

D-particle bound states and generalized instantons

Gregory Moore,¹ Nikita Nekrasov,^{2,3} and Samson Shatashvili^{1,4*}

¹ Department of Physics, Yale University, New Haven, CT 06520, Box 208120, USA

² Institute of Theoretical and Experimental Physics, 117259, Moscow, Russia

³ Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

⁴ Theory Division, CERN, CH-1211, Geneve 23, Switzerland

moore@castalia.physics.yale.edu

nikita@string.harvard.edu

samson@euler.physics.yale.edu

We compute the principal contribution to the index in the supersymmetric quantum mechanical systems which are obtained by reduction to 0+1 dimensions of $\mathcal{N} = 1$, $D = 4, 6, 10$ super-Yang-Mills theories with gauge group $SU(N)$. The results are: $\frac{1}{N^2}$ for $D = 4, 6$, $\sum_{d|N} \frac{1}{d^2}$ for $D = 10$. We also discuss the $D = 3$ case.

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* On leave of absence from St. Petersburg Branch of Steklov Mathematical Institute, Fontanka, St. Petersburg, Russia.

1. Introduction

The existence of M -theory depends crucially on the existence within type- IIA string theory of a tower of massive BPS particles electrically charged with respect to the RR 1-form. These particles, originally described as black holes in IIA supergravity [1], can be interpreted as Kaluza-Klein particles of eleven-dimensional M -theory compactified on a circle [2][3]. Later, these particles were identified with “ $D0$ -branes” [4]. In the D -brane formulation it becomes clear that in certain energy regimes the dynamics of N such particles can be described by the supersymmetric quantum mechanics of $N \times N$ Hermitian matrices obtained from dimensional reduction of $\mathcal{N} = 1$, $D = 10$ super-Yang-Mills theory [5] (the quantum mechanical model was originally studied in [6]). The existence of the M -theoretic Kaluza-Klein tower of states is equivalent to the statement that this quantum mechanics has exactly one bound state for each N . Consequently, proving the existence of these bound states has been the focus of several recent papers of which [7][8][9] are the most relevant to the present work. In particular, we note that the existence of the bound state in the case of $N = 2$ was proven in [7], but the case $N > 2$ remains open. The results of the present paper will help complete the proof for all N .

The existence of bound states in susy quantum mechanics can be detected by computing the Witten index:

$$\lim_{\beta \rightarrow \infty} \text{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H} = N_B - N_F \quad (1.1)$$

where $N_{B,F}$ are the numbers of bosonic and fermionic zero eigen-states of the Hamiltonian H respectively. The expression $\text{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H}$ is β -independent in theories with a discrete spectrum, but may be rather complicated if the spectrum is continuous. In fact, the densities of fermionic and bosonic eigen-states may differ, leading to nontrivial β -dependence. Nevertheless, supersymmetry allows us to relate the index of interest to the easier-to-access quantity:

$$\lim_{\beta \rightarrow 0} \text{Tr}_{\mathcal{H}} (-1)^F e^{-\beta H} \quad . \quad (1.2)$$

In the case of the quantum mechanics of N $D0$ -branes (1.2) can be expressed very explicitly as a matrix integral

$$\frac{1}{\text{Vol}(G)} \int d^{10} X d^{16} \Psi e^{-S} \quad (1.3)$$

where S is the reduction to zero dimensions of the action of the $\mathcal{N} = 1$ $d = 10$ super-Yang-Mills theory with the gauge group $G = SU(N)/\mathbf{Z}_N$. More generally, we are aiming at computing the integral

$$I_D(N) \equiv \left(\frac{\pi}{g}\right)^{\frac{(N^2-1)(D-3)}{2}} \frac{1}{\text{Vol}(G)} \int d^D X d^{2^{D/2-1}} \Psi e^{-S} \quad (1.4)$$

for $D = 3 + 1, 5 + 1, 9 + 1$ respectively, where

$$S = \frac{1}{g} \left(\frac{1}{4} \sum_{\mu, \nu=1, \dots, D} \text{Tr}[X_\mu, X_\nu]^2 + \frac{i}{2} \sum_{\mu=1}^D \text{Tr}(\bar{\Psi} \Gamma^\mu [X_\mu, \Psi]) \right), \quad (1.5)$$

and Γ^μ are the Clifford matrices for $Spin(D)$. The integrals (1.3)(1.4) are not the full contribution to the Witten index (indeed, as we will see, they are not integral). The difference (also called the boundary term)

$$\lim_{\beta \rightarrow \infty} \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} - \lim_{\beta \rightarrow 0} \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} = \int_0^\infty d\beta \frac{d}{d\beta} \text{Tr}_{\mathcal{H}}(-1)^F e^{-\beta H} \quad (1.6)$$

may be analysed separately and is beyond the scope of this paper. See [7][8][9] for further discussion.

The paper is organized as follows. In section 2 we reinterpret the integrals (1.3)(1.4) as those appearing in the CohFT approach to the studies of the moduli space of susy gauge configurations, reduced to 0 dimensions.¹ The susy gauge configurations obey flatness, instanton and complexified (or octonionic) instanton equations in 3+1, 5+1 and 9+1 cases respectively. (Quantum mechanics on the moduli spaces of such susy gauge configurations on compact manifolds was studied recently in [10].)

In section 3 we deform the integral using the global symmetries of the equations. The symmetry groups are $Spin(2)$, $Spin(4)$ and $Spin(6)$ (or $Spin(7)$) in $D = 4, 6, 10$, respectively. We simplify the deformed integrals by the method of “integrating out BRST quartets” and get contour integrals over the eigenvalues of one of the matrices, denoted ϕ below. This brings the integrals to the form given in equations (3.6), (3.7), and (3.8) below for the cases $D = 10, 6, 4$. The method used to arrive at these expressions is a direct extension of methods we used to integrate over Higgs branches in [11].

The expressions (3.6), (3.7), (3.8) are one of the main results of this paper. Nevertheless, we must note at the outset that the result is incomplete. As Lebesgue integrals

¹ “CohFT” = “Cohomological field theory.”

these expressions do not make sense. Rather, they should be regarded as contour integrals, which do make sense once a prescription is adopted for picking up the poles of the integrand. We are confident that a more careful implementation of the quartet mechanism will lead to a definite pole prescription. In this paper we will take the pragmatic route and simply find a pole prescription which gives the desired answer. In particular, in section 4 we perform an explicit evaluation of (1.4) for $G = SU(2), SU(3)$. In sections 5, 6, 7 we evaluate the integrals for the general case $G = SU(N)$. Each case, $D = 4, 6, 10$, requires a different trick in order to carry out the intricate sum over poles. In the $D = 3 + 1$ case we use an identity familiar from bosonization in two-dimensions. In the $D = 5 + 1$ case we use fixed-point techniques for a certain torus action on the Hilbert scheme of N points on \mathbb{C}^2 . In section 7 we deform the octonionic instanton equations and reduce the $D = 9 + 1$ case to a sum over answers for $D = 3 + 1$ with the sum running over all possible unbroken gauge groups of $\mathcal{N} = 4$ super-Yang-Mills theory broken down to $\mathcal{N} = 1$ by mass terms. This final reduction leads to an answer for the index computation, predicted by M. Green and M. Gutperle in [8], building on the work of [7].

Finally, in section 8 we relate our computations to the partition functions of the SYM theories on $T^4, K3, T^3$ and discusses some subtleties of the latter case.

As our paper was nearing completion a related paper appeared [12]. This paper describes a complementary (numerical) approach to the evaluation of the integrals $I_D(N)$ and in particular evaluates the integral for the $SU(3)$ case. Also, the paper [9] studied the mass deformations of the quantum mechanical problems we consider here, for the case when N is prime. It would be interesting to understand better the relation to these works. It was brought to our attention that the CohFT reformulation of IKKT model has been also considered in [13].

2. CohFT reinterpretation

To map to CohFT formalism we choose two matrices, say X_D and X_{D-1} , and arrange them into a complex matrix ϕ :

$$\phi = X_{D-1} + iX_D$$

The rest of the matrices can be written as $B_j = X_{2j-1} + iX_{2j}$ for $j = 1, \dots, D/2 - 1$. Sometimes we simply denote them as $\mathbf{X} = \{X_a, a = 1, \dots, D - 2\}$. We also rearrange the fermions: $\Psi \rightarrow \Psi_a = (\psi_j, \psi_j^\dagger), \vec{\chi}, \eta$ and add bosonic auxiliary fields \vec{H} . Then we rewrite the bosonic part of the action as:

$$S = \frac{1}{16g} \text{Tr}[\phi, \bar{\phi}]^2 - i \text{Tr} \vec{\mathcal{E}}(\mathbf{X}) \vec{H} + g \text{Tr} \vec{H}^2 - \frac{1}{4g} \sum_{a=1}^{D-2} \text{Tr} |X_a, \phi|^2 \quad (2.1)$$

where the “equations” $\vec{\mathcal{E}}$ are:

$$D = 4 : \quad \vec{\mathcal{E}} = [B_1, B_1^\dagger] \quad (2.2a)$$

$$D = 6 : \quad \vec{\mathcal{E}} = \left([B_1, B_1^\dagger] + [B_2, B_2^\dagger], [B_1, B_2], [B_2^\dagger, B_1^\dagger] \right) \quad (2.2b)$$

$$D = 10 : \quad \vec{\mathcal{E}} = \left([B_i, B_j] + \frac{1}{2} \epsilon_{ijkl} [B_k^\dagger, B_l^\dagger], i < j, \quad \sum_i [B_i, B_i^\dagger] \right) \quad (2.2c)$$

It is worth noting that one can also write the equations (2.2b) as a three-vector $\mathcal{E}_A = [X_A, X_4] + \frac{1}{2} \epsilon_{ABC} [X_B, X_C]$, $A = 1, 2, 3$. Similarly, we can also write the equations (2.2c) as a seven-vector: $\mathcal{E}_{\mathbf{A}} = [X_{\mathbf{A}}, X_8] + \frac{1}{2} c_{\mathbf{ABC}} [X_{\mathbf{B}}, X_{\mathbf{C}}]$ using octonionic structure constants: $\mathbf{A} = \mathbf{1}, \dots, \mathbf{7}$.

The integral (1.4) has the following important nilpotent symmetry:

$$\begin{aligned} QX_a &= \Psi_a & Q\Psi_a &= [\phi, X_a] \\ Q\vec{\chi} &= \vec{H} & Q\vec{H} &= [\phi, \vec{\chi}] \\ Q\bar{\phi} &= \eta & Q\eta &= [\phi, \bar{\phi}] \\ Q\phi &= 0 \end{aligned} \quad (2.3)$$

In fact, the action (2.1) together with fermions can be represented as:

$$S = Q \left(\text{Tr} \frac{1}{16g} \eta [\phi, \bar{\phi}] - i \text{Tr} \vec{\chi} \cdot \vec{\mathcal{E}} + g \text{Tr} \vec{\chi} \cdot \vec{H} + \frac{1}{4g} \sum_{a=1}^{D-2} \text{Tr} \Psi_a [X_a, \bar{\phi}] \right) \quad (2.4)$$

As usual, there is a ghost charge. It is equal to +2 for ϕ , +1 for Ψ_a , 0 for \vec{H} , X_a , -1 for $\vec{\chi}, \eta$ and -2 for $\bar{\phi}$.

All the bosonic fields except ϕ are paired with the fermions. Therefore, in order to fix the normalization of the integral one need only fix the measure $D\phi$ on the Lie algebra of G . Since $\text{Lie}G$ is a simple Lie algebra, there is a unique Killing form up to a constant multiple. This form determines the measure both on the Lie algebra and on the group G . The measure

$$\frac{D\phi}{\text{Vol}(G)}$$

is thus independent of the choice of the Killing form. However, the measure depends on whether the gauge group contains the center or not. We ought to use the measure normalized against $G = SU(N)/\mathbf{Z}_N$, since it is G which is the actual gauge group of the problem. When we reduce the computation to an integral over the Lie algebra of the maximal torus $T \subset SU(N)$ the measure $\mathcal{D}\phi$ will be normalized in such a way that the measure on T obtained by the exponential map integrates to one. Therefore there is an extra factor $\#Z$ in front of the integral since in passing to the measure on \mathfrak{t} we get as a factor a volume of the generic adjoint orbit:

$$\frac{\text{Vol}(G/(T/Z))}{\text{Vol}(G)} = \frac{\#Z}{\text{Vol}(T)}. \quad (2.5)$$

Finally, upon eliminating the auxilliary fields \vec{H} by taking the Gaussian integral the extra factor $\left(\frac{\pi}{g}\right)^{\frac{(D-3)(N^2-1)}{2}}$ appears. It is related to the β -dependent factor appearing in the index computation [7].

3. Global symmetries and deformation

The global symmetries are

$$4 : K = Spin(2), \vec{\mathcal{E}} \in \mathbf{1};$$

$$6 : K = Spin(4), \vec{\mathcal{E}} \in \mathbf{3}_L;$$

$$10 : K = Spin(6), \vec{\mathcal{E}} \in (\mathbf{6} \oplus \bar{\mathbf{6}})_r \oplus \mathbf{1}$$

Alternatively, in the last case we can use the octonionic representation with $K = Spin(7), \vec{\mathcal{E}} \in \mathbf{7}$;

We will simplify our integrals by deforming the BRST operator. The deformation will involve a choice of a generic element ϵ in the Cartan subalgebra of the global symmetry group K . We therefore choose elements $\epsilon \in \text{Lie}(Spin(2)), \text{Lie}(Spin(4)),$ and $\text{Lie}(Spin(6))$ for $D = 4, 6, 10$ respectively. Explicitly we will write these elements as:

$$\begin{aligned} D = 4 \quad \epsilon &= \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \\ D = 6 \quad \epsilon &= \begin{pmatrix} 0 & E_1 & & & & \\ -E_1 & 0 & & & & \\ & & 0 & E_2 & & \\ & & -E_2 & 0 & & \\ & & & & 0 & E_3 \\ & & & & -E_3 & 0 \end{pmatrix} \\ D = 10 \quad \epsilon &= \begin{pmatrix} 0 & E_1 + E_2 & & & & & & & & & \\ -E_1 - E_2 & 0 & & & & & & & & & \\ & & 0 & E_2 + E_3 & & & & & & & \\ & & -E_2 - E_3 & 0 & & & & & & & \\ & & & & 0 & E_1 + E_3 & & & & & \\ & & & & -E_1 - E_3 & 0 & & & & & \end{pmatrix} \end{aligned} \quad (3.1)$$

for sufficiently generic real constants E_i .

Using the global symmetry one may deform the nilpotent charge (2.3) to the differential of K -equivariant cohomology:

$$\begin{aligned}
Q_\epsilon X_a &= \Psi_a & Q_\epsilon \Psi_a &= [\phi, X_a] + X_b T_v(\epsilon)_a^b \\
Q_\epsilon \vec{\chi} &= \vec{H} & Q_\epsilon \vec{H} &= [\phi, \vec{\chi}] + T_s(\epsilon) \cdot \vec{\chi} \\
Q_\epsilon \bar{\phi} &= \eta & Q_\epsilon \eta &= [\phi, \bar{\phi}] \\
Q_\epsilon \phi &= 0
\end{aligned} \tag{3.2}$$

where we denote T_v for the action of $\text{Lie}(K)$ on X 's and T_s for the action of $\text{Lie}(K)$ on the equations. Now deform the action (2.4) to

$$S_\epsilon = Q_\epsilon \left(\frac{1}{16\tilde{g}} \text{Tr} \eta[\phi, \bar{\phi}] - i \text{Tr} \vec{\chi} \cdot \vec{\mathcal{E}} + g \text{Tr} \vec{\chi} \cdot \vec{H} + \frac{1}{4\hat{g}} \sum_{a=1}^{D-2} \text{Tr} \Psi_a[X_a, \bar{\phi}] \right) \tag{3.3}$$

At this point the couplings \hat{g} , \tilde{g} and g are all equal but in the sequel we shall treat them separately. In particular we will first take $\tilde{g} \rightarrow \infty$. The new integral

$$\int \dots e^{-S_\epsilon}$$

is convergent if the original (1.4) integral is convergent. In fact the added piece $S_\epsilon - S_0$ is equal to

$$g \text{Tr} (\vec{\chi} \cdot T_s(\epsilon) \vec{\chi}) + \frac{1}{4\hat{g}} T_v(\epsilon)^{ab} \text{Tr} \bar{\phi}[X_a, X_b]$$

which has ghost charge -2 (if we temporarily assign a charge zero to ϵ). This means that the value of the integral (whose measure has net ghost charge zero) is not changed. Now a closer look at the eigenvalues of T_s reveals that there is always one zero eigen-value for the mass matrix of χ but the rest is non-vanishing for generic ϵ . We denote this massless mode by χ_0 , and consider adding to the action a Q_ϵ -exact term

$$s Q_\epsilon \text{Tr}(\chi_0 \bar{\phi}) \tag{3.4}$$

with a large coefficient s . It has ghost charge -2 . This term together with $g\vec{H}^2$ produces masses for all the fermions of negative ghost charge. Integrating them out (by taking the limit $s \rightarrow \infty$, $g \rightarrow \infty$) would produce a very simple action but without a ‘‘kinetic’’ term for Ψ_a 's. To cure this problem we add a *positive* ghost charge operator

$$\frac{1}{2} t Q_\epsilon \left(\sum_{i=1}^{D/2-1} B_i \Psi_i^\dagger - B_i^\dagger \Psi_i \right)$$

If we assign the standard ghost charge +2 to ϵ , then the insertions of the coupling t must be compensated by the insertions of the coupling s , so the answer may only depend on the combination st . On the other hand, it is easy to repeat the derivation of [14] by first taking the limit $s \rightarrow \infty$ with g much smaller than s . In this way one gets an effective action which is schematically of the form:

$$S_{eff} \sim \frac{1}{s} \{Q_\epsilon, \text{Tr} \Psi_a [X, \mathcal{E}]\} \quad (3.5)$$

and which has ghost charge two. ² As we shall see momentarily, in the limit $s, t \rightarrow \infty$ the dependence on either variable actually vanishes, therefore the value of integral which we get is equal to the original integral (1.4).

As discussed in [14][15][11], one can now proceed to do the integrals in the semiclassical approximation for large s, t, g . We first do the Gaussian integrals to eliminate the BRST quartet $(\eta, \bar{\phi}, \bar{\chi}, \bar{H})$. This results in a determinant in the numerator of the measure of the form $\text{Det}(\epsilon + ad(\phi))$ where the determinant is evaluated in the representation space of the equations. Proceeding with the Gaussian integrals on (B_i, Ψ_i) produces determinants of the form $\text{Det}(\epsilon + ad(\phi))$ in the denominator. Finally, taking into account the Vandermonde factor in reducing the integral on ϕ from $\text{Lie}(G)$ to $\mathfrak{t} = \text{Lie}(T)$ we obtain the integral:

$$I_{D=10}(N) = \left(\frac{(E_1 + E_2)(E_2 + E_3)(E_3 + E_1)}{E_1 E_2 E_3 E_4} \right)^{N-1} \frac{N}{N!} \int_{\mathfrak{t}} \mathcal{D}\phi \prod_{i \neq j} \frac{P(\phi_{ij})}{Q(\phi_{ij})}$$

$$P(x) = x(x + E_1 + E_2)(x + E_3 + E_2)(x + E_1 + E_3) \quad (3.6)$$

$$Q(x) = \prod_{\alpha=1}^4 (x + E_\alpha + i0)$$

for the case $D = 10$. Here $\sum_\alpha E_\alpha = 0$ and the integral is taken along the real line. Similarly, the same procedure gives the integral:

$$I_{D=6}(N) = \left(\frac{E_1 + E_2}{E_1 E_2} \right)^{N-1} \frac{N}{N!} \int_{\mathfrak{t}} \mathcal{D}\phi \prod_{i \neq j} \frac{\phi_{ij}(\phi_{ij} + E_1 + E_2)}{\prod_{\alpha=1}^2 (\phi_{ij} + E_\alpha + i0)} \quad (3.7)$$

² One might worry that the original integrals and the ones we are getting at this point differ by exponentially small terms, as in [14]. The difference with the situation of [14] is that due to the absence of topologically non-trivial solutions to the equations on finite-dimensional matrices there are no extra contributions to the integral coming from infinity.

for $D = 6$, and can be obtained from (3.6) by taking a *formal* limit $E_3 \rightarrow \infty$. Finally, for $D = 4$ the integral is:

$$I_{D=4}(N) = \frac{N}{N!E_1^{N-1}} \int_{\underline{\mathbf{t}}} \mathcal{D}\phi \prod_{i \neq j} \frac{\phi_{ij}}{(\phi_{ij} + E_1 + i0)} \quad (3.8)$$

and can be obtained from (3.7) by taking a *formal* limit $E_2 \rightarrow \infty$.

The factor $\frac{N}{N!}$ has the following origin. The denominator is the order of the Weyl group of $SU(N)$ which enters in passing to the integral over the conjugacy classes of ϕ . We then rewrite this integral as an integral over $\underline{\mathbf{t}}$, divided by $|W(G)| = N!$. The numerator N is the order of the center of \mathbf{Z}_N which appears in comparing the volumes of $SU(N)$ and G . The measure $\mathcal{D}\phi$ is defined as follows. The maximal Cartan subalgebra of $SU(N)$ can be identified with \mathbb{R}^{N-1} by means of the imbedding:

$$(\phi_1, \dots, \phi_{N-1}) \rightarrow \text{diag}(\phi_1, \dots, \phi_{N-1}, -\phi_1 - \dots - \phi_{N-1})$$

into the space of traceless hermitian matrices. The measure $\mathcal{D}\phi$ is simply the normalized Euclidean measure on \mathbb{R}^{N-1} :

$$\mathcal{D}\phi = \prod_{k=1}^{N-1} \frac{d\phi_k}{2\pi i} \quad . \quad (3.9)$$

Finally, as mentioned in the introduction, it might appear that the integrals (3.8)(3.7)(3.6) are ill-defined since they are integrals along \mathbb{R}^{N-1} with a measure that generically approaches 1 at ∞ . This is an illusion. They should be regarded as contour integrals and become convergent once a contour deformation prescription is adopted. We will find such prescription in every case. The prescription $E \rightarrow E + i0$ is required for the validity of the Gaussian integrations, but we still must give a prescription for closing the contours. We expect that the contour prescriptions found below will follow from a more careful implementation of the technique of integrating out BRST quartets than we have yet performed.

4. Detailed evaluation for low values of N

4.1. Two-body problem

We begin by evaluating the integral (3.6) for the $D = 10$ case:

$$I = \frac{1}{2\pi i} \frac{P'(0)}{Q(0)} \int_{\mathbb{R}} d\phi \frac{P(2\phi)P(-2\phi)}{Q(2\phi)Q(-2\phi)}, \quad (4.1)$$

$$P(x) = x(x + E_1 + E_2)(x + E_3 + E_2)(x + E_1 + E_3)$$

$$Q(x) = \prod_{\alpha=1}^4 (x + E_\alpha + i0)$$

In order to evaluate it we close the contour in the upper half plane (this is an example of the “prescription” alluded to above) and pick up the contribution of four poles, at $\phi = \frac{1}{2}E_\alpha + i0$. The residue at E_α turns out to be

$$\text{Res}_{\frac{1}{2}E_\alpha + i0} = \frac{1}{12} \frac{R(-2E_\alpha)}{E_\alpha R'(E_\alpha)} \quad (4.2)$$

where

$$R(x) = \prod_{\alpha=1}^4 (x - E_\alpha)$$

and the sum over the residues can be evaluated using an auxiliary contour integral:

$$\sum_{\alpha=1}^4 \text{Res}_{\frac{1}{2}E_\alpha} = \frac{1}{12} \left(\oint \frac{R(-2x)}{xR(x)} dx - 1 \right) = 5/4 \quad (4.3)$$

For lower D 's the same formula (4.2) holds, and the equation (4.3) gives

$$\frac{1}{12} \left(2^{D/2-1} + (-1)^{D/2} \right) \quad (4.4)$$

i.e. the famous 5/4, 1/4, 1/4 for $D = 10, 6, 4$ respectively originally computed in [16][7].

4.2. Three-body problem

The formalism we have developed so far is rather powerful. In fact, it is still possible to evaluate the integral for $N = 3$ directly. Let $x = \phi_1 - \phi_2$, $y = \phi_2 - \phi_3 = 2\phi_2 + \phi_1$. The measure can be rewritten as:

$$d\phi_1 \wedge d\phi_2 = \frac{1}{3} dx \wedge dy$$

Specializing (3.6)(3.7) to this case we find 3 sets of possible poles. The first set is given by:

$$x \in \{E_1 + i0, E_2 + i0, E_3 + i0, E_4 + i0\} \quad \text{and} \quad y \in \{E_1 + i0, E_2 + i0, E_3 + i0, E_4 + i0\}, \quad (4.5)$$

The second set is

$$x \in \{E_1 + i0, E_2 + i0, E_3 + i0, E_4 + i0\} \quad \text{and} \quad x + y \in \{E_1 + i0, E_2 + i0, E_3 + i0, E_4 + i0\}, \quad (4.6)$$

and the third set is:

$$x + y \in \{E_1 + i0, E_2 + i0, E_3 + i0, E_4 + i0\} \quad \text{and} \quad y \in \{E_1 + i0, E_2 + i0, E_3 + i0, E_4 + i0\}. \quad (4.7)$$

We order the $+i0$'s appropriately so that $\text{Im}(E_\alpha - E_\beta) > 0$ for $\alpha > \beta$. In the $D = 6$ case we have similar sets of poles but with E_1, E_2 present without E_3, E_4 .

In evaluating the integral we choose poles from the first set but only take the second or third set (but not both). It is straightforward to evaluate the residues. For example, for the 5+1 case $x = E_1, y = E_1$ gives:

$$\left(\frac{E_1 + E_2}{E_1 E_2}\right)^2 \frac{E_1^2 E_2^2 (2E_1 + E_2)(3E_1 + E_2)}{3(E_1 + E_2)^2 (E_1 - E_2)(2E_1 - E_2)} \quad (4.8)$$

while the residue vanishes for $x = E_1, y = E_2$, with a similar contribution with $1 \leftrightarrow 2$. Thus the sum of the first set of poles gives:

$$\left(\frac{E_1 + E_2}{E_1 E_2}\right)^2 \frac{2E_1^2 E_2^2 (4E_1^2 + 5E_1 E_2 + 4E_2^2)}{3(E_1 + E_2)^2 (E_1 - 2E_2)(2E_1 - E_2)} \quad (4.9)$$

Choosing (4.6), and not (4.7), the contribution $x + y = E_2, x = E_1$ gives:

$$-\left(\frac{E_1 + E_2}{E_1 E_2}\right)^2 \frac{E_1^2 E_2^2 (2E_1 + E_2)(E_1 + 2E_2)}{(E_1 + E_2)^2 (E_1 - 2E_2)(2E_1 - E_2)} \quad (4.10)$$

The sum of (4.9) and (4.10) is $1/3$, which leads to $1/3^2$ for the net answer. With a little more work one can check that in the 9+1 case we obtain $Z = \frac{1}{3} (3 + 1/3)$ (again, with $1/3$ coming from the factor $\frac{d\phi_1 \wedge d\phi_2}{dx \wedge dy}$).

5. $SU(N)$, $D = 4$

For the $D = 4$ case we may use the Bose-Cauchy identity:

$$\frac{1}{E_1^N} \prod_{i \neq j} \frac{\phi_{ij}}{(\phi_{ij} + E_1 + i0)} = \sum_{\sigma \in \mathcal{S}_N} (-1)^\sigma \prod_{i=1}^N \frac{1}{\phi_i - \phi_{\sigma(i)} + E_1 + i0} \quad (5.1)$$

Of all the terms in (5.1) only the cycles of maximal length N can contribute to the residue evaluation (and there are $(N - 1)!$ of those). The integral (3.8) will pick up a residue for all i except one (let us denote it by j) provided that

$$\phi_{\sigma(i)} = \phi_i + E_1 + i0, \quad \text{for all } i \neq j.$$

By relabelling the indices with the help of the Weyl group we can assume that $j = N$ and the permutation σ is a long cycle $\sigma(i) = i + 1$. The pole is at

$$\phi_i = \frac{1}{2}(2i - N - 1)E_1 \quad (5.2)$$

and the residue is equal to: $\frac{1}{N^2}$.³

We prove this fact by taking the integral over the variables ϕ in the following order: $\phi_{N-1}, \phi_{N-2}, \dots, \phi_1$. In the sequel E_1 should read as $E_1 + i0$.

Given the fact that $\sigma = (123 \dots N)$ we need to evaluate:

$$\frac{N \frac{E_1(-1)^{N-1}}{N(2\pi i)^{N-1}} \oint \prod_{i=1}^{N-2} \frac{d\phi_i}{\phi_i - \phi_{i+1} + E_1}}{(2\phi_{N-1} + \phi_1 + \dots + \phi_{N-2} + E_1)(-2\phi_1 - \phi_2 - \dots - \phi_{N-1} + E_1)} \quad (5.3)$$

(the factor N in the denominator is the order of the stabilizer of σ in the Weyl group: $N!/(N-1)!$ and the sign $(-1)^{N-1}$ is $(-1)^\sigma$ for the long cycle). Let us prove by induction that the integral (5.3) reduces to

$$\frac{\frac{kE_1(-1)^{N-k}}{(k+1)^2(2\pi i)^{N-k}} \oint \prod_{i=1}^{N-k-1} \frac{d\phi_i}{\phi_i - \phi_{i+1} + E_1}}{\left(\phi_{N-k} + \frac{1}{k+1}(\phi_1 + \dots + \phi_{N-k-1}) + \frac{k}{2}E_1\right) \left(-\phi_1 - \frac{1}{k+1}(\phi_2 + \dots + \phi_{N-k}) + \frac{k}{2}E_1\right)} \quad (5.4)$$

For $k = 1$ this expression is identical to (5.3). Now let us take the ϕ_{N-k} integral. By closing the contour in either the upper or the lower half plane (it doesn't matter) we pick up either one or two residues. For simplicity we always close the integral in the lower half-plane, meaning that:

$$\phi_{N-k} = -\frac{k}{2}E_1 - \frac{1}{k+1}(\phi_1 + \dots + \phi_{N-k-1}) \quad (5.5)$$

By evaluating the residue we immediately see that the declared form of the integral is reproduced with the replacement $k \rightarrow k+1$. Finally, for $k = N-1$ we get

$$\frac{E_1(N-1)(-1)}{N^2 2\pi i} \oint \frac{d\phi_1}{\left(\phi_1 + \frac{N-1}{2}E_1\right) \left(-\phi_1 + \frac{N-1}{2}E_1\right)} = \frac{1}{N^2} \quad (5.6)$$

Hence, the $D = 4$ integral is equal to

$$I_{D=4}(N) = \frac{1}{N^2} \quad (5.7)$$

³ Notice that $\phi \in \underline{\mathfrak{t}}$ can be expressed as $\phi = \rho \cdot E_1$ where ρ is half the sum of the positive roots.

Note that the integral has been localized to the fixed point of the \mathbb{C}^* action on the quotient of the space of regular traceless matrices B by the adjoint action of the group $SL_N(\mathbb{C})$.⁴ Indeed, the ϕ from (5.2) solves the equation

$$[B, \phi] = E_1 B \quad (5.8)$$

for $B_{ij} = \delta_{i,j-1}$. On general principles we expect the integral to localize to the Q_ϵ fixed-points. Of course, the equation (5.8) has other, more non-trivial, solutions. In fact, for every Jordan cell decomposition

$$B = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & J_k \end{pmatrix} \quad (5.9)$$

for J_l being a Jordan block of length n_l , $\sum_l n_l = N$ we get a solution to (5.8) of the form:

$$\phi = \begin{pmatrix} \varphi_1 & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \varphi_k \end{pmatrix} \quad (5.10)$$

where $\varphi_l = f_l \text{Id}_{n_l} + \text{diag}(\frac{1}{2}(2i - n_l - 1)E_1)$, $i = 1, \dots, n_l$. The parameters f_l are only constrained by the requirement that $\text{Tr}\phi = 0$, which leaves $k - 1$ free zero modes. But the presence of extra zero modes is equivalent to the statement that the integrand in (3.8) can't pick up sufficiently many residues. Before eliminating the redundant fields every mode of ϕ came together with a bunch of superpartners, fermionic modes among them. By supersymmetry the unlifted modes, the f_l 's, correspond to the extra fermionic modes which make the integral vanish. We thus obtain the following important principle: *The fixed points with extra $U(1)$'s left unbroken don't contribute to the index.* It is interesting to compare this principle with the one derived in [17] in a seemingly different context.

6. $SU(N)$, $D = 6$

In this case we rewrite the integral (3.7) as:

$$\frac{1}{(N-1)!} \left(\frac{E_1 + E_2}{E_1 E_2} \right)^{N-1} \frac{1}{(2\pi i)^N} \oint \frac{d\phi_1 \wedge \dots \wedge d\phi_N}{\phi_1 + \dots + \phi_N} \prod_{i \neq j} \frac{\phi_{ij}(\phi_{ij} + E_1 + E_2)}{\prod_{\alpha=1}^2 (\phi_{ij} + E_\alpha + i0)} \quad (6.1)$$

⁴ A group element is *regular* if its centralizer in $SL_N(\mathbb{C})$ has dimension $N - 1$.

We next perform the change of variables:

$$\phi_i \mapsto \tilde{\phi}_i = \phi_i + \sum_{j=1}^{N-1} \phi_j \quad i = 1, \dots, N. \quad (6.2)$$

The measure gets an extra factor $\frac{1}{N}$:

$$\frac{d\phi_1 \wedge \dots \wedge d\phi_N}{\phi_1 + \dots + \phi_N} = \frac{1}{N} \frac{d\tilde{\phi}_1 \wedge \dots \wedge d\tilde{\phi}_N}{\tilde{\phi}_N} \quad (6.3)$$

and we may rewrite (6.1) as:

$$\begin{aligned} & \frac{(E_1 + E_2)^{N-1}}{N(2\pi i)^N (E_1 E_2)^{N-1}} \oint \frac{d\tilde{\phi}_1 \wedge \dots \wedge d\tilde{\phi}_N}{\tilde{\phi}_N} \times \prod_{i \neq j} \frac{\phi_{ij}(\phi_{ij} + E_1 + E_2)}{\prod_{\alpha=1}^2 (\phi_{ij} + E_\alpha + i0)} = \\ & \frac{E_1 E_2}{N(2\pi i)^N (E_1 + E_2)} \oint d\phi_1 \wedge \dots \wedge d\phi_N \prod_{i < N} (-\phi_i) \prod_i (\phi_i + E_1 + E_2) \times \\ & \times \frac{\prod_{i \neq j} \phi_{ij}}{\prod_i (-\phi_i)(\phi_i + E_1 + E_2)} \prod_{i,j} \frac{(\phi_{ij} + E_1 + E_2)}{(\phi_{ij} + E_1)(\phi_{ij} + E_2)} \end{aligned} \quad (6.4)$$

where in the second line we made a substitution: $\tilde{\phi} \rightarrow \phi$ and in the denominators $E_\alpha \rightarrow E_\alpha + i0$. The factor $(N-1)!$ disappears for the following reason. The choice of ϕ_N breaks the permutation group to \mathcal{S}_{N-1} . We can fix the latter symmetry by ordering the eigenvalues ϕ_i . As we shall see later, in assigning the poles of the integral (6.1) to Young tableaux each tableau yields a definite way of ordering the eigenvalues which takes up the whole of \mathcal{S}_{N-1} .

Despite the seemingly senseless manipulation we have arrived at an integral we can make sense of and in fact evaluate. In order to explain its meaning we recall that the solutions to the equation $[B_1, B_2] = 0$ modulo conjugation describe the symmetric product of \mathbb{C}^2 away from singularities and in fact provide a certain resolution of singularities, once appropriate stability conditions are imposed. These stability conditions can be formulated by introducing an auxiliary vector $I \in \mathbb{C}^N$. Then the stable data consists of a triple $Z = (B_1, B_2, I)$, such that $[B_1, B_2] = 0$ and there is no proper B_1, B_2 invariant subspace of \mathbb{C}^N which contains I . The triples (B_1, B_2, I) and $(g^{-1}B_1g, g^{-1}B_2g, g^{-1}I)$ are considered equivalent for any $g \in \text{GL}_N(\mathbb{C})$. It can be shown that the equivalence classes of such data Z are in one-one correspondence with codimension N ideals \mathcal{I}_Z in $\mathbb{C}[z_1, z_2]$.⁵ The

⁵ Briefly, $V_Z \equiv \mathbb{C}[z_1, z_2]/\mathcal{I}_Z$ is an N -dimensional complex vector space. The linear operators B_1, B_2 are the operations of multiplication by z_1, z_2 , respectively, projected to endomorphisms of V_Z . The vector I is the image of $1 \in \mathbb{C}[z_1, z_2]$. The inverse map proceeds by identifying the span of $\{B_1^n B_2^m \cdot I\}_{n,m \geq 0}$ with V_Z . This is explained in details in Theorem 1.14, page 10, of [18].

set of all codimension N ideals in the polynomial ring $\mathbb{C}[z_1, z_2]$ forms what is called the “Hilbert scheme of N points on \mathbb{C}^2 ,” and is denoted by $\mathcal{H}_N = (\mathbb{C}^2)^{[N]}$. The quotients $V_Z = \mathbb{C}[z_1, z_2]/\mathcal{I}_Z$ are the fibers of a rank N vector bundle \mathcal{E} over \mathcal{H}_N . The Chern roots of \mathcal{E} are nothing but $-\phi_i$'s. The space \mathcal{H}_N is acted on by the complex torus $\mathbf{T} = \mathbb{C}^* \times \mathbb{C}^*$ by rotation of the coordinates (z_1, z_2) :

$$(z_1, z_2) \mapsto (e^{E_1} z_1, e^{E_2} z_2). \quad (6.5)$$

This action lifts to the action on the data (B_1, B_2, I) as follows:

$$(B_1, B_2, I) \mapsto (e^{E_1} B_1, e^{E_2} B_2, I) \quad (6.6)$$

The action of \mathbf{T} on \mathcal{E} is defined through the identification of the fiber $\mathcal{E}_{(B_1, B_2, I)}$ with the vector space $\mathbb{C}[B_1, B_2]I$. Let Q be the topologically trivial \mathbf{T} -equivariant rank 2 vector bundle over \mathcal{H}_N whose isotypical decomposition coincides with that of the space \mathbb{C}^2 with coordinates z_1, z_2 . The integral (6.4) computes the Euler character of a certain \mathbf{T} -equivariant bundle \mathcal{F}_N over \mathcal{H}_N . To be more precise, we need the virtual bundle given by:

$$\mathcal{F}_N = (Q \oplus \mathcal{E} \oplus \mathcal{E}^* \otimes \wedge^2 Q) \ominus (\det \mathcal{E} \oplus \wedge^2 Q). \quad (6.7)$$

This bundle has virtual dimension $2N$. The Euler classes of the various factors can be recognized in the integrand of (6.4). For example, the Euler class of $\mathcal{E}^* \otimes \wedge^2 Q$ is the product $\prod_i (\phi_i + E_1 + E_2)$, while the incomplete product $\prod_{i < N} (-\phi_i)$ gives, roughly speaking, the class of $\mathcal{E} - \det \mathcal{E}$. The factors involving Q lead to the overall factors involving E_i , and the third line of (6.4) is a measure factor for integration over \mathcal{H}_N .

The evaluation of (6.4) by residues is equivalent to the use of fixed point techniques (see [19][15] for more examples of such techniques). We now make a slight detour and remind the reader of the ideology behind such computations [20][14]. Suppose one wishes to compute the integral

$$\int e^{-\frac{S}{\hbar}} DX$$

in the quasiclassical approximation $\hbar \rightarrow 0$. In general one has to take into account a certain set of critical points of S and include the determinants of the matrix of second derivatives of S . Some integrals have the property of having exact quasiclassics. One should take into account all critical points of S and compute the determinants which would have in

general ± 1 signs for unstable critical points. One famous example of such an integral is the Duistermaat-Heckmann formula:

$$\int_{M^{2m}} \frac{\omega^m}{m!} e^{-tH} = \sum_{p: dH(p)=0} \frac{e^{-tH(p)}}{\prod_{i=1}^m tm_i(p)} \quad (6.8)$$

where (M, ω) is a symplectic manifold with Hamiltonian $U(1)$ action generated by H , p 's are the fixed points of the $U(1)$ action (assuming they are isolated) and $m_i(p)$ are the weights of the $U(1)$ action in the tangent space to M at the fixed point p . Of course, there exist generalizations of this formula for other manifolds, groups other than $U(1)$, non-isolated fixed points and so on. In the problem of present interest it turns out that the fixed points are enumerated by Young tableaux D with $\#D = N$ boxes.⁶ In other words, consider the partition $N = \nu_1 + \dots + \nu_{\nu'_1} = \nu'_1 + \dots + \nu'_{\nu_1}$. Let (α, β) denote the position of a box in the Young tableau. There is the one-to-one correspondence between the labels $i \in \{1, \dots, N\}$ and the allowed pairs (α, β) : $1 \leq \alpha \leq \nu_\beta$, $1 \leq \beta \leq \nu'_\alpha$, given by the lexicographic order $((\alpha, \beta) > (\alpha', \beta')$ if $\alpha < \alpha'$ or $\beta < \beta'$ for $\alpha = \alpha'$). In particular, $(1, 1) \leftrightarrow N$. The corresponding eigenvalues ϕ_i are given by:

$$\phi_{(\alpha, \beta)} = (\alpha - 1)E_1 + (\beta - 1)E_2 \quad (6.9)$$

One can evaluate the residue at (6.9) using the results of [18]. Namely, in [18] it is proven that for a Young tableau D and the set ϕ_i given by (6.9) the following sum:

$$\sum_{i, j \in D} [e^{\phi_{ij}} + e^{\phi_{ij} + E_1 + E_2} - e^{\phi_{ij} + E_1} - e^{\phi_{ij} + E_2}] - \sum_{i \in D} [e^{-\phi_i} + e^{\phi_i + E_1 + E_2}] \quad (6.10)$$

is equal to:

$$- \sum_{(\alpha, \beta) \in D} e^{(\nu_\beta - \alpha + 1)E_1 + (\beta - \nu'_\alpha)E_2} + e^{(\alpha - \nu_\beta)E_1 + (\nu'_\alpha - \beta + 1)E_2} \quad (6.11)$$

In fact, in [18] the weight decomposition of the tangent space to $(\mathbb{C}^2)^{[N]}$ at the fixed point corresponding to D is computed. It is encoded in the formula (6.11). We simply have to take the product of those weights, which will go into the denominator. In addition we need to take into account the decomposition of the bundle \mathcal{F}_N into weight subspaces and

⁶ We thank V. Ginzburg for very clear explanation of this fact. In the language of ideals \mathcal{I}_Z the fixed points are the ideals which are spanned by $z_1^a z_2^b$ with $a \geq \nu_b, b \geq \nu'_a$ [21]. It explains the formula for the weights $\phi_{(\alpha, \beta)}$ below.

compute the product of those weights, which will go into the numerator. We simply use the fact that the weights of \mathcal{E} are given by ϕ_i 's. Combining these two products we arrive at:

$$Y_D \equiv \text{contribution of } D = (-)^{N-1} E_1 E_2 \frac{\prod_{(\alpha,\beta) \neq (1,1)} ((\alpha-1)E_1 + (\beta-1)E_2) (\alpha E_1 + \beta E_2)}{\prod_{(\alpha,\beta)} ((\nu_\beta - \alpha + 1)E_1 + (\beta - \nu'_\alpha)E_2) ((\alpha - \nu_\beta)E_1 + (\nu'_\alpha - \beta + 1)E_2)} \quad (6.12)$$

Now what remains is to sum over all Young tableaux D .

One can check that the explicit pole prescriptions found above for the $SU(2)$, $SU(3)$ cases are reproduced by the poles associated to Young diagrams. Moreover, as a further illustration (and to have a look at the case with N non-prime) we write out all the residues for the $SU(4)$ case: there are five Young tableaux, a column (4), a hook (3, 1), a box (2, 2), the mirror hook (2, 1, 1) and a row (1, 1, 1, 1) (in the brackets we listed the values of ν_α 's). Let $x = E_2/E_1$. The contributions are:

column	(4)	$-\frac{1}{4} \frac{(1+2x)(1+3x)(1+4x)}{(1-x)(1-2x)(1-3x)}$	
hook	(3, 1)	$-\frac{1}{2} \frac{(1+2x)(1+3x)(x+2)}{(1-x)^2(-1+3x)}$	
box	(2, 2)	$-\frac{1}{2} \frac{(1+2x)(x+2)(1+x)^2}{(1-2x)(2-x)(1-x)^2}$	
hook	(2, 1, 1)	$-\frac{1}{2} \frac{(1+2x)(x+3)(x+2)}{(1-x)^2(-x+3)}$	(6.13)
row	(1, 1, 1, 1)	$-\frac{1}{4} \frac{(x+2)(x+3)(x+4)}{(x-1)(x-2)(x-3)}$	
the sum		$\frac{1}{4}$	

which together with the $1/4$ factor from the measure gives $1/16$ as the answer.

The general answer is also expected to be E_1, E_2 independent. Looking at (6.12) we see that the factors which contain single E_1 's cancel out. Indeed, in the numerator these come from $\beta = 1$ in the factors $(\alpha - 1)E_1 + (\beta - 1)E_2$, producing

$$E_1^{\nu_1} (\nu_1 - 1)! \quad (6.14)$$

In the denominator the single E_1 's come from $\beta = \nu'_\alpha$ in the factors $(\nu_\beta - \alpha + 1)E_1 + (\beta - \nu'_\alpha)E_2$, giving rise to the product:

$$E_1^{\nu_1} \prod_{\alpha=1}^{\nu_1} (\nu_{\nu'_\alpha} - \alpha + 1) = E_1^{\nu_1} (\nu_1 - 1)! \quad (6.15)$$

Hence, single E_1 's cancel out and the limit $E_1 \rightarrow 0$ is well-defined. It is easy to see that all other factors cancel out except for the overall sign $(-)^{\nu_1 - N}$, coming from comparing the products $\prod_{\beta < \nu'_\alpha} (\beta - \nu'_\alpha)$ and $\prod_{\nu'_\alpha \geq \beta > 1} (\beta - 1)$. Thus we are left with:

$$Y_D = \frac{(-)^{\nu_1 - 1}}{N} \frac{(\nu_1 - 1)!}{\prod_\alpha (\nu'_{\nu'_\alpha} - \alpha + 1)} \quad (6.16)$$

Scary as it seems, the expression (6.16) can be represented in a very simple form. The way to do it is to combine the factors in the denominator into the groups with constant ν'_α . A little mental exercise shows that the result can be represented as follows:

$$Y(q) = \sum_D q^{\#D} Y_D = \sum_{\{\ell_\gamma\}, \sum_\gamma \ell_\gamma > 0, \gamma=1,2,\dots, \ell_\gamma \geq 0} \frac{q^{\sum_\gamma \gamma \ell_\gamma}}{\sum_\gamma \gamma \ell_\gamma} (-)^{\sum_\gamma \ell_\gamma} \frac{(\sum_\gamma \ell_\gamma - 1)!}{\prod_\gamma \ell_\gamma!} \quad (6.17)$$

Here ℓ_γ represent yet another way of partition N into the sum of positive integers:

$$N = \sum_{\gamma=1}^{\infty} \gamma \ell_\gamma$$

and $\ell_\gamma = \#\{\alpha | \nu'_\alpha = \gamma\}$, in particular $\nu_1 = \sum_\gamma \ell_\gamma$. The rest is easy: represent the factorial in the numerator of (6.17) and $\sum_\gamma \gamma \ell_\gamma$ in the denominator with the help of integrals:

$$\begin{aligned} Y(q) &= - \int_0^\infty ds \int_0^\infty \frac{dt}{t} e^{-t} \sum_{\{\ell_\gamma\}} \prod_{\gamma=1}^{\infty} \frac{(-tq^\gamma e^{-s\gamma})^{\ell_\gamma}}{\ell_\gamma!} = \\ &= - \int \int_0^\infty \frac{dsdt}{t} e^{-t} \left(e^{-t \frac{qe^{-s}}{1-qe^{-s}}} - 1 \right) = - \int \int_0^\infty \frac{dsdt}{t} \left(e^{-\frac{t}{1-qe^{-s}}} - e^{-t} \right) = \\ &- \int ds \log(1 - qe^{-s}) = \text{Li}_2(q) = \sum_{N=1}^{\infty} \frac{q^N}{N^2} \end{aligned} \quad (6.18)$$

So we get:

$$I_{D=6}(N) = \frac{1}{N^2} \quad (6.19)$$

just as in the $3 + 1$ case.

It is probably worth pointing out that the last stage of computations is very similar to those performed in [22] in the course of proving that the contribution to a prepotential of an isolated rational curve sitting in Calabi-Yau manifold equals $\text{Li}_3(q)$.

Another important remark is that a faster way of getting the equality $I_{D=6} = I_{D=4}$ is by taking the limit $E_2 \rightarrow \infty$. One might also attempt to take the limit $E_3 \rightarrow \infty$ in the $D = 10$ integral. This needn't (and in fact doesn't) work because the sum rule $\sum E_\alpha = 0$ then forces $E_4 \sim -E_3 \rightarrow \infty$ too, and the contour integration is "pinched" between the poles. Pinching poles in a contour integral is a well-known source of discontinuity.

7. $SU(N)$, $D = 10$

This section concludes our tour of the matrix integrals. In principle the integral (3.6) may be computed by summing over a set of generalized Young tableaux (as we did above for $N = 2, 3$). It turns out, however, that there is a shorter route to the answer, which avoids working with any new integrals. The strategy is to reduce the number of matrices by enforcing deformed octonionic instanton equations. As opposed to section 2 where we were basically taking strong coupling limits here we are taking mixed weak and strong coupling limits, imposing the weak coupling limit to enforce some of the equations.

Let us take $E_\alpha = 0$ for all α . Introduce the formal variable m . Consider the expression

$$\Phi_{ij} = [B_i, B_j] - m\epsilon_{ijk4}B_k \quad 1 \leq i, j \leq 4 \quad (7.1)$$

The instanton equations may now be deformed to

$$\mathcal{E}_{ij} = \Phi_{ij} - \frac{1}{2}\epsilon_{ijkl}\Phi_{kl}^\dagger \quad (7.2)$$

Note that

$$\frac{1}{2} \sum_{1 \leq i, j \leq 4} \text{Tr} \mathcal{E}_{ij} \mathcal{E}_{ij}^\dagger = \sum_{1 \leq i, j \leq 4} \text{Tr} \Phi_{ij} \Phi_{ij}^\dagger \quad (7.3)$$

Hence the equations $\mathcal{E}_{ij} = 0$ imply:

$$[B_i, B_j] = m\epsilon_{ijk4}B_k, \quad (7.4)$$

$$[B_4, B_k] = 0. \quad (7.5)$$

The equations (7.4) are formally the equations for the vacua of $\mathcal{N} = 4$ broken down to $\mathcal{N} = 1$ (see [23]). Equation (7.5) implies that B_4 generates the gauge transformations in the complexified unbroken group.

Now let us take separate couplings g', g'' for the equations \mathcal{E}_{ij} and for the equation $\sum_{i=1}^4 [B_i, B_i^\dagger]$ respectively (we can do this without spoiling Q -symmetry). Take the limit $g' \rightarrow 0$. This limit enforces equations (7.4)(7.5). We also split the coupling $\frac{1}{\hat{g}} \text{Tr}[X_a, \phi]^2$ as follows:

$$\frac{1}{\hat{g}} \sum_{a=1}^4 \text{Tr}[X_a, \phi]^2 \rightarrow \frac{1}{\hat{g}'} \sum_{i=1}^3 \text{Tr}[B_i, \phi]^2 + \frac{1}{\hat{g}''} \text{Tr}[B_4, \phi]^2 \quad (7.6)$$

Upon taking the limit $\hat{g}' \rightarrow 0$ we enforce the equations $[B_i, \phi] = 0, i = 1, 2, 3$.

Adopting the argument that extra $U(1)$'s kill the contributions to the partition function we only have to count the vacua where the adjoint gauge group is broken down to $SU(d)/\mathbb{Z}_d$, for $N = ad$. For these vacua:

$$B_\alpha = \|L_\alpha\|_{a \times a} \otimes \text{Id}_{d \times d} \quad (7.7)$$

for $\alpha = 1, 2, 3$, L_α being $SU(2)$ generators in the a -dimensional irreducible representation of $SU(2)$. Also, we have

$$(B_4)_{N \times N} = \text{Id}_{a \times a} \otimes (B_4)_{d \times d}, \quad (\phi)_{N \times N} = \text{Id}_{a \times a} \otimes (\phi)_{d \times d}. \quad (7.8)$$

In the limit we are taking we can integrate out the $B_\alpha, \alpha = 1, 2, 3$ degrees of freedom, leaving behind B_4, ϕ . Accordingly, we recognize that we have exactly the degrees of freedom present in the integral $I_{D=4}(d)$ for gauge group $SU(d)/\mathbb{Z}_d$. Moreover, due to supersymmetry, not only the degrees of freedom but also the measure is appropriate to interpret the integral as $I_{D=4}(d)$. Now, we showed above that $I_{D=4}(d) = 1/d^2$. Thus, we conclude that the answer is:

$$I_{D=10}(N) = \sum_{d|N} \frac{1}{d^2} \quad (7.9)$$

and in particular is equal to $1 + 1/N^2$ only for N prime. The term with $d = 1$ comes from the vacuum with completely broken gauge group.

8. Comparison with partition functions of susy gauge theory on T^4 and $K3$

There are some interesting relations of the integrals $I_D(N)$ with other well-studied partition functions. First there is a relation with 5-branes. It is worth noting that the q^0 term in the partition function of N fivebranes wrapped on $K3$ proposed in [17] reproduces the answer (7.9) for all N . One must divide by 24, which is the Euler characteristics of the moduli space of a center of mass of $D0$ branes moving on $K3$. The partition function is computed by wrapping the worldvolume of the fivebranes on $K3 \times T^2$, which by a series of T - and S -dualities can be mapped to the problem of N $D4$ -branes wrapped on $K3$ and N $D0$ branes bound to it. The q^0 term counts the zero $D0$ -brane charge sector in the effective gauge theory. Presumably, by a Fourier-Mukai-Nahm-duality of $K3$ surface one can map this problem to the problem of N $D0$ branes in ten dimensions, by taking the limit of very large $K3$ surface on which N $D0$ branes propagate.

A more direct connection is that between $I_D(N)$ and partition functions of SYM on tori. Consider $SU(N)/\mathbb{Z}_N$ $\mathcal{N} = 4$ SYM on T^4 , viewed as the theory of N $D3$ -instantons wrapped on T^4 with the center of mass motion factored out (otherwise the partition function vanishes). Again, the mass perturbation breaks the theory to $\mathcal{N} = 1$ with unbroken groups without $U(1)$'s being⁷

$$SU(d)/\mathbb{Z}_d, \quad ad = N \quad (8.1)$$

The $SU(d)$ $\mathcal{N} = 1$ theory has d vacua, each contributing 1 to the partition function and their total contribution is d . The partition function of $\mathcal{N} = 1$ $SU(d)/\mathbb{Z}_d$ theory is d^3 times smaller, since the partition function of $SU(d)$ contained as a factor the number of \mathbb{Z}_d flat connections (d^4) and the volume of $SU(d)/\mathbb{Z}_d$ is d times smaller, see [23] for more detailed explanations. Hence the partition function of $\mathcal{N} = 1$ $SU(d)/\mathbb{Z}_d$ gauge theory on the four-torus⁸ is equal to

$$\frac{1}{d^2} \quad (8.2)$$

and the partition function of the $SU(N)/\mathbb{Z}_N$ $\mathcal{N} = 4$ theory is given by:

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_N}^{\mathcal{N}=4}(T^4) = \sum_{d|N} \frac{1}{d^2} \quad (8.3)$$

Then the T -duality presumably relates the partition function of N $D3$ branes wrapped on T^4 to that of $D(-1)$ instantons in ten dimensions. This concludes the proof of the conjecture of [8].

For lower numbers of supersymmetries the partition functions of the $SU(N)/\mathbb{Z}_N$ are easy. For $\mathcal{N} = 1$ as we argued we get $\frac{1}{N^2} = N/N^3$, where N in the numerator is Witten's index [24] and the factor N^3 is the effect of the center $\mathbb{Z}_N \subset SU(N)$. For $\mathcal{N} = 2$, standard lore says that by the mass perturbation the theory reduces to $\mathcal{N} = 1$ and this perturbation does not affect the value of the partition function [25]. So, we get:

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_N}^{\mathcal{N}=1,2}(T^4) = \frac{1}{N^2} \quad (8.4)$$

For the minimal supersymmetric three-dimensional gauge theory with the gauge group $SU(N)$ Witten's index is equal to 1. The effect of flat \mathbb{Z}_N connections is now $N^{3-1} = N^2$ thus leading to the same answer

$$\mathcal{Z}_{SU(N)/\mathbb{Z}_N}^{\mathcal{N}=1,2}(T^3) = \frac{1}{N^2} \quad (8.5)$$

⁷ We thank C. Vafa for the clarifying discussion on this point

⁸ In the zero 't Hooft magnetic flux sector

One could also get this answer by adding a Chern-Simons term to the SYM Lagrangian (suitably accompanied by the fermions so as to preserve some susy, see [26][10]) and then analytically continuing in k - the coefficient in front of the CS term. It would be interesting to see whether the above answer could be reproduced by the finite dimensional integral of the sort we have considered in the paper. As has been pointed out in [7] for even N and subsequently argued in [12] for all N the $D = 3$ integral should vanish. The reason (at least for even N) being that the fermionic Pfaffian is odd under the parity reversal $X \rightarrow -X$. On the other hand, by adding the Chern-Simons-like term:

$$k (\text{Tr} X[\phi, \bar{\phi}] + \psi\eta)$$

and integrating out all massive modes we arrive at the integral of the same form as the one for $D = 4$, which should be equal to $\frac{1}{N^2}$ thus providing an agreement with the field theory computation. Clearly, this CS-like term violates parity. On the other hand, the original integral is not obviously absolutely convergent, therefore the parity arguments may be invalid. It would be interesting to resolve this puzzle.

Another interesting question, but one which is beyond the scope of this paper, is the applications to the IKKT model [27]. In fact, our technique allows for the derivation of regularized correlation functions of the operators $\text{Tr}\phi^{n_1} \dots \text{Tr}\phi^{n_k}$.

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