-valence (sea) components of boundstates. This phenomena was recently analysed for QCD in ref. [2], where it was shown to be responsible, at least perturbatively, for the rising Regge behaviour of hadronic structure functions at small momentum fraction x. In this work some implications for non-perturbative calculations will be demonstrated. The relevant effects may prove significant in precision calculations with variational bases. The basic physical ideas can be illustrated by two-dimensional gauge theories. Implications for four-dimensional gauge theories will be discussed later.

II. TWO-DIMENSIONAL GAUGE THEORIES

The results to be discussed in this section will in fact be true for any two-dimensional gauge theory exhibiting pair production. Rather than attempting to cover all particular cases at once, we proceed by using familiar examples. The generalisation will easily be seen to follow from the simple structure of two-dimensional gauge theories. Therefore, consider the standard two-dimensional gauge theory with fermion fields Ψ and action

$$S = \int dx^0 dx^1 \ \mathrm{i}\overline{\Psi}\gamma_\alpha D^\alpha \Psi - m\overline{\Psi}\Psi - \frac{1}{4e^2}F_{\alpha\beta}F^{\alpha\beta} \ , \quad (2.1)$$

where representation and colour indices are suppressed for the moment, for generality. Using a chiral representation of the two-dimensional γ matrices in the light-cone gauge $A_- = (A_0 - A_1)/\sqrt{2} = 0$, we are left with $A_+ = (A_0 + A_1)/\sqrt{2}$ and the left-moving ψ_L and right-moving ψ_R components of Ψ. A_+ and ψ_L are found to be constrained variables with respect to light-cone time $x^+ = (x^0 + x^1)/\sqrt{2}$, and can be eliminated by their equations of motion to yield a light-cone hamiltonian

$$P^- = \int dx^- \ \frac{m^2}{2}\psi_R^* \frac{1}{\mathrm{i}\partial_-}\psi_R + \frac{e^2}{2}J^+ \frac{1}{(\mathrm{i}\partial_-)^2}J^+ \ . \quad (2.2)$$

Here $x^- = (x^0 - x^1)/\sqrt{2}$ and J^+ is the right-moving current $J^+ = \psi_R \psi_R^*$. (The same form of hamiltonian results for bosons ϕ, with an appropriate current $J^+ = \mathrm{i}\phi^* \overleftrightarrow{\partial}_- \phi$.) These theories generically exhibit linear confinement, since the Coulomb potential is linear in one space dimension. The hamiltonian (2.2) can be diagonalised in the light-cone Fock space at fixed x^+, constructed from the Fourier modes of ψ_R,

$$\psi_R(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \ \left(b(k)\mathrm{e}^{-\mathrm{i}kx^-} + d^\dagger(k)\mathrm{e}^{\mathrm{i}kx^-} \right) \ . \qquad (2.3)$$

A (non-baryonic) eigenfunction of P^- with total momentum P^+ is of the form

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\begin{aligned}
|\Psi(P^+)\rangle = &\int_0^{P^+} dk_1 dk_2 \ \delta(P^+ - k_1 - k_2) \\
& \times \ f_2(k_1, k_2)b^\dagger(k_1)d^\dagger(k_2)|0\rangle \\
& + \frac{1}{2}\int_0^{P^+} dk_1 dk_2 dk_3 dk_4 \ \delta\left(P^+ - \sum_{i=1}^4 k_i\right) \\
& \times \ f_4(k_1, k_2, k_3, k_4)b^\dagger(k_1)b^\dagger(k_2)d^\dagger(k_3)d^\dagger(k_4)|0\rangle \\
& + \cdots \ .
\end{aligned}
\qquad (2.4)
$$

Ellipses indicate higher Fock states f_n, $n > 4$.

Projecting $2P^+P^-|\Psi(P^+)\rangle = M^2|\Psi(P^+)\rangle$ onto specific Fock states, where the eigenvalue M^2 is the invariant mass squared of the system, one derives coupled integral equations for the wavefunction components f_n. Let us consider the familiar example of two-dimensional QED (massive Schwinger Model at $\theta = 0$ [1]), which in the approximation that retains only f_2 and f_4 reads [3,4]

$$M^2 f_2(x, 1-x) = \frac{m^2}{x(1-x)} f_2(x, 1-x) \tag{2.5}$$

$$+\frac{e^2}{\pi} \int_0^1 dy \frac{f_2(x, 1-x) - f_2(y, 1-y)}{(x-y)^2} \tag{2.6}$$

$$+\frac{e^2}{\pi} \int_0^1 dy f_2(y, 1-y) \tag{2.7}$$

$$+\frac{e^2}{\pi} \int_0^1 dy_1 dy_2 dy_3 \delta(y_1 + y_2 + y_3 - x)$$
$$\times \frac{f_4(y_1, y_2, y_3, 1-x)}{(x-y_1)^2} \tag{2.8}$$

$$-\frac{e^2}{\pi} \int_0^1 dy_2 dy_3 dy_4 \delta(y_2 + y_3 + y_4 - (1-x))$$
$$\times \frac{f_4(x, y_2, y_3, y_4)}{(1-x-y_4)^2} \,, \tag{2.9}$$

$$M^2 f_4(x_1, x_2, x_3, x_4) = m^2 \sum_{i=1}^4 \frac{1}{x_i} f_4(x_1, x_2, x_3, x_4)$$
$$+\frac{e^2}{2\pi} A(x_1, x_2, x_3)$$
$$+\frac{e^2}{\pi} \int_0^1 dy_1 dy_2 \left\{ \delta(y_1 + y_2 - (x_1 + x_2)) \right.$$
$$\times \frac{f_4(y_1, y_2, x_3, x_4) - f_4(x_1, x_2, x_3, x_4)}{(x_1 - y_1)^2}$$
$$\left. + \text{Similar} \right\} \,, \tag{2.10}$$

where and $x_i = k_i/P^+$, $x_4 = 1 - x_1 - x_2 - x_3$ and

$$A(x_1, x_2, x_3) = \frac{f_2(1-x_3, x_3) - f_2(x_2, 1-x_2)}{(1-x_2-x_3)^2}$$
$$+\frac{f_2(x_1 + x_2 + x_3, 1-(x_1+x_2+x_3)) - f_2(x_1, 1-x_1)}{(x_2 + x_3)^2}$$
$$+\frac{f_2(x_2, 1-x_2) - f_2(x_1+x_2+x_3, 1-(x_1+x_2+x_3))}{(x_1 + x_3)^2}$$
$$+\frac{f_2(x_1, 1-x_1) - f_2(1-x_3, x_3)}{(1-x_1-x_3)^2} \,. \tag{2.11}$$

The many 'similar' contributions to the last term in (2.10) are enumerated in ref. [4], but are unimportant

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[1]Some observations related to those made in this paper have recently been given in connection with the θ-dependence of the massive Schwinger model [5]

for our purposes. At this point it is important to stress the distinction between the component f_2 to the solution of such equations in the valence approximation, which sets $f_n = 0$ for $n > 2$, and the component f_2 to the solution when some or all of the higher Fock states f_n are retained. As is shown below, these have different high-energy behaviour in general.

Like any wavefunction, the components f_n are subject to high energy boundary conditions. From (1.1), this is a small light-cone momentum boundary. A typical variational ansatz used for the solution of the light-cone boundstate integral equations is

$$f_n(x_1, \ldots, x_n) \sim x_1^\beta x_2^\beta \cdots x_n^\beta \cdot P(x_1, \cdots, x_n) \tag{2.12}$$

where $x_n = (1 - \sum_{i=1}^{n-1} x_i)$ and P is polynomial. The safest procedure is to leave $\beta > 0$ as a variational parameter. However, often the following formula is used

$$m^2 = e^2(1 - \pi\beta \cot \pi\beta). \tag{2.13}$$

This choice is motivated by a study of the high-energy 'endpoint' behaviour of $f_n(x_1, \ldots, x_n)$ as one of the arguments vanishes, $x_i \to 0$. Ignoring terms resulting from a change in the number of partons (pair production (2.8)(2.9)), one deduces that for finite M^2, as $x \to 0$ the divergent behaviour of the mass (2.5) and Coulomb terms (2.6) must cancel;

$$\lim_{x\to 0} \left\{ \frac{m^2}{x} f_2(x, 1-x) + \frac{e^2}{\pi} \int_0^1 dy \frac{f_2(x, 1-x) - f_2(y, 1-y)}{(x-y)^2} \right\}$$
$$= \text{finite} \tag{2.14}$$

(The Coulomb term is dominated by the region of asymptotically small momentum transfer $y \sim x$). This depends only upon the endpoint behaviour of f_2 and leads to (2.13), as was first shown by t' Hooft in large-N_c two-dimensional QCD [6] (for which only f_2 exists). The same endpoint behaviour can be deduced under similar assumptions for any f_n when one of its arguments vanishes. The usual justification given for ignoring pair production (see e.g. [8]) is based on the assumption that f_n does not diverge if three or more momenta vanish. In this case, it is easy to see from (2.8)(2.9) by power counting that pair production would be subleading as $x \to 0$.

In fact, the ansatz (2.12) with (2.13) fails in general to account correctly for the behaviour of f_n when one momentum vanishes, except for f_2 in the valence approximation. Furthermore, (2.12) fails to account for the correct behaviour as two or more momenta vanish in f_n. This follows by examining the behaviour of eqs.(2.5)-(2.10) when two or more momenta vanish. To illustrate these points, let us perform a one-loop calculation in e/m. In this case, it is easy to see that to leading order in e/m, the solution to the four-parton equation (2.10) is

$$f_4(x_1, x_2, x_3, x_4) = \frac{e^2}{2\pi} \frac{A(x_1, x_2, x_3)}{M^2 - m^2 \sum_{i=1}^4 \frac{1}{x_i}} \tag{2.15}$$

(Note that $M^2 \sim O(m^2)$.) This shows that as two momenta vanish, say x_1 and x_3, such that $\lim_{z\to 0} x_1, x_3 \propto z$, then f_4 is non-zero and finite, which is incompatible with (2.12). Furthermore, as three momenta vanish, say $\lim_{z\to 0} x_1, x_2, x_3 \propto z$, then f_4 diverges as $z^{\beta-1}$ if $f_2(z, 1-z) \sim z^{\beta}$, which is also incompatible with (2.12). Therefore, rather than vanishing at the corners of phase space, we find relations between wavefunction components in each Fock sector, showing in particular that the valence and the sea are not analytically independent. Substituting f_4 (2.15) into the two-parton equation (2.5)-(2.9), one obtains an effective valence equation. In the limit of small x the singular part of this equation reads

$$\lim_{x\to 0} \left\{ \frac{e^2}{\pi} \int_0^1 dy \frac{f_2(x,1-x) - f_2(y,1-y)}{(x-y)^2} \left(1 - \frac{e^2}{2m^2\pi} I(x/y)\right) \right.$$
$$\left. + \frac{m^2}{x} f_2(x, 1-x) \right\} = \text{finite} \qquad (2.16)$$

$$I(w) = \int_0^1 dv \left\{ \frac{1}{\frac{1}{v(1-v)} + w - 1} \left(1 - \frac{(1-w)^2}{(1-v(1-w))^2}\right) \right.$$
$$\left. + \frac{1}{\left(1 + \frac{w}{v(1-w)}\right)^2 \left(w + \frac{1}{1-v}\right)} \right\} \qquad (2.17)$$

Comparing with (2.14), this demonstrates that the neglect of pair production is unjustified in the determination of β, and that eq.(2.13) will not be correct in general.

To summarize, the ansatz (2.12) with (2.13) correctly accounts for the behaviour of $f_n(x_1, \ldots x_n)$ as one $x_i \to 0$ only in the approximation which sets $f_m = 0$ for $m > n$. More generally β will be renormalised from the value given by (2.13). Eq.(2.12) fails to account for the behaviour as two or more momenta vanish.

First order e/m perturbation theory and a 4-particle Tamm-Dancoff Fock-space truncation were used above, but it is easy to check that the phenomenon is not special to these choices. It follows by power counting in the boundstate integral equations when various momenta x_i are made to vanish, i.e. one does not need to solve the equations to reach the conclusions. Moreover, it is also easy to check that since it only depends upon the current-current structure of interactions given in eq.(2.2), it will occur in *any two-dimensional gauge theory* with pair production; in particular, for fundamental representations at finite N_c [10] and adjoint representations [11]. In the massive Schwinger model, typically a one percent error in β can lead to a similar size error in M^2 estimated from variational ansatz, which is significant at the level of precision of current computations [4,12]. The error is small in this case because the valence approximation becomes exact in the large and small m limits. In cases where the latter is not true, such as boson or Majorana fermion representations, the effects can be somewhat larger, especially for small m.

The algebraic manipulations have a simple underlying physical explanation. Higher Fock states renormalise the Coulomb coupling at asymptotically small momentum transfer through vacuum polarisation and vertex renormalisation. In general, the parton mass m^2 may also be renormalised. These (sector-dependent) renormalisations are well-known in the Tamm-Dancoff treatment of light-cone boundstate integral equations [9]. However, it does not seem to have been generally recognised that formulae such as (2.13) are subject to renormalisation as a result. Although a suitable variational ansatz may just include (renormalised) β as one of the variational parameters, an ansatz which correctly incorporates the behaviour as two or more momenta vanish is not so simple.

A significant effect of these renormalisations might be expected in theories which exhibit complete screening in some limit. This is the case for $m \to 0$ in two-dimensional QED, for example. In this limit the string tension is known to vanish as $\sigma \sim em$ [7].[2] Pair production effects now completely screen the linear potential as $m \to 0$, resulting in the Higgs mechanism and the production of Schwinger's massive photon in the physical spectrum. If we assume a similar renormalisation, in the boundstate integral equations, of the Coulomb coupling at asymptotically small momentum transfer $e^2 \to e_R^2 \sim em$ (this is not entirely justified if the sources are not heavy), one naively finds from (2.13) that $\beta \sim \sqrt{m}$ as $m \to 0$, rather than $\beta \sim m$. In any case, since it is e_R which is relevant to the determination of β, rather than e, this emphasizes the need to properly account for renormalisation effects due to higher Fock states, before attempting to accurately solve light-cone gauge theories by variational ansatz. It may have some bearing on the small discrepancies in results found between bosonization and light-cone quantisation of two-dimensional QED [12].

III. FOUR-DIMENSIONAL GAUGE THEORIES.

The considerations above are relevant in QCD to the behaviour of partons carrying small light-cone momentum fraction x of the hadron. Let us first consider nonperturbative formulations. The transverse lattice colour-dielectric formulation of pure QCD is conceptually similar to two-dimensional gauge theories with adjoint boson representations [13]. This is especially true in the large-N_c limit, when an *exact* dimensional reduction to

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[2]This result was established by bosonization. In principle there can be a renormalisation of the parameter m/e in the bosonization approach relative to a fermionic basis, which prevents direct comparison with light-cone results in such a basis. To the author's knowledge, the result has not been established in light-cone quantisation in the fermionic basis independently of the groundstate mass M obtained from variational argument using (2.13).

two-dimensional gauge theory takes place [14]. Use of variational wavefunctions in some shape or form seems unavoidable to obtain satisfactory numerical convergence with this method. The job of implementing the subtleties described in the previous section for higher Fock states looks quite formidable. However, a hybrid method introduced in ref. [14], which combines a variational approach with discretized light-cone quantisation (DLCQ) [15], offers a compromise. In this method, the parton momentum fractions $x_i = t_i/K$ are artificially discretized into integers t_i, such that $\sum_{i=1}^{n} t_i = K$. When computing matrix elements of P^- in Fock space, spectator partons retain their discrete momentum t_i, while partons involved in an elementary interaction have their momenta projected onto a continous-momentum wavefunction basis (see Appendix C of ref. [14] for details), for evaluation by variational ansatz. For a theory with elementary interactions of order $\leq q$ in the number of fields, no more than q partons are involved in such interactions in initial plus final states. This means that that an appropriate variational basis need only be found for a few partons, however many higher Fock states are included in the calculations. We are currently incorporating the correct behaviour of variational wavefunctions, discussed in the last section, into our calculations on the transverse lattice [16]. Other variational approaches to light-cone QCD break gauge invariance explicitly by giving large constituent masses to quarks and gluons [17]. It is hoped that higher Fock states are strongly suppressed as a result and that perturbative renormalisation theory can be performed.

Finally, it should be emphasized that the singular behaviour of light-cone wavefunctions, when two or more light-cone momentum fractions x_i vanish, has observable implications for the small Bjorken-x region of hadronic structure functions. This behaviour is determined from the high-energy boundary condition on light-cone wavefunctions in QCD [2], which can be easily solved perturbatively and again relates Fock space sectors with differing numbers of partons. The definition of the distribution function for a parton of momentum fraction x_1 is

$$Q(x_1) = \sum_n \int [d\mathbf{k}] \int dx_2 \cdots dx_n \delta \left(1 - \sum_{i=1}^{n} x_i \right)$$
$$\times |f_n(x_1, x_2, \ldots, x_n)|^2 . \quad (3.1)$$

$[d\mathbf{k}]$ indicates the integration over all parton transverse momenta upon which f_n also depends (this is absent in two dimensions). Generalising the analysis of the previous section to all n, one finds that the wavefunctions of two-dimensional gauge theories diverge slower than $x^{-(s-2)}$ when $s > 2$ light-cone momenta are of order $x \to 0$, and are otherwise finite. By power counting, this means that they are not singular enough to prevent $Q(x \to 0)$ from vanishing in two dimensions. In four-dimensional gauge theories however, it is possible (via the quark–gluon–anti-quark vertex for example) for f_n to diverge already when only two parton momenta are

vanishing, leading to $\log x$ contributions to $Q(x \to 0)$ [2,18]. The accumulation of such logarithms is thought to lead to the power law rise (Regge behaviour) exhibited by hadronic structure functions at small x.

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