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LONGITUDINAL INSTABILITIES OF CHARGED PARTICLE BEAMS INSIDE CYLINDRICAL WALLS OF FINITE THICKNESS

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The stability of coherent longitudinal oscillations of coasting particle beams in accelerators or storage rings can be described in terms of the dispersion relation coefficients which are a measure of the force exerted by an oscillating beam upon itself. These coefficients are calculated with the help of an algorithm valid for beams of circular cross-section inside a concentric vacuum chamber consisting of an arbitrary number of layers of various materials. Two cases of practical importance are discussed in more detail: (1) the metal wall of finite thickness, for which the dispersion relation coefficients at all frequencies are found to be less than or of the same order of magnitude as those for an infinite thick wall at the lowest mode number, and (2) the ceramic wall, where dielectric resonances may increase the dispersion relation coefficients by several orders of magnitude at certain frequencies, for which the wall thickness is approximately an odd multiple of a quarter of a wavelength in the dielectric. These resonances can be damped effectively by thin layers of metal deposited on the inside of the ceramic.

INTRODUCTION

The object of this investigation is to study the stability of coasting charged particle beams oscillating longitudinally inside a vacuum chamber. The material properties of a surrounding vacuum chamber have a strong influence on the fields exited by the beam, and, for example, metallic walls have only been investigated for the case where they are thick compared to the skindepth.(1) In this paper, we present an algorithm that allows the calculation of the fields for walls consisting of an arbitrary number of layers of arbitrary material properties. This algorithm is then applied to study the effect of thin metal walls and of ceramic walls in more detail.

We investigate only beams of circular cross section inside a concentric vacuum chamber. Since the radius of curvature of a particle accelerator or storage ring is usually very large compared to the transverse dimensions of the vacuum chamber, the curvature is neglected and the actual beam and chamber are replaced by infinitely long, straight cylinders. The cyclic nature of the machine then is taken into account by the requirement that all relevant quantities must be periodic with the circumference of the machine.

For simplicity, we further assume that the charge density and velocity of the beam are uniform over its cross section. The study is limited to small amplitudes, which restriction allows linearization of all equations in first order.

The oscillations of charged particles generate electric and magnetic fields, which in turn react on the motion of the particles and change their oscillation parameters. The interdependence of wavelength and frequency of an oscillating system is in general given by the dispersion relation. Since the wavelength is determined by the periodicity condition, we can thereby calculate the frequency. It will turn out to be in general a complex number, where the imaginary part corresponds to a growing or decaying wave (depending on its sign).

The dispersion relation may be written as a product of two parts, where one depends mainly on beam and machine parameters, and the other on the force acting on the particles. This part is conveniently expressed in terms of the dispersion relation coefficients U and V, which are proportional to the real and imaginary part of the average of the force over the beam cross section (the original definition(1) was in terms of the force on the beam axis, but the average seems more appropriate for oscillations of the beam as a whole—although the difference is usually of minor importance).

For given beam and machine parameters, we can express the stability limit as upper limits of the dispersion relation coefficients, for which the solution of the dispersion relation is still entirely real.

For the convenience of the reader we repeat below the dispersion relation and the definition of the coefficients U and V originally given in Ref. (1). In terms of the normalized particle distribution function f_0, for which

$$\int f_0(W)\,\mathrm{d}W = 1$$

the dispersion relation can be written

$$-1 = (U - iV) \int \frac{\mathrm{d}f_0}{\mathrm{d}W} \frac{\mathrm{d}W}{\omega - n\dot{\theta}}$$

The momentum W is defined as

$$W = 2\pi \int \frac{\mathrm{d}E}{\dot{\theta}(E)}$$

where E is the total energy and $\dot{\theta}$ the angular velocity of a particle. The coefficients U and V are given by

$$U - iV = \frac{iNe^2}{\pi a^2 \rho} \langle E_z \rangle$$

where N is the total number of particles of charge e, a the beam radius, and ρ the charge density. $\langle E_z \rangle$ is the average of the longitudinal component of the electric field strength over the beam cross section.

The main part of the paper consists of the calculation of E_z, which can be found by solving the wave equation with appropriate boundary conditions.

In the algorithm presented in section 1 and derived in the appendix, the electric field and the dispersion relation coefficients are expressed in terms of a normalized wave impedance Y_3 at the inner edge of the vacuum chamber. This dimensionless quantity is generally defined at the inner edge of an arbitrary region with index p by

$$Y_p = \frac{i\omega\epsilon_p'}{\nu_p} \frac{E_z}{H_\varphi} \bigg|_{r=a_{p-1}}$$

where ϵ_p' and ν_p are the (complex) dielectric constant and the radial propagation constant in that region. However, the advantage of the algorithm lies just in the fact that one does not need to calculate the fields in every region, but that one can express Y_p in terms of the quantity Y_q at the outer edge of the vacuum chamber. The quantity Y_q is given there by simple expressions depending only on the material surrounding the chamber, which may be either air to infinity, a perfect conductor or a perfect magnet.

In the derivation of the approximate formulas for U and V valid in the long wavelength limit, we further utilize the dimensionless quantity L which is simply related to the wave impedance at the inner edge of the vacuum chamber by the equation

$$L = \frac{\nu_3}{\nu_2 \epsilon_3'} Y_3$$

where $\nu_2 = k/\gamma$ is the radial propagation constant in vacuum ($\epsilon_2' = 1$). The results of the approximations for thin metal and for ceramic walls are summarized in Tables I and II.

1. ALGORITHM FOR THE CALCULATION OF THE DISPERSION RELATION COEFFICIENTS

Since we have assumed that all relevant equations may be linearized in the small amplitude approximation, it is sufficient to study perturbations at a single frequency. Any arbitrary perturbation may then be expressed from single frequency solutions with the help of the superposition principle.

A perturbation of circular frequency ω travels along the beam with wavenumber k, which may be expressed in complex notation by the factor $\exp i(kz - \omega t)$ (this factor is not explicitly written, but should be understood for all perturbed quantities). The periodicity condition requires that the circumference of the machine must be a multiple of the wavelength, which yields for the wave number

$$k = n/R \tag{1.1}$$

where 'n' is an arbitrary integer usually called the mode number, and R is the equivalent radius of the machine. The phase velocity of the wave is given by

$$\beta c = \omega/k \tag{1.2}$$

while the unperturbed particle velocity is related to the revolution frequency Ω by

$$\beta_0 c = \Omega R \tag{1.3}$$

When the wave velocity is equal to the beam velocity, we may combine the last three equations to find

$$\omega = n\Omega \tag{1.4}$$

In reality, the wave velocity differs slightly from the beam velocity, and the departure of the actual oscillation frequency from the value given by Eq. (1.4) is referred to as frequency shift. It is in general complex, corresponding to growing (or decaying) waves on the beam.

In analogy to the energy factors of the particles

$$\gamma_0 = (1 - \beta_0^2)^{-1/2} \tag{1.5}$$

we shall define the quantity

$$\gamma = (1 - \beta^2)^{-1/2} \tag{1.6}$$

Both β and γ are in general complex quantities when ω is complex.

The dispersion relation coefficients for a circular cylindrical beam in a concentric vacuum chamber of arbitrary many layers of various material properties then can be found by a simple algorithm,[2] which is derived in the Appendix. In order to perform the indicated calculations, we need to specify the following quantities for each region (including the beam and the vacuum between beam and wall):

a_p	outer radius
ϵ_p	relative permittivity
$\tan \delta_p$	electric loss tangent
σ_p	conductivity
μ_p	relative permeability
$\tan \delta'_p$	magnetic loss tangent

Further we require only the total number of particles N, their charge e, and some universal constants such as the light velocity c and the free space permittivity ϵ_0.

From the material constants, we first calculate the complex dielectric constant and permeability. For nonconducting media we find

$$\left.\begin{aligned} \epsilon'_p &= \epsilon_p(1 + i \tan \delta_p) \\ \mu'_p &= \mu_p(1 + i \tan \delta)'_p \end{aligned}\right\} \tag{1.7}$$

while for conductors the first of these equations should be replaced by

$$\epsilon'_p = \epsilon_p + \frac{i\sigma_p}{\omega\epsilon_0} \tag{1.8}$$

The radial propagation constants ν_p in each region are then given by

$$\nu_p = k(1 - \beta^2\epsilon'_p\mu'_p)^{1/2} \tag{1.9}$$

For consistency, we shall always choose the sign of the square root in such a way that the real part of ν_p is positive. In the beam and vacuum regions, where $\epsilon' = \mu' = 1$, we get

$$\nu_1 = \nu_2 = k(1 - \beta^2)^{1/2} = \frac{k}{\gamma} \equiv \tau \tag{1.10}$$

The algorithm now consists of the calculation of a single (complex) quantity Y_p for each region which is proportional to the wave impedance at the inner surface. If the outermost region $(p = q)$ extends to infinity, we have

$$Y_q = -\frac{K_0}{K_1}\Big| \nu_q a_q - 1 \tag{1.11}$$

The problem is also defined mathematically if the outermost region extends to a perfect conductor or a perfect magnet. In these cases we must replace the right-hand side of Eq. (1.11) by R_q/R'_q or S_q/S'_q, respectively (the symbols are explained below). We can then calculate the values of Y_p in consecutive regions from

$$Y_p = \frac{X_{p+1}R_p - Y_{p+1}S_p}{X_{p+1}R'_p - Y_{p+1}S'_p} \tag{1.12}$$

We continue this until $p = 3$, and then get the dispersion relation coefficients from

$$U - iV = \frac{Ne^2}{\pi\epsilon_0 k a_1^2} \frac{X_3 N_0 - Y_3 N_1}{(X_3 I_0 - Y_3 I_1)\,\tau a_2} \tag{1.13}$$

The quantities X_p used in these equations are simply abbreviations defined by

$$X_p = \frac{\epsilon'_p \nu_{p-1}}{\epsilon'_{p-1}\nu_p} \tag{1.14}$$

The quantities R_p and S_p, as well as R'_p and S'_p, are superpositions of the modified Bessel functions $I_{0,1}$ and $K_{0,1}$ of general complex argument

$$\begin{aligned} R_p^{(\prime)} &= K_0(\nu_p a_p) I_{0,1}(\nu_p a_{p-1}) \pm I_0(\nu_p a_p) K_{0,1}(\nu_p a_{p-1}) \\ S_p^{(\prime)} &= -K_1(\nu_p a_p) I_{0,1}(\nu_p a_{p-1}) \pm I_1(\nu_p a_p) K_{0,1}(\nu_p a_{p-1}) \end{aligned} \tag{1.15}$$

Finally, the quantities N_0 and N_1 are given by

$$\begin{aligned} N_{0,1} = I_{0,1}(\tau a_2)[1 - 2I_1(\tau a_1)K_1(\tau a_1)] \\ \pm 2K_{0,1}(\tau a_2)I_1^2(\tau a_1) \end{aligned} \tag{1.16}$$

Equations (1.9) to (1.11) have been programmed for computer evaluation.

For not too complicated walls consisting of only a few layers, we can also use the algorithm to derive approximate formulae valid under certain restrictions.

2. SIMPLE WALLS OF FINITE THICKNESS

In this geometry we need consider only four regions: beam, vacuum, wall, and air. The configuration is simple enough to allow the derivation of explicit expressions for the dispersion relation coefficients, from which we may then derive approximations valid over limited ranges of interest.

We get from Eqs. (1.11) to (1.13), with $a_1 \equiv b$ and $a_2 \equiv a$

$$U - iV = F \frac{N_0 - N_1 L}{(I_0 - LI_1)_{\tau a}} \tag{2.1}$$

with

$$F = \frac{Ne^2}{\pi \epsilon_0 k b^2} \tag{2.2}$$

$$L = \frac{X_4 R_3 - Y S_3}{X_3 (X_4 R_3' - Y S_3')} \tag{2.3}$$

and

$$Y = -\frac{K_0}{K_1}\Big|_{\nu_4 a_3} \tag{2.4}$$

when the air extends to infinity. When a perfect conductor or magnet are at $r = a_4$, we replace Eq. (2.4) by R_4/R_4' or S_4/S_4'.

The quantity L is in general complex. When

$$ReL \approx \frac{I_0}{I_1}\Big|_{\tau a},$$

and $ImL \ll ReL$, the denominator of Eq. (2.1) may become quite small, and the dispersion relation coefficients have a maximum. This 'resonance condition' will be discussed in Sec. 4.

For low mode numbers, for which the wavelength of the oscillation is large compared to transverse dimensions of the vacuum chamber ('*long wavelength limit*') we have $ka \ll 1$. We actually only require $\tau a \ll 1$ to use the small argument approximations for the modified Bessel functions[3] of argument τa and τb. In first order N_0 vanishes, and we have only a term proportional L in the denominator. Since we know nothing yet about the value of L, we include second order terms to evaluate N_0 and get

$$U - iV = \frac{G}{\gamma^2} \frac{(\frac{1}{2} + 2 \ln \kappa) - (2/\tau a) L}{1 - (\tau a/2) L} \tag{2.5}$$

$$\text{with } G = \frac{Ne^2 k}{4\pi\epsilon_0} \tag{2.6}$$

and where $\kappa = a/b$ is the ratio of wall-to-beam radii. When $\tau a/2 \,|\, L \,| \ll 1$, we may simplify to

$$U - iV = \frac{G}{\gamma^2}\left[\left(\frac{1}{2} + 2 \ln \kappa\right) - \frac{2}{\tau a} L\right] \tag{2.7}$$

Since we have assumed that the outermost layer consists of air, with electrical properties close to those of vacuum†, we take $\epsilon_4' = \mu_4' = 1$, hence $\nu_4 = \tau$ and $X_4 = \nu_3/\tau\epsilon_3' = 1/X_3$.

We now distinguish two different cases. When $|\, \nu_3 a_3 \,| \ll 1$, we may use the small argument approximations also in Eq. (2.3) and find with $\eta = a_3/a_{(2)}$

$$R_3 = -\ln \eta \quad R_3' = \frac{1}{\nu a}$$
$$S_3 = -\frac{1}{\eta \nu a} \quad S_3' = \frac{\eta^2 - 1}{2\eta} \tag{2.8}$$

and hence

$$L = -\frac{(\nu^2 a/\tau\epsilon') \ln \eta - (Y/\eta)}{1 - Y\epsilon'\tau a(\eta^2 - 1/2\eta)} \tag{2.9}$$

The index is left off in $\epsilon_3' \equiv \epsilon'$ and $\nu_3 \equiv \nu$ since all other ϵ_p' and ν_p have been specified. In the other extreme, when $|\, \nu a \,| \gg 1$, we may use the asymptotic expansions of the modified Bessel functions to get with $d = a_3 - a_2$

$$R_3 = -S_3' = -\frac{\text{Sinh}\,(\nu d)}{\nu\sqrt{aa_3}}$$
$$S_3 = -R_3' = -\frac{\text{Cosh}\,(\nu d)}{\nu\sqrt{aa_3}} \tag{2.10}$$

and hence

$$L = -\frac{(\nu/\tau\epsilon')\,\text{Tanh}\,(\nu d) - Y}{1 - (\tau\epsilon'/\nu)\,Y\,\text{Tanh}\,(\nu d)} \tag{2.11}$$

When the wall thickness is not excessively large we may further assume that $\tau a_3 \ll 1$ in the long wavelength limit.

The small argument approximations then yield for air extending to infinity

$$Y = \tau a_3 \ln (\tau a_3) \tag{2.12}$$

and

$$Y = \begin{cases} -\tau a_3 \ln \xi \\ -\dfrac{2}{\tau a_3(\xi^2 - 1)} \end{cases} \tag{2.13}$$

for air extending to a perfect conductor, respectively magnet, at $a_4 = \xi a_3$.

For a more detailed investigation of the long wavelength limit of the dispersion relation coefficients we now have to specify the wall material.

3. METAL WALLS OF FINITE THICKNESS

The complex dielectric constant for a conducting medium is given by Eq. (1.8). However, the

† For high beam energies, the small deviation from unity in the permittivity of air ($\epsilon = 1.0006$) is sufficient to make ν_4 imaginary. From Eq. (1.13) we see that this happens for $\gamma > 40$. This effect is neglected here.

conductivity of practically all metals is so large that the permittivity is entirely negligible in the frequency range of interest, and we may write

$$\epsilon' = \frac{i\sigma}{\omega\epsilon_0} \tag{3.1}$$

Similarly, we may neglect a small term in Eq. (1.9) to get

$$\nu = \beta k \left(-i \frac{\mu\sigma}{\omega\epsilon_0} \right)^{1/2} \tag{3.2}$$

Introducing the skin depth

$$\delta = \left(\frac{2}{\omega\sigma\mu\mu_0} \right)^{1/2} \tag{3.3}$$

allows us to write the radial propagation constant as

$$\nu = \frac{1-i}{\delta} \tag{3.4}$$

We may further combine Eqs. (3.1) and (3.2) to express the dielectric constant by

$$\epsilon' = -\frac{1}{\mu}\left(\frac{\nu}{\beta k}\right)^2 \tag{3.5}$$

In the frequency range of interest, the skin depth is usually quite small compared to the transverse dimensions of the vacuum chamber. The assumption $|\nu a| = a/\delta \gg 1$ is well justified, and we may use Eq. (2.11) to evaluate L

$$L = \frac{\mu(\beta^2\gamma k/\nu)\,\mathrm{Tanh}\,(\nu d) + Y}{1 + (Y\nu/\mu\beta^2\gamma k)\,\mathrm{Tanh}\,(\nu d)} \tag{3.6}$$

The absolute value of the argument of the hyperbolic functions is $|\nu d| = d/\delta$. Depending on the ratio of wall thickness to skin depth (which in turn depends on frequency) we may use different approximations.

(a) *Metal Thickness Large Compared to Skin Depth*

For this case $|\nu d| \gg 1$, and therefore

$$\mathrm{Tanh}\,\nu d \approx 1.$$

Eq. (3.6) becomes

$$L = \frac{\mu(\beta^2\gamma k/\nu) - Y}{1 - (Y/\mu)(\nu/\beta^2\gamma k)} = \mu\frac{\beta^2\gamma k}{\nu} = \frac{1+i}{2}\mu\beta^2\gamma^2\tau\delta \tag{3.7}$$

We see that that $\tau a\,|L|$ is small compared to unity under these assumptions, and we may use Eq. (2.7) to find the dispersion relation coefficients.

Separation of real and imaginary parts results in

$$U = G\left[\frac{1}{\gamma^2}(\tfrac{1}{2} + 2\ln\kappa)\mu\beta^2\frac{\delta}{a}\right]$$
$$V = G\mu\beta^2\frac{\delta}{a} \tag{3.8}$$

in reasonable agreement with former results for the (infinitely) thick metal wall.[1] The small difference in the first term of U($\frac{1}{2}$ instead of 1) is due to the averaging of the force over the beam cross section, which was used here in the definition of the dispersion relation coefficients. The second term in U, due to the skin effect, was considered negligible in Ref. (1). However, for large values of γ, the skin effect contribution may be even larger than the first term, as is the case for the CERN Intersecting Storage Rings. We have further included the effect of the permeability, which shows that a vacuum chamber formed directly by polepieces with high μ may increase the dispersion relation coefficients considerably. The dependence of V (and of the skin effect term in U) is actually only proportional to $\sqrt{\mu}$, as δ decreases with $1/\sqrt{\mu}$. On the other hand, magnetic losses may increase V, which effect has not been taken into account here.

(b) *Very Thin Metal Walls*

For $|\nu d| \ll 1$, we may use Tanh $\nu d \approx \nu d$ and find

$$L = \frac{\mu\beta^2\gamma^2\tau\, d + Y}{1 - i(2Y\tau\, d/\mu(\beta k\,\delta)^2)} \tag{3.9}$$

where we may use the approximation (2.12) for Y. Assuming further that the second term in the denominator is small compared to unity, or

$$d \ll \frac{\mu\beta^2\gamma^2\delta^2}{2a\,|\ln\tau a|} \tag{3.10}$$

we find from Eq. (2.7)

$$U = G\left[\frac{1}{\gamma^2}(\tfrac{1}{2} + 2\,|\ln\tau b\,|) - 2\mu\beta^2\frac{d}{a}\right.$$
$$V = G\frac{4\,|\ln\tau a\,|}{\mu\beta^2\gamma^4}\frac{ad}{\delta^2}\left(|\ln\tau a\,| - \mu\beta^2\gamma^2\frac{d}{a}\right) \tag{3.11}$$

In the limit of $d \to 0$, V vanishes correctly (as there is no more lossy material) while U tends towards a finite limit due to the reactance of the beam in free space

$$\lim_{d\to 0} U = U_0 = \frac{G}{\gamma^2}(\tfrac{1}{2} + 2\,|\ln\tau b\,|) \tag{3.12}$$

For very small values of τb, the value of U_0 may be appreciably larger than the value of U for a thick metal wall.

TABLE I

Dispersion relation coefficients for thin metal walls (Long wavelength limit)

Air extending to	$U/G - A/\gamma^2$	V/G
(1) *infinity*		
(a) $d \gg \delta$	$-\beta^2 \frac{\delta}{a}$	$\beta^2 \frac{\delta}{a}$
(b) $d_1 \ll d \ll \delta$	$-\left(\frac{\gamma^2}{2 \ln \tau a}\right)\left(\frac{\beta^4 \delta^4}{a^2 d^2}\right)\left(1 + \frac{\beta^2 \gamma^2}{\ln \tau a}\frac{d}{a}\right)$	$\left(\frac{\beta^2 \delta^2}{ad}\right)\left(1 + \frac{\beta^2 \gamma^2}{\ln \tau a}\frac{d}{a}\right)$
(c) $d \ll d_1 = \frac{-\beta^2 \gamma^2 \delta^2}{2 \ln \tau a \cdot a}$	$-\frac{2 \ln \tau a}{\gamma^2} - 2\beta^2 \frac{d}{a}$	$\left(\frac{2 \ln \tau a}{\gamma^2}\right)\left(\frac{ad}{\beta^2 \delta^2}\right)$
(2) *perfect conductor*		
(a) $d \gg \delta$	$-\beta^2 \frac{\delta}{a}$	$\beta^2 \frac{\delta}{a}$
(b) $d_2 \ll d \ll \delta$	$\left(\frac{\gamma^2}{2 \ln \xi}\right)\left(\frac{\beta^4 \delta^4}{a^2 d^2}\right)\left. 1 - \frac{\beta^2 \gamma^2}{\ln \xi}\frac{d}{a}\right)$	$\left(\frac{\beta^2 \delta^2}{ad}\right)\left(1 - \frac{\beta^2 \gamma^2}{\ln \xi}\frac{d}{a}\right)$
(c) $d \ll d_2 = \frac{\beta^2 \gamma^2 \delta}{2 \ln \xi . a}$	$\frac{2 \ln \xi}{\gamma^2} - 2\beta^2 \frac{d}{a}$	$\left(\frac{2 \ln \xi}{\gamma^2}\right)\left(\frac{ad}{\beta^2 \delta^2}\right)$
(3) *perfect magnet*		
(a) $d \gg \delta$	$-\beta^2 \frac{\delta}{a}$	$\beta^2 \frac{\delta}{a}$
(b) $d_3 \ll d \ll \delta$	$-\frac{A}{\xi^2 \gamma^2} + \left(\xi \frac{\beta^2 k \delta^2}{2d}\right)^2$	$\frac{\beta^2 \delta^2}{ad}$
(c) $d \ll d_3 = a\left(\xi \frac{\beta k \delta}{2}\right)^2$	$\left(\frac{2}{\xi k a}\right)^2$	$\left(\frac{2}{\xi k a}\right)\left(\frac{ad}{\beta^2 \delta^2}\right)$

where $G = \frac{Ne^2 k}{4\pi\epsilon_0}$, $A = \frac{1}{2} + 2 \ln \kappa$, $\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$, $b = a_1$, $a = a_2$, $d = a_3 - a_2$, $\kappa = \frac{a_2}{a_1}$, $\xi = \frac{a_4}{a_3}$

TABLE II

Dispersion relation coefficients for a ceramic wall

(A) *Long wavelength limit* $\quad |\nu a| \ll 1$

Ceramic extending to	$U/G - A/\gamma^2$	V/G
(1) infinity	$2(\beta^2 - 1/\epsilon) \ln \lvert \nu a \rvert$	$\pi(\beta^2 - 1/\epsilon) - \frac{2}{\epsilon} \ln \lvert \nu a \rvert \tan \delta$
(2) perfect conductor	$-2(\beta^2 - 1/\epsilon) \ln \eta$	$\frac{2}{\epsilon} \ln \eta \tan \delta$
(3) perfect magnet	$\left(\frac{2}{\eta_1 k a}\right)^2$	$\left(\frac{2}{n_1 k a}\right)^2 \left(1 - \frac{1}{\eta_1^2}\right) \tan \delta$
(4) air to infinity	$-\frac{2}{\gamma^2} \ln \eta \tau a - 2(\beta^2 - 1/\epsilon) \ln \eta$	$\frac{2}{\epsilon} \ln \eta \tan \delta$
(5) air to perfect conductor	$\frac{2}{\gamma^2} - 1/\epsilon \ln \xi - 2(\beta^2 - 1/\epsilon) \ln \eta$	$\frac{2}{\epsilon} \ln \eta \tan \delta$
(6) air to perfect magnet	$\left(\frac{2}{\eta_2 k a}\right)^2$	$\left(\frac{2}{n_2 k a}\right)^2 \left(\frac{\epsilon(\eta^2 - 1)}{\eta_2^2}\right) \tan \delta$

where $\quad |\nu| = k\sqrt{\beta^2\epsilon - 1}, \quad \eta = \frac{a_3}{a_2}, \quad \xi = \frac{a_4}{a_3}, \quad \eta_1^2 = \epsilon(\eta^2 - 1) + 1, \quad \eta_2^2 = (\eta^2 - 1) + \eta^2(\xi^2 - 1) + 1$

(B) *Resonance region*

resonance condition $\quad \cot |\nu d| = \frac{a}{2\epsilon d} |\nu d|, \quad |\nu d| = z_p$

at resonance $U = 0$ $\quad V_{\text{res}} = \left(\frac{16G}{\tan \delta}\right)\left(\frac{\beta^2 d^3}{z_p^2 a^3}\right)\frac{(\epsilon - 1/\beta^2)^2}{z_p + \psi}$

where $\quad \psi = \frac{2\epsilon d}{a}\left(\frac{2\epsilon d}{a} + \frac{2}{\epsilon\beta^2} - 1\right)$

From the formulae above we see that U is proportional to the mode number n due to the factor $k = n/R$ in G. Only when the skin effect term is dominant (for a thick wall), U will be proportional to $\sqrt{n}$ just as V for a thick wall, due to the factor $1/\sqrt{n}$ in δ. V is proportional to n^2 for thin walls.

Since the stability limits are proportional to n (see Sec. 5), we are interested in the behaviour of $\bar{U} = U/n$ and $\bar{V} = V/n$ for increasing mode number. $\bar{U}$ is approximately constant, and may decrease as $1/\sqrt{n}$ when the skin effect term dominates. $\bar{V}$ increases linearly with frequency for a thin wall until the wall thickness is no longer small compared to the skin depth which decreases with frequency, respectively until Eq. (3.10) is no longer fulfilled. Then it decreases with frequency as $1/\sqrt{n}$, until the long wavelength limit is no longer applicable (it decreases exponentially for even higher frequencies). In between those two regions the quantity $\bar{V}$ will have a maximum (see Fig. 2).

(c) *Intermediate Metal Thickness*

When the metal thickness is comparable to the skin depth, such as is the case for the lowest modes in the CERN-ISR (metal thickness about 2 mm, skin depth of stainless steel at 100 kHz about 1.5 mm) we have to use the complete expression for L to evaluate the dispersion relation coefficients. In order to separate real and imaginary parts we need the expression

$$\mathrm{Tanh}\,(1-i)x = \frac{\mathrm{Sinh}\,2x - i\sin 2x}{\mathrm{Cosh}\,2x + \cos 2x} \tag{3.13}$$

Still assuming that $\tau a\,|\,L\,| \ll 1$, we find the dispersion relation coefficients from Eq. (2.7) after some arithmetic

$$U = G\left[\frac{1}{\gamma^2}\left(\tfrac{1}{2} + 2\ln\kappa\right) - \mu\beta^2\frac{\delta}{a}\frac{B^2(S+s) - 2B(C+c) + 2(S-s)}{B^2(C+c) - 2B(S-s) + 2(C-c)}\right] \tag{3.14}$$

$$V = G\mu\beta^2\frac{\delta}{a}\frac{B^2(S-s) - 2B(C-c) + s(S+s)}{B^2(C+c) - 2B(S-s) + 2(C-c)}$$

where
$$B = \frac{\mu}{Y}\beta^2\gamma^2\tau\,\delta \approx \frac{\mu\beta^2\gamma^2}{|\ln\tau a|}\frac{\delta}{a} \tag{3.15}$$

and

$$S = \mathrm{Sinh}\frac{2d}{\delta}, \quad s = \sin\frac{2d}{\delta} \tag{3.16}$$

$$C = \mathrm{Cosh}\frac{2d}{\delta}, \quad c = \cos\frac{2d}{\delta}$$

A5

These approximations can be compared with the exact values of U and V as function of wall thicknesses shown in Fig. 1† for the lowest mode number and typical parameters for the ISR. Similar approximations can be derived for the case where the air (respectively vacuum) outside the metal wall does not extend to infinity, but only to a perfect conductor or to a perfect magnet at a finite radius a_4. The resulting formulae are listed in Table I.

When the air extends to infinity or to a perfect conductor, we find that the quantity $(U^2 + V^2)^{1/2}$ has its maximum value for a thick metal wall (close to the skin depth), contrary to the case of transverse oscillations where the maximum value is found at some small fraction of the skin depth. However, when the air extends to a perfect magnet, the situation is entirely changed, and the values of the dispersion relation coefficients can become very large for very thin metal walls (see Fig. 1B, curve c). The presence of magnetic material outside a thin-walled metal vacuum chamber thus can have a profound influence on the longitudinal stability of the beam.

4. DIELECTRIC WALLS

In order to reduce eddy-current losses in fast cycling particle accelerators, it is advantageous to make the walls of non-conducting material such as ceramic. Usually a thin metal layer is deposited on the inside of the ceramic to avoid static charging. We shall first discuss the effects of a ceramic wall without that metallization.

(a) *Long Wavelength Limit for Ceramic Wall*

When $|\,\epsilon'\,| > 1/\beta^2$, the radial propagation constant becomes imaginary (assuming $\mu' = 1$). The negative sign has to be taken for the square root in Eq. (1.9), as one can easily convince oneself by including a small loss factor in ϵ' (see Eq. 4.17)

$$\nu = -ik(\beta^2\epsilon' - 1)^{1/2} = -i\,|\,\nu\,| \tag{4.1}$$

Since the square root of the permittivity is not very large compared to unity for most ceramic materials useful as vacuum chamber walls the condition $|\,\nu a\,| \ll 1$ will be only slightly more

† (Figures 1–6) The dispersion relation coefficients have been computed for a ring radius of 150 m, a beam radius of 1 cm, and an inner wall radius of 3 cm. We have assumed 4.10^{14} protons with $\gamma = 30$ in the ring (corresponding to a current of 20 A). For all metals we take a conductivity $\sigma = 1.1 \times 10^6$ Sm/m (typical for stainless steel), and for the ceramic we use a permittivity $\epsilon = 5.5$ with a loss-tangent of $\delta = 10^{-4}$ (typical for alumina).

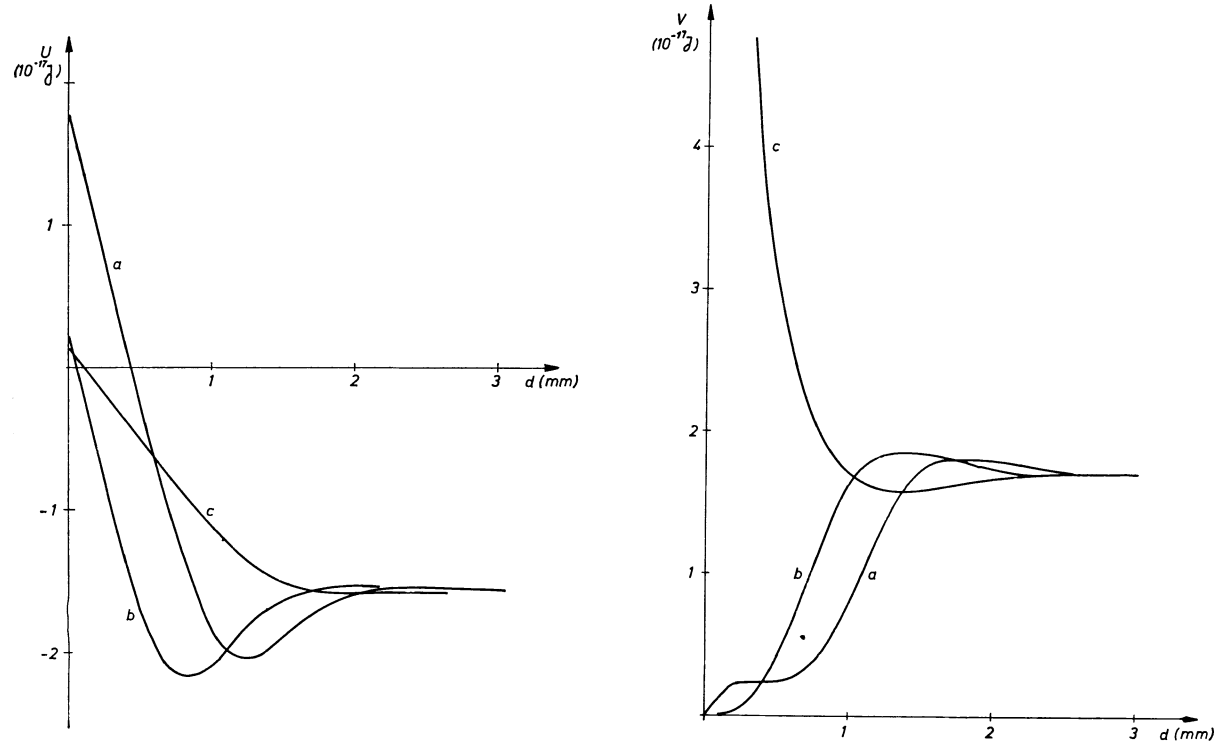

FIG. 1. Thin metal wall: U and V versus wall-thickness d for the lowest mode number $n=1$ (318 kHz) for the three cases. (a) air (vacuum) outside metal to infinity, (b) air extending to perfect conductor at $r = 5$ cm, and (c) air extending to perfect magnet at r = 5 cm. (A) U versus d, (B) V versus d.

stringent than the long wavelength condition $ka \ll 1$. Under this assumption we may use Eq. (2.9) to calculate the quantity L, where we can neglect the second term in the denominator to get

$$L = \tau a[\gamma^2(\beta^2 - 1/\epsilon') \ln \eta + \ln \tau a_3] \tag{4.2}$$

Usually we may also neglect the second term in the brackets, and get approximately

$$U - iV = G\left[\frac{1}{\gamma^2}(\tfrac{1}{2} + 2 \ln \kappa) - 2\left(\beta^2 - \frac{1}{\epsilon'}\right)\ln \eta\right] \tag{4.3}$$

For a lossless ceramic, $\epsilon' = \epsilon$ is real and thus $V = 0$, while Eq. (4.3) gives the expression for U. If the ceramic is infinitely thick, we find a contribution to V which is due to the Cerenkov losses in the dielectric. For finite thicknesses, we only need to include dielectric losses in the ceramic wall. We find a contribution to V, which for $\tan \delta \ll 1$ is approximately given by

$$V = G \cdot \frac{2 \ln \eta}{\epsilon} \tan\delta \tag{4.4}$$

while U can be found from Eq. (4.3), with ϵ' replaced by ϵ.

(b) *Resonance Region*

At higher frequencies, the denominator in Eq. (2.1) for the dispersion relation coefficient may become quite small. The condition for this to happen is

$$\mathrm{Re}\, L = \frac{I_0}{I_1}\Big|_{\tau a} \tag{4.5}$$

when we assume that $\mathrm{Im}\, L \ll \mathrm{Re}\, L$ and that $\mathrm{Im}\, L$ varies only slowly with frequency. With

$$N_1 I_0(\tau a) - N_0 I_1(\tau a) = \frac{2}{\tau a} I_1^2(\tau b) \tag{4.6}$$

we find the resonance values

$$(U - iV)_{\mathrm{res}} = F\left(N_1 \frac{I_0}{I_1}\right)_{\tau a} - \frac{2i}{\tau a\, \mathrm{Im}\, L} \frac{I_1^2(\tau b)}{I_1^2(\tau a)} \tag{4.7}$$

If the resonant mode number given by Eq. (4.5) is low enough that $\tau a \ll 1$ is valid, we may replace

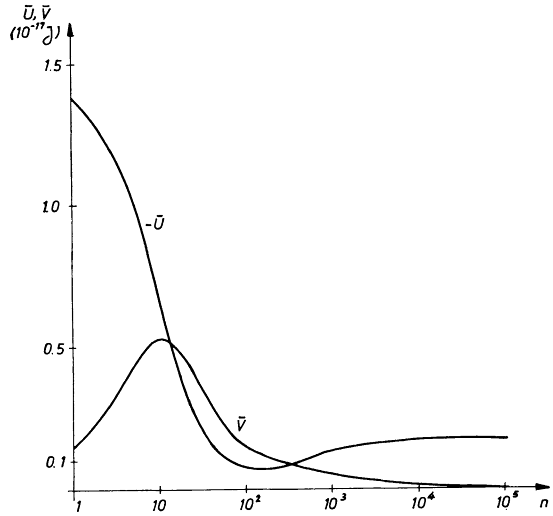

FIG. 2. Thin metal wall: $|\bar{U}| = U/n$ and $\bar{V} = V/n$ versus mode number n for a wall thickness of 0.1 mm (see Fig. 5 for other thicknesses).

the modified Bessel functions by their small argument approximations and get the approximate resonance condition

$$\operatorname{Re} L = \frac{2}{\tau a} \tag{4.8}$$

and the resonance values

$$(U - iV)_{\text{res}} = \frac{4G}{(ka)^2}\left(1 - \frac{2i}{\tau a \operatorname{Im} L}\right) \tag{4.9}$$

Normally, $\tau a/2 \operatorname{Im} L$ will be small compared to unity, and V_{res} will be very large compared to U_{res}. Actually, a closer investigation shows that U passes through zero at a mode number very close to the maximum of V.

For a lossless dielectric, the radial propagation constant is purely imaginary $\nu = -i|\nu|$. Since $|\nu a| \gg 1$ for the frequency range of interest, we use Eq. (2.12) to calculate the quantity L. With $\operatorname{Tanh} \nu d = -i \tan|\nu| d$ we find

$$L = \frac{(|\nu|/\tau\epsilon)\tan|\nu| d + Y}{1 - Y(\tau\epsilon/|\nu|)\tan|\nu| d} \tag{4.10}$$

which is a purely real quantity, so V_{res} will become infinite. Since Y is of the order τa, we may neglect the second terms in both numerator and denominator, and get approximately

$$\operatorname{Re} L = \frac{|\nu|}{\tau\epsilon}\tan|\nu| d = \frac{2}{\tau a} \tag{4.11}$$

as the condition for resonance. We may rewrite it in the form

$$\cot|\nu| d = \frac{a}{2\epsilon d}|\nu| d \tag{4.12}$$

The solutions $z_p = |\nu_p| d$ of this transcendental equation are tabulated.[3] When $(a/2\epsilon d) \ll 1$, the solutions become approximately

$$z_p = \frac{\pi}{2}(2p+1) \tag{4.13}$$

and therefore

$$|\nu_p| = \frac{\pi}{2d}(2p+1) \tag{4.14}$$

Substituting for $|\nu|$, we find for the wavelength $\lambda = 2\pi/k$

$$\frac{\lambda}{(\beta^2\epsilon - 1)^{1/2}} = \frac{4d}{2p+1} \tag{4.15}$$

Since $\lambda/(\epsilon)^{1/2}$ is the reduced wavelength in the dielectric, we find for $\beta \approx 1, \epsilon \gg 1$ the resonance condition

$$(2p+1)\frac{\lambda_{\text{red}}}{4} = d \tag{4.16}$$

We see that the first resonance appears approximately when the thickness of the dielectric equals a quarter of the reduced wavelength, and the higher resonances at odd multiples thereof. Although the assumptions have been rather restrictive, this criterion is quite useful for its simplicity.

Including the losses of practical ceramic materials results in finite values of V_{res}. For $\tan\delta \ll 1$ we find approximately

$$\nu = -i|\nu|(1 + ir) \tag{4.17}$$

with $$r = \frac{\tan\delta}{2(1 - 1/\beta^2\epsilon)}$$

We further need the expression

$$\operatorname{Tanh} \nu d = -i(1 + is)\tan|\nu| d \tag{4.18}$$

with $$s = r|\nu| d\frac{1 + \tan^2|\nu| d}{\tan|\nu| d}$$

to get approximately

$$L = \frac{|\nu|}{\tau\epsilon}\tan|\nu| d(1 + i(r + s - \tan\delta)) \tag{4.19}$$

The requirement $\operatorname{Re} L = 2/\tau a$ yields the same resonance conditions as found above. However, the finite imaginary part of L now leads to

$$V_{\text{res}} = \frac{4G}{(ka)^2} \frac{1}{r+s-\tan\delta}$$
$$= \frac{16G}{\tan\delta} \frac{\beta^2}{z_p^2 a^3} \frac{d^3(\epsilon - 1/\beta^2)^2}{z_p^2 + \phi} \tag{4.20}$$

with
$$\phi = \frac{2\epsilon\, d}{a}\left(\frac{2\epsilon\, d}{a} + \frac{2}{\beta^2\epsilon} - 1\right) \tag{4.21}$$

In this last form we see that V_{res} (or rather V_{res}/n as G still contains the factor $k = n/R$) decreases rapidly with increasing z_p, and we need only be concerned with the first resonance. We also see that V_{res} is inversely proportional to $\tan\delta$, and the small damping of low loss ceramics may lead to a catastrophic increase of the dispersion relation coefficients. They become several orders of magnitude larger than the values for metal walls (see Fig. 3).

(c) *Metallization on the inside of a Ceramic Wall*

The explicit formulae become rather complicated for a five layer geometry. However, we may treat the problem of a metallized ceramic extending to a perfect conductor, which corresponds to four layers. For a metal thickness D we find the resonance condition

$$|\nu|\, d \tan|\nu| = \frac{2\epsilon\, d}{a} \cdot \frac{1 - ((\beta ka)^2/2)\,(D/a)}{1 - ((z_p^2 + \phi)\tan\delta)/((\epsilon - 1/\beta^2)(\beta k\,\delta)^2)\,(D/a)} \tag{4.22}$$

with the solutions $|\nu|\, d = Z_p$ for the pth resonance (d is the thickness of the ceramic as before). For vanishing metal thickness D, the condition agrees with Eq. (4.12). The value of the dispersion relation coefficient V at resonance becomes approximately

$$V_{\text{res}} = \beta^2 G \bigg/ \left(\frac{z_p^2(z_p^2 + \phi)\tan\delta}{16(d/a)^3(\epsilon - 1/\beta^2)^2} + \frac{aD}{\delta^2}\right) \tag{4.23}$$

which agrees with Eq. (4.20) for $D = 0$. For practical thicknesses of the metal layer ($D > 100$ Å) the second term usually dominates, and we get simply

$$V_{\text{res}} = G \,.\, \frac{\beta^2\, \delta^2}{aD} \tag{4.24}$$

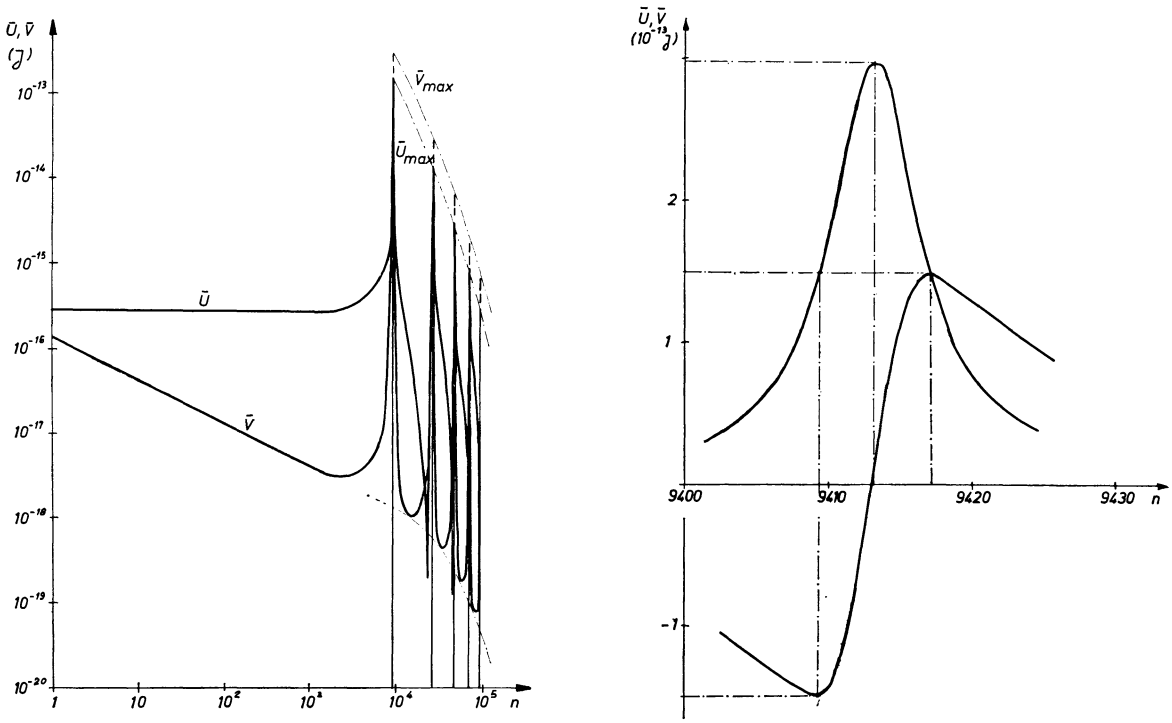

FIG. 3. Ceramic wall of 1 cm thickness (outside air to infinity). (A) $|\bar{U}|$ and $\bar{V}$ versus mode number n, (B) blow-up of the first resonance region.

This is the same as the long wavelength limit expression, valid for a metal thickness small compared to the skin depth, but larger than d, given by

$$d_1 = \frac{\beta^2\,\delta^2/2a}{(\beta^2 - 1/\epsilon)n\tau a} \tag{4.25}$$

(see Table II).

The resonant behaviour thus disappears when the metal thickness is large compared to d_1, as the formulae for the maximum and for the low mode number coincide. For a representative set of values taken from the ISR geometry, the thickness d_1 is of the order of 100 Å (at the first resonance).

The exact formulae have been evaluated by computer, and the results are shown in Figs. 4 and 6. The resonances and peak values agree well with the approximations (4.22) and (4.23). For metal thicknesses above 1000 Å, the resonance peaks of V have virtually disappeared, as predicted by Eq. (4.25). The application of a metallization of very modest thickness thus is an efficient means to counteract the deleterious effects of dielectric resonances.

5. STABILITY DIAGRAMS

It has been shown recently[4] that the longitudinal stability limit for a particle beam can be represented by a curve in the normalized U', V' plane, which is usually well approximated by a circle of radius $2/\pi$. The normalized quantities U', V' are related to the dispersion relation coefficients U, V by

$$U' - iV' = \frac{U - iV}{n\,|\,k_0\,|\,\Delta^2} \tag{5.1}$$

where

$$k_0 = \frac{\Omega}{2\pi}\frac{\mathrm{d}\Omega}{\mathrm{d}E} = \frac{\Omega^2}{2\pi\beta_0^2\gamma_0 E_0}\left(\frac{1}{\gamma_\tau^2} - \frac{1}{\gamma_0^2}\right) \tag{5.2}$$

is a constant of the accelerator (γ_τ transition energy factor, $E_0 = m_0 c^2$ rest energy of particle), and where

$$\Delta = \pi\frac{\Delta E}{\Omega} = \frac{\pi\gamma_0 E_0}{\Omega}\left(\frac{\Delta E}{E}\right) \tag{5.3}$$

is a measure of the spread of the canonical angular momentum (ΔE is the total energy spread at half height).

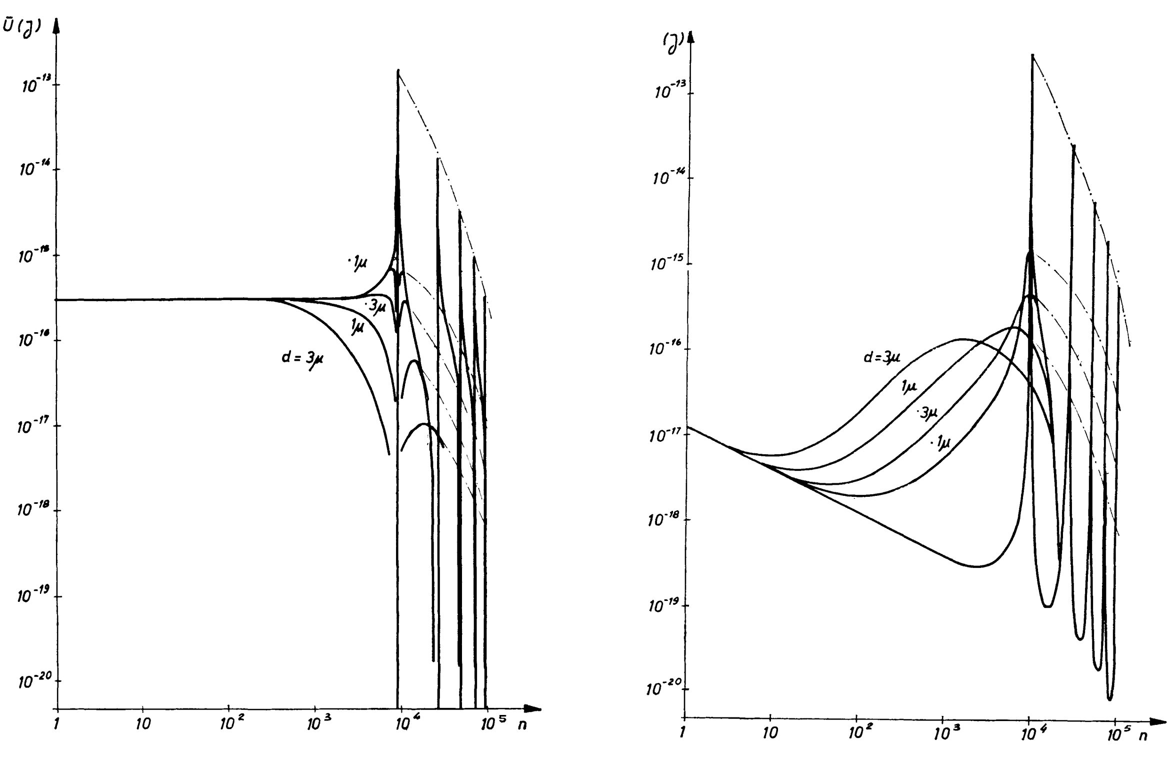

FIG. 4. Metallized ceramic wall: metal thickness $d = 0.1$ to $3\ \mu$ (A) $\bar{U}$ versus mode number n, (B) $\bar{V}$ versus mode number n (the unlabelled curves are those for vanishing metal thickness).

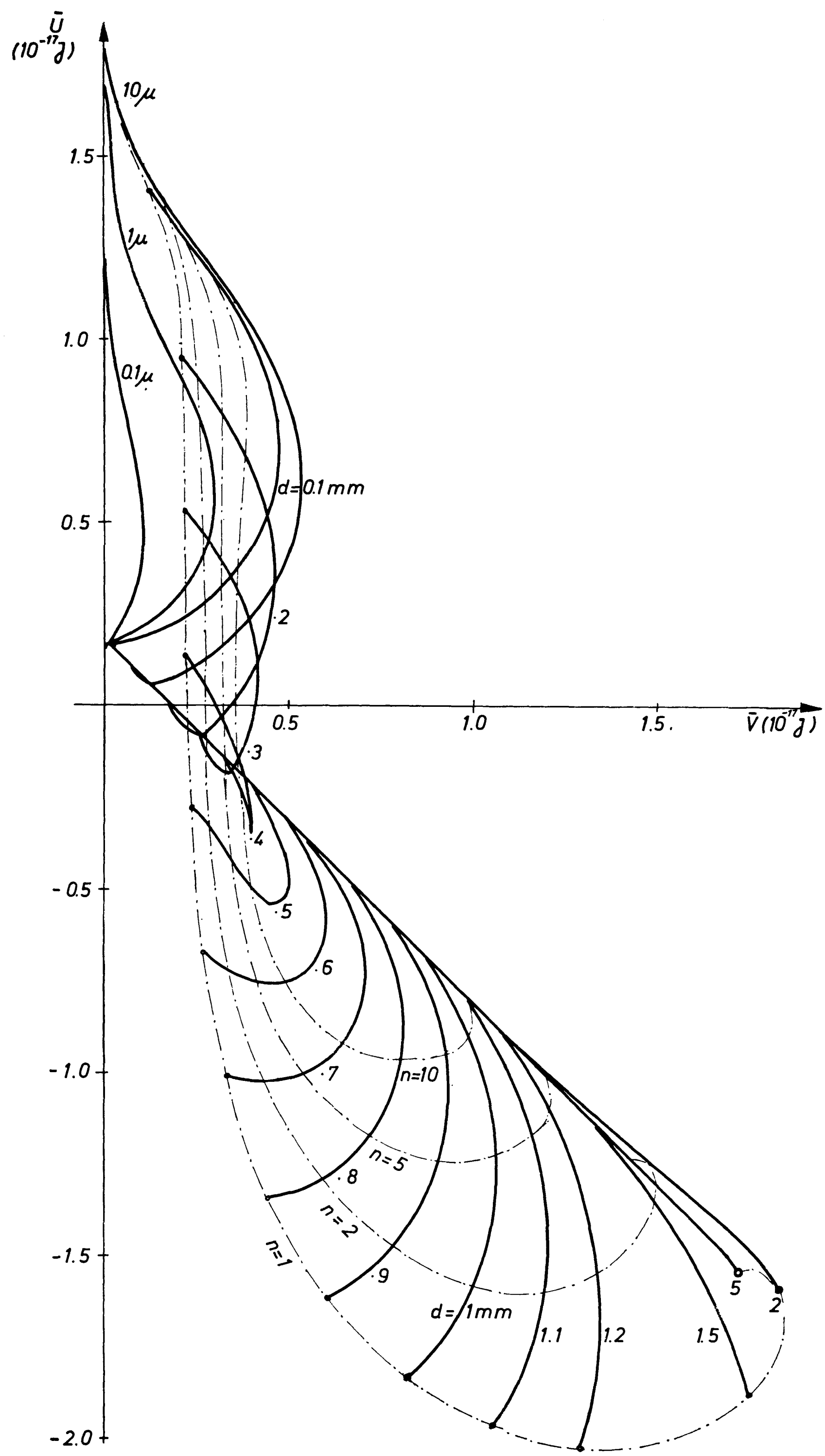

FIG. 5. Thin metal wall: $\bar{U}$ versus $\bar{V}$ diagram for various wall thickness, with mode number n as running parameter.

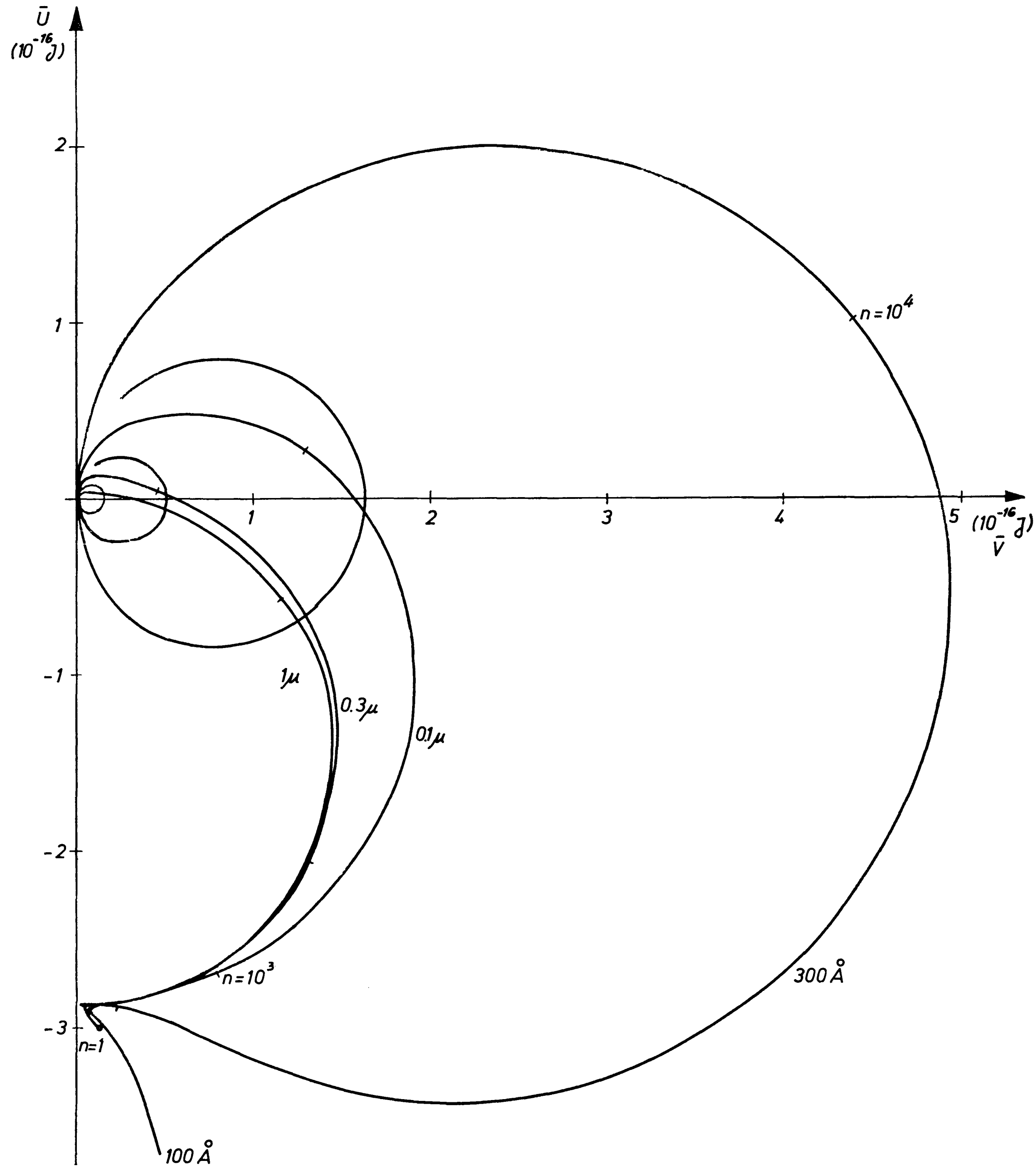

FIG. 6. Metallized ceramic wall: $\overline{U}$ versus $\overline{V}$ diagram for various metal thicknesses.

We may write the simplified stability criterion as

$$(U'^2 + V'^2)^{1/2} \leqslant \frac{2}{\pi} \tag{5.4}$$

or

$$\left(\left(\frac{U}{n}\right)^2 + \left(\frac{V}{n}\right)^2\right)^{1/2} \leqslant \frac{2}{\pi} | k_0 | \Delta^2 \tag{5.5}$$

To get a feeling for the order of magnitude of the constants involved, we calculate k_0 and Δ for typical values of the CERN-ISR. With $\gamma_0 = 30$, $\gamma_\tau = 8.97$, $\Omega = 2.10^6$ sec^{-1}, $E_0 = 1.503 \,.\, 10^{-10}$J, and

$$\frac{\Delta E}{E} = 5 \times 10^{-5},$$

we find $| k_0 | = 1.60 \times 10^{18}$(J^{-1} sec^{-1}, $\Delta = 3.53 \times 10^{-19}$ (J sec), and the stability criterion becomes

$$\left(\left(\frac{U}{n}\right)^2 + \left(\frac{V}{n}\right)^2\right)^{1/2} \leqslant 1.27 \times 10^{-19} \tag{5.6}$$

Typical values for U/n and V/n for $N = 1.5 \times 10^{12}$ particles in a chamber with a wall of stainless steel are

$$U/n = 6.8 \times 10^{-21}\left(1 - \frac{9.3}{\sqrt{n}}\right),$$
$$V/n = 6.3 \times 10^{-20}/\sqrt{n} \mid \mathrm{J} \mid \tag{5.7}$$

We see that both U/n and V/n at $n = 1$ come dangerously close to the stability limit for the first injected turn, while stability is better for the full stack due to the factor Δ^2 in Eq. (5.5).

Any large increase of the dispersion relation coefficients can therefore endanger the stability of coherent longitudinal oscillations. We can best illustrate the situation by drawing both the stability limit and the frequency locus of the dispersion relation coefficients in the U/n versus V/n plane. The U/n versus V/n diagrams for a metal and for a metallized ceramic wall are shown in Figs. 5 and 6 for several values of metal thickness.

Alternately, the stability criterion may be expressed in terms of the coupling impedance, which has been defined[4] as the (negative) ratio of the integral over the average electric field strength in the beam region to the total beam current. Its relation to the dispersion relation coefficients then can be expressed as

$$Z = -\frac{2\pi R\langle E_Z^I\rangle}{I} = i\frac{2\pi R}{Ne^2\beta_0 c}(U - iV) \tag{5.8}$$

For a machine radius $R = 150$ m, and $N = 1.5 \times 10^{12}$ particles in the beam, the factor of $i(U - iV)$ becomes 8.16×10^{19} (for large γ_0), and the stability criterion may be written

$$\left|\frac{Z}{n}\right| < 10\Omega \tag{5.9}$$

The values of dispersion relation coefficients in the example above (Eq. 5.7) correspond to an impedance of approximately $5 \cdot (1 - i)\Omega$ for the lowest mode number.

6. CONCLUSIONS

The dispersion relation coefficients for a thin metal wall and for a ceramic wall are calculated from a general algorithm. Approximate formulae are given for both cases, and graphs are presented that have been evaluated by computer. Except for the case where magnetic material surrounds a very thin metal wall, the dispersion relation coefficients are largest for metal walls of large thickness (compared to the skin depth) at the lowest mode of oscillation. For ceramic walls, however, dielectric resonances at certain frequencies can increase the dispersion relation coefficients by many orders of magnitude, and may seriously endanger stability. A thin layer of metal on the inside of the ceramic damps the resonances effectively.

We have tacitly assumed that the whole particle accelerator has the same wall material and geometry all around its circumference. Practical machines have many different sections, which fact can be taken into account by adding the individual contributions multiplied by the respective circumference factors. In addition, contributions due to resonant elements such as cavities or clearing electrodes have to be added to get a complete picture of the stability of longitudinal conditions.

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APPENDIX

Derivation of the Algorithm for the Calculation of the Dispersion Relation Coefficients

1. *Wave Equation* in cylindrical coordinates for the longitudinal electric field component E_z, proportional exp $i(kz - \omega t)$, phase velocity $\beta c = \omega/k$.

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dE_z}{dr}\right) - \nu^2 E_z = \frac{ik\rho}{\epsilon_0} - i\omega\mu_0 J_z \tag{I.1}$$

where $\nu = k(1-\beta^2\epsilon'\mu')^{1/2}$ is the radial propagation constant.

Solutions

(a) *beam region:* sources ρ and $J_z = \omega\rho/k$,

$$\mu' = \epsilon' = 1 \quad E_z^I = E_1 \times I_0(\tau r) - (i\rho/k\epsilon_0)$$

with $\quad \tau = k(1-\beta^2)^{1/2} = k/\gamma$

(b) *intermediate regions:* $(2 \leqq p \leqq j-1)$: no sources

$$E_z^p = E_p R_p(\nu_p r) + F_p S_p(\nu_p r)$$

where

$$R_p(\nu_p r) = K_0(\nu_p a_p) I_0(\nu_p r) - I_0(\nu_p a_p) K_0(\nu_p r)$$

$$S_p(\nu_p r) = -K_1(\nu_p a_p) I_0(\nu_p r) - I_1(\nu_p a_p) K_0(\nu_p r)$$

(c) *outermost region,* extending to infinity ($p = m$): no sources

$$E_z^p = F_q K_0(\nu_q r)$$

2. *Transverse Components* of fields can be found from E_z and H_z. For rotational symmetry $E_\phi = H_r = H_z = 0$. With $J_\phi = J_r = 0$

$$E_r = \frac{ik}{\nu^2}\frac{\partial E_z}{\partial r}$$

$$Z_0 H_\phi = \frac{ik}{\nu^2}\beta\epsilon'\frac{\partial E_z}{\partial r}$$

3. *Matching* at the interfaces: no surface charges or currents anywhere, thus E_z and H_ϕ must be continuous (condition on E_r redundant). From the properties of the functions $R_p(\nu_p r)$ and $S_p(\nu_p r)$

$$R_p(\nu_p a_p) = S_p'(\nu_p a_p) = 0,$$

$$R_p'(\nu_p a_p) = -S_p(\nu_p a_p) = \frac{1}{\nu_p a_p}$$

one finds

$$E_z^p(r = a_p) = -\frac{F_p}{\nu_p a_p},$$

$$Z_0 H_\phi^p(r = a_p) = ik\beta\frac{\epsilon_p}{\nu_p}\frac{E_p}{\nu_p a_p}$$

definition:

$$R_p^{(\prime)} \equiv R_p^{(\prime)}(\nu_p a_{p-1}), \quad S_p^{(\prime)} \equiv S_p^{(\prime)}(\nu_p a_{p-1})$$

(a) *beam surface* $r = a_1$ (assuming $\epsilon_2' = \mu_2' = 1$, i.e. vacuum outside)

$$E_1 I_0(\tau a_1) - \frac{i\rho}{k\epsilon_0} = E_2 R_2 + F_2 S_2$$

$$E_1 I_1(\tau a_1) \quad = E_2 R_2' + F_2 S_2'$$

(b) *intermediate surfaces* $r = a_{p-1}$, $3 \leqq p \leqq q-1$

$$-\frac{F_{p-1}}{\nu_{p-1}a_{p-1}} = E_p R_p + F_p S_p$$

$$\epsilon_{p-1}'\frac{E_{p-1}}{\nu_{p-1}^2 a_{p-1}} = \frac{\epsilon_p'}{\nu_p}(E_p R_p' + F_p S_p')$$

(c) *outermost surface* $r = a_{q-1}$

$$-\frac{F_{q-1}}{\nu_{q-1}a_{q-1}} = F_q K_0(\nu_q a_{q-1})$$

$$\epsilon_{q-1}'\frac{E_{q-1}}{\nu_{q-1}^2 a_{q-1}} = F_q K_1(\nu_q a_{q-1})$$

4. *Solution for field coefficient* E_1. Divide top by bottom equation at all interfaces

(a) *beam surface*

$$\left.\frac{I_0}{I_1}\right|_{\tau a_1} - \frac{i\rho/k\epsilon_0}{E_1 I_1(\tau a_1)} = \frac{R_2 + (F/E)_2 S_2}{R_2' + (F/E)_2 S_2'}$$

solve for $\quad E_1 = \dfrac{i\rho/k\epsilon_0}{I_0(\tau a_1) - I_1(\tau a_1)\dfrac{R_2 + (F/E)_2 S_2}{R_2' + (F/E)_2 S_2'}}$

$$= \frac{i\rho}{k\epsilon_0}\frac{R_2' + (F/E)_2 S_2'}{(I_0 + (F/E)_2 I_1)_{\tau a_2}}\tau a_1$$

(b) *intermediate surfaces*

$$\left(\frac{F}{E}\right)_{p-1} = -\frac{\epsilon_{p-1}'\nu_p}{\epsilon_p'\nu_{p-1}}\frac{R_p + (F/E)_p S_p}{R_p' + (F/E)_p S_p'}$$

(c) *outermost surface*

$$\left(\frac{F}{E}\right)_{j-1} = \frac{\epsilon_{j-1}'\nu_j}{\epsilon_j'\nu_{j-1}}\left.\frac{K_0}{K_1}\right|_{\nu_j a_{j-1}}$$

define $\quad X_p \equiv \dfrac{\epsilon_p'\nu_{p-1}}{\epsilon_{p-1}'\nu_p}, \qquad Y_{p+1} = X_{p+1}(F/E)_p$

rewrite above as

$$E_1 = \frac{i\rho}{\gamma\epsilon_0}\frac{X_3 R_2' - Y_3 S_2'}{(X_3 I_0 - Y_3 I_1)_{\tau a_2}}\tau a_1$$

$$Y_p = \frac{X_{p+1}R_p - Y_{p+1}S_p}{X_{p+1}R_p' - Y_{p+1}S_p'}$$

$$Y_q = -\left.\frac{K_0}{K_1}\right|_{\nu_q a_{q-1}}$$

5. *Average force on beam particle of charge* e

$$\langle F_z\rangle = \frac{e}{\pi a_1^2}\int_0^{2\pi} d\phi \int_0^{a_1} E_z\, dr = e\left(E_1\frac{2I_1(\tau a_1)}{\tau a_1} - \frac{i\rho}{k\epsilon_0}\right)$$

$$= -\frac{ie\rho}{k\epsilon_0}\frac{N_0 - N_1 L}{(I_0 - I_1 L)_{\tau a_2}}$$

with $L = Y_3/X_3$

$$N_{0,1} = I_{0,1}(\tau a_2)[1 - 2I_1(\tau a_1)K_1(\tau a_1)]$$

$$\mp 2K_{0,1}(\tau a_2)I_1^2(\tau a_1)$$

6. Dispersion relation coefficients

$$U - iV = \frac{ieN}{\pi\rho a_1^2}\langle F_z\rangle = \frac{Ne^2}{\pi\epsilon_0 k a_1^2}\frac{N_0 - N_1 L}{(I_0 - I_1 L)_{\tau a_2}}$$

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