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Euclidean-signature Supergravities, Dualities and Instantons[◇]

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ABSTRACT

We study the Euclidean-signature supergravities that arise by compactifying $D = 11$ supergravity or type IIB supergravity on a torus that includes the time direction. We show that the usual T-duality relation between type IIA and type IIB supergravities compactified on a spatial circle no longer holds if the reduction is performed on the time direction. Thus there are two inequivalent Euclidean-signature nine-dimensional maximal supergravities. They become equivalent upon further spatial compactification to $D = 8$. We also show that duality symmetries of Euclidean-signature supergravities allow the harmonic functions of any single-charge or multi-charge instanton to be rescaled and shifted by constant factors. Combined with the usual diagonal dimensional reduction and oxidation procedures, this allows us to use the duality symmetries to map any single-charge or multi-charge p-brane soliton, or any intersection, into its near-horizon regime. Similar transformations can also be made on non-extremal p-branes. We also study the structures of duality multiplets of instanton and $(D-3)$-brane solutions.

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1 Introduction

The study of dimensional reduction from eleven-dimensional supergravity or type IIB supergravity is of great interest for a variety of reasons. In particular, the U-duality symmetries [1] become more apparent in lower dimensions. Not only are the lower-dimensional supergravity theories of intrinsic interest in their own right, but they also provide an organised way of studying classes of solutions to the higher-dimensional equations of motion that possess certain continuous symmetries. This is particularly relevant for the study of the various p-brane solitons that play such a central rôle in some of the recent advances in understanding duality symmetries in string theory and M-theory. The most commonly considered symmetries are translational symmetries along the spatial directions in the p-brane world-volume. Since the standard kind of p-brane solution is Poincaré invariant on the world-volume, it follows that the solution can be diagonally dimensionally reduced, from a p-brane in $D+1$ dimensions to a $(p-1)$ brane in D dimensions. The consistency of the Kaluza-Klein dimensional reduction procedure ensures that if the p-brane solves the $(D+1)$-dimensional equations of motion, then the $(p-1)$-brane will solve the D-dimensional equations of motion. Thus the spatially dimensionally reduced D-dimensional supergravities provide the arena within which the dimensionally reduced, or "wrapped," brane configurations can be described.

It is also, of course, the case that the standard p-brane solutions are *static* (or *stationary* in the case of rotating configurations). Thus there is also an isometry in the time direction, and so it is possible to interpret such configurations from a lower-dimensional point of view as solutions of Euclidean-signature supergravities. These theories are obtained from the usual Minkowskian-signature eleven-dimensional supergravity, or type IIB supergravity, by performing a sequence of Kaluza-Klein reductions that includes a reduction on the time direction. Such Euclidean-signature theories have been much less well studied than the Minkowskian-signature ones (but see [2]) , and it is with various aspects of the former that we shall be principally concerned in this paper.

Among our results, one of the more striking is that in nine dimensions there are actually two inequivalent Euclidean-signature maximal supergravities, one that is obtained from the reduction of type IIA supergravity on the time direction, and the other that comes from the reduction of type IIB supergravity on the time direction. Usually, if a spatial reduction is performed, the two nine-dimensional theories are equivalent, up to real field redefinitions. This is the field-theory precursor of the T-duality of the type IIA and type IIB strings. However, in the time reduction that we are considering here, the two nine-dimensional theories are distinct, and cannot be related to one another by any valid field redefinition. It is only after a further reduction of the two Euclidean-signature theories to $D = 8$ that an equivalence emerges.

One of the motivations for investigating Euclidean-signature supergravities is to study the instanton states, which necessarily live in Euclidean-signature space. Unlike p-branes with $p \geq 0$, which are supported by higher-degree field strengths, and which form linear representations under U-duality group, the instantons are supported by axionic scalars, which transform non-linearly under U-duality. The orbits of the higher p-branes in M-theory are much better understood, and were obtained in [3, 4]. In this paper, we shall study the U-duality transformations of instanton solutions, and also the orbits of their charges, which are the Noether charges of the global symmetry group.

Another of our results is concerned with the properties of the instanton solutions that are the natural end-points of a sequence of diagonal reductions of p-branes, when the reduction has encompassed the entire world-volume including the time direction. We show that all instanton solutions, including multi-charge ones and even non-extremal ones, have the property that they can be transformed, using $SL(2, \mathbb{R})$ global duality symmetries of the lower-dimensional theories, into solutions where the harmonic functions characterising the solutions are shifted and scaled by constants. In particular, the shifts can be chosen so as to remove the constant terms in the harmonic functions altogether, with the result that for extremal p-branes the entire solution is of the form that was previously approached only asymptotically in the near-horizon limit. The solutions can then be oxidised back to higher dimensions, by retracing the sequence of reduction steps. They then describe p-branes again, but now with similarly shifted harmonic functions. Thus the asymptotic structure of *any* extremal p-brane can be modified, by such duality transformations, to have their near-horizon form. In the case of p-branes where the dilatons are finite on the horizon, this means that the solutions are mapped into the AdS$\times$Sphere form, where the supersymmetry is enhanced. For non-extremal p-branes, the structure of the outer horizon is governed by a function that is not transformed under the $SL(2, \mathbb{R})$ symmetry, and so the effect of the modifications to the harmonic functions is more complicated. A similar idea for transforming the asymptotic structure of solutions was first developed in [5], using a different procedure in which a sequence of T-duality and S-duality transformations were used to map the p-brane to a wave, on which a general coordinate transformation was then performed, followed by a retracing of the steps of the duality transformations. This again has the effect of shifting and scaling the harmonic functions. More recently, another approach was given in which the $SL(2,\mathbb{R})$ duality of the Euclideanised type IIB theory was used instead of the general coordinate transformation on a wave [6]. Our approach is simpler than either of these, since it just involves diagonal reduction and oxidation, with an $SL(2,\mathbb{R})$ transformation on the instanton at the bottom of the chain.

The paper is organised as follows. In section 2, we construct the bosonic sectors of the Euclidean-signature maximal supergravities that are obtained by dimensional reduction on a torus that includes the time direction. We also give an explicit demonstration that the resulting lower-dimensional theories are insensitive to the order in which the time and spatial reduction steps are performed. In section 3, we discuss the global symmetries of the D-dimensional Euclidean-signature theories, showing that they have the same $E_{n(+n)}$ form as in the case of Minkowskian signature, where $n = 11 - D$. However, the denominator groups in the description of the scalar cosets are no longer the maximal compact subgroups of $E_{n(+n)}$, but instead certain non-compact forms of the previous denominator groups, and we determine these for all D. In section 4, we consider the nine-dimensional Euclidean-signature theory obtained by reducing type IIB supergravity on the time direction, and we show that it is inequivalent to the nine-dimensional theory obtained by reducing type IIA supergravity on the time direction. In section 5, we examine extremal instanton solutions in an $SL(2,\mathbb{R})$-invariant Euclidean-signature theory, discussing in detail how the symmetry acts on the solutions. We also comsider non-extremal instantons, and show that solutions exist only in an enlarged theory with at least a $GL(2,\mathbb{R})$ global symmetry. We also study the effects of the global symmetry transformations on the asymptotic structures of p-branes. In section 6, we consider instantons in an $SL(3,\mathbb{R})$-invariant theory. The action of the global symmetries on instanton solutions in this case gives a better understanding of the general situation when a number of different axions are capable of supporting the solution. In section 7, we consider $(D-3)$-brane solutions. Although these are in some sense the magnetic duals of the instantons, their structure is very different for a variety of reasons, including the fact that they live in Minkowskian-signature theories, and that their transverse spaces are only two-dimensional. The paper ends with a discussion in section 8.

2 Kaluza-Klein reduction on time

The standard categories of p-brane soliton solutions in supergravities can be extended to the case of $p = -1$. These (-1)-branes have "world-volumes" of dimension $p+1 = 0$, and so all the dimensions are occupied by the transverse space. This means that there is no longer any timelike dimension, and the solution is an instanton in a purely Euclidean-signature space. There are two ways that such Euclidean-signature theories can arise. The first is if we take a standard supergravity theory in a D-dimensional spacetime, and perform a Wick rotation of the time coordinate and reformulate the theory in a D-dimensional space of Euclidean metric signature. This is a potentially problematic procedure; it might well be that the original Lorentzian-signature supergravity involved the use of fermions satisfying a Majorana condition, which can no longer be covariantly imposed if the spacetime signature is altered. Or, as in the the case of the type IIB theory in $D = 10$, the self-duality constraint on the 5-form field strength cannot be imposed if the spacetime is Euclideanised.

A much more satisfactory situation obtains in cases where a supergravity theory is dimensionally reduced on its time direction. In such a case, the resulting lower-dimensional theory naturally arises with a Euclidean-signature metric, and the consistency of the reduction procedure guarantees that any Majorana or self-duality constraints will be compatible with the Euclidean signature. From the point of view of the p-brane solutions, the instantons can be viewed as the final stage of a sequence of diagonal dimensional reductions, in which the time dimension of the "world-volume" of a 0-brane, or static black hole, is dimensionally reduced in the final reduction step.

In this paper, we shall principally focus our attention on Euclidean-signature supergravities of this latter type, which are obtained by dimensional reduction on the time coordinate. In order to study these theories in detail, it is useful to repeat an analysis given in [7], for the single-step Kaluza-Klein reduction of a the metric tensor and a generic gauge potential of degree $(n-1)$, but where we now take the reduction to be on the time coordinate. Combined with the usual rules for spacelike reductions, we can follow a route from $D = 11$ or $D = 10$ to any desired lower dimension D, where the time reduction occurs at any desired stage in the process.

2.1 Bosonic Lagrangians

Let us suppose that we start with the Lagrangian

$$\mathcal{L} = \hat{e}\,\hat{R} - \tfrac{1}{2}\hat{e}\,(\partial\hat{\phi})^2 - \frac{1}{2n!}\hat{e}\,e^{\hat{a}\hat{\phi}}\hat{F}_{(n)}^2\ , \tag{2.1}$$

in $(D+1)$ spacetime dimensions, where $\hat{F}_{(n)} = d\hat{A}_{(n-1)}$. We now perform a Kaluza-Klein reduction on the time coordinate, making the ansätze

$$\begin{aligned} d\hat{s}^2 &= e^{-2\alpha\varphi}\,ds^2 - e^{2(D-2)\alpha\varphi}\,(dt + \mathcal{A}_{(1)})^2\ ,\\ \hat{A}_{(n-1)}(x,t) &= A_{(n-1)}(x) + A_{(n-2)}(x)\wedge dt\ , \end{aligned}$$

$$\hat{\phi}(x,t) = \phi(x) \ , \tag{2.2}$$

where $\alpha = (2(D-1)(D-2))^{-1/2}$. Substituting into (2.1), we obtain the reduced Lagrangian in D spatial dimensions:

$$\begin{aligned}\mathcal{L} &= e\,R - \tfrac{1}{2} e\,(\partial\phi)^2 - \tfrac{1}{2} e\,(\partial\varphi)^2 + \tfrac{1}{4} e\, e^{2(D-1)\alpha\varphi}\,\mathcal{F}_{(2)}^2 \\ &\quad - \tfrac{1}{2\,n!}\, e\, e^{2(n-1)\alpha\varphi + \hat{a}\phi}\, F_{(n)}^2 + \tfrac{1}{2\,(n-1)!}\, e\, e^{-2(D-n)\alpha\varphi+\hat{a}\phi}\, F_{(n-1)}^2 \ ,\end{aligned} \tag{2.3}$$

where

$$\begin{aligned} F_{(n)} &= dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} \ , \\ F_{(n-1)} &= dA_{(n-2)} \ . \end{aligned} \tag{2.4}$$

The reduced Lagrangian (2.3) differs from the usual one that arises from a reduction on a spacelike coordinate in the signs of the kinetic terms for $\mathcal{F}_{(2)}$ and $F_{(n-1)}$.

We are now in a position to present the general results for the form of the D-dimensional maximal supergravity, where one of the dimensional reduction steps may be on the timelike coordinate. The Kaluza-Klein ansatz for the reduction of the metric will be [9]

$$ds_{11}^2 = e^{-\frac{1}{3}\vec{a}\cdot\vec{\phi}}\, ds_D^2 + \sum_i \varepsilon_i\, e^{2\vec{\gamma}_i\cdot\vec{\phi}}\,(dz^i + \mathcal{A}_{(1)}^i + \mathcal{A}_{(0)j}^i\, dz^j)^2 \ , \tag{2.5}$$

where $\vec{\gamma}_i = -\frac{1}{6}\vec{a} + \frac{1}{2}\vec{b}_i$, with $\vec{a}$ and $\vec{b}_i$ being the dilaton vectors for $F_{(4)}$ and $\mathcal{F}_{(2)}^i$, as defined in [8, 9], as discussed below. The constants ε_i are $+1$ for spacelike coordinate reduction steps, and -1 for a timelike step. In the notation of [8, 9], the D-dimensional Lagrangian will therefore be

$$\begin{aligned}\mathcal{L} &= eR - \tfrac{1}{2} e\,(\partial\vec{\phi})^2 - \tfrac{1}{48} e\, e^{\vec{a}\cdot\vec{\phi}}\, F_{(4)}^2 - \tfrac{1}{12} e \sum_i \varepsilon_i\, e^{\vec{a}_i\cdot\vec{\phi}}\,(F_{(3)i})^2 \\ &\quad -\tfrac{1}{4} e \sum_{i<j} \varepsilon_i\,\varepsilon_j\, e^{\vec{a}_{ij}\cdot\vec{\phi}}\,(F_{(2)ij})^2 - \tfrac{1}{4} e \sum_i \varepsilon_i\, e^{\vec{b}_i\cdot\vec{\phi}}\,(\mathcal{F}_{(2)}^i)^2 \\ &\quad -\tfrac{1}{2} e \sum_{i<j<k} \varepsilon_i\,\varepsilon_j\,\varepsilon_k\, e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,(F_{(1)ijk})^2 - \tfrac{1}{2} e \sum_{i<j} \varepsilon_i\,\varepsilon_j\, e^{\vec{b}_{ij}\cdot\vec{\phi}}\,(\mathcal{F}_{(1)j}^i)^2 + \mathcal{L}_{FFA} \ ,\end{aligned} \tag{2.6}$$

where the dilaton vectors $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$, $\vec{a}_{ijk}$, $\vec{b}_i$, $\vec{b}_{ij}$ are constants that characterise the couplings of the dilatonic scalars $\vec{\phi}$ to the various gauge fields. Their detailed expressions, together with the Kaluza-Klein modifications in the various field strengths, are given in [8,9]. $\mathcal{L}_{FFA}$ is the Wess-Zumino term, whose detailed expression can be found in [8,9]; since this is written without the use of the metric, it is the same as in the usual purely spatial reductions. Note that the indices $i, j \ldots$ range over the internal compactified dimensions, starting with $i = 1$

for the reduction step from $D = 11$ to $D = 10$. Thus in the situation where the N'th reduction step is on the time coordinate, the signs of certain of the kinetic terms will be reversed, relative to the Minkowskian-signature case, as is evident in (2.6). Specifically, we can see that these are the kinetic terms for all field strengths that carry an internal index equal to the value N.

2.2 Commutativity of time and space reductions

We have seen that the D-dimensional Euclidean-signature theory that is obtained by compactifying on the time coordinate at the step i, and on spatial coordinates at all other steps, is given by eqn (2.6) with the signs of all the kinetic terms whose fields involve the index value "i" reversed. It is not *a priori* obvious that the Euclidean-signature theory in D dimensions is the same regardless of which step is chosen for the time reduction. In this section, we present a proof which demonstrates that all such D-dimensional theories are in fact related by valid field redefinitions. Specifically, we shall concentrate on the scalar subsectors of the D-dimensional theories. The proof extends to the full bosonic Lagrangians.

To do this, we note that the Lagrangian (2.6) was obtained directly from the dimensional reduction of $D = 11$ supergravity without any dualisations. This Lagrangian has a $GL(11-D, \mathbb{R}) \ltimes R^q$ global symmmetry, where $q = \frac{1}{6}(11-D)(10-D)(9-D)$ [10,9]. (The $E_{n(+n)}$ global symmetry [11] is achieved by performing all Hodge dualisations that turn higher-degree fields into lower-degree ones.) The $GL(11-D, \mathbb{R})$ symmetry is generated by the full set of dilatonic and axionic scalars coming from the metric; the higher-degree fields and the remaining axionic scalars coming from the reduction of the 3-form potential in $D = 11$ form linear representations under $GL(11-D, \mathbb{R})$. We recall from [9] that the scalar coset manifold for $SL(11-D, \mathbb{R})$ can be parameterised in the Borel gauge as

$$\mathcal{V} = e^{\frac{1}{2}\vec{\phi}\cdot\vec{H}}\, h\ , \tag{2.7}$$

where $\vec{H}$ is the set of Cartan generators for the $SL(11-D, \mathbb{R})$ global symmetry group, $\vec{\phi}$ are the dilatons, and h is a parameterisation of the exponential of the positive-root algebra of $SL(11-D, \mathbb{R})$, with the axionic fields $\mathcal{A}^i_{(0)j}$ as the parameters, *i.e.*

$$h = \prod_{i<j} e^{\mathcal{A}^i_{(0)j} E_i{}^j}\ , \tag{2.8}$$

with the terms in the product arranged in anti-lexical order, namely

$$(i, j) = \cdots (3, 4), (2, 4), (1, 4), (2, 3), (1, 3), (1, 2)\ . \tag{2.9}$$

The scalar Lagrangian for the $SL(11-D,\mathbb{R})$ part is given by

$$\mathcal{L} = \tfrac{1}{4} e\, \mathrm{tr}(\partial_\mu \mathcal{M}^{-1}\, \partial^\mu \mathcal{M})\ , \tag{2.10}$$

where the matrix $\mathcal{M}$ is defined by

$$\mathcal{M} = \mathcal{V}^T\, \eta\, \mathcal{V}\ , \tag{2.11}$$

and η is a metric tensor. In the usual case where one compactifies $D = 11$ supergravity on a set of spatial directions, η is just the identity. If instead the i'th compactification coordinate is the time coordinate, the results of section 2.1 show that the metric η will have the form $\eta = \eta(i)$, where

$$\eta(i) \equiv \mathrm{diag}(1, \cdots, 1, -1, 1 \cdots, 1)\ , \tag{2.12}$$

and the -1 occurs at the i'th position. Note that as we shall shown in section 3, the Lagrangian (2.10) describes the coset of $SL(11-D,\mathbb{R})/O(10-D,1)$ when the time is one of the internal coordinates, rather than $SL(11-D,\mathbb{R})/O(11-D)$ when the internal directions are all spatial.

We wish to show that the D-dimensional Euclidean-signature theories where the time compactification occurs at the i'th position are equivalent, up to field redefinitions, for all i. We may show this in the following way. The i'th theory is characterised completely by the fact that the matrix $\mathcal{M}$ in (2.11) is constructed using $\eta = \eta(i)$, where $\eta(i)$ is defined in (2.12). In order to show that the theories for all i are equivalent, we need to show that there exist field redefinitions that relate them all. To do this, consider the i'th theory, and then make the following field redefinition

$$\vec\phi \longrightarrow \vec\phi' = \vec\phi + \frac{i\pi}{2}\, \vec b_{ij}\ . \tag{2.13}$$

It is evident from (2.7) that this will transform the matrix $\mathcal{M}(i) = \mathcal{V}^T\, \eta(i)\, \mathcal{V}$, defined in (2.11), according to

$$\mathcal{M}(i) \longrightarrow \mathcal{M}'(i) = \mathcal{V}^T\, e^{\frac{i\pi}{4}\vec b_{ij}\cdot \vec H}\, \eta(i)\, e^{\frac{i\pi}{4}\vec b_{ij}\cdot \vec H}\, \mathcal{V}\ . \tag{2.14}$$

In fact we may take $\vec H$ and $\eta(i)$ to be diagonal, and so we simply have

$$\mathcal{M}'(i) = \mathcal{V}^T\, \eta'(i)\, \mathcal{V}\ . \tag{2.15}$$

where

$$\eta'(i) = e^{\frac{i\pi}{2}\vec b_{ij}\cdot \vec H}\, \eta(i)\ . \tag{2.16}$$

To evaluate the expression $e^{\frac{i\pi}{2}\vec{b}_{ij}\cdot\vec{H}}$, we may make use of the fact that the positive-root generators $E_i{}^j$ associated with the $SL(11-D,\mathbb{R})$ subalgebra of the $E_{n(+n)}$ global symmetry algebra satisfy the commutation relations [9]

$$[\vec{H}, E_i{}^j] = \vec{b}_{ij}\, E_i{}^j \ . \tag{2.17}$$

Furthermore, the dilaton vectors $\vec{b}_{ij}$ have the property that $\vec{b}_{ij}\cdot\vec{b}_{k\ell}$ is equal to 0 if i and j are different from k and ℓ, while it equals 4 if $i=k$ and $j=\ell$, and ± 2 if there is just one index in common between i, j, and k, ℓ. Since $E_i{}^j$ can be represented by the matrix consisting of zeroes everywhere except for a "1" at the i'th row and j'th column, we can deduce that the diagonal matrix $\vec{b}_{ij}\cdot\vec{H}$ has entries equal to 2 mod 4 at the i'th and j'th positions, and entries equal to 0 mod 4 at all other positions, on the diagonal. Thus we have that

$$e^{\frac{i\pi}{2}\vec{b}_{ij}\cdot\vec{H}} = \mathrm{diag}\,(1,\ldots 1,-1,1\ldots,1,-1,1,\ldots 1)\ , \tag{2.18}$$

where the -1 entries are at positions i and j. We see that the metric $\eta'(i)$ defined in (2.16) is therefore simply given by

$$\eta'(i) = \eta(j)\ . \tag{2.19}$$

The conclusion from this is that if we start from the D-dimensional Euclidean-signature Lagrangian in which the time reduction was performed at step i, and make the field redefinition (2.13), we end up with the Lagrangian that would be obtained by making the time reduction instead at step j.

So far we have concentrated on the scalars coming from the metric, which generate the $SL(11-D,\mathbb{R})$ global symmetry. The field redefintion (2.13) also provides proper sign changes for the kinetic terms of the rest of the scalars and the higher forms as well. This can be seen from the fact that the dot products of the dilaton vectors $(\vec{a},\vec{a}_i,\vec{a}_{ij},\vec{a}_{ijk},\vec{b}_i)$ for all the other fields with the dilaton vector $\vec{b}_{\ell m}$ gives either ± 2 if there is one common index, or 0 or 4 if there is either no common index or two common indices. Note that the field redefition (2.13) does not alter the signs of the kinetic terms for all dilatonc scalars $\vec{\phi}$. It has the effect of shafling the signs of the signs of kinetic terms of axions, and higher-form potentials. However, the total number of fields with positive kinetic terms for each degree (and hence that of fields with negative knietic terms) is preserved under this field redefinition.

One might wonder about the validity of this construction, in view of the fact that the field redefinition (2.13) involves making an imaginary constant shift of the dilatons. It is indeed true that in general complex field redefinitions on real fields are not permissable as a way of demonstrating the equivalence of ostensibly different theories. However, the crucial point here is that the scalar manifolds in question are coset spaces, and provided that the reality of the coset matrices $\mathcal{M}$ that are used in the construction of the Lagrangians (2.10) is maintained, then the redefined $\vec{\phi}$ fields, even though subjected to imaginary shifts, still provide a valid parameterisation, and the imaginary parts have no physically-observable consequences. And indeed, we have seen that the redefinition (2.13) simply has the effect of replacing the real metric $\eta(i)$ in (2.15) by the real metric $\eta(j)$, thus making manifest the continued reality of the redefined matrix $\mathcal{M}'(i)$.

In fact the transformation relating the step-i and the step-j Lagrangians can be viewed abstractly as a transformation between the coset matrices $\mathcal{M}(i)$ and $\mathcal{M}(j)$, rather than a transformation implemented explicitly on the coset coordinates $\vec{\phi}$ and χ_a. In general, we can allow any transformation of the form

$$\mathcal{M}(i) \longrightarrow \mathcal{M}(j) = \Lambda\,\mathcal{M}(i)\,\Lambda^{-1}\ , \tag{2.20}$$

where Λ is in the $E_{n(+n)}$ numerator group, since the Λ factors will cancel out in the Lagrangian (2.10). If Λ is taken to be the identity, then this transformation happens to be implementable in the form (2.13), in terms of the parameterisation of $\mathcal{V}$ given by (2.7). For other parameterisations, or for other choices of Λ, the specific form that the transformation (2.20) induces on the coordinates of the coset will be different, and can, for example, be arranged to be real, at least in some coordinate patch. We give an example of this later, in the case of an $SL(3,\mathrm{I\!R})/O(2,1)$ coset. It should be emphasised, however, that the equivalence of the two Lagrangians is proved once the existence of a real transformation between their respective coset matrices $\mathcal{M}$ is established. Exhibiting explicit real, rather than complex, coset-coordinate relations that implement this real transformation between the coset matrices may be desirable for some purposes, but it is an inessential part of the proof of equivalence of the Lagrangians.

It is worth mentioning also that the transformation (2.13) not only preserves the reality of the coset matrices $\mathcal{M}$, but it also preserves the reality of the original eleven-dimensional metric. This is an important point, since the various fields in the D-dimensional theory, including the dilatons $\vec{\phi}$, all originate from real fields in eleven dimensions. To be specific, the Kaluza-Klein ansatz for the D-dimensional metric that was used in obtaining (2.6) is given by (2.5). We now observe from [8] that $\vec{a}\cdot\vec{b}_{ij} = 0$, and $\vec{b}_k\cdot\vec{b}_{ij} = 2\delta_{ik} - 2\delta_{jk}$. Consequently, the effect of performing the field redefinition $\vec{\phi} \to \vec{\phi} + \frac{i\pi}{2}\vec{b}_{ij}$ is to leave the entire eleven-dimensional metric (2.5) unchanged, except for the replacements

$$\varepsilon_i \longrightarrow -\varepsilon_i \ , \qquad \varepsilon_j \longrightarrow -\varepsilon_j \ , \qquad \varepsilon_k \longrightarrow \varepsilon_k \ , \qquad k \neq i \ , k \neq j \ . \tag{2.21}$$

In other words, the effect of the transformation is precisely to interchange which of i and j is the compactified time-like direction. The fact that the eleven-dimensional metric remains real under the transformation (2.13) re-emphasises the fact that it is not the $\vec{\phi}$ fields themselves that are physically meaningful, but only the various exponentials of them that occur in the metric and the D-dimensional Lagrangian.

Having established that the field redefinition (2.13) maps the D-dimensional theory obtained by reducing on t at the i'th step to the theory obtained by instead reducing on t at the j'th step, it is evident that by choosing all possible dilaton vectors $\vec{b}_{ij}$ in (2.13), we can establish the equivalence of all the D-dimensional Euclidean-signature theories obtained by this method. In other words, the order in which the time and space reductions are performed is immaterial.

So far we have considered the bosonic Lagrangians obtained from dimensional reduction of the $D = 11$ Lagrangian without performing any dualisation of the fields. In order for the theories to have the $E_{n(+n)}$ global symmetry groups of the maximal supergravities, it is necessary to Hodge dualise all field strengths with degrees $> D/2$ to give fields of lesser degrees. It is well known that this dualisation procedure and Kaluza-Klein reduction on a torus commute. In fact, dualisation commutes also with the reduction on the time coordinate. Let us illustrate this by a simple example. Consider a field strength $\hat{F}_{(n)}$ in $(D+1)$-dimensional spacetime. After a dimensional reduction on the time direction, this field strength gives rise to a compact field $F_{(n)}$ and a non-compact field $F_{(n-1)}$. Here we are calling fields with the "standard" sign for their kinetic terms compact fields, while those with the non-standard sign are called non-compact fields. Now let us dualise the $\hat{F}_{(n)}$ field strength to $\hat{F}_{(D+1-n)}$ in $(D+1)$ dimensions. After dimensional reduction on the time direction, this field gives rise to a compact field $F_{(D+1-n)}$ and a non-compact field $F_{(D-n)}$. Now, Hodge dualisation in a Euclidean-signature space always has the effect of changing the sign of the kinetic term for any field of any degree, whilst dualisation in a Minkowskian-signature spacetime always leaves the sign unaltered. (Note that this dualisation property implies in particular that the signs of the kinetic terms for n-form field strengths in a Euclidean-signature space of even dimension $D = 2n$ can be reversed at will by dualisation.) Thus in a D-dimensional Euclidean-signature theory, the compact field strengths $F_{(n)}$ and $F_{(D+1-n)}$ are dual to the non-compact fields $F_{(D-n)}$ and $F_{(n-1)}$ respectively.

3 Cosets in Euclidean-signature spaces

In section 2, we obtained the bosonic Lagrangians for all the maximal supergravities in Euclidean-signature spaces that come from the dimensional reduction of eleven-dimensional supergravity with time as one of the internal dimensions. We have observed that the Lagrangians are similar to those in Minkowskian-signature spacetime, except that the signs of the kinetic terms for certain fields are reversed. In this section, we show that the sign changes in these kinetic terms do not alter the fact that these theories have $E_{n(+n)}$ global symmetries, just as in the Minkowskian-signature spacetimes. However, the denominator group H of the coset $E_{n(+n)}/H$ is no longer the maximal compact subgroup of $E_{n(+n)}$. It becomes instead a certain non-compact subgroup of the same dimension as the maximal compact subgroup. In this section, we shall determine these denominator groups for $D \geq 3$.

3.1 An $SL(2,\mathbb{R})$ example

Let us first examine the simplest non-trivial example, namely the $SL(2,\mathbb{R})$ system. The nine-dimensional scalar Lagrangian in a Euclidean-signature space is given by

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}e^{2\phi}\,(\partial\chi)^2\ , \tag{3.1}$$

together with a decoupled $O(1,1)$-invariant term $-\frac{1}{2}(\partial\varphi)^2$ that does not concern us here. This is to be contrasted with the Minkowskian-signature Lagrangian

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}\,(\partial\chi)^2\ , \tag{3.2}$$

Note that the axionic scalar χ in (3.1) comes from the dimensional reduction of the R-R vector in the $D = 10$ type IIA theory, and hence its kinetic term is ghost-like in the Euclidean-signature space. (The situation is different in the time reduction of the type IIB theory, which we shall discussion in section 4). We now show that the Lagrangian (3.1) is described by the coset $SL(2,\mathbb{R})/O(1,1)$. To see this, we note that the Lagrangian can be parameterised by the Borel subgroup of $SL(2,\mathbb{R})$. Following [9], we can parameterise an $SL(2,\mathbb{R})/O(1,1)$ coset representative $\mathcal{V}$, in the Borel gauge, as

$$\mathcal{V} = e^{\frac{1}{2}\phi H}e^{\chi E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix}\ , \tag{3.3}$$

where H and E_+ are the Cartan and positive-root generators of $SL(2,\mathbb{R})$. The Lagrangian (3.1) can then be expressed as

$$e^{-1}\mathcal{L} = \tfrac{1}{4}\mathrm{tr}(\partial_\mu\mathcal{M}^{-1}\partial^\mu\mathcal{M})\ , \tag{3.4}$$

where $\mathcal{M}$ is given by

$$\mathcal{M} = \mathcal{V}^{\mathrm{T}}\eta\mathcal{V} = \begin{pmatrix} e^{\phi} & \chi e^{\phi} \\ \chi e^{\phi} & \chi^2 e^{\phi} - e^{-\phi} \end{pmatrix} , \qquad \eta = \mathrm{diag}(1,-1) \ . \tag{3.5}$$

Note that $\mathrm{Det}(\mathcal{M}) = \mathrm{Det}(\eta) = -1$, so $\mathcal{M}$ is no longer an $SL(2,\mathbb{R})$ matrix. (In the case of the usual $SL(2,\mathbb{R})$ coset, where χ has the standard sign for its kinetic term, η would be $\mathrm{diag}(1,1)$, and hence $\mathcal{M}$ would be an $SL(2,\mathbb{R})$ matrix.)

The global $SL(2,\mathbb{R})$ transformations on the scalar fields can be implemented by acting on the right of $\mathcal{V}$ with a constant $SL(2,\mathbb{R})$ matrix Λ, and on the left with a field-dependent compensating $O(1,1)$ transformation $\mathcal{O}$, whose job is to restore the transformed $\mathcal{V}$ to the Borel gauge:

$$\mathcal{V} \longrightarrow \mathcal{V}' = \mathcal{O}\mathcal{V}\Lambda \ . \tag{3.6}$$

It is manifest that provided $\mathcal{O}$ satisfies $\mathcal{O}^{\mathrm{T}}\eta\mathcal{O} = \eta$, this will leave the Lagrangian (3.4) invariant for any global $SL(2,\mathbb{R})$ transformation. Note that if the axionic field χ had had the standard sign for its kinetic term, as in the Minkowskian-signature Lagrangian (3.2), then we would instead have $\eta = \mathrm{diag}(1,1)$, and so $\mathcal{O}$ would be an element of the compact group $O(2)$, implying that the coset would be $SL(2,\mathbb{R})/O(2)$. In our Euclidean-signature Lagrangian (3.1), however, the opposite sign for the kinetic term for χ implies that $\eta = \mathrm{diag}(1,-1)$, and hence $\mathcal{O}$ is an element of the non-compact group $O(1,1)$. Thus the Lagrangian (3.1) is described by the coset $SL(2,\mathbb{R})/O(1,1)$.

If we now introduce the pseudo-imaginary unit j, with $j^2 = 1$ and $\bar{j} = -j$, the fields χ and ϕ in this $SL(2,\mathbb{R})/O(1,1)$ system can be grouped together as the double-number valued field $\tau = \chi + je^{-\phi}$. The $SL(2,\mathbb{R})$ global symmetry transformations can then be expressed as the fractional linear transformation [12]

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} \ , \tag{3.7}$$

where $ad - bc = 1$. In the more usual Minkowskian-signature $SL(2,\mathbb{R})/O(2)$ system, j would be replaced by the unit imaginary number i.

In section 2.2, we showed that the signs of the kinetic terms of certain of the scalar fields can be altered by making the field redefinition (2.13), and by this means we established that the processes of making dimensional reductions on time and space coordinates commute. In this $SL(2,\mathbb{R})$ example, the difference between the Lagrangians (3.2) and (3.1) for the cosets $SL(2,\mathbb{R})/O(2)$ and $SL(2,\mathbb{R})/O(1,1)$ is that the sign of the kinetic term for the χ field is negative in the former case, and positive in the latter. This sign reversal could be achieved by sending $\phi \to \phi + \frac{i}{2}\pi$. One might naively deduce from this field redefinition that the cosets $SL(2,\mathbb{R})/O(2)$ and $SL(2,\mathbb{R})/O(1,1)$ were equivalent, a conclusion that is actually false. The reason for this is that under the redefinition $\phi \to \phi + \frac{i}{2}\pi$ the $\mathcal{M}$ matrix, which parameterises the points in the scalar manifold, does not remain real, unlike the situation in the cases we described in section 2.2. In particular the string coupling constant $g = e^{-\phi}$ would become imaginary. In fact, for this reason, the field redefinition precisely establishes the *inequivalence* of the two theories. Furthermore, when higher-degree field strengths $F_n = dA_{n-1}$ are included in the Lagrangian, they couple to the scalars through terms of the form of $F_n^{\rm T}\mathcal{M}F_n$. By causing the matrix $\mathcal{M}$ to become complex, the field redefinition $\phi \to \phi + \frac{i}{2}\pi$ would also have the effect of making the Lagrangian complex. This emphasises the distinction between the valid complex transformations of the kind we used in section 2.2 to show that two ostensibly different Lagrangians are actually equivalent, and more general kinds of complex transformation that change the structure of the theory. Note that the analogue of the transformation (2.13) in this $D = 9$ example is $\phi \to \phi + i\pi$, which does not change the sign of the kinetic term for χ.

The field redefinition (2.13) does have the effect of making $\mathcal{V}$ become complex, but it leaves $\mathcal{M}$ real. Of course this field redefinition is not a symmetry of the theory, since it changes the form of the Lagrangian. In fact even transformations under the global symmetries of the theory can also have the effect of causing $\mathcal{V}$ to become complex, while again leaving $\mathcal{M}$ real. The reality of $\mathcal{M}$ is guaranteed by the form of the global transformation, namely $\mathcal{M} \to \Lambda^{\rm T}\mathcal{M}\Lambda$, where Λ is a real-valued matrix in the global symmetry group G. In a Euclidean-signature space, $\mathcal{M}$ is not positive definite, and hence $\mathcal{V}$, which can be viewed as a square-root of $\mathcal{M}$, can be complex. In a Minkowskian-signature spacetime, by contrast, $\mathcal{M}$ *is* positive definite, and so $\mathcal{V}$ itself remains real under the global transformations.

3.2 Cosets for maximal supergravities in Euclidean-signature spaces

The above demonstration can easily be generalised to lower dimensions D, where the global symmetry groups are $E_{n(+n)}$ with $n = 11 - D$. The structure of $\mathcal{M}$ and (3.5), and the transformations (3.6), imply that only the denominator local compensating group elements $\mathcal{O}$ will "see" the η matrix, whilst the global group elements Λ will be unaffected by the signature change of η. This shows that changing η will not affect the global symmetry group, but it will change the local denominator group H.

To determine H, one can start by counting the numbers of scalars that have standard or non-standard signs for their kinetic terms. We shall call the fields that have standard-sign kinetic terms C-fields (compact fields), and those that have the non-standard sign

NC-fields (non-compact fields). It is easy to verify that the number of NC-scalars in the coset G/H is the same as the number of NC-generators (non-compact generators) in H. For example, if there are no NC-scalars at all in the coset, as is the case for the standard maximal supergravities in Minkowskian-signature spacetime, then H has no NC-generators, and hence it will be the maximal compact subgroup of G. In the above $SL(2,\mathbb{R})$ example, we have one NC-scalar, and hence one NC-generator in the denominator group, implying that $H = O(1,1)$. We list in Table 1 all the scalars in all the $D \geq 3$ supergravities in Euclidean-signature spaces that come from the dimensional reduction of eleven-dimensional supergravity (including the scalars that are dualisations of all $(D-2)$-form potentials).

	NC-scalars	C-scalars	$\begin{pmatrix}\text{Total scalars}\\ = Dim(G/H)\end{pmatrix}$	$Dim(G)$	$Dim(H)$
$D=10$	0	1	1	1	0
$D=9$	1	2	3	4	1
$D=8$	3	4	7	11	4
$D=7$	6	8	14	24	10
$D=6$	10	15	25	45	20
$D=5$	16	26	42	78	36
$D=4$	27	43	70	133	63
$D=3$	56	72	128	248	120

Table 1: Scalars in Euclidean-signature maximal supergravities

It is now straightforward to determine the local denomninator groups H. For example, in $D = 7$ we have that the dimension of H is 10, comprising 6 NC-generators and 4 C generators, so we have $H = O(3,2)$. In Table 2, we summarise the cosets for all the Euclidean-signature maximal supergravities in $D \geq 3$ (these results can be found also in [13, 14]).

	Minkowskian	Euclidean
$D=10$	O(1,1)	O(1,1)
$D=9$	$\frac{GL(2,\mathbb{R})}{O(2)}$	$\frac{GL(2,\mathbb{R})}{O(1,1)}$
$D=8$	$\frac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{O(3)\times O(2)}$	$\frac{SL(3,\mathbb{R})\times SL(2,\mathbb{R})}{O(2,1)\times O(1,1)}$
$D=7$	$\frac{SL(5,\mathbb{R})}{O(5)}$	$\frac{SL(5,\mathbb{R})}{O(3,2)}$
$D=6$	$\frac{O(5,5)}{O(5)\times O(5)}$	$\frac{O(5,5)}{O(5,C)}$
$D=5$	$\frac{E_{6(+6)}}{USp(8)}$	$\frac{E_{6(+6)}}{USp(4,4)}$
$D=4$	$\frac{E_{7(+7)}}{SU(8)}$	$\frac{E_{7(+7)}}{SU^*(8)}$
$D=3$	$\frac{E_{8(+8)}}{SO(16)}$	$\frac{E_{8(+8)}}{SO^*(16)}$

Table 2: Cosets for maximal supergravities in Minkowkian and Euclidean signatures

As was discussed in [10,9], maximal supergravities have global symmetries $E_{n(+n)}$ when all dualisations that lower the degrees of field strengths are performed. If instead we dimensionally reduce $D=11$ supergravity to D dimensions without performing any dualisations, then the resulting theory will have a $GL(11-D,\mathbb{R})\ltimes R^q$ global symmetry, where $q=\frac{1}{6}(11-D)(10-D)(9-D)$ [10,9]. From the point of view of perturbative string theory, another natural possibility is to dualise only R-R fields, since, unlike the NS-NS fields, they couple to the world-sheet through their field strengths only. If this is done, the global symmetry becomes $O(10-D,10-D)\ltimes R^{8-D}$. The coset structures of these theories in Minkowskian and Euclidean-signature spaces are given by

	Minkowskian	Euclidean
No-dual	$\frac{GL(11-D)\ltimes R^{\frac{1}{6}(11-D)(10-D)(9-D)}}{O(11-D)}$	$\frac{GL(11-D)\ltimes R^{\frac{1}{6}(11-D)(10-D)(9-D)}}{O(10-D,1)}$
RR-dual	$\frac{O(10-D,10-D)\ltimes R^{8-D}}{O(10-D)\times O(10-D)}$	$\frac{O(10-D,10-D)\ltimes R^{8-D}}{O(10-D,C)}$

Table 3: Cosets for non-dualised or RR-dualised maximal supergravities

4 Time reduction and type IIA/type IIB T-duality

So far we have discussed the dimensional reduction of eleven-dimensional supergravity, in cases where one of the internal directions is the time coordinate. In this section we shall give an analogous discussion for the type IIB theory, and re-examine the type IIA/IIB T-duality when a time reduction is involved.

The bosonic sector of type IIB supergravity comprises the metric, a dilaton, a self-dual 5-form (with potential B_4), NS-NS and R-R 2-form potentials ($A_2^{\rm NS}, A_2^{\rm R}$), and one axion χ. The nine-dimensional Lagrangian that results from dimensionally reducing this on a spatial S^1 can be found in [15]. In the case of a time reduction instead, it follows from the discussion in section 2.1 that we need only modify the signs of the kinetic terms for the 3-form potential coming from the reduction of B_5, and all the vector potentials, since they are NC-fields.

4.1 Type IIA/type IIB T-duality

First let us review the standard type II/type IIB T-duality when the two theories are compactified on a spatial circle S^1. The relations between the gauge potentials of the two theories reduced to $D = 9$ are summarised in Table 4.

	IIA			IIB	
	$D = 10$	$D = 9$	T-duality	$D = 9$	$D = 10$
R-R fields	$A_{(3)}$	$A_{(3)}$	$\longleftrightarrow$	$\underline{A_{(3)}}$	$B_{(4)}$
		$\underline{A_{(2)2}}$	$\longleftrightarrow$	$A^{\rm R}_{(2)}$	$A^{\rm R}_{(2)}$
	$\mathcal{A}^1_{(1)}$	$\mathcal{A}^1_{(1)}$	$\longleftrightarrow$	$\underline{A^{\rm R}_{(1)}}$	
		$\underline{\mathcal{A}^1_{(0)2}}$	$\longleftrightarrow$	χ	χ
NS-NS fields	$G_{\mu\nu}$	$\underline{\mathcal{A}^2_{(1)}}$	$\longleftrightarrow$	$\underline{A^{\rm NS}_{(1)}}$	$A^{\rm NS}_{(2)}$
	$A_{(2)1}$	$A_{(2)1}$	$\longleftrightarrow$	$A^{\rm NS}_{(2)}$	
		$\underline{A_{(1)12}}$	$\longleftrightarrow$	$\underline{\mathcal{A}_{(1)}}$	$G_{\mu\nu}$

Table 4: Gauge potentials of type II theories in $D = 10$ and $D = 9$

Note that the underlined fields are NC-fields (and therefore have plus signs in front of their kinetic terms) if the reduction from $D = 10$ to $D = 9$ is performed instead on the time coordinate. The relation between the dilatonic scalars of the two nine-dimensional theories is given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{\rm IIA} = \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{7}}{4} \\ -\frac{\sqrt{7}}{4} & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{\rm IIB} \equiv M \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{\rm IIB} . \tag{4.1}$$

Note that we have $M^{-1} = M$. The dimensional reduction of the ten-dimensional string metric to $D = 9$ is given by

$$\begin{aligned} ds^2_{\rm str} &= e^{\frac{1}{2}\phi_1}\, ds^2_{10} \\ &= e^{\frac{1}{2}\phi_1}\, (e^{-\phi_2/(2\sqrt7)}\, ds_9^2 + e^{\sqrt7\phi_2/2}\, (dz_2 + \mathcal{A})^2)\ , \end{aligned} \tag{4.2}$$

where ds^2_{10} and ds^2_9 are the Einstein-frame metrics in $D = 10$ and $D = 9$. The radius of the compactifying circle, measured using the ten-dimensional string metric, is therefore given by $R = e^{\frac14\phi_1 + \frac{\sqrt7}{4}\phi_2}$. Note that the dilaton vector $\{\frac14, \frac14\sqrt7\}$ of the radius is the eigenvector of M with eigenvalue -1. It follows that the radii $R_{\rm IIA}$ and $R_{\rm IIB}$ of the compactifying circles, measured using their respective ten-dimensional string metrics, are related by $R_{\rm IIA} = 1/R_{\rm IIB}$.

This picture of type IIA/IIB T-duality breaks down when the theory is compactified instead on the time direction. In fact it is non-perturbative states, such as D-branes, that can be held responsible for this breakdown. To see this, we first note that the scalar coset manifold for the type IIB theory is $SL(2,\mathbb{R})/O(2)$, and this will remain as a factor in the complete scalar sector in $D = 9$, regardless of whether the compactification is on the time or a space direction. On the other hand, as we saw in section 3, the coset for the Euclidean-signature $D = 9$ type IIA theory has an $SL(2,\mathbb{R})/O(1,1)$ factor. (In each case, there is an additional scalar field that is decoupled from the $SL(2,\mathbb{R})$-invariant factor.) In other words, the axionic scalars of the type IIA and type IIB theories in $D = 9$ Euclidean-signature space have opposite signs for their kinetic terms. In fact, it is easy to verify that all the R-R fields of the type IIA theory in $D = 9$ Euclidean-signature space (see Table 4) will have opposite signs for their kinetic terms, in comparison to the kinetic terms for the R-R fields of the Euclidean-signature $D = 9$ type IIB theory. On the other hand in the NS-NS sector, the signs are in agreement, since the vectors associated both with the Kaluza-Klein and the winding modes (which are interchanged on passing between type IIA and type IIB) acquire minus signs when the theories are compactified on the time direction.[1]

This sign discrepancy in the kinetic terms of the R-R fields in the type IIA and type IIB theories, and hence the breakdown of the T-duality, can also be understood from the point of view of D-brane physics. In the spatial S^1 compactification, a Dp-brane in one theory is

[1]It is, of course, possible to make a *complex* field redefinition in order to relate the two nine-dimensional Euclidean-signature theories, by following the rules given in Table 4, but with a factor of i in the identification for each R-R field. Since this would therefore relate real solutions to complex solutions both in $D = 9$ and $D = 10$, any "T-duality" would have the undesirable consequence of requiring the existence of complex solutions, even in the original Minkowskian-signature ten-dimensional theories.

dual to a D($p+1$)-brane in the other theory, due to the fact that in type IIA, Dp-branes arise only for even p, while in type IIB, they arise only for odd p. In particular, this implies that in going from $D = 10$ to $D = 9$, a D($p+1$)-brane undergoes a diagonal (double) dimensional reduction, where both the world-volume and the spacetime dimension are reduced, while a Dp-brane undergoes a vertical reduction, in which both the transverse space of the brane and the spacetime dimension are reduced. If, on the other hand, we compactify the theory on the time coordinate, then this means that only diagonal dimensional reduction of p-brane solitons is performed, since the time coordinate is always part of the world-volume. In other words, the time direction can participate only in a diagonal reduction step, but not in a vertical reduction. Since the T-duality of Dp-branes requires both double and vertical reductions, it follows that the existence of these *non-perturbative* states leads to a breakdown of T-duality in the case of a dimensional reduction in the time direction.

As a cautionary note, it should be remarked that there do in fact exist static p-brane solutions in which time is one of the coordinates of the transverse space. Such solutions can be vertically dimensionally reduced on the time direction, giving rise to p-branes in a Euclidean-signature space. However, these solutions, and hence also the original higher-dimensional solutions, will be complex. The reason for this can be seen most easily by looking in the reduced theory; the field strength supporting the p-brane will have the "wrong sign" for its kinetic term. Specifically, if the solution is supported by an electric charge, then the reduced field strength has the same degree as in the higher dimension, and so, by the results of section 2.1, it will have a minus sign in its kinetic term. On the other hand, if the p-brane carries a magnetic charge, then the reduced field strength will have a degree that is 1 less than in the higher dimension, and hence its will have a plus sign in its kinetic term. In each case, the sign is the opposite of what is needed for a real solution in a Euclidean-signature space. For convenience, a summary of the signs needed in order to have real solutions in Minkowskian and Euclidean signature spaces is given in Table 5:

	Minkowskian	Euclidean
Electric	$-F^2$	$+F^2$
Magnetic	$-F^2$	$-F^2$

Table 5: Signs of kinetic terms for real p-branes

In the case of real solutions, an extremal p-brane satisfies the usual relation $m = Q$ between the mass and the charge. When the solutions are instead complex, as a result of a wrong sign for the kinetic term, the mass and charge are instead related by $m = iQ$. It would be natural in such cases to take the mass to be real, so that the metric would be real, and therefore the charge would be imaginary.

If one takes the point of view that all lower-dimensional solutions are ultimately to be interpreted as solutions of the original ten or eleven-dimensional theories, then it would be natural to insist that all the solutions should be real. On the other hand, complex solutions might be regarded as being acceptable in Euclidean-signature theories in their own right, since in fact any electric/magnetic dual pair of p-branes in a Euclidean-signature space will necessarily have one member that is complex, as can be seen from Table 5.

However, to return to our discussion of the two inequivalent nine-dimensional Euclidean-signature theories, even if states with imaginary charge were admitted in the spectrum, this would still not imply a T-duality between the two theories, because it would require the identification of states carrying real charges with states carrying imaginary charges. This would imply that the ten-dimensional type IIA (or type IIB) theory would have complex states. Note that U-duality symmetries, which act transitively on the charge lattice, will never map a solution with a real charge to a solution with an imaginary charge. In the case of the type IIA/IIB T-duality, we should likewise expect that real solutions of the one theory should map into real solutions of the other. (Note again that as we discussed in sections 2 and 3, the reality of a solution should be judged by the reality of the charges and $\mathcal{M}$ (which are physically observable), and does not necessarily require the reality of the scalar fields themselves.)

Purely within the NS-NS sector, this problem does not arise, even for the duality between non-perturbative states such as 5-branes and NUTs. At the field-theory level, this is related to the fact that the vector potentials coming from the 2-form potential and the metric both have kinetic terms that undergo sign reversals when the reduction is on the time direction. From the point of view of the p-brane spectrum, T-duality requires that NS-NS strings or 5-branes (or waves or NUTs) in the type IIA and type IIB theories either both undergo vertical reduction, or both undergo diagonal reduction in order to match up in $D = 9$, and so again no incompatibility arises.

We may now examine how the NS-NS and R-R strings in the two nine-dimensional Euclidean-signature theories transform under the $SL(2, \mathbb{R})$ global symmetry ($SL(2, \mathbb{Z})$ at the quantum level). This symmetry acts transitively on the two-dimensional charge space of the NS-NS and R-R strings. However, it also has the effect in general of transforming the scalar moduli. There exists a (point-dependent) denominator subgroup that leaves any chosen point in the modulus space fixed. Let us first consider the nine-dimensional Euclidean-signature theory coming from the reduction of the type IIB theory on the time direction, for which the scalar coset is $SL(2,\mathbb{R})/O(2)$. Note that in this case, the electric charges for NS-NS or R-R strings will be imaginary, while their dual 4-branes will carry real magnetic charges. Without loss of generality, we may consider the string solutions at the self-dual point $\tau_0 = i$. The $O(2)$ denominator group is then of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{4.3}$$

It has the effect of continously rotating the NS-NS and R-R string (or 4-brane) charges (q_1, q_2). At the quantum level, the $O(2)$ group reduces to its Weyl group Z_2 [17], and its group elements are given by (4.3) with $\theta = 0$ and $\theta = \frac{1}{2}\pi$. The Weyl group, which leaves the moduli invariant, has the effect of making a discrete interchange between the NS-NS and R-R strings (or 4-branes).

The picture is different for the nine-dimensional theory coming from the reduction of the type IIA theory on the time direction. In this case, the NS-NS string (or the R-R 4-brane) carries an imaginary charge, while the the R-R string (or the NS-NS 4-brane) carries a real charge. It is no longer the O(2) group (4.3) that leaves the self-dual point $\tau_0 = j$ invariant, but instead the $O(1,1)$ group

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} . \tag{4.4}$$

This $O(1,1)$ group no longer acts as a rotation between NS-NS and R-R charges. In particular, a pure NS-NS solution can never be rotated to a pure R-R solution, or *vice versa*. At the quantum level, the only surviving element of the $O(1,1)$ is just the identity, corresponding to $t = 0$ in (4.4). Thus we see that the behaviour of the NS-NS and R-R strings (or the magnetic dual 4-branes) under the global $SL(2,\mathbb{R})$ symmetry is very different in the two nine-dimensional Eucludean-signature theories, coming from the reduction of either type IIA or type IIB supergravity on the time direction.

Having shown that the type IIA and type IIB theories are not equivalent when they are reduced on the time direction, it is worth pointing out that they do become equivalent when they are further reduced to $D = 8$, by compactification on a spatial circle. An easy way to understand this follows from the fact that, as we showed in section 2, the orders of time reduction and spatial reduction commute. So the reduction first on time and then on a spatial S^1 is equivalent to a reduction first on a spatial S^1 and then on time. Since the spatially-reduced theories are already equivalent in $D = 9$, this equivalence is then inherited by all the maximal supergravities in $D \le 8$ Euclidean-signature spaces. Nevetheless, it is instructive to look in detail at how the two inequivalent Euclidean-signature $D = 9$ type IIA and type IIB theories become the equivalent when they are further reduced on S^1 to $D = 8$.

To make the comparison, let us begin by considering the dimensional reduction of type IIA and type IIB first spatially on z_2, followed by a reduction on time $t = z_3$. (Note that we reserve z_1 as the internal coordinate in the compactification of $D = 11$ supergravity from $D = 11$ to $D = 10$.) Since the fields are already identified in $D = 9$, as given in Table 4, it follows that the descendents of these fields are also identified in a one-to-one fashion. If instead we first compactify the type IIA and type IIB theories on the time direction $z_2 = t$, and then on the spatial coordinate z_3, the identifications listed in the Table 4 might seem no longer to be applicable, since the R-R fields of the two theories have opposite signs for their kinetic terms. For example, the axion χ of type IIB is a C-scalar whilst the field $\mathcal{A}^1_{(0)2}$ is an NC-scalar. This seems to suggest that χ should be identified with $\mathcal{A}^1_{(0)3}$, which is also a C-scalar. However, the field strength for $\mathcal{A}^1_{(0)3}$ has a Kaluza-Klein modification, namely $\mathcal{F}^1_{(1)3} = d\mathcal{A}^1_{(0)3} - \mathcal{A}^2_{(0)3}\, d\mathcal{A}^1_{(0)2}$, whilst the field strengths for χ and $\mathcal{A}^1_{(0)2}$ have no such modifications. The Kaluza-Klein modification implies that χ can only be identified with $\mathcal{A}^1_{(0)2}$. In order to resolve the sign discrepancy of the two kinetic terms, we need to perform a field redefinition of the type given in (2.13), which has the effect of reversing the sign of the kinetic term for $\mathcal{A}^1_{(0)2}$. (Note that the sign of the kinetic term for χ cannot be changed, as discussed in section 3, since it is simply the one inherited from the χ kinetic term in $D = 10$ type IIB.) This transformation on ϕ effectively interchanges the order of the reduction on the two coordinates, so that it becomes first a spatial reduction, followed by the time reduction.

We should therefore identify the fields χ and $\mathcal{A}^1_{(0)2}$ in $D = 8$, even though the former is a C-scalar while the latter is an NC-scalar in the reduction from $D = 10$ first on time and then on space. This immediately raises an apparent paradox, in which one might think that we could just as well have done the same thing already in nine dimensions, since $\mathcal{A}^1_{(0)2}$ is a field that already exists in $D = 9$. However, there is a subtle difference because in $D = 8$, owing to the existence of the extra dilaton, there is a freedom to perform a field redefinition (2.13) whose effect is to reverse the sign of the kinetic term of $\mathcal{A}^1_{(0)2}$. This is not possible in $D = 9$, because although any field redefinition of the form (2.13) has the effect of altering which fields are compact and which non-compact, it is always in a way that preserves the total number of NC field strengths and the total number of C field strengths of each degree.[2] In $D \le 8$ there exists more than one axion, and hence (2.13) can be used to redefine which axions are NC and which are C, while keeping the total numbers of each fixed. In $D = 9$ however, $\mathcal{A}^1_{(0)2}$ is the only axion, and it is non-compact in the Euclidean-signature space. Thus a field redefinition of the form (2.13), (which preserves the reality of the physical quantities such as the matrix $\mathcal{M}$) cannot alter this non-compactness, and hence $\mathcal{A}^1_{(0)2}$ cannot be identified with the compact scalar χ coming from the type IIB theory. So it is only by descending one step further, with a spatial compactification to $D = 8$, that the identification of the type IIA and type IIB fields can be effected.

5 Instantons in $SL(2,\mathbb{R})/O(1,1)$ Lagrangians

Scalar cosets coupled to gravity can support real electric instanton solutions in Euclidean-signature spaces. In this section, we study instantons in the simplest non-trivial scalar coset, namely $SL(2,\mathbb{R})/O(1,1)$. We study the orbits of the extremal instanton solution, and show that an $SL(2,\mathbb{R})$ transformation does not alter its essential form, except for a constant shift and rescaling of the harmonic function that characterises the solution. Since such an instanton in $SL(2,\mathbb{R})/O(1,1)$ can be obtained from dimensional reduction of any p-brane on its world-volume, it follows that this duality symmetry relates any p-brane solution to its near-horizon limit. We also construct non-extremal instanton solutions and conclude that their existence requires at least a $GL(2,\mathbb{R})$ invariant scalar-manifold, extending the results described above for extremal instantons.

5.1 Orbits of extremal instantons

The Lagrangian (3.1) for the scalar coset $SL(2,\mathbb{R})/O(1,1)$, together with the Einstein-Hilbert term, admits an extremal instanton solution in D-dimensional Euclidean space [16, 7, 12]:

$$\begin{aligned} ds^2 &= dr^2 + r^2 d\Omega^2 , \\ e^{\phi} &= H , \qquad \chi = H^{-1} , \end{aligned} \tag{5.1}$$

[2]In the case of field strengths of degree n in $D = 2n$ dimensions, the counting of the C and NC fields should include their Hodge duals, since the field strengths and their duals are both included in a single irreducible representation of the global symmetry group. For example, $F_{(4)}$ and its Hodge dual in Euclidean-signature $D = 8$ supergravity form a doublet under $SL(2,\mathbb{R})$, one component of which is C, while the other is NC. For this reason although a field redefinition of the type given in (2.13), but for the case of the type IIB reduction to $D = 8$, reverses the sign of the kinetic term for $F_{(4)}$, this does not contradict the rule that the total numbers of C and NC fields are preserved.

where H is an harmonic function on the Euclidean space. For the purpose of our discussion, we may consider an isotropic solution, namely $H = 1 + Q/r^{D-2}$. The asymptotic values of the scalars $\tau_0 = \chi_0 + j e^{-\phi_0}$ for this solution are given by $\tau_0 = 1 + j$. (The solution at the self-dual point $\tau_0 = j$ can be obtained by shifting the axion χ to $\chi = H^{-1} - 1$, and indeed solutions at any other modulus point can be obtained by making constant shifts of the dilaton ϕ and the axion χ, using the Borel subgroup of $SL(2,\mathbb{R})$ transformations.) The χ and ϕ fields in the solution (5.1) can be combined to give $\tau = H^{-1}(1+j)$. Applying the $SL(2,\mathbb{R})$ transformation (3.7), we find

$$\tau' = \frac{ad + bc + bdH + j}{2cd + d^2 H} \ . \tag{5.2}$$

Thus we obtain the new solution

$$e^{\phi} = H' \equiv 2cd + d^2 H \ , \qquad \chi = {H'}^{-1} + \frac{b}{d} \ . \tag{5.3}$$

We see that the structure of the solution is unchanged, except for a constant shift and rescaling of the harmonic function H.[3] Later, in section 5.3, we shall show that this ability to shift the constant term in the harmonic function can be used in order to relate any extremal p-brane to its near-horizon limit, using the relevant $SL(2,\mathbb{R})$ subgroup of the duality group, in the dimension where the p-brane has been reduced to an instanton. We shall show that it can also be done for multi-charge p-branes, and intersections, and also that similar transformations can be made in the case of non-extremal p-branes.

The trivial modification of the harmonic function of the instanton solution under the full $SL(2,\mathbb{R})$ transformation suggests that the instanton is a singlet. In fact, we may now show that the instanton is a singlet under the discretised $SL(2,\mathbb{Z})$ symmetry of the quantum theory. We may illustrate this by examining the orbits of the charges of the instanton under the $SL(2,\mathbb{R})$ and $SL(2,\mathbb{Z})$ transformations. Instantons carry electric charges, which can be defined to be the integrals of the duals of the Noether currents for the global symmetries of the scalar coset. There are three Noether currents for the Lagrangian (3.1), with its $SL(2,\mathbb{R})/O(1,1)$ coset:

$$J_0 = -d\phi - e^{2\phi}\chi d\chi \ , \quad J_+ = e^{2\phi} d\chi \ , \quad J_- = -d\chi - 2\chi d\phi - e^{2\phi}\chi^2 d\chi \ . \tag{5.4}$$

(See appendix A for a derivation of Noether currents for arbitrary scalar coset manifolds.) J_0 and J_+ can be called Borel currents, since they are associated with the shift symmetries of the two scalars, which are generated by the Borel subgroup of $SL(2,\mathbb{R})$. The J_- current, which can be expressed as a linear combination of J_0 and J_+ with scalar-dependent

[3] This result was also obtained for D-instantons in [6].

coefficients, is associated with transformations generated by the negative root. These three Noether currents transform linearly under the adjoint representation of $SL(2,\mathbb{R})$:

$$\mathcal{J} \longrightarrow \mathcal{J}' = \Lambda^{-1}\,\mathcal{J}\,\Lambda\ , \tag{5.5}$$

where

$$\mathcal{J} = \begin{pmatrix} J_0 & J_- \\ J_+ & -J_0 \end{pmatrix}\ , \tag{5.6}$$

and Λ is a constant $SL(2,\mathbb{R})$ matrix. The charges of the instanton then can be defined as

$$\mathcal{Q} = \int *\mathcal{J} = \begin{pmatrix} Q_0, & Q_- \\ Q_+ & -Q_0 \end{pmatrix}\ , \tag{5.7}$$

which therefore transform in the same way as the Noether currents $\mathcal{J}$.

The standard global symmetry group $SL(2,\mathbb{R})$ transforms not only the charges, but also the scalar moduli, *i.e.* the asymptotic values of the scalar fields at infinity. For any point in the modulus space, there exists a (modulus-dependent) $O(1,1)$ stability subgroup that leaves the modulus fixed. We shall examine how the charges transform under this demoninator subgroup. Without loss of generality, we may consider the instanton solution at the self-dual point $\tau_0 = j$, *i.e.* $e^{\phi} = H$ and $\chi = H^{-1} - 1$. Substituting this into the expression (5.6) for the Noether currents we find that

$$\mathcal{J} = \begin{pmatrix} -dH & dH \\ -dH & dH \end{pmatrix}\ , \tag{5.8}$$

and hence the Noether charges are

$$\mathcal{Q} = \begin{pmatrix} -Q & Q \\ -Q & Q \end{pmatrix}\ , \tag{5.9}$$

where $Q = \int *dH$. Note that the $SL(2,\mathbb{R})$-invariant quadratic quantity $\mathrm{Det}(\mathcal{Q})$ vanishes for the instanton solutions. The $O(1,1)$ transformation at the self-dual point $\tau_0 = j$ is given by

$$\Lambda_{O(1,1)} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}\ , \tag{5.10}$$

and it has the effect of simply rescaling charges:

$$\mathcal{Q} \longrightarrow \mathcal{Q}' = e^{-2t}\,\mathcal{Q}\ . \tag{5.11}$$

Thus we see the classical symmetry group $O(1,1)$ does not rotate the charges in the charge lattice; rather it merely rescales the charges. In fact it has the same effect on the charge lattice as does the "trombone" symmetry [18], under which the metric is rescaled: $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$.

At the quantum level, the $O(1,1)$ degenerates to the identity group, and hence the charge cannot be changed. This implies that the instanton solution is a singlet under the $SL(2,\mathbb{Z})$ symmetry. At this point, it is instructive to compare the instanton solution with the usual p-branes supported by higher-degree field strengths. For such p-branes, the charges carried by the participating field strengths are independent parameters. In other words, for any given choice of scalar moduli, there exist solutions whose charges fill out a charge lattice. In this case, it is necessary to find a spectrum-generating symmetry that maps between the solutions whose charges lie at different points in the charge lattice, while holding the moduli fixed. It was shown in [18] that this can be done by means of a non-linearly realised duality symmmetry whose action is quite distinct from that of the standard linear action of the global supergravity symmetry on the higher-degree fields, and in particular it makes essential use of the trombone rescaling symmetry too. In our present case where we are considering instead instanton solutions, the charges are not independent parameters; instead they are related to the modulus parameters (up to trombone rescalings). Thus for any given choice of scalar moduli, there is only one charge configuration. For example, at the self-dual point the charge of any instanton solution has the same form as the one given in (5.9). Any $SL(2,\mathbb{R})$ transformation that had the effect of rotating the charges would necessarily also change the values of the scalar moduli. This shows that the instanton solutions in the $SL(2,\mathbb{R})/O(1,1)$ theory are singlets under the spectrum-generating symmetries.

5.2 Non-extremal instantons

In the previous subsection, we discussed extremal instanton solutions in the $SL(2,\mathbb{R})/O(1,1)$ system in a Euclidean-signature space. Naively one would expect that as in the cases of general p-branes, these solutions should be straightforwardly generalisable to non-extremal instanton solutions. In this subsection we show that in fact the $SL(2,\mathbb{R})/O(1,1)$ coset cannot support a non-extremal instanton that is *isotropic* in the transverse space. In fact an isotropic non-extremal instanton requires the use of an additional scalar field, meaning that we require a $GL(2,\mathbb{R})/O(1,1)$ scalar manifold in order to be able to describe it.

A non-extremal instanton in D dimensions can be obtained from the dimensional reduction on the time direction of a non-extremal static black hole in $D+1$ dimensions. But first let us discuss the dimensional-reduction properties of more general non-extremal p-branes. For simplicity, we consider first the single-charge non-extremal p-brane solution in maximal supergravity in $(D+1)$ dimensions that involves a single n-form field strength with dilaton vector $\hat{\vec{c}}$. The relevant part of Lagrangian describing the solution is then given by

$$\hat{e}^{-1}\mathcal{L}_{D+1} = \hat{R} - \tfrac{1}{2}(\partial\hat{\phi})^2 - \frac{1}{2n!}e^{\hat{a}\hat{\phi}}\hat{F}_n^2 \ , \tag{5.12}$$

with $\hat{\phi} = \hat{\vec{c}}\cdot\hat{\vec{\phi}}$ and $\hat{a} = |\hat{\vec{c}}|$. Note that for all such single-charge p-branes, we have $\hat{a}^2 = 4 - 2(n-1)(D-n)/(D-1)$. The Lagrangian allows electric non-extremal $(n-2)$-brane solutions, given by

$$\begin{aligned}
&ds^2_{D+1} = -e^{2A'}dt^2 + e^{2A}dx^i dx^i + e^{2B}(e^{-2f}dr^2 + r^2 d\Omega^2) \ , \\
&e^{2A'} = e^{2A+2f} \ , \qquad e^{2A} = H^{-\frac{\tilde{d}}{D-1}} \ , \qquad e^{2B} = H^{\frac{d}{D-1}} \ , \\
&F_n = \coth\mu \, dH^{-1}\wedge d^d x \ , \qquad \hat{\phi} = \tfrac{1}{2}\hat{a}\log H \ ,
\end{aligned} \tag{5.13}$$

where

$$H = 1 + \frac{k\sinh^2\mu}{r^{\tilde{d}}} \ , \qquad e^{2f} = 1 - \frac{k}{r^{\tilde{d}}} \ , \tag{5.14}$$

and $d = n-1$, $\tilde{d} = D-n$. Here k and μ are constants, parameterising the charge, $Q = \frac{1}{2}k\,\sinh 2\mu$ and mass per unit p-volume $m = k\,(\tilde{d}\,\sinh^2\mu + \tilde{d}+1)$ (the extremal limit is achieved by sending μ to infinity and k to zero, holding $k\,e^{2\mu}$ constant). Note that we can write $\hat{F}_n$ in terms of its potential $\hat{A}_{n-1}$, with

$$\hat{A}_{n-1} = \cosh\mu \ H^{-1}\wedge d^d x \ . \tag{5.15}$$

The solution (5.13) has an important property, namely

$$A\,d + B\,\tilde{d} = 0 \ . \tag{5.16}$$

Here we shall continue to refer the function H in (5.14) as an 'harmonic' function. The Lagrangian (5.12) also admits a magnetic $(D-n-2)$-brane in $(D+1)$ dimensions which we shall not consider further, since the discussion is analogous to that for the electric solution. Let us now consider the diagonal dimensional reduction on a world-volume coordinate of the non-extremal p-brane in $(D+1)$ dimensions to a $(p-1)$-brane in D dimensions.[4] If we dimensionally reduce the electrically-charged solution on one of the world-volume spatial

[4] *Vertical* dimensional reduction of non-extremal p-branes requires the construction of an infinite number of non-extremal p-brane in $(D+1)$ dimensions, periodically arrayed along the internal coordinate z that is to be compactified. The symmetry associated with the periodicity implies the equilibrium of the configuration, and the compactification of z implies the stability. See [21].

coordinates x^i, the supporting D-dimensional field strength will become an $(n-1)$-form, and so the relevant D-dimensional Lagrangian will be

$$e^{-1}\mathcal{L}_D = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2(n-1)!}e^{-2(D-n)\alpha\varphi+\hat{a}\hat{\phi}}F_{n-1}^2 \tag{5.17}$$

where $\alpha = 1/(2(D-1)(D-2))^{1/2}$. It was observed in [22] that when the condition $A\,d+B\,\tilde{d} = 0$ is satisfied by the p-brane solution in $D+1$ dimension, the linear combination of dilatons $a\phi = -2(D-n)\alpha\varphi + \hat{a}\hat{\phi}$ (with $a^2 = \hat{a}^2 + 4(D-n)^2\alpha^2$) that couples to $F_{(n-1)}$ in D dimensions is non-vanishing, whilst the orthogonal linear combination vanishes. This means that a single-charge non-extremal p-brane in $D+1$ dimensions reduces to a standard single-charge single-scalar non-extremal $(p-1)$-brane in D dimensions. (The discussion for the diagonal reduction of the magnetic solution is analogous, in which case the field strength $F_{(n)}$ in the lower-dimensional theory will be the relevant one that supports the $(p-1)$-brane, and its dilaton coupling will be non-vanishing whilst the orthogonal dilaton combination will vanish, leading again to a standard single-charge single-scalar solution.) If we instead reduce the p-brane solution on the time direction, then in the *extremal* case we have $e^{2f} = 1$, and hence the conclusion is the same as for reduction on a spatial world-volume direction. However, if we start the timelike dimensional reduction from a *non-extremal* p-brane solution, we have, from (5.13), $A'\,d + B\,\tilde{d} \neq 0$, which implies that the other combination of the two dilatons, orthogonal to the combination that couples to the field strength, will also be non-vanishing in D dimensions. Thus non-extremal p-brane solutions in Euclidean-signature spaces are supported by a set of fields that includes an additional scalar, which does not couple to the field strength that carries the charge.

In our present case, we are particularly interested in the non-extremal instanton solutions that can be obtained from the dimensional reduction of non-extremal black holes, which arise as solutions for the $(D+1)$-dimensional Lagrangian (5.12) with $n=2$, for example, D0-branes in $D=10$ type IIA theory. The non-extremal black hole solution is given by (5.13) with $d=1$, $\tilde{d} = D-2$. From the Kaluza-Klein ansatz

$$ds_{D+1}^2 = e^{-2\alpha\varphi}ds_D^2 - e^{2\alpha(D-2)\varphi}dt^2\ , \qquad \alpha = \frac{1}{\sqrt{2(D-1)(D-2)}}\ , \tag{5.18}$$

(the Kaluza-Klein vector is zero in this case) and from (5.13), we see that

$$e^{2(D-2)\alpha\varphi} = e^{2f}\,H^{-\frac{D-3}{D-2}} \tag{5.19}$$

for the reduction of the black hole to an instanton in D dimensions. This instanton will be of a solution of the equations following from the reduced Lagrangian

$$e^{-1}\mathcal{L}_D \;=\; R - \tfrac{1}{2}(\partial\hat{\phi})^2 - \tfrac{1}{2}(\partial\varphi)^2 + \tfrac{1}{2}e^{-2(D-2)\alpha\varphi+\hat{a}\hat{\phi}}(\partial\chi)^2$$

$$= \quad R - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{2}(\partial\phi_1)^2 + e^{2\phi_1}(\partial\chi)^2 \ , \tag{5.20}$$

where

$$\phi_1 = \tfrac{1}{2}\hat{a}\hat{\phi} - (D-2)\alpha\,\varphi \ , \qquad \phi_2 = (D-2)\alpha\,\hat{\phi} + \tfrac{1}{2}\hat{a}\varphi \ . \tag{5.21}$$

The axion χ results from the dimensional reduction of A_1, according to $A_1 \to \chi dt$. Thus in terms of these new variables the non-extremal instanton solution in D-dimensions for the Lagrangian (5.20) is given by

$$\begin{aligned} &ds^2 = e^{\frac{2f}{D-2}}(e^{-2f}dr^2 + r^2 d\Omega^2) \ , \\ &\phi_1 = -f + \log H \ , \qquad \chi = H^{-1}\coth\mu \ , \qquad \phi_2 = f\sqrt{\tfrac{D}{D-2}} \ . \end{aligned} \tag{5.22}$$

We see that the existence of a non-extremal instanton requires a Lagrangian containing at least two dilatons, and thus at least a $GL(2,\mathbb{R}) \sim \mathbb{R}\times SL(2,\mathbb{R})$ invariant scalar manifold, although the $\mathbb{R}$ factor decouples in the extremal limit. This phenomenon may be of significance in the understanding of an F-theory interpretation [23] of the type IIB theory. The ten-dimensional type IIB theory has only an $SL(2,\mathbb{R})$-invariant scalar Lagrangian. The extremal instanton solution of the Euclideanised theory was constructed in [12]. Its twelve-dimensional interpretation as a pp-wave was put forward in [25]. In this case, the scalar field associated with the volume of the two-torus in the compactification of F-theory to type IIB was considered to be non-dynamical [23][5], and indeed it decouples in the extremal instanton solution. However, this scalar associated with volume of the 2-torus would have to be non-zero in a non-extremal instanton.

In [25], a non-extremal instanton solution within the $SL(2,\mathbb{R})$ system was constructed for the type IIB theory, using the T-duality that maps D0-branes in type IIA to instantons in type IIB. Putting aside for now the previously-noted obstacles to the implementation of type IIA/IIB T-duality on the time direction, we may note also that the non-extremal instanton constructed in [25] is not isotropic in the ten-dimensional Euclidean-signature space. In other words, the solution has a $U(1)$ isometry along the Euclideanised time axis y_0, and hence the harmonic function H is given by $1 + Q(y_1^2 + \cdots y_9^2)^{-7/2}$ rather than $1 + Q(y_0^2 + \cdots y_9^2)^{-4}$. As we showed above, the *isotropic* non-extremal instanton does not exist in a system with only an $SL(2,\mathbb{R})/O(1,1)$ scalar manifold; it requires an additional independent scalar in order to support the solution. If the type IIA/IIB T-duality also implied a relationship between non-BPS states such as non-extremal p-branes, then the

[5]It was shown in [24] that it cannot simply be taken to be non-dynamical once the higher-degree fields of the theory are included.

consequent existence of a non-extremal isotropic instanton in type IIB would give supporting evidence for the existence of F-theory, since the emergence of the necessary extra scalar could easily be understood from a twelve-dimensional point of view, in parallel to the relation of nine-dimensional non-extremal instantons to non-extremal pp-waves in $D = 11$.

Since the non-extremal instanton solution is described by a $GL(2,\mathbb{R})/O(1,1)$ scalar Lagrangian, we may examine how it transforms under the $GL(2,\mathbb{R}) \sim \mathbb{R}\times SL(2,\mathbb{R})$ global symmetry. The $\mathbb{R}$ factor of transformation is straightforward, implying simply a constant shift of the scalar ϕ_2. On the other hand, the global $SL(2,R)$ transformations act on the (ϕ_1,χ) system in the standard way, while leaving ϕ_2 invariant. Defining the double-number valued field $\tau = \chi + j\, e^{-\phi_1}$, then under fractional linear transformations $\tau \to (a\,\tau + b)/(c\,\tau + d)$, we find that the fields in the instanton solution transform to

$$\phi_1' = -f + \log H' \ , \qquad \chi' = \Big(\coth\mu + \frac{c}{d}\,\mathrm{cosech}^2\mu\Big)\, H'^{-1} + \frac{b}{d} \ , \tag{5.23}$$

where the transformed harmonic function H' is given by

$$H' = d^2\, H + c^2\,\mathrm{cosech}^2\mu + 2c\,d\,\coth\mu \ , \tag{5.24}$$

while the field ϕ_2 remains unchanged. Thus we see that as in the extremal case, the form of the solution is the same as before except that the harmonic function H is rescaled and shifted by a constant. This extends the previous result for extremal D-instantons [6] to include arbitrary non-extremal instantons.

So far we have considered just single-charge instanton solutions, obtained by dimensionally reducing single-charge non-extremal p-branes on the entire set of $d = p+1$ world-volume directions. More generally, if the Lagrangian in D dimensions contains a number of n-form field strengths, for which a subset of $F^\alpha_{(n)}$ $(\alpha = 1,\ldots,N)$ have dilaton vectors satisfying the dot-product relations [26]

$$\hat{\vec{c}}_\alpha \cdot \hat{\vec{c}}_\beta = 4\delta_{\alpha\beta} - \frac{2d\,\tilde{d}}{D-2} \ , \tag{5.25}$$

then there exist N-charge non-extremal p-branes [19]. These solutions can be diagonally reduced to give N-charge non-extremal instantons, which are solutions of the equations of motion following from the Lagrangian

$$e^{-1}\mathcal{L} = R - \tfrac12 \partial\varphi^2 - \tfrac12(\partial\vec{\phi})^2 + \tfrac12\sum_{\alpha=1}^{N} e^{\vec{c}_\alpha\cdot\vec{\phi}}(\partial\chi_\alpha)^2 \ , \tag{5.26}$$

where we have $\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta}$. Thus we see that the dilatons φ and $\varphi_\alpha \equiv \vec{c}_\alpha \cdot \vec{\phi}$ are completely decoupled from each other, and the pairs $(\varphi_\alpha, \chi_\alpha)$ form a total of N independent

$SL(2,\mathbb{R})/O(1,1)$ cosets. The non-extremal instanton solution in D dimensions is then given by

$$
\begin{aligned}
ds^2 &= e^{\frac{2f}{D-2}}\,(e^{-2f}\,dr^2 + r^2\,d\Omega^2)\ ,\\
\varphi_\alpha &= -f + \log H_\alpha\ , \qquad \chi_\alpha = H_\alpha^{-1}\,\coth\mu_\alpha\ , \qquad \varphi = f\sqrt{\tfrac{D}{D-2}}\ ,
\end{aligned}
\tag{5.27}
$$

with $H_\alpha = 1 + (k\sinh\mu_a)\,r^{-(D-2)}$. Acting with the independent $SL(2,\mathbb{R})$ transformations on the N cosets, we are able to make indepedentent transformations of the form (5.24) on each of the harmonic functions H_α. Note that in the extremal limit $k\to 0$ we have $f\to 0$, and hence the extra scalar φ decouples from the system. In this case, the harmonic functions H_α can each be independently shifted and scaled, as in (5.3), so that they become

$$
H'_\alpha = 2c_\alpha\,d_\alpha\ + d_\alpha^2\,H_\alpha\ , \tag{5.28}
$$

under the $SL(2,\mathbb{R})$ transformations

$$
\Lambda_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}\ . \tag{5.29}
$$

5.3 Instanton transformation and p-brane asymptotic geometry

As we have seen in the previous two subsections, a generic $SL(2,\mathbb{R})$ transformation does not change the essential structure of the instanton solution, but it does have the effect of modifying the harmonic function H by a constant shift and a constant rescaling. This is true both for the extremal and the non-extremal instantons. For particular choices of the $SL(2,\mathbb{R})$ transformation parameters, however, this can have the effect of completely altering the asymptotic behaviour of the harmonic function in the limit where $r\to\infty$. Thus if the $SL(2,\mathbb{R})$ parameters are chosen so that $c=-\frac12 d$, the harmonic function H' defined in (5.3) becomes

$$
H' = \frac{Q\,d^2}{r^{D-2}}\ . \tag{5.30}
$$

This duality transformation has therefore had the effect of expanding the near-horizon structure of the instanton, where the $Q\,r^{2-D}$ term in H dominates over the constant term, to the entire range of r values.

Let us now consider this phenomenon in more general situations, in which we begin by considering a p-brane solution in a higher dimension. To begin with, we shall take the example of a single-charge extremal p-brane. This solution can be diagonally dimensionally reduced through a succession of steps, until we arrive at a single-charge instanton, after having compactified the entire $(p+1)$-dimensional world-volume of the p-brane. This will be a solution that involves only a subsector of the lower-dimensional supergravity fields, namely the metric, a certain single combination of the dilatons, and the axion whose field strength carries the charge supporting the instanton. This dilaton/axion pair will be described by a standard $SL(2,\mathbb{R})/O(1,1)$ coset. Since the reduction was performed on the world-volume, the harmonic function $H = 1 + Qr^{-\tilde{d}}$ of the higher-dimensional p-brane solution, with $\tilde{d} = D - 3 - p$, is exactly the same as the harmonic function of the instanton solution in the lower dimension.

We now perform the $SL(2,\mathbb{R})$ transformation on the instanton, as described above, and obtain the new harmonic function $H' = Q'R^{-\tilde{d}}$. Having done so, we retrace the previous steps and diagonally oxidise this transformed instanton solution back to the original higher dimension. Thus we arrive at an extremal single-charge p-brane solution that differs from the original one only in having the orginal harmonic function H replaced by H'. The asymptotic structure of the new p-brane solution with H' is therefore altered, and now takes the same form as the near-horizon limit of the original solution, in the regime where the constant term in H was negligable in comparison to the $Qr^{-\tilde{d}}$. Cases of particular interest arise when the dilaton in the original D-dimensional p-brane solution is finite on the horizon, since the near-horizon structure then approaches $\mathrm{AdS}_{(p+2)} \times S^{D-p-2}$ and the supersymmetry is enhanced [27–31]. In such cases, the $SL(2,\mathbb{R})$-transformed solution has the global structure of $\mathrm{AdS}_{(p+2)} \times S^{D-p-2}$ everywhere. (See also [32, 33] for non-standard intersections [34–36] that give rise to AdS structures.) At first sight, there seems to be a paradox regarding this enhancement of supersymmetry, since the $SL(2,\mathbb{R})$ symmetry of the instantons is part of the U-duality groups, which commute with supersymmetry, and hence one would expect that the transformed solution should have the same supersymmetry. This paradox is resolved by the observation that the AdS space can be viewed as a domain wall solution in holospherical coordinates, and the half of the Killing spinors depends on the world-volume coordinates [37]. Thus half of the supersymmetry of the AdS space is lost when the solution is reduced to an instanton. This explains that the preserved fractions of supersymmetry of a p-branes with AdS structure in its near horizon is always doubled on its near horizon.

This discussion can be extended also to the discretised $SL(2,\mathbb{Z})$ U-duality group of the quantum theory. This shows that the near-horizon geometry captures the essence of any p-brane, since it is dual to the p-brane itself.

A generalised discussion can be given for any harmonic intersection of p-branes, waves and NUTs. The solution for the intersection of N basic objects involves N independent harmonic functions H_α. These solutions can be dimensionally reduced to N-charge p-branes, which can then be further reduced to N-charge instantons. The nature of the harmonic intersection implies that the participating fields that support this N-charge instanton can be described by a Lagrangian of the form (5.26), where φ decouples in the extremal limit. In particular the dilaton vectors satisfy $\vec{c}_\alpha \cdot \vec{c}_\beta = 4\delta_{\alpha\beta}$. This implies that the dilatons $\varphi_\alpha = \vec{c}_\alpha \cdot \vec{\phi}$ are decoupled from each other, and the system is described by N independent $SL(2,\mathbb{R})/O(1,1)$ cosets. Each $SL(2,\mathbb{R})$ factor in the total global symmetry group can therefore be used to transform the associated harmonic function. This means that for appropriate choices of the various $SL(2,\mathbb{R})$ parameters, the constant terms in all N harmonic functions can be independently adjusted, and in particular, caused to vanish. If the the harmonic functions are chosen to have charges that are equal and coincident, then the solution becomes a bound-state p-brane with $\Delta = 4/N$. Thus we have seen that using dimensional reduction on the time direction, and $SL(2,\mathbb{R})$ transformations of the instantons, we can alter the asymptotic geometry for all p-branes, intersections and bound states.

The above discussion applies equally well to non-extremal p-branes, whose dimensional reduction on it world-volume gives rise to non-extremal instantons. The $SL(2,\mathbb{R})$ transformation leave the non-extremal factor e^{2f} invariant, but can rescale and shift the 'harmonic' function H by constants, as given in (5.24).

The use of duality symmetries to alter the asymptotic structure of p-branes was first proposed in [5], by utilising a combination of general coordinate transformations, the S-duality symmetry in type IIB, and the T-duality transformation between the type IIA and type IIB theories. In general, the prescription in [5] is to start from a p-brane in D dimensions, map it to $D = 10$ by oxidation or reduction, and then by then by means of a sequence of T-duality transformations (together with an S-duality symmetry transformation if the starting point was an R-R p-brane), map it to a ten-dimensional wave solution. Next, a linear coordinate transformation mixing the time and the longitudinal wave directions is performed, which has the effect of shifting and rescaling the harmonic function by constants. Finally, the previous sequence of duality and oxidation or reduction steps is retraced, eventually giving back a p-brane with a shifted and rescaled harmonic function. A succession of such processes can be performed in order to make independent shifts and rescalings of all the harmonic functions in a multi-charge p-brane or an intersection. In particular, this procedure was used to relate non-dilatonic black holes in $D = 5$ and $D = 4$ to their near-horizon $AdS_3 \times$ sphere $\times$ torus structures. This provides a conformal field theoretic understanding of the entropy of non-dilatonic black holes [38].

Recently, a different procedure for shifting and scaling the harmonic functions in certain p-brane solutions was proposed [6]. Effectively, the idea is to follow similar steps to those described above, except that the goal is to map the p-brane into the instanton of the type IIB theory, rather than to map it to a wave in $D = 10$. Now, one uses the $SL(2,\mathbb{R})$ symmetry of the type IIB theory to shift and scale the harmonic function, in a similar manner to the procedure we described in section 5.1. Again, by retracing the dualisation and oxidation or reduction steps, one arrives at a p-brane with whose harmonic function is shifted and scaled relative to the starting point. There is an interesting question that can be raised about this procedure, since in order to describe instantons in the type IIB theory in $D = 10$, it is necessary to Euclideanise the theory. As we discussed in section 2, the issue arises of how the Majorana condition on the fermions, and more especially, how the self-duality condition on the 5-form, is to be handled. In fact the self-duality condition will force the 5-form to be complex, implying that the T-dual type IIA Euclidean-signature theory will also be complex. Thus T-duality would have to be discussed in situations where some of the relevant solutions, and indeed the theories themselves, are complex.

By contrast, our prescription is simply to diagonally reduce a p-brane until it becomes an instanton, perform an $SL(2,\mathbb{R})$ duality transformations to shift and rescale the harmonic function, and then diagonally oxidise back to the p-brane again. N-charge p-branes, or intersections (*i.e.* with N independent harmonic functions), are handled similarly, by reducing down to the dimension where they become N-charge instantons, which are described by a Euclidean-signature theory with N independent $SL(2,\mathbb{R})$ factors in the the global symmetry group that allow the N harmonic functions to be independently shifted and scaled. Although the instantons are described by Euclidean-signature theories, these arise naturally from reduction on the time coordinate, and no act of Euclideanisation is performed. Also, our discussion extends to the case of non-extremal solutions, which would not be possible in the D-instanton approach described in [6], since type IIB supergravity does not have the $GL(2,\mathbb{R})/O(1,1)$ scalar manifold that would be needed for constructing non-extremal instantons.

6 $SL(3,\mathbb{R})/O(2,1)$ Lagrangians, and instantons

In this section, we give some explicit results for the $SL(3,\mathbb{R})$-symmetric part of the scalar Lagrangian for eight-dimensional Euclidean-signature supergravity. This is a useful example because it is exhibits more “generic” behaviour than is seen in the $SL(2,\mathbb{R})$ example in type IIB or in $D = 9$. In particular, there are three axions (plus two dilatons) involved in the $SL(3,\mathbb{R})$ scalar manifold, and the axions can undergo rotations under $SL(3,\mathbb{R})$, for which there is no analogue in the single-axion $SL(2,\mathbb{R})$ system, in addition to non-linear transformations of a kind that are familiar in $SL(2,\mathbb{R})$. At the same time, $SL(3,\mathbb{R})$ is still sufficiently simple that explicit formulae can be presented. We shall present results for the case where the eight-dimensional space is of Euclidean signature. We shall shall obtain this theory by taking the time reduction to be at the $D = 10$ to $D = 9$ stage of the dimensional reduction process.

6.1 $SL(3,\mathbb{R})$-invariant scalar Lagrangians

The Lagrangian can be obtained from the general results in section 2, with the sign reversal occurring for the kinetic terms of all field strengths carrying the index value "2." . Thus we find that the relevant part of the Lagrangian is

$$e^{-1}\,\mathcal{L} = -\tfrac{1}{2}(\partial\vec\phi)^2 - \tfrac{1}{2}e^{\vec b_{13}\cdot\vec\phi}\,(\partial\chi_{13} - \chi_{23}\,\partial\chi_{12})^2 + \tfrac{1}{2}e^{\vec b_{12}\cdot\vec\phi}\,(\partial\chi_{12})^2 + \tfrac{1}{2}e^{\vec b_{23}\cdot\vec\phi}\,(\partial\chi_{23})^2\ . \tag{6.1}$$

We have suppressed the extra $SL(2,\mathbb{R})$ invariant part of the full $D = 8$ scalar Lagrangian. This comprises a 1-dilaton, 1-axion system. In fact in the full scalar Lagrangian in $D = 8$, there will be a 3-vector ϕ of dilatons, and the extra axion χ_{123} of the $SL(2,\mathbb{R})$ system, coming from the dimensional reduction of $A_{(3)}$ in $D = 11$. Its dilaton coupling in the Lagrangian, $\frac{1}{2}e^{\vec a_{123}\cdot\vec\phi}\,(\partial\chi_{123})^2$, involves a dilaton vector $\vec a_{123}$ that is orthogonal to all three of the $\vec b_{ij}$ dilaton vectors in (6.1), and in fact in writing the pure $SL(3,\mathbb{R})$ system in (6.1), we are taking $\vec\phi$ to be just a 2-component vector of dilatons in the directions orthogonal to $\vec a_{123}$. In fact in this basis, we are taking the dilaton vectors $\vec b_{ij}$ to be

$$\vec b_{12} = (\sqrt3, -1)\ ,\qquad \vec b_{23} = (-\sqrt3, -1)\ ,\qquad \vec b_{13} = (0, -2)\ . \tag{6.2}$$

Following [9], we can parameterise an $SL(3,\mathbb{R})/O(2,1)$ coset representative $\mathcal{V}$, in the Borel gauge, as

$$\begin{aligned}\mathcal{V} &= e^{\frac12\vec\phi\cdot\vec H}\,e^{\chi_{23}\,E_{23}}\,e^{\chi_{13}\,E_{13}}\,e^{\chi_{12}\,E_{12}}\ ,\\ &= \begin{pmatrix} e^{\frac{1}{2\sqrt3}\phi_1-\frac12\phi_2} & \chi_{12}\,e^{\frac{1}{2\sqrt3}\phi_1-\frac12\phi_2} & \chi_{13}\,e^{\frac{1}{2\sqrt3}\phi_1-\frac12\phi_2}\\ 0 & e^{-\frac{1}{\sqrt3}\phi_1} & \chi_{23}\,e^{-\frac{1}{\sqrt3}\phi_1}\\ 0 & 0 & e^{\frac{1}{2\sqrt3}\phi_1+\frac12\phi_2}\end{pmatrix}\ ,\end{aligned} \tag{6.3}$$

where $\vec H$ represents the two Cartan generators, and E_{ij} denote the positive-root generators of $SL(3,\mathbb{R})$. Defining

$$\mathcal{M} = \mathcal{V}^T\,\eta\,\mathcal{V}\ ,\qquad\qquad \eta = \mathrm{diag}\,(1,-1,1)\ , \tag{6.4}$$

the Lagrangian (6.1) can be written as

$$e^{-1}\,\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\,(\partial_\mu \mathcal{M}^{-1}\,\partial^\mu \mathcal{M})\ . \tag{6.5}$$

Global $SL(3,\mathbb{R})$ transformations on the scalar fields can be implemented by acting on the right of $\mathcal{V}$ with a constant $SL(3,\mathbb{R})$ matrix Λ, and on the left with a local compensating $O(2,1)$ transformation $\mathcal{O}$, whose job is to restore the transformed $\mathcal{V}$ to the Borel gauge:

$$\mathcal{V} \longrightarrow \mathcal{V}' = \mathcal{O}\,\mathcal{V}\,\Lambda\ . \tag{6.6}$$

It is manifest that provided $\mathcal{O}$ satisfies $\mathcal{O}^T\,\eta\,\mathcal{O} = \eta$, this will leave the Lagrangian (6.5) invariant for any global $SL(3,\mathbb{R})$ transformation matrix Λ.

It is of particular interest to study the transformations of the scalar fields under the $O(2,1)$ subgroup of $SL(3,\mathbb{R})$, since this is the subgroup that preserves a given set of values for the scalars. Thus we may choose the particular $O(2,1)$ subgroup that preserves the values of the scalar moduli, *i.e.* the asymptotic values of the scalar fields at infinity. The simplest choice is to take all the moduli to be zero, since then the coset representative (6.3) is simply the identity, and so the required $O(2,1)$ subgroup will consist just of matrices Λ that satisfy

$$\Lambda^T\,\eta\,\Lambda = \eta\ . \tag{6.7}$$

It is somewhat involved, even in this special case, to give a parameterisation of all such $O(2,1)$ matrices, and the resulting expressions for the transformed scalar fields will be quite complicated. However, it suffices that we derive the transformations for two different 1-parameter subgroups, namely the $O(2)$ subgroup of matrices

$$\Lambda_1 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\ , \tag{6.8}$$

and the $O(1,1)$ subgroup of matrices

$$\Lambda_2 = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}\ . \tag{6.9}$$

Any desired $O(2,1)$ transformation can be obtained by composing these two basic transformations.

In each case, we may obtain the transformation rules for the scalar fields by substituting Λ_1 or Λ_2 into (6.6), solving for the compensator $\mathcal{O}$ that restores the Borel gauge, and then reading off the transformed fields from the resulting Borel matrix $\mathcal{V}'$. For the $O(2)$ transformation Λ_1 given in (6.8), it is useful first to define the function

$$f_1 = e^{\sqrt{3}\phi_1}\,(\cos\theta - \chi_{13}\,\sin\theta)^2 + e^{\sqrt{3}\phi_1+2\phi_2}\,\sin^2\theta - \chi_{23}^2\,e^{\phi_2}\,\sin^2\theta\ . \tag{6.10}$$

We find that the $O(2)$ transformations then take the form

$$\begin{aligned}
e^{-2\phi_1'/\sqrt{3}} &= f_1^{-1}\,e^{\phi_1/\sqrt{3}}\Big((\cos\theta - (\chi_{13}-\chi_{12}\,\chi_{23})\sin\theta)^2 + e^{2\phi_2}\,\sin^2\theta \\
&\qquad -\chi_{12}^2\,e^{\sqrt{3}\phi_1+\phi_2}\,\sin^2\theta)\Big)\ ,\\
e^{\phi_1'/\sqrt{3}-\phi_2'} &= f_1\,e^{-2\phi_1/\sqrt{3}-\phi_2}\ ,\\
e^{\phi_1'/\sqrt{3}-\phi_2'}\,\chi_{12}' &= \chi_{12}\,e^{\phi_1/\sqrt{3}-\phi_2}\,(\cos\theta-\chi_{13}\,\sin\theta) + \chi_{23}\,e^{-2\phi_1/\sqrt{3}}\,\sin\theta\ ,\\
e^{\phi_1'/\sqrt{3}-\phi_2'}\,\chi_{13}' &= (\chi_{13}\cos\theta+\sin\theta)(\cos\theta-\chi_{13}\sin\theta)\,e^{\phi_2/\sqrt{3}-\phi_2}\\
&\qquad -e^{\phi_2/\sqrt{3}+\phi_2}\,\sin\theta\cos\theta + \chi_{23}^2\,e^{-2\phi_1/\sqrt{3}}\,\sin\theta\cos\theta\ ,\\
e^{-2\phi_1'/\sqrt{3}}\,\chi_{23}' &= f_1^{-1}\,e^{\phi_1/\sqrt{3}}\,\Big(\chi_{23}(\cos\theta-(\chi_{13}-\chi_{12}\,\chi_{23})\sin\theta) - \chi_{12}\,e^{\sqrt{3}\phi_1+\phi_2}\,\sin\theta\Big)\ .
\end{aligned} \tag{6.11}$$

For the $O(1,1)$ transformations given by Λ_2 in (6.9), we define

$$f_2 = (\cosh t + \chi_{12}\,\sinh t)^2\,e^{\sqrt{3}\phi_1} - e^{\phi_2}\,\sinh^2 t\ , \tag{6.12}$$

and we find that

$$\begin{aligned}
e^{-2\phi_1'/\sqrt{3}} &= f_2^{-1}\,e^{\phi_1/\sqrt{3}}\ ,\\
e^{\phi_1'/\sqrt{3}-\phi_2'} &= f_2\,e^{-2\phi_1/\sqrt{3}-\phi_2}\ ,\\
e^{\phi_1'/\sqrt{3}-\phi_2'}\,\chi_{12}' &= (\chi_{12}\,\cosh t+\sinh t)(\cosh t+\chi_{12}\,\sinh t)\,e^{\phi_1/\sqrt{3}-\phi_2}\\
&\qquad -e^{-2\phi_1/\sqrt{3}}\,\sinh t\cosh t\ ,\\
e^{\phi_1'/\sqrt{3}-\phi_2'}\,\chi_{13}' &= \chi_{13}\,(\cosh t+\chi_{12}\,\sinh t)\,e^{\phi_1/\sqrt{3}-\phi_2} - \chi_{23}\,e^{-2\phi_1/\sqrt{3}}\,\sinh t\ ,\\
e^{-2\phi_1'/\sqrt{3}}\,\chi_{23}' &= f_2^{-1}\,e^{\phi_1/\sqrt{3}}\,(\chi_{23}\,\cosh t - (\chi_{13}-\chi_{12}\,\chi_{23})\,\sinh t)\ .
\end{aligned} \tag{6.13}$$

Let us now turn to an explicit demonstration for the $SL(3,\mathrm{IR})/O(2,1)$ scalar manifold of the claim that we made in section 2.2, that the order in which the time and the space reductions are performed does not affect the final form of the lower-dimensional Lagrangian. In particular, we shall show that the Lagrangian (6.1) obtained by reducing on t at the second step is the same as the Lagrangian obtained by reducing on t instead at the first step. This latter Lagrangian is

$$e^{-1}\,\widetilde{\mathcal{L}} = -\tfrac12(\partial\vec{\phi}')^2 + \tfrac12 e^{\vec{b}_{13}\cdot\vec{\phi}'}\,(\partial\chi_{13}'-\chi_{23}'\,\partial\chi_{12}')^2 + \tfrac12 e^{\vec{b}_{12}\cdot\vec{\phi}'}\,(\partial\chi_{12}')^2 - \tfrac12 e^{\vec{b}_{23}\cdot\vec{\phi}'}\,(\partial\chi_{23}')^2\ , \tag{6.14}$$

in terms of a primed set of field variables. At first sight, it is far from obvious that this is equivalent to (6.1), especially in view of the fact that the "distinguished" axion χ_{13} whose field strength receives the Kaluza-Klein modification has a kinetic term with opposite signs in the two cases. To show that in fact the Lagrangians are the same, but written in different field variables, we shall give a slightly different proof from the general one that we presented in section 2.2. In particular, we shall derive an explicit purely real field transformation here (*i.e.* real within a neighbourhood). To do this, we note that just as (6.1) can be written as $\mathcal{L} = \frac{1}{4} e\, \mathrm{tr}(\partial_\mu \mathcal{M}^{-1}\, \partial^\mu \mathcal{M})$, where $\mathcal{M} = \mathcal{V}^T\, \eta\, \mathcal{V}$, and $\eta = \mathrm{diag}(1,-1,1)$, so (6.14) can be written as $\widetilde{\mathcal{L}} = \frac{1}{4} e\, \mathrm{tr}(\partial_\mu (\widetilde{\mathcal{M}'})^{-1}\, \partial^\mu \widetilde{\mathcal{M}'})$, where $\widetilde{\mathcal{M}'} = \mathcal{V}'^T\, \tilde\eta\, \mathcal{V}'$, and $\tilde\eta = \mathrm{diag}(-1,1,1)$. Now let us consider an $SL(3,\mathbb{R})$ transformation Λ, and define a transformed Borel-gauge coset representative by $\mathcal{V}' = \mathcal{C}\, \mathcal{V}\, \Lambda$, where the "compensating transformation" $\mathcal{C}$ is required to satisfy

$$\mathcal{C}^T\, \tilde\eta\, \mathcal{C} = \eta\ . \tag{6.15}$$

Then we find that the Lagrangian (6.14), *i.e.* $\widetilde{\mathcal{L}} = \frac{1}{4} e\, \mathrm{tr}(\partial_\mu (\widetilde{\mathcal{M}'})^{-1}\, \partial^\mu \widetilde{\mathcal{M}'})$, is mapped by this transformation into the Lagrangian (6.1), expressed in terms of the unprimed fields. Taking the $SL(3,\mathbb{R})$ transformation to be

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ , \tag{6.16}$$

we find that the explicit transformation between the primed fields in (6.14) and the unprimed fields in (6.1) is

$$\begin{aligned} \phi_1' &= -\tfrac{1}{2}\phi_1 + \frac{\sqrt{3}}{2}\log f\ , \\ \phi_2' &= \frac{\sqrt{3}}{2}\phi_1 + \phi_2 - \tfrac{1}{2}\log f\ , \\ \chi_{12}' &= f^{-1}\, e^{\sqrt{3}\phi_1}\, \chi_{12}\ , \\ \chi_{13}' &= -\chi_{23}\, f^{-1}\, e^{\phi_2} + f^{-1}\, \chi_{12}\, \chi_{13}\, e^{\sqrt{3}\phi_1}\ , \\ \chi_{23}' &= \chi_{13} - \chi_{12}\, \chi_{23}\ , \end{aligned} \tag{6.17}$$

where

$$f = e^{\phi_2} - e^{\sqrt{3}\phi_1}\, \chi_{12}^2\ . \tag{6.18}$$

Note that this transformation is real in the patch where $f > 0$. This is an example of the general result that we described in section 2.2, where the transformation between the $\mathcal{M}$ matrices parameterising the cosets in the two equivalent Lagrangians can be arranged to induce a real transformation between the two sets of coset coordinates, by appropriate choice of parameterisation. In the notation of section 2.2, our example here corresponds to taking $\mathcal{M}(2) = \Lambda\, \mathcal{M}'(1)\, \Lambda^{-1}$, with Λ given by (6.16).

6.2 $SL(3, \mathbb{R})$ transformation of instantons

Having studied the full global $SL(3, \mathbb{R})$ symmetry of the Lagrangian (6.1) for the coset $SL(3, \mathbb{R})/O(2, 1)$, we are in a position to investigate how the instanton solutions transform under the $SL(3, \mathbb{R})$ global symmetry. There are a total of three axions, namely χ_{12}, χ_{23} and χ_{13}, each of which can support a simple single-charge instanton of the form

$$
\begin{aligned}
ds^2 &= dr^2 + r^2 \delta\Omega^2 , \\
\vec{\phi} &= \tfrac{1}{2}\vec{b}_{ij} \log H , \qquad \chi_{ij} = H^{-1} .
\end{aligned} \tag{6.19}
$$

Note that for the instantons suported by either χ_{12} or χ_{23}, the charge Q in the harmonic function $H = 1 + Q r^{-\tilde{d}}$ is real since these axions are NC-scalars. For χ_{13}, on the other hand, the charge Q is imaginary. The scalar coset matrices $\mathcal{M}$ for these three solutions are given by

$$
\chi_{12} = H^{-1} : \qquad \mathcal{M} = \begin{pmatrix} H & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.20}
$$

$$
\chi_{23} = H^{-1} : \qquad \mathcal{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -H & -1 \\ 0 & -1 & 0 \end{pmatrix} \tag{6.21}
$$

$$
\chi_{13} = H^{-1} : \qquad \mathcal{M} = \begin{pmatrix} H & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2H^{-1} \end{pmatrix} \tag{6.22}
$$

Thus it is straightforward derive how the scalar fields in these three solutions transform under the $SL(3, \mathbb{R})$ transformation $\mathcal{M} \to \Lambda^{\mathrm{T}} \mathcal{M} \Lambda$.

We shall now examine the transformations of the Noether currents for the instanton solutions under the $SL(3, \mathbb{R})$ symmetry. In Appendix A, we present results for the eight Noether currents associated with the parameters of the global $SL(3, \mathbb{R})$ symmetry. These are the analogues of the three $SL(2, R)$ Noether currents given in (5.4). We also show that these transform linearly under $SL(3, \mathbb{R})$.

It is a straightforward matter to substitute the above instanton solutions, but with the scalar moduli chosen to be zero for simplicity, into the set of eight Noether currents given in (A.7) in Appendix A. (The solutions with zero moduli are obtained from those given in (6.19) by performing a shift Borel transformation so that now $\chi_{ij} = H^{-1} - 1$, with all other fields unchanged.) We find that the instanton supported by the axion χ_{12} has Noether currents given by

$$\mathcal{J}(\chi_{12}) = \begin{pmatrix} -dH & dH & 0 \\ -dH & dH & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{6.23}$$

The instanton supported instead by χ_{23} has Noether currents given by

$$\mathcal{J}(\chi_{23}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -dH & dH \\ 0 & -dH & dH \end{pmatrix} . \tag{6.24}$$

Finally, the Noether currents for the complex solution supported by χ_{13} are given by

$$\mathcal{J}(\chi_{13}) = \begin{pmatrix} (1-2H^{-1})dH & 0 & (-1+4H^{-1}-2H^{-2})dH \\ 0 & 0 & 0 \\ dH & 0 & (-1+2H^{-1})dH \end{pmatrix} . \tag{6.25}$$

In each case, dH is the exterior derivative of the harmonic function H characterising the solution.

It is easy to verify that the $O(2)$ transformation Λ_1 given in (6.8) rotates the two sets of Noether currents for the real solutions using χ_{12} and χ_{23} into one another, and in fact if the rotation parameter θ is chosen to be $3\pi/2$, then we find that $\Lambda_1^{-1}\,\mathcal{J}(\chi_{12})\,\Lambda_1 = \mathcal{J}(\chi_{23})$. It is also evident that the $O(1,1)$ transformation (6.9) acts on the χ_{12} solution in the same way as did the $O(1,1)$ transformation (5.10) in the $SL(2,\mathbb{R})$ instanton solution discussed in section 5.1, namely as an overall rescaling of the Noether currents. The other $O(1,1)$ subgroup transformation that we did not write down, corresponding to a Lorentz rotation in the 2-3 plane rather than the 1-2 plane of the transformation (6.9), would act similarly on the χ_{23} instanton solution.

Thus we see that the orbits of the modulus-preserving $O(2,1)$ subgroup of the $SL(3,\mathbb{R})$ symmetry group of the scalar Lagrangian (6.1) include an $O(2)$ subgroup that rotates between the pair of instanton solutions supported by χ_{12} and by χ_{23}. It is important to emphasise, however, that we have only a doublet, and not a triplet, of instanton solutions in this example, despite the occurrence of three axions in the Lagrangian. The reason for this is that only two out of the three axions have kinetic terms with the necessary sign to allow them to support real instanton solutions, and only these two can rotate into one another under the action of the real global symmetry transformations.

7 $(D-3)$-branes

Our principal focus so far in this paper has been on the investigation of Euclidean-signature maximal supergravities, and the real instanton solutions that can be supported by those axions whose kinetic terms have undergone a sign reversal in the reductions to the Euclidean-signature theories.

Axions can also support $(D-3)$-brane solitons in D-dimensional Minkowskian-signature spacetimes. One might think that they can simply be viewed as the magnetic duals of the instantons, but we shall shortly see that their relationship is more complicated than that. There are different types of $(D-3)$-branes. First let us consider the ones that can be viewed as coming from the vertical dimensional reduction of standard p-branes, until the point is reached where the transverse space becomes two-dimensional. Such solutions have the following structure

$$\begin{aligned} ds^2 &= dx^\mu dx_\mu + (1 + \tfrac{Q}{2\pi}\log r)(dr^2 + r^2 d\theta^2)\ , \\ e^{-\phi} &= 1 + \tfrac{Q}{2\pi}\log r\ , \qquad \chi = \tfrac{Q}{2\pi}\theta\ . \end{aligned} \tag{7.1}$$

It was argued in [39] in the context of strings in four-dimensional theories that such solutions break the classical $SL(2,\mathbb{R})$ duality symmetry down to the quantum S-duality group $SL(2,\mathbb{Z})$, since the periodicity of the angular coordinate $\theta = \theta + 2\pi$ implies that χ also must become periodic, with $\chi = \chi + 1$ in the case of a string carrying a unit charge Q. The magnetic charge of the solution can be defined as $Q_m = \int J_m$, where $J_m = d\chi$ is the current dual to $e^{2\phi} {*d\chi} = {*J_+}$, whose integral gives the electric charge of the instanton. It is conserved, by the virtue of the Bianchi identity $dJ_m = 0$. However, J_m is not invariant under $SL(2,\mathbb{R})$, and in fact acting on J_m with $SL(2,\mathbb{R})$ generates an infinite number of currents

$$\begin{aligned} J_m^1 &= d\chi\ , \\ J_m^2 &= \chi d\chi + e^{-2\phi} d\phi\ , \\ J_m^3 &= \chi^2 d\chi + 2e^{-2\phi}\chi d\phi - e^{-2\phi} d\chi\ , \\ J_m^4 &= \chi^3 d\chi + 3e^{-2\phi}\chi^2 d\phi - 3e^{-2\phi}\chi d\chi - e^{-4\phi} d\phi\ . \\ &\cdots\cdots \end{aligned} \tag{7.2}$$

(An analogous calculation of the action of $SL(2,\mathbb{R})$ on the Noether current J_+ just gives the closed system of three Noether currents.) The currents (7.2) form an infinite-dimensional representation of $SL(2,\mathbb{R})$. The $(D-3)$-brane solution obviously cannot be viewed as the dual of the instanton solution discussed earlier, since in that case the Noether charges are in the adjoint representation of $SL(2,\mathbb{R})$. In fact the manifest occurence of bare undifferentiated χ fields in (7.2) implies that the charges of the $(D-3)$-brane, calculated from (7.2), except for $\int J^1_m$ itself, are ill-defined, since $\chi = \frac{Q}{2\pi}\,\theta$ is not a periodic function of θ in the solution. This manifests itself in the fact that quantities such as $\oint \theta\, d\theta$ are ill-defined. In fact the $(D-3)$-brane solution breaks the $SL(2,\mathbb{R})$ symmetry group down to the group of integer-valued strict Borel transformations,

$$\Lambda = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\ . \tag{7.3}$$

There also exists an $SL(2,\mathbb{R})$-invariant $(D-3)$-brane [40, 12]. It might be this solution that is dual to the instanton we discussed earlier. Athough the solution (7.1) may be incompatible with the U-duality, it provides a starting-point for obtaining a domain-wall solution of a massive supergravity in one dimension lower. This massive supergravity, which does not inherit the full U-duality from the higher dimension, is obtained by making a Scherk-Schwarz reduction on the θ coordinate. And indeed, the domain-wall solution has no magnetic dual.

In the rest of this section, we shall consider in some detail a subset of the set of all $(D-3)$-brane solitons for which the complications described above can be avoided. Specifically, we shall consider exclusively $(D-3)$ branes that are supported by axions coming from the dimensional reduction of the $A_{(3)}$ potential of eleven-dimensional supergravity. Furthermore, we shall consider the action on such solutions of only the $SL(11-D,\mathbb{R})$ subgroups of the full global supergravity symmetry groups, which we shall call the "restricted symmetry groups." Consequently, we shall be considering axions that undergo only linear transformations under the restricted global symmetry group.

The method that we shall use in order to study the $(D-3)$-brane multiplets is analogous to the one used in [4] for studying the multiplet structures for p-branes supported by higher-degree field strengths. Here, we are concerned only with the global $SL(11-D,\mathbb{R})$ symmetry, and the associated positive-root generators $E_i{}^j$. Since we are considering only $(D-3)$-branes that are supported by axions derived from the potential $A_{(3)}$ of $D=11$ supergravity, the highest dimension in which any such solution can exist is $D=8$. Furthermore, since there is only one axion of this kind in $D=8$, namely $A_{(0)123}$, the associated 5-brane is a singlet. Thus we must descend to $D=7$ before encountering an interesting multiplet structure.

In $D=7$ the restricted symmetry group is $SL(4,\mathbb{R})$, with simple roots $\vec{b}_{i,i+1}$ where $i=1,2,3$. The full root system is given by $\pm\vec{b}_{ij}$, where $i,j=1,2,3,4$, associated with the generators $E_{\pm\vec{b}_{ij}}$. In the standard basis for $SL(n,\mathbb{R})$, the Cartan generators can be written as $H_i = E_i{}^i - E_{i+1}{}^{i+1}$. We wish to consider 4-brane solutions supported by the 1-form field strengths $F_{(1)ijk}$. These fields form a 4-dimensional representation of the $SL(4,\mathbb{R})$ algebra. We can determine the orbits of the 4-brane solutions by picking a representative solution, and considering the stability subgroup $\mathcal{H}$ of $SL(4,\mathbb{R})$ that leaves the solution invariant. The orbits will then be given by the coset $SL(4,\mathbb{R})/\mathcal{H}$. Let us, for definiteness, pick the solution supported by $F_{(1)123}$ as our starting point. The dilaton vector $\vec{a}_{123}$ for this field is the highest-weight vector in the 4-dimensional representation. Thus we now need to find the subset of $SL(4,\mathbb{R})$ generators that annihilate the highest-weight state $|\vec{a}_{123}\rangle$. It is straightforward to check that from the three Cartan generators there are just two combinations that annihilate this state, namely $H_1 = E_1{}^1 - E_2{}^2$, and $H_2 = E_2{}^2 - E_3{}^3$. Of the remaining $SL(4,\mathbb{R})$ generators, the following annihilate $|\vec{a}_{123}\rangle$:

$$E_1{}^2,\quad E_2{}^1,\quad E_1{}^3,\quad E_3{}^1,\quad E_2{}^3,\quad E_3{}^2,\quad E_4{}^1,\quad E_4{}^2,\quad E_4{}^3 \tag{7.4}$$

Under the two Cartan combinations (H_1, H_2) , the weights of the first six generators in (7.4), which form three conjugate pairs $\{E_i{}^j, E_j{}^i\}$, are

$$(2,-1);\ (-2,1);\ (1,1);\ (-1,-1);\ (-1,2);\ (1,-2)\ . \tag{7.5}$$

From the new Cartan generators one can construct the Killing metric

$$g_{ij} = \mathrm{tr}(H_i\, H_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\ . \tag{7.6}$$

Defining the sign of a root by the sign of its first non-zero component found working in from the left, one can easily see that the simple roots are $\alpha_1 = (1,1)$ and $\alpha_2 = (1,-2)$. Their dot products, defined using the Killing metric (7.6), are given by

$$\alpha_1\cdot\alpha_1 = 2\ ,\quad \alpha_2\cdot\alpha_2 = 2\ ,\quad \alpha_1\cdot\alpha_2 = -1\ . \tag{7.7}$$

We therefore see that $E_1{}^2, E_2{}^1, E_1{}^3, E_3{}^1, E_2{}^3$ and $E_3{}^2$, together with H_1 and H_2, generate an $SL(3,\mathbb{R})$ algebra. Furthermore, the remaining three generators $E_4{}^1, E_4{}^2$ and $E_3{}^4$ mutually commute, and form a vector representation under $SL(3,\mathbb{R})$. Thus the coset space parameterising the single-charge 1-form solutions in $D=7$ is

$$\frac{SL(4,\mathbb{R})}{SL(3,\mathbb{R})\ltimes\mathbb{R}^3}\ . \tag{7.8}$$

We shall not present the analogous detailed calculations in lower dimensions, and instead we shall just give the results. The cosets describing the orbits under the restricted symmetry groups for single-charge $(D-3)$-branes are listed in Table 6.

	Coset
$D=7$	$\frac{SL(4,\mathbb{R})}{SL(3,\mathbb{R})\ltimes\mathbb{R}^3}$
$D=6$	$\frac{SL(5,\mathbb{R})}{SL(3,\mathbb{R})\times SL(2,\mathbb{R})\ltimes\mathbb{R}^6}$
$D=5$	$\frac{SL(6,\mathbb{R})}{SL(3,\mathbb{R})\times SL(3,\mathbb{R})\ltimes\mathbb{R}^6}$
$D=4$	$\frac{SL(7,\mathbb{R})}{SL(3,\mathbb{R})\times SL(4,\mathbb{R})\ltimes\mathbb{R}^{12}}$
$D=3$	$\frac{SL(8,\mathbb{R})}{SL(3,\mathbb{R})\times SL(5,\mathbb{R})\ltimes\mathbb{R}^{15}}$

Table 6: Cosets for single-charge $(D-3)$-brane orbits

In addition to these single-charge $(D-3)$-branes, there are also, in lower dimensions, multi-charge $(D-3)$-brane solutions for which the natures of the orbits are different. We shall just present one example here, to illustrate the procedure. The simplest example that illustrates the point occurs in $D=6$. We see from Table 6 that the single-charge 3-brane solution has orbits of dimension 7, while the number of 1-form field strengths $F_{(1)ijk}$ is 10. (By contrast, in $D=7$ the orbits have dimension 4, which is equal to the number of field strengths $F_{(1)ijk}$.) The fact that in $D=6$ the single-charge orbits have a smaller dimension than the number of available field strengths that could support the solutions suggests that there should exist new classes of solution, that would "fill out" orbits of higher dimension. Indeed, in $D=6$ the possibility arises for the first time of having 2-charge 3-brane solutions, carrying two independent charges. An example is a solution whose two charges are carried by the field strengths $F_{(1)123}$ and $F_{(1)145}$. The orbits of this solution can then be determined by the same methods as above, namely by first identifying the stability group that leaves both of the associated root vectors $\vec{a}_{123}$ and $\vec{a}_{145}$ simultaneously invariant. This turns out to be $Sp(4)\ltimes\mathbb{R}^4$. Thus the coset describing the 2-charge orbits is

$$\frac{SL(5,\mathbb{R})}{Sp(4)\ltimes\mathbb{R}^4}\,. \tag{7.9}$$

This coset has dimension 10, and so one can expect that the orbits for these solutions indeed fill out the entire solution space. And indeed, there do not exist any more general 3-charge solutions in $D=6$.

8 Discussion

In this paper, we have obtained the Euclidean-signature supergravities that result from compactifying $D=11$ supergravity or type IIB supergravity on a torus that includes the time direction. These Euclidean-signature theories are automatically compatible with any Majorana or self-duallity conditions on fields, since they are obtained by a consistent dimensional-reduction procedure. We showed that there are two inequivalent nine-dimensional theories, coming from the reduction of the type IIA and type IIB supergravities on their time directions. The two nine-dimensional Euclidean-signature theories become equivalent upon further compactification on a spatial circle. This can also be understood from the general result that the same Euclidean-signature theory is obtained regardless of the order in which the time reduction and spatial reductions are performed. We studied the global symmetry groups of the Euclidean-signature theories, and the structure of their scalar cosets. We also investigated the orbits of instanton solutions under the global symmetry groups in the examples of $SL(2,\mathbb{R})$ and $SL(3,\mathbb{R})$-invariant Lagrangians.

We showed that the $SL(2,\mathbb{R})$ symmetry of the Euclidean-signature theory which describes the instanton coming from the diagonal dimensional reduction of a p-brane on its entire world-volume[6] can transform the p-brane into its near-horizon structure. In the case of non-dilatonic p-branes the curvature, and the singularity structure of the p-brane, can be completely different from its near-horizon behaviour. For example, the eleven-dimensional membrane [41] has a curvature singularity singularity, and it requires the inclusion of the membrane action [42] as a source term [16]. On the other hand, its near-horizon structure is $\mathrm{AdS}_4 \times S^7$, which is an exact supergravity solution without any singularity and with no need for a source term. This emphasises that the lower-dimensional U-duality groups must be more than just the residues of the general-coordinate symmetries and gauge symmetries of the eleven-dimensional theory.[7] For example, the eleven-dimensional membrane becomes an instanton in $D = 8$, after it is reduced on its 3-dimensional world-volume. The instanton is supported by the axion $A_{(0)123}$, coming from the reduction of $A_{(3)}$ in $D = 11$, and by a dilaton $\phi = \frac{1}{2}\vec{a}_{123} \cdot \vec{\phi}$, which comes from the metric. The $SL(2,\mathbb{R})$ symmmetry of this system, which we used in order to transform the structure of the harmonic function, is the $SL(2,\mathbb{R})$ factor of the $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ U-duality group, and it is therefore distinct from the $SL(3,\mathbb{R})$ which comes from the general coordinate symmetry of the 3-torus. In $D = 11$, it is a symmetry that mixes the metric and the 3-form gauge potential.

On the other hand, in cases where the $SL(2,R)$ symmetry of the theory describing the instanton does come from the general coordinate symmetry in the internal space, the

[6] It has recently been argued that it is necessary to consider the wrapping of p-branes on the time as well as spatial world-volume directions in a full discussion of their singularity structure [43].

[7] For example, it is known that the global homogeneous scaling transformation of the eleven-dimensional theory plays an essential rôle in the global symmetry transformations in $D \leq 10$ [9].

constant shift of the harmonic function in the higher-dimensional solution will not affect its curvature. For example, the D0-brane in the type IIA theory, which can be viewed as a wave in $D = 11$, can be reduced on its time direction to an instanton in $D = 9$. The $SL(2,\mathbb{R})$ symmetry in $D = 9$ is just the general-coordinate symmetry of the internal torus, and so in this case there should be no change in the curvature or singularity structure. Indeed, any constant shift or rescaling of the harmonic function of the wave solution can be achieved by a general-coordinate transformation [5]).

The fact that the U-duality groups in $D \le 8$ dimensions can alter the singularity structure of M-branes suggests that a better understanding of U-duality from the higher-dimensional viewpoint is needed.

A $SL(3,\mathbb{R})$ Noether currents

In this appendix, we present the detailed expressions for the eight Noether currents corresponding to the eight parameters of the global $SL(3,\mathbb{R})$ symmetry of the scalar Lagrangian (6.1). In fact it is convenient first to present a more general derivation of the Noether currents for an arbitrary scalar Lagrangian of the form

$$\mathcal{L} = \tfrac{1}{4} e\, \mathrm{tr}\, (\partial_\mu \mathcal{M}^{-1}\, \partial^\mu \mathcal{M}) \ . \tag{A.1}$$

This is invariant under the global G transformations

$$\mathcal{M} \longrightarrow \mathcal{M}' = \Lambda^T\, \mathcal{M}\, \Lambda \ . \tag{A.2}$$

Infinitesimally, where we write $\Lambda = 1\!\!1 + \lambda$, and λ is infinitesimal, we have

$$\delta \mathcal{M} = \lambda^T\, \mathcal{M} + \mathcal{M}\, \lambda \ . \tag{A.3}$$

By the usual procedure, we calculate the Noether currents by varying the Lagrangian with respect to a spacetime-dependent transformation, keeping only those terms where a derivative falls on the parameters λ. Thus we have

$$\begin{aligned} \delta\mathcal{L} &= -\tfrac{1}{2} e \Big(\mathcal{M}^{-1} (\partial_\mu \lambda^T\, \mathcal{M} + \mathcal{M}\, \partial_\mu \lambda) \mathcal{M}^{-1}\, \partial^\mu \mathcal{M} \Big) \ , \\ &= -\tfrac{1}{2} \mathrm{tr} \Big(\partial_\mu \lambda (\mathcal{M}^{-1}\, \partial^\mu \mathcal{M} + (\mathcal{M}^T)^{-1} \partial^\mu \mathcal{M}^T) \Big) \\ &= -\mathrm{tr}\, (\partial_\mu \lambda\, \mathcal{M}^{-1}\, \partial^\mu \mathcal{M}) \ . \end{aligned} \tag{A.4}$$

Thus we obtain the G-valued Noether currents

$$\mathcal{J} = -\mathcal{M}^{-1}\, d\mathcal{M} \ . \tag{A.5}$$

It is easily verified that under global G transformations (A.2), the Noether currents transform as

$$\mathcal{J} \longrightarrow \mathcal{J}' = \Lambda^{-1}\, \mathcal{J}\, \Lambda\ . \tag{A.6}$$

Applying this to the $SL(3,\mathbb{R})$ Lagrangian (6.1), described by (A.1) with $\mathcal{M}$ given by (6.3) and (6.4), we find the Lie algebra $SL(3,\mathbb{R})$-valued Noether currents

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} & \mathcal{J}_{13} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & \mathcal{J}_{23} \\ \mathcal{J}_{31} & \mathcal{J}_{32} & \mathcal{J}_{33} \end{pmatrix}\ , \tag{A.7}$$

where the components are given by

$$\begin{aligned}
\mathcal{J}_{11} &= -\tfrac{1}{\sqrt{3}}\, d\phi_1 + d\phi_2 - e^{\sqrt{3}\phi_1-\phi_2}\, \chi_{12}\, d\chi_{12} - e^{-2\phi_2}\, \chi_{13}\, \chi_{23}\, d\chi_{12} \\
&\quad + e^{-2\phi_2}\, \chi_{12}\, \chi_{23}^2\, d\chi_{12} + e^{-2\phi_2}\, \chi_{13}\, d\chi_{13} - e^{-2\phi_2}\, \chi_{12}\, \chi_{23}\, d\chi_{13} \\
\mathcal{J}_{22} &= \tfrac{2}{\sqrt{3}}\, d\phi_1 + e^{\sqrt{3}\phi_1-\phi_2}\, \chi_{12}\, d\chi_{12} - e^{-2\phi_2}\, \chi_{12}\, \chi_{23}^2\, d\chi_{12} \\
&\quad + e^{-2\phi_2}\, \chi_{12}\, \chi_{23}\, d\chi_{13} - e^{-\sqrt{3}\phi_1-\phi_2}\, \chi_{23}\, d\chi_{23} \\
\mathcal{J}_{12} &= -\sqrt{3}\, \chi_{12}\, d\phi_1 + \chi_{12}\, d\phi_2 - d\chi_{12} - e^{\sqrt{3}\phi_1-\phi_2}\, \chi_{12}^2\, d\chi_{12} - e^{-2\phi_2}\, \chi_{12}\, \chi_{13}\, \chi_{23}\, d\chi_{12} \\
&\quad + e^{-2\phi_2}\, \chi_{12}^2\, \chi_{23}^2\, d\chi_{12} + e^{-2\phi_2}\, \chi_{12}\, \chi_{13}\, d\chi_{13} - e^{-2\phi_2}\, \chi_{12}^2\, \chi_{23}\, d\chi_{13} \\
&\quad - e^{-\sqrt{3}\phi_1-\phi_2}\, \chi_{13}\, d\chi_{23} + e^{-\sqrt{3}\phi_1-\phi_2}\, \chi_{12}\, \chi_{23}\, d\chi_{23} \\
\mathcal{J}_{13} &= -\sqrt{3}\, \chi_{12}\, \chi_{23}\, d\phi_1 + 2\chi_{13}\, d\phi_2 - \chi_{12}\, \chi_{23}\, d\phi_2 - e^{\sqrt{3}\phi_1-\phi_2}\, \chi_{12}\, \chi_{13}\, d\chi_{12} \\
&\quad - e^{-2\phi_2}\, \chi_{13}^2\, \chi_{23}\, d\chi_{12} + e^{-2\phi_2}\, \chi_{12}\, \chi_{13}\, \chi_{23}^2\, d\chi_{12} - d\chi_{13} \\
&\quad + e^{-2\phi_2}\, \chi_{13}^2\, d\chi_{13} - e^{-2\phi_2}\, \chi_{12}\, \chi_{13}\, \chi_{23}\, d\chi_{13} \\
&\quad + \chi_{12}\, d\chi_{23} - e^{-\sqrt{3}\phi_1-\phi_2}\, \chi_{13}\, \chi_{23}\, d\chi_{23} + e^{-\sqrt{3}\phi_1-\phi_2}\, \chi_{12}\, \chi_{23}^2\, d\chi_{23} \\
\mathcal{J}_{21} &= e^{\sqrt{3}\phi_1-\phi_2}\, d\chi_{12} - e^{-2\phi_2}\, \chi_{23}^2\, d\chi_{12} + e^{-2\phi_2}\, \chi_{23}\, d\chi_{13} \\
\mathcal{J}_{23} &= \sqrt{3}\chi_{23} d\phi_1 + \chi_{23}\, d\phi_2 + e^{\sqrt{3}\phi_1-\phi_2}\, \chi_{13}\, d\chi_{12} - e^{-2\phi_2}\, \chi_{13}\, \chi_{23}^2\, d\chi_{12} \\
&\quad + e^{-2\phi_2}\, \chi_{13}\, \chi_{23}\, d\chi_{13} - d\chi_{23} - e^{-\sqrt{3}\phi_1-\phi_2}\, \chi_{23}^2\, d\chi_{23} \\
\mathcal{J}_{31} &= e^{-2\phi_2}\, \chi_{23}\, d\chi_{12} - e^{-2\phi_2}\, d\chi_{13} \\
\mathcal{J}_{32} &= e^{-2\phi_2}\, \chi_{12}\, \chi_{23}\, d\chi_{12} - e^{-2\phi_2}\, \chi_{12}\, d\chi_{13} + e^{-\sqrt{3}\phi_1-\phi_2}\, d\chi_{23}
\end{aligned} \tag{A.8}$$

and $\mathcal{J}_{33} = -\mathcal{J}_{11} - \mathcal{J}_{22}$.

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