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A remarkable connection between Yangians and finite $\mathcal{W}$-algebras

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Abstract

For a large class of finite $\mathcal{W}$ algebras, the defining relations of a Yangian are proved to be satisfied. Therefore such finite $\mathcal{W}$ algebras appear as realisations of Yangians. This result is useful to determine properties of such $\mathcal{W}$ algebra representations.

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1 Introduction: general considerations on Yangians and $\mathcal{W}$ algebras

$\mathcal{W}$ algebras first showed up in the context of two dimensional conformal field theories[1]. They benefited of development owing in particular to their property to be algebras of constant of motion for Toda field theories, themselves defined as constrained WZNW models[2]. Yangians were first considered and defined in connection with some rational solutions of the quantum Yang-Baxter equation[3]. Later, their relevance in integrable models with non Abelian symmetry was remarked[4]. Yangian symmetry has been proved for the Haldane-Shastry $SU(n)$ quantum spin chains with inverse square exchange, as well as for the embedding of this model in the $\hat{SU}(2)_1$ WZNW one; this last approach leads to a new classification of the states of a conformal field theory in which the fundamental quasi-particles are the spinons[5] (see also[6]). Let us finally emphasize on the Yangian symmetry determined in the Calogero-Sutherland-Moser models[5, 7]. Coming back to $\mathcal{W}$ algebras, it can be shown that their zero modes provide algebras with a finite number of generators and which close polynomially. Such algebras can also be constructed by symplectic reduction of finite dimensional Lie algebras in the same way usual -or affine- $\mathcal{W}$ algebras arise as reduction of affine Lie algebras: they are called finite $\mathcal{W}$ algebras[8] (F.W.A.), this definition extending to any algebra which satisfies the above properties of finiteness and polynomiality[9]. Some properties of such FWA's have been developed[9]-[12] and in particular a large class of them can be seen as the commutant, in a generalization of the enveloping algebra $\mathcal{U}(\mathcal{G})$, of a subalgebra $\tilde{\mathcal{G}}$ of a simple Lie algebra $\mathcal{G}$[11]. This feature fo FWA's has been exploited in order to get new realizations of a simple Lie algebra $\mathcal{G}$ once knowing a $\mathcal{G}$ differential operator realization. In such a framework, representations of a FWA are used for the determination of $\mathcal{G}$ representations. This method has been applied to reformulate the construction of the unitary, irreducible representations of the conformal algebra $SO(4,2)$ and of its Poincaré subalgebra, and compared it to the usual induced representation technics[12]. It has also been used for building representations of observable algebras for systems of two identical particles in $d=1$ and $d=2$ dimensions, the $\mathcal{G}$ algebra under consideration being then symplectic ones; in each case, it has then been possible to relate the anyonic parameter to the eigenvalues of a $\mathcal{W}$-generator[10].

In this note, we will show that the defining relations of a Yangian are satisfied for a family of FWA's. In other words, such $\mathcal{W}$-algebras provide Yangian realizations. This remarkable connection between two a priori different types of symmetry deserves in our opinion to be considered more precisely. Meanwhile, we will use results on the representation theory of Yangians and adapt them to this class of FWA's. We will also show on a special example -the algebra $\mathcal{W}(sl(4), 2sl(2))$- how to get the classification of all its irreducible finite dimensional representations.

This report is a condensed version of [13]. More results on $\mathcal{W}$ representations will also be given in [14].

2 Finite $\mathcal{W}(sl(np), p.sl(n))$ algebras

The usual notation for a $\mathcal{W}$ algebra obtained by the Hamiltonian reduction procedure is $\mathcal{W}(\mathcal{G},\mathcal{H})$[2, 15]. More precisely, given a simple Lie algebra $\mathcal{G}$, there is a one-to-one correspondance between the finite $\mathcal{W}$ algebras one can construct in $\mathcal{U}(\mathcal{G})$ and the $sl(2)$ subalgebras in $\mathcal{G}$. We note that any $sl(2)$ $\mathcal{G}$-subalgebra is principal in a subalgebra $\mathcal{H}$ of $\mathcal{G}$. It is rather usual to denote the corresponding $\mathcal{W}$ algebra as $\mathcal{W}(\mathcal{G},\mathcal{H})$.

As an example, let us consider the $\mathcal{W}(sl(4), sl(2)\oplus sl(2))$ algebra. It is made of seven generators J_i, S_i ($i=1,2,3$) and a central element C_2 such that:

$$\begin{array}{ll} [J_i,J_j] = i\epsilon_{ij}^k J_k & (i,j,k=1,2,3) \\ [J_i,S_j] = i\epsilon_{ij}^k S_k & \\ [S_i,S_j] = -i\epsilon_{ij}^k J_k(2\vec{J}^2 - C_2 - 4) & \\ [C_2,J_i] = [C_2,S_i] = 0 & \text{with } \vec{J}^2 = J_1^2+J_2^2+J_3^2 \end{array} \tag{2.1}$$

We recognize the $sl(2)$ subalgebra generated by the J_i's as well as a vector representation (i.e. S_i generators) of this $sl(2)$ algebra. We note that the S_i's close polynomially on the other generators.

The same type of structure can be remarked, at a higher level, for the class of algebras $\mathcal{W}(sl(np), p.sl(n))$ where $p.sl(n)$ stands for: $sl(n) \oplus \cdots \oplus sl(n)$ (p-times). A careful study leads to gather the generators as follows:

i. an $sl(p)$ algebra with generators $W_0^a \quad a = 1, \cdots, p^2 - 1$.

ii. $n-1$ sets of W_k^a generators of "conformal spin" $(k+1)$, with $k = 1, \cdots, n-1$, each set transforming as the adjoint of $sl(p)$ under the W_0^a's.

iii. and finally $(n-1)$ central elements C_i $(i = 2, \cdots, n)$, i.e.:

$$\begin{array}{ll} [W_0^a, W_0^b] = f^{ab}{}_c W_0^c & (a = 1, \cdots, p^2 - 1) \\ [W_0^a, W_k^b] = f^{ab}{}_c W_k^c & (k = 1, 2, \cdots, n-1) \\ [C_i, W_0^a] = [C_i, W_k^a] = 0 & (i = 2, 3, \cdots, n) \end{array} \tag{2.2}$$

Let us emphasize that the algebras $\mathcal{W}(sl(2n), nsl(2))$ can also be defined as the commutant in the enveloping algebra $\mathcal{U}(\mathcal{G})$ of some $\mathcal{G}$-subalgebra $\widetilde{\mathcal{G}}$[11].

3 Yangians $Y(\mathcal{G})$: a definition

Yangians are one of the two well-known families of infinite dimensionnal quantum groups[16] (the other one being quantum affine algebras) that correspond to deformation of the universal enveloping algebra of some finite-dimensional Lie algebra, called $\mathcal{G}$. As such, it is a Hopf algebra, topologically generated by elements Q_0^a and Q_1^a, $a = 1, \ldots, \dim\mathcal{G}$ which satisfy the defining relations:

$$\text{the } Q_0^a\text{'s generate } \mathcal{G} \ : \ [Q_0^a, Q_0^b] \ = \ f^{ab}{}_c \, Q_0^c \tag{3.1}$$

$$\text{the } Q_1^a\text{'s form an adjoint rep. of } \mathcal{G} \ : \ [Q_0^a, Q_1^b] \ = \ f^{ab}{}_c \, Q_1^c \tag{3.2}$$

$$f^{bc}{}_d[Q_1^a, Q_1^d] + f^{ca}{}_d[Q_1^b, Q_1^d] + f^{ab}{}_d[Q_1^c, Q_1^d] \ = \ f^a{}_{pd} f^b{}_{qx} f^c{}_{ry} f^{xy}{}_e \eta^{de} \ s_3(Q_0^p, Q_0^q, Q_0^r) \tag{3.3}$$

$$f^{cd}{}_e[[Q_1^a, Q_1^b], Q_1^e] + f^{ab}{}_e[[Q_1^c, Q_1^d], Q_1^e] \ = \ \left(f^a{}_{pe} f^b{}_{qx} f^{cd}{}_y f^y{}_{rz} f^{xz}{}_g + f^c{}_{pe} f^d{}_{qx} f^{ab}{}_y f^y{}_{rz} f^{xz}{}_g \right) \eta^{eg} \ s_3(Q_0^p, Q_0^q, Q_1^r) \tag{3.4}$$

where $f^{ab}{}_c$ are the totally antisymmetric structure constant of $\mathcal{G}$, η^{ab} is the Killing form, and $s_n(., ., \ldots, .)$ is the totally symmetrized product of n terms. It can be shown that for $\mathcal{G} = sl(2)$, (3.3) is a consequence of the other relations, while for $\mathcal{G} \neq sl(2)$, (3.4) follows from (3.1–3.3). The coproduct on $Y(\mathcal{G})$ is given by

$$\Delta(Q_0^a) = 1 \otimes Q_0^a + Q_0^a \otimes 1 \text{ and } \Delta(Q_1^a) = 1 \otimes Q_1^a + Q_1^a \otimes 1 + f_{bc}{}^a Q_0^b \otimes Q_0^c \tag{3.5}$$

In the following, we will focus on the Yangians $Y(sl(p))$.

4 $\mathcal{W}(sl(np), p.sl(n))$ as a realisation of $Y(sl(p))$.

Let us look again at the algebra $\mathcal{W}(sl(4), 2sl(2))$. Identifying in (2.1) the generators J_i with Q_0^i and S_i with Q_1^i, one checks that the relations (3.1)-(3.4) are satisfied. As a consequence, this $\mathcal{W}$-algebra appears as a realisation of $Y(sl(2))$: we will denote this realisation $Y_2(sl(2))$ for reasons which will become clear soon.

Actually such an identification can be extended to any algebra of the type $\mathcal{W}(sl(np), psl(n))$. Then each W_0^a will be identified with Q_0^a and the W_k^a's with the Q_k^a's, the Q_k^a's (for $k = 1, \cdots, n-1$) being naturally obtained from the Q_1^a's by repeated C.R.'s. Then in this Yangian realisation, the $Q_{n+l}^a \equiv W_{n+l}^a, l \geq 0$, are polynomial functions of the Q_k^a (i.e. W_k^a): we will denote this realisation as $Y_n(sl(p))$. Let us mention that the proof for this property is developed in [13]. We summarize this assertion in :

Proposition 1: *Identifying the generators W_k^a ($k = 0, 1, \cdots, n-1$) of the finite dimensional $\mathcal{W}(sl(np), p.sl(n))$ algebra with the elements Q_k^a of the Yangian $Y(sl(p))$, one verifies that the defining relations of a Yangian are satisfied for this $\mathcal{W}$ algebra, which therefore appears as a realisation denoted $Y_n(sl(p))$ of the Yangian $Y(sl(p))$.*

5 Application: the irreducible finite dimensional representations of $\mathcal{W}(sl(4), 2sl(2))$

Owing to the above identification, it is possible to adapt some known properties on Yangian representation theory to finite $\mathcal{W}$ representations. We illustrate this assertion on the case of $\mathcal{W}(sl(4), 2sl(2))$ inviting the reader to consult [13, 14] for the proof, more details and generalisation.

Before summarising our result, let us first define the evaluation module[16] $V_a(r), a \in \mathbb{C}$ of the $\mathcal{W}(sl(4), 2sl(2))$ algebra : it is the representation of dimension $(r+1)$ which on the canonical basis $\{v_0, v_1, \cdots, v_r\}$ the action of the generators $J_\pm = J_1 \pm iJ_2$, $S_\pm = S_1 \pm iS_2$, $J_0 = 2J_3$ and $S_0 = 2S_3$ is:

$$\begin{array}{lll} J_+ v_s = (r-s+1)v_{s-1} & J_- v_s = (s+1)v_{s+1} & J_0 v_s = (r-2s)v_s \\ S_+ v_s = a(r-s+1)v_{s-1} & S_- v_s = a(s+1)v_{s+1} & S_0 v_s = a(r-2s)v_s \end{array} \tag{5.1}$$

Then one can prove:

Proposition 2: *Any irreducible finite dimensional representation of the algebra $\mathcal{W}(sl(4), 2sl(2))$ is either an evaluation module $V_a(r)$ or the tensor product of two evaluation modules $V_a(r) \otimes V_{-a}(s)$ with $\pm 2a \neq \frac{1}{2}(r+s) - m + 1$ for any m such that $0 < m \leq$ min (r, s) the tensor product being calculated via the Yangian coproduct defined in (3.5).*

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