CERN-TH/98-80
Imperial/TP/97-98/30
hep-th/9803116

BPS BRANES IN SUPERGRAVITY*

K.S. STELLE
The Blackett Laboratory, Imperial College,
Prince Consort Road, London SW7 2BZ, UK
and
TH Division, CERN
CH-1211 Geneva 23, Switzerland

DEDICATED TO THE MEMORY OF ABDUS SALAM

This review considers the properties of classical solutions to supergravity theories with partially unbroken supersymmetry. These solutions saturate Bogomol'ny-Prasad-Sommerfield bounds on their energy densities and are the carriers of the p-form charges that appear in the supersymmetry algebra. The simplest such solutions have the character of $(p+1)$-dimensional Poincaré-invariant hyperplanes in spacetime, *i.e.* p-branes. Topics covered include the relations between mass densities, charge densities and the preservation of unbroken supersymmetry; interpolating-soliton structure; diagonal and vertical Kaluza-Klein reduction families; multiple-charge solutions and the four $D = 11$ elements; duality-symmetry multiplets; charge quantisation; low-velocity scattering and the geometry of world-volume supersymmetric σ-models; and the target-space geometry of BPS instanton solutions obtained by the dimensional reduction of static p-branes.

Contents

*Based upon lectures given at the 1996 and 1997 ICTP Summer Schools in High Energy Physics and Cosmology.

1 Introduction

Let us begin from the bosonic sector of $D = 11$ supergravity,[1]

$$I_{11} = \int d^{11}x \left\{ \sqrt{-g}(R - \tfrac{1}{48}F_{[4]}^2) + \tfrac{1}{6}F_{[4]} \wedge F_{[4]} \wedge A_{[3]} \right\} . \tag{1.1}$$

In addition to the metric, one has a 3-form antisymmetric-tensor gauge potential $A_{[3]}$ with a gauge transformation $\delta A_{[3]} = d\Lambda_{[2]}$ and a field strength $F_{[4]} = dA_{[3]}$. The third term in the Lagrangian is invariant under the $A_{[3]}$ gauge transformation only up to a total derivative, so the action (1.1) is invariant under gauge transformations that are continuously connected to the identity. This term is required, with the coefficient given in (1.1), by the $D = 11$ local supersymmetry that is required of the theory when the gravitino-dependent sector is included.

The equation of motion for the $A_{[3]}$ gauge potential is

$$d\,{}^*F_{[4]} + \tfrac{1}{2}F_{[4]} \wedge F_{[4]} = 0 \ ; \tag{1.2}$$

this equation of motion gives rise to the conservation of an "electric" type charge [2]

$$U = \int_{\partial\mathcal{M}_8} ({}^*F_{[4]} + \tfrac{1}{2}A_{[3]} \wedge F_{[4]}) \ , \tag{1.3}$$

where the integral of the 7-form integrand is over the boundary at infinity of an arbitrary infinite spacelike 8-dimensional subspace of $D = 11$ spacetime. Another conserved charge relies on the Bianchi identity $dF_{[4]} = 0$ for its conservation,

$$V = \int_{\partial\widetilde{\mathcal{M}}_5} F_{[4]} \ , \tag{1.4}$$

where the surface integral is now taken over the boundary at infinity of a spacelike 5-dimensional subspace.

Charges such as (1.3, 1.4) can occur on the right-hand side of the supersymmetry algebra,[3]

$$\{Q, Q\} = C(\Gamma^A P_A + \Gamma^{AB} U_{AB} + \Gamma^{ABCDE} V_{ABCDE}) \ , \tag{1.5}$$

where C is the charge conjugation matrix, P_A is the energy-momentum 11-vector and U_{AB} and V_{ABCDE} are 2-form and 5-form charges that we shall find to be related to the charges U and V (1.3, 1.4) above. Note that since the supercharge Q in $D = 11$ supergravity is a 32-component Majorana spinor, the LHS of (1.5) has 528 components. The symmetric spinor matrices $C\Gamma^A$,

$C\Gamma^{AB}$ and $C\Gamma^{ABCDE}$ on the RHS of (1.5) also have a total of 528 independent components: 11 for the momentum P_A, 55 for the "electric" charge U_{AB} and 462 for the "magnetic" charge V_{ABCDE}.

Now the question arises as to the relation between the charges U and V in (1.3, 1.4) and the 2-form and 5-form charges appearing in (1.5). One thing that immediately stands out is that the Gauss' law integration surfaces in (1.3, 1.4) are the boundaries of integration volumes $\mathcal{M}_8$, $\widetilde{\mathcal{M}}_5$ that do not fill out a whole 10-dimensional spacelike hypersurface in spacetime, unlike the more familiar situation for charges in ordinary electrodynamics. A rough idea about the origin of the index structures on U_{AB} and V_{ABCDE} may be guessed from the 2-fold and 5-fold ways that the corresponding 8 and 5 dimensional integration volumes may be embedded into a 10-dimensional spacelike hypersurface. We shall see in Section 4 that this is too naïve, however: it masks an important topological aspect of both the electric charge U_{AB} and the magnetic charge V_{ABCDE}. The fact that the integration volume does not fill out a full spacelike hypersurface does not impede the conservation of the charges (1.3, 1.4); this only requires that no electric or magnetic currents are present at the boundaries $\partial\mathcal{M}_8$, $\partial\widetilde{\mathcal{M}}_5$. Before we can discuss such currents, we shall need to consider in some detail the supergravity solutions that carry charges like (1.3, 1.4). The simplest of these have the structure of $p+1$-dimensional Poincaré-invariant hyperplanes in the supergravity spacetime, and hence have been termed "p-branes" (see, *e.g.* Ref.[4]). In Sections 2 and 3, we shall delve in some detail into the properties of these solutions.

Let us recall at this point some features of the relationship between supergravity theory and string theory. Supergravity theories originally arose from the desire to include supersymmetry into the framework of gravitational models, and this was in the hope that the resulting models might solve some of the outstanding difficulties of quantum gravity. One of these difficulties was the ultraviolet problem, on which early enthusiasm for supergravity's promise gave way to disenchantment when it became clear that local supersymmetry is not in fact sufficient to tame the notorious ultraviolet divergences that arise in perturbation theory.[a] Nonetheless, supergravity theories won much admiration for their beautiful mathematical structure, which is due to the stringent constraints of their symmetries. These severely restrict the possible terms that can occur in the Lagrangian. For the maximal supergravity theories, such as those descended from the $D = 11$ theory (1.1), there is simultaneously a great wealth of fields present and at the same time an impossibility of coupling any independent external field-theoretic "matter." It was only occasionally noticed in this early period that this impossibility of coupling to matter fields does not,

[a] For a review of ultraviolet behavior in supergravity theories, see Ref.[5]

however, rule out coupling to "relativistic objects" such as black holes, strings and membranes.

The realisation that supergravity theories do not by themselves constitute acceptable starting points for a quantum theory of gravity came somewhat before the realisation sunk in that string theory might instead be the sought-after perturbative foundation for quantum gravity. But the approaches of supergravity and of string theory are in fact strongly interrelated: supergravity theories arise as long-wavelength effective-field-theory limits of string theories. To see how this happens, consider the σ-model action [10] that describes a bosonic string moving in a background "condensate" of its own massless modes (g_{MN}, A_{MN}, ϕ):

$$\begin{aligned} I = \frac{1}{4\pi\alpha'}\int d^2z\sqrt{\gamma}\,[\gamma^{ij}\partial_i x^M\partial_j x^N g_{MN}(x) \\ +\mathrm{i}\epsilon^{ij}\partial_i x^M\partial_j x^N A_{MN}(x) + \alpha' R(\gamma)\phi(x)]\ . \end{aligned} \quad (1.6)$$

Every string theory contains a sector described by fields (g_{MN}, A_{MN}, ϕ); these are the only fields that couple directly to the string worldsheet. In superstring theories, this sector is called the Neveu-Schwarz/Neveu-Schwarz (NS–NS) sector.

The σ-model action (1.6) is classically invariant under the worldsheet Weyl symmetry $\gamma_{ij} \to \Lambda^2(z)\gamma_{ij}$. Requiring cancellation of the anomalies in this symmetry at the quantum level gives differential-equation restrictions on the background fields (g_{MN}, A_{MN}, ϕ) that may be viewed as effective equations of motion for these massless modes.[11] This system of effective equations may be summarized by the corresponding field-theory effective action

$$\begin{aligned} I_{\text{eff}} = \int d^D x\sqrt{-g}e^{-2\phi}\Big[(D-26) - \tfrac{3}{2}\alpha'(R + 4\nabla^2\phi - 4(\nabla\phi)^2 \\ -\tfrac{1}{12}F_{MNP}F^{MNP} + \mathcal{O}(\alpha')^2\Big]\ , \end{aligned} \quad (1.7)$$

where $F_{MNP} = \partial_M A_{NP} + \partial_N A_{PM} + \partial_P A_{MN}$ is the 3-form field strength for the A_{MN} gauge potential. The $(D-26)$ term reflects the critical dimension for the bosonic string: flat space is a solution of the above effective theory only for $D = 26$. The effective action for the superstring theories that we shall consider in this review contains a similar (NS–NS) sector, but with the substitution of $(D-26)$ by $(D-10)$, reflecting the different critical dimension for superstrings.

The effective action (1.7) is written in the form directly obtained from string σ-model calculations. It is not written in the form generally preferred by relativists, which has a clean Einstein-Hilbert term free from exponential prefactors like $e^{-2\phi}$. One may rewrite the effective action in a different *frame* by making a Weyl-rescaling field redefinition $g_{MN} \to e^{\lambda\phi} g_{MN}$. $I_{\rm eff}$ as written in (1.7) is in the *string frame*; after an integration by parts, it takes the form, specialising now to $D = 10$,

$$I^{\,\rm string} = \int d^{10}x \sqrt{-g^{(\rm s)}} e^{-2\phi} \Big[R(g^{(\rm s)}) + 4\nabla_{M}\phi\nabla^{M}\phi - \tfrac{1}{12}F_{MNP}F^{MNP} \Big] \ . \quad (1.8)$$

After making the transformation

$$g^{(\rm e)}_{MN} = e^{-\phi/2} g^{(\rm s)}_{MN} \ , \quad (1.9)$$

one obtains the *Einstein frame* action,

$$I^{\,\rm Einstein} = \int d^{10}x \sqrt{-g^{(\rm e)}} \Big[R(g^{(\rm e)}) - \tfrac{1}{2}\nabla_{M}\phi\nabla^{M}\phi - \tfrac{1}{12}e^{-\phi}F_{MNP}F^{MNP} \Big] \ , \quad (1.10)$$

where the indices are now raised and lowered with $g^{(\rm e)}_{MN}$. To understand how this Weyl rescaling works, note that under x-independent rescalings, the connection $\Gamma_{MN}{}^{P}$ is invariant. This carries over also to terms with ϕ undifferentiated, which emerge from the $e^{\lambda\phi}$ Weyl transformation. One then chooses λ so as to eliminate the $e^{-2\phi}$ factor. Terms with ϕ undifferentiated do change, however. As one can see in (1.10), the Weyl transformation is just what is needed to unmask the positive-energy sign of the kinetic term for the ϕ field, despite the apparently negative sign of its kinetic term in $I^{\,\rm string}$.

Now let us return to the maximal supergravities descended from (1.1). We shall discuss in Section 5) the process of Kaluza-Klein dimensional reduction that relates theories in different dimensions of spacetime. For the present, we note that upon specifying the Kaluza-Klein ansatz expressing ds^2_{11} in terms of ds^2_{10}, the Kaluza-Klein vector $\mathcal{A}_{M}$ and the dilaton ϕ,

$$ds^2_{11} = e^{-\phi/6} ds^2_{10} + e^{4\phi/3}(dz + \mathcal{A}_{M}dx^{M})^2 \qquad M = 0, 1, \ldots, 9, \quad (1.11)$$

the bosonic $D = 11$ action (1.1) reduces to the Einstein-frame type IIA bosonic action [14]

$$\begin{aligned} I^{\rm Einstein}_{\rm IIA} \ = \ & \int d^{10}x \sqrt{-g^{(\rm e)}} \Big\{ \big[R(g^{(\rm e)}) - \tfrac{1}{2}\nabla_{M}\phi\nabla^{M}\phi - \tfrac{1}{12}e^{-\phi}F_{MNP}F^{MNP} \big] \\ & - \frac{1}{48}e^{\phi/2}F_{MNPQ}F^{MNPQ} - \frac{1}{4}e^{3\phi/2}\mathcal{F}_{MN}\mathcal{F}^{MN} \Big\} + \mathcal{L}_{FFA} \ , \quad (1.12) \end{aligned}$$

where $\mathcal{F}_{MN}$ is the field strength for the Kaluza-Klein vector $\mathcal{A}_{M}$.

The top line in (1.12) corresponds to the NS–NS sector of the IIA theory; the bottom line corresponds the R–R sector (plus the Chern-Simons terms, which we have not shown explicitly). In order to understand better the distinction between these two sectors, rewrite (1.12) in string frame using (1.9). One finds

$$
\begin{aligned}
I_{\rm IIA}^{\rm string} \quad = \quad & \int d^{10}x\sqrt{-g^{(\rm s}}\Big\{e^{-2\phi}\big[R(g^{(\rm e)}) + 4\nabla_M\phi\nabla^M\phi - \tfrac{1}{12}e^{-\phi}F_{MNP}F^{MNP}\big] \\
& -\frac{1}{48}F_{MNPQ}F^{MNPQ} - \frac{1}{4}\mathcal{F}_{MN}\mathcal{F}^{MN}\Big\} + \mathcal{L}_{FFA}\ . \qquad (1.13)
\end{aligned}
$$

Now one may see the distinguishing feature of the NS–NS sector as opposed to the R–R sector: the dilaton coupling is a uniform $e^{-2\phi}$ in the NS–NS sector, and it does not couple (in string frame) to the R–R sector field strengths. Comparing with the familiar g^{-2} coupling-constant factor for the Yang-Mills action, one sees that the asymptotic value e^{ϕ_infty} plays the rôle of the string-theory coupling constant. Since in classical supergravity theory, one will encounter transformations that have the effect of flipping the sign of the dilaton, $\phi \to -\phi$, the study of classical supergravity will contain decidedly non-perturbative information about string theory. In particular, this will arise in the study of p-brane solitons, to which we shall shortly turn.

In this review, we shall mostly consider the descendants of the type IIA action (1.12). This leaves out one important case that we shall have to consider separately: the chiral type IIB theory in $D = 10$. In the type IIB theory,[15] one has $F_{[1]} = d\chi$, where χ is a R–R zero-form (*i.e.* a pseudoscalar field), $F_{[3]}^{\rm R} = dA_{[2]}^{\rm R}$, a second 3-form field strength making a pair together with $F_{[3]}^{\rm NS}$ from the NS–NS sector, and $F_{[5]} = dA_{[4]}$, which is a *self-dual* 5-form in $D = 10$, $F_{[5]} = {}^*F_{[5]}$.

Thus one naturally encounters field strengths of ranks 1–5 in the supergravity theories deriving from superstring theories. In addition, one may use $\epsilon_{[10]}$ to dualize certain field strengths; *e.g.* the original $F_{[3]}$ may be dualized to the 7-form ${}^*F_{[7]}$. The upshot is that antisymmetric-tensor gauge field strengths of diverse ranks need to be taken into account when searching for solutions to string-theory effective field equations. These field strengths will play an essential rôle in supporting the p-brane solutions that we shall now describe.

2 The p-brane ansatz

2.1 Single-charge action and field equations

We have seen that one needs to consider effective theories containing gravity, various ranks of antisymmetric-tensor field strengths and various scalars. To obtain a more tractable system to study, we shall make a *consistent truncation* of the action down to a simple system in D dimensions comprising the metric

g_{MN}, a scalar field ϕ and a single $(n-1)$-form gauge potential $A_{[n-1]}$ with corresponding field strength $F_{[n]}$; the whole is described by the action

$$I = \int D^D x\sqrt{-g}\Big[R - \tfrac{1}{2}\nabla_M\phi\nabla^M\phi - \tfrac{1}{2n!}e^{a\phi}F_{[n]}^2\Big] \ . \tag{2.1}$$

We shall consider later in more detail how (2.1) may be obtained by a consistent truncation from a full supergravity theory in D dimensions. The notion of a consistent truncation will play a central rôle in our discussion of the BPS solutions of supergravity theories. A consistent truncation is one for which solutions of the truncated theory are also perfectly good, albeit specific, solutions of the original untruncated theory. Truncation down to the system (2.1) with a single scalar ϕ and a single field strength $F_{[n]}$ will be consistent except for certain special cases when $n = D/2$ that we shall have to consider separately. In such cases, one can have *dyonic* solutions, and in such cases it will generally be necessary to retain an axionic scalar χ as well. Note that in (2.1) we have not included contributions coming from the FFA Chern-Simons term in the action. These are also consistently excluded in the truncation to the single-charge action (2.1). The value of the important parameter a controlling the interaction of the scalar field ϕ with the field strength $F_{[n]}$ in (2.1) will vary according to the cases considered in the following.

Varying the action (2.1) produces the following set of equations of motion:

$$R_{MN} = \tfrac{1}{2}\partial_M\phi\partial_N\phi + S_{MN} \tag{2.2a}$$

$$S_{MN} = \frac{1}{2(n-1)!}e^{a\phi}(F_{M\dots}F_N{}^{\dots} - \frac{n-1}{n(D-2)}F^2 g_{MN}) \tag{2.2b}$$

$$\nabla_{M_1}(e^{a\phi}F^{M_1\cdots M_n}) = 0 \tag{2.2c}$$

$$\Box\phi = \frac{a}{2n!}e^{a\phi}F^2 \ . \tag{2.2d}$$

2.2 Electric and magnetic ansätze

In order to solve the above equations, we shall make a simplifying ansatz. We shall be looking for solutions preserving certain unbroken supersymmetries, and these will in turn require unbroken translational symmetries as well. For simplicity, we shall also require isotropic symmetry in the directions “transverse” to the translationally-symmetric ones. These restrictions can subsequently be relaxed in generalizations of the basic class of p-brane solutions that we shall discuss here. For this basic class of solutions, we make an ansatz requiring $(\text{Poincaré})_d \times \text{SO}(D-d)$ symmetry. One may view the sought-for solutions as flat $d = p+1$ dimensional hyperplanes embedded in the ambient D-dimensional spacetime; these hyperplanes may in turn be viewed as the histories, or worldvolumes, of p-dimensional spatial surfaces. Accordingly, let the spacetime coordinates be split into two ranges: $x^M = (x^\mu, y^m)$, where x^μ ($\mu = 0, 1, \cdots, p = d-1$) are coordinates adapted to the $(\text{Poincaré})_d$ isometries on the worldvolume and where y^m ($m = d, \cdots, D-1$) are the coordinates "transverse" to the worldvolume.

An ansatz for the spacetime metric that respects the $(\text{Poincaré})_d \times \text{SO}(D-d)$ symmetry is [12]

$$
\begin{aligned}
ds^2 &= e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^n \delta_{mn} \\
\mu &= 0, 1, \ldots, p \qquad m = p+1, \ldots, D-1 \ ,
\end{aligned}
\tag{2.3}
$$

where $r = \sqrt{y^m y^m}$ is the isotropic radial coordinate in the transverse space. Since the metric components depend only on r, translational invariance in the worldvolume directions x^μ and $\text{SO}(D-d)$ symmetry in the transverse directions y^m is guaranteed.

The corresponding ansatz for the scalar field $\phi(x^M)$ is simply $\phi = \phi(r)$.

For the antisymmetric tensor gauge field, we face a bifurcation of possibilities for the ansatz, the two possibilities being related by duality. The first possibility is naturally expressed directly in terms of the gauge potential $A_{[n-1]}$. Just as the Maxwell 1-form naturally couples to the worldline of a charged particle, so does $A_{[n-1]}$ naturally couple to the worldvolume of a $p = d-1 = (n-1)-1$ dimensional "charged" extended object. The "charge" here will be obtained from Gauss'-law surface integrals involving $F_{[n]}$, as we shall see later. Thus, the first possibility for $A_{[n-1]}$ is to support a $d_{\text{el}} = n-1$ dimensional worldvolume. This is what we shall call the "elementary," or "electric" ansatz:

$$
A_{\mu_1 \cdots \mu_{n-1}} = \epsilon_{\mu_1 \cdots \mu_{n-1}} e^{C(r)} \ , \qquad \text{others zero.} \tag{2.4}
$$

$\text{SO}(D-d)$ isotropicity and $(\text{Poincaré})_d$ symmetry are guaranteed here because the function $C(r)$ depends only on the transverse radial coordinate r. Instead of the ansatz (2.4), expressed in terms of $A_{[n-1]}$, we could equivalently have given just the $F_{[n]}$ field strength:

$$
F^{(\text{el})}_{m\mu_1 \cdots \mu_{n-1}} = \epsilon_{\mu_1 \cdots \mu_{n-1}} \partial_m e^{C(r)} \ , \qquad \text{others zero.} \tag{2.5}
$$

The worldvolume dimension for the elementary ansatz (2.4, 2.5) is clearly $d_{\text{el}} = n-1$.

The second possible way to relate the rank n of $F_{[n]}$ to the worldvolume dimension d of an extended object is suggested by considering the dualized field strength ${}^*F$, which is a $(D-n)$ form. If one were to find an underlying gauge potential for ${}^*F$ (locally possible by courtesy of a Bianchi identity), this would naturally couple to a $d_{\rm so} = D - n - 1$ dimensional worldvolume. Since such a dualized potential would be nonlocally related to the fields appearing in the action (2.1), we shall not explicitly follow this construction, but shall instead take this reference to the dualized theory as an easy way to identify the worldvolume dimension for the second type of ansatz. This "solitonic" or "magnetic" ansatz for the antisymmetric tensor field is most conveniently expressed in terms of the field strength $F_{[n]}$, which now has nonvanishing values only for indices corresponding to the transverse directions:

$$F^{(\rm mag)}_{m_1\cdots m_n} = \lambda\epsilon_{m_1\cdots m_n p}\frac{y^p}{r^{n+1}}\ , \qquad \text{others zero,} \tag{2.6}$$

where the magnetic-charge parameter λ is a constant of integration, the only thing left undetermined by this ansatz. The power of r in the solitonic/magnetic ansatz is determined by requiring $F_{[n]}$ to satisfy the Bianchi identity.[b] Note that the worldvolume dimensions of the elementary and solitonic cases are related by $d_{\rm so} = \tilde{d}_{\rm el} \equiv D - d_{\rm el} - 2$; note also that this relation is idempotent, *i.e.* $\widetilde{(\tilde{d})} = d$.

2.3 *Curvature components and p-brane equations*

In order to write out the field equations after insertion of the above ansätze, one needs to compute the Ricci tensor for the metric.[13] This is most easily done by introducing vielbeins, *i.e.,* orthonormal frames,[16] with tangent-space indices denoted by underlined indices:

$$g_{MN} = e_M{}^{\underline{E}} e_N{}^{\underline{F}} \eta_{\underline{E}\,\underline{F}}\ . \tag{2.7}$$

Next, one constructs the corresponding 1-forms: $e^{\underline{E}} = dx^M e_M{}^{\underline{E}}$. Splitting up the tangent-space indices $\underline{E} = (\underline{\mu}, \underline{m})$ similarly to the world indices $M = (\mu, m)$, we have for our ansätze the vielbein 1-forms

$$e^{\underline{\mu}} = e^{A(r)}dx^\mu\ , \qquad\qquad e^{\underline{m}} = e^{B(r)}dy^m\ . \tag{2.8}$$

The corresponding spin connection 1-forms are determined by the condition that the torsion vanishes, $de^{\underline{E}} + \omega^{\underline{E}}{}_{\underline{F}} \wedge e^{\underline{F}} = 0$, which yields

$$\begin{aligned} \omega^{\underline{\mu}\,\underline{\nu}} &= 0\ , \qquad\qquad \omega^{\underline{\mu}\,\underline{n}} = e^{-B(r)}\partial_n A(r) e^{\underline{\mu}} \\ \omega^{\underline{m}\,\underline{n}} &= e^{-B(r)}\partial_n B(r) e^{\underline{m}}\ -\ e^{-B(r)}\partial_m B(r) e^{\underline{n}}\ . \end{aligned} \tag{2.9}$$

[b]Specifically, one finds $\partial_q F_{m_1\cdots m_n} = r^{-(n+1)}\left(\epsilon_{m_1\cdots m_n q} - (n+1)\epsilon_{m_1\cdots m_n p} y^p y_q / r^2\right)$; upon taking the totally antisymmetrised combination $[qm_1\cdots m_n]$, the factor of $(n+1)$ is evened out between the two terms and then one finds from cycling a factor $\sum_m y^m y_m = r^2$, thus obtaining cancellation.

The curvature 2-forms are then given by

$$R_{[2]}^{\underline{E}\,\underline{F}} = d\omega^{\underline{E}\,\underline{F}} + \omega^{\underline{E}\,\underline{D}} \wedge \omega_{\underline{D}}{}^{\underline{F}} \ . \tag{2.10}$$

From the curvature components so obtained, one finds the Ricci tensor components

$$\begin{aligned} R_{\mu\nu} &= -\eta_{\mu\nu} e^{2(A-B)}(A'' + d(A')^2 + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r}A') \\ R_{mn} &= -\delta_{mn}(B'' + dA'B' + \tilde{d}(B')^2 + \frac{(2\tilde{d}+1)}{r}B' + \frac{d}{r}A') \\ &\quad -\frac{y^m y^n}{r^2}(\tilde{B}'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 - \frac{\tilde{d}}{r}B' - \frac{d}{r}A') \ , \end{aligned} \tag{2.11}$$

where again, $\tilde{d} = D - d - 2$, and the primes indicate $\partial/\partial r$ derivatives.

Substituting the above relations, one finds the set of equations that we need to solve to obtain the metric and ϕ:

$$\begin{array}{rcll} A'' + d(A')^2 + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r}A' &=& \frac{\tilde{d}}{2(D-2)}S^2 & \{\mu\nu\} \\ B'' + dA'B' + \tilde{d}(B')^2 + \frac{(2\tilde{d}+1)}{r}B' + \frac{d}{r}A' & & -\frac{d}{2(D-2)}S^2 & \{\delta_{mn}\} \\ \tilde{B}'' + dA'' - 2dA'B' + d(A')^2 - \tilde{d}(B')^2 & & & \\ -\frac{\tilde{d}}{r}B' - \frac{d}{r}A' + \frac{1}{2}(\phi')^2 &=& \frac{1}{2}S^2 & \{y_m y_n\} \\ \phi'' + dA'\phi' + \tilde{d}B'\phi' + \frac{(\tilde{d}+1)}{r}\phi' &=& -\frac{1}{2}\varsigma a S^2 \ , & \end{array} \tag{2.12}$$

where $\varsigma = \pm 1$ for the elementary/solitonic cases and the source appearing on the RHS of these equations is

$$S = \begin{cases} (e^{\frac{1}{2}a\phi - dA + C})C' & \text{electric: } d = n-1,\ \varsigma = +1 \\ \lambda(e^{\frac{1}{2}a\phi - \tilde{d}B})r^{-\tilde{d}-1} & \text{magnetic: } d = D-n-1,\ \varsigma = -1. \end{cases} \tag{2.13}$$

2.4 p-brane solutions

The p-brane equations (2.12, 2.13) are still rather daunting. Before we embark on solving these equations, let us first note a generalisation. Although Eqs (2.12) have been specifically written for an isotropic p-brane ansatz, one may recognise more general possibilities by noting the form of the Laplace operator, which for isotropic scalar functions of r is

$$\nabla^2\phi = \phi'' + (\tilde{d}+1)r^{-1}\phi' \ . \tag{2.14}$$

We shall see later that more general solutions of the Laplace equation than the simple isotropic ones considered here will also play important rôles in the story.

In order to reduce the complexity of Eqs (2.12), we shall refine the p-brane ansatz $(2.3, 2.5, 2.6)$ by looking ahead a bit and taking a hint from the requirements for supersymmetry preservation, which shall be justified in more detail later on in Section 4. Accordingly, we shall look for solutions satisfying the linearity condition

$$dA' + \tilde{d}B' = 0 \ . \tag{2.15}$$

After eliminating B using (2.15), the independent equations become [17]

$$\begin{aligned} \nabla^2\phi &= -\tfrac{1}{2}\varsigma a S^2 && (2.16a) \\ \nabla^2 A &= \frac{\tilde{d}}{2(D-2)} S^2 && (2.16b) \\ d(D-2)(A')^2 + \tfrac{1}{2}\tilde{d}(\phi')^2 &= \tfrac{1}{2}\tilde{d} S^2 \ , && (2.16c) \end{aligned}$$

where, for spherically-symmetric (*i.e.* isotropic) functions in the transverse $(D-d)$ dimensions, the Laplacian is $\nabla^2\phi = \phi'' + (\tilde{d}+1)r^{-1}\phi'$.

Equations (2.16a,b) suggest that we now further refine the ansätze by imposing another linearity condition:

$$\phi' = \frac{-\varsigma a(D-2)}{\tilde{d}} A' \ . \tag{2.17}$$

At this stage, it is useful to introduce a new piece of notation, letting

$$a^2 = \Delta - \frac{2d\tilde{d}}{(D-2)} \ . \tag{2.18}$$

With this notation, equation (2.16c) gives

$$S^2 = \frac{\Delta(\phi')^2}{a^2} \ , \tag{2.19}$$

so that the remaining equation for ϕ becomes $\nabla^2\phi + \frac{\varsigma\Delta}{2a}(\phi')^2 = 0$, which can be re-expressed as a Laplace equation,[c]

$$\nabla^2 e^{\frac{\varsigma\Delta}{2a}\phi} = 0 \ . \tag{2.20}$$

[c]Note that Eq. (2.20) can also be more generally derived; for example, it still holds if one relaxes the assumption of isotropicity in the transverse space.

Solving this in the transverse $(D - d)$ dimensions with our assumption of transverse isotropicity (*i.e.* spherical symmetry) yields

$$e^{\frac{\varsigma\Delta}{2a}\phi} \equiv H(y) = 1 + \frac{k}{r^{\tilde{d}}} \qquad\qquad k > 0 \ , \tag{2.21}$$

where the constant of integration $\phi_{|_{r\to\infty}}$ has been set equal to zero here for simplicity: $\phi_\infty = 0$. The integration constant k in (2.21) sets the mass scale of the solution; it has been taken to be positive in order to ensure the absence of naked singularities at finite r. This positivity restriction is similar to the usual restriction to a positive mass parameter M in the standard Schwarzschild solution.

In the case of the elementary/electric ansatz, with $\varsigma = +1$, it still remains to find the function $C(r)$ that determines the antisymmetric-tensor gauge field potential. In this case, it follows from (2.13) that $S^2 = e^{a\phi - 1dA}(C'e^C)^2$. Combining this with (2.16), one finds the relation

$$\frac{\partial}{\partial r}(e^C) = \frac{-\sqrt{\Delta}}{a} e^{-\frac{1}{2}a\phi + dA}\phi' \tag{2.22}$$

(where it should be remembered that $a < 0$). Finally, it is straightforward to verify that the relation (2.22) is consistent with the equation of motion for $F_{[n]}$:

$$\nabla^2 C + C'(C' + \tilde{B}' - dA' + a\phi') = 0 \ . \tag{2.23}$$

In order to simplify the explicit form of the solution, we now pick values of the integration constants to make $A_\infty = B_\infty = 0$, so that the solution tends to flat empty space at transverse infinity. Assembling the result, starting from the Laplace-equation solution $H(y)$ (2.21), one finds [7,13]

$$ds^2 = H^{\frac{-4\tilde{d}}{\Delta(D-2)}} dx^\mu dx^\nu \eta_{\mu\nu} + H^{\frac{4d}{\Delta(D-2)}} dy^m dy^m \tag{2.24a}$$

$$e^\phi = H^{\frac{2a}{\varsigma\Delta}} \qquad \varsigma = \begin{cases} +1, & \text{elementary/electric} \\ -1, & \text{solitonic/magnetic} \end{cases} \tag{2.24b}$$

$$H(y) = 1 + \frac{k}{r^{\tilde{d}}} \ , \tag{2.24c}$$

and in the elementary/electric case, $C(r)$ is given by

$$e^C = \frac{2}{\sqrt{\Delta}} H^{-1} \ . \tag{2.25}$$

In the solitonic/magnetic case, the constant of integration is related to the magnetic charge parameter λ in the ansatz (2.6) by

$$k = \frac{\sqrt{\Delta}}{2\tilde{d}}\lambda \ . \tag{2.26}$$

In the elementary/electric case, this relation may be taken to *define* the parameter λ.

The harmonic function $H(y)$ (2.21) determines all of the features of a p-brane solution (except for the choice of gauge for the $A_{[n-1]}$ gauge potential). It is useful to express the electric and magnetic field strengths directly in terms of H:

$$\begin{aligned} F_{m\mu_1\ldots\mu_{n-1}} &= \epsilon_{\mu_1\ldots\mu_{n-1}}\partial_m(H^{-1}) \quad m = d,\ldots,D-1 \quad \text{electric} & (2.27a)\\ F_{m_1\ldots m_n} &= -\epsilon_{m_1\ldots m_n r}\partial_r H \quad m = d,\ldots,D-1 \quad \text{magnetic,} & (2.27b)\end{aligned}$$

with all other independent components vanishing in either case.

3 $D = 11$ examples

Let us now return to the bosonic sector of $D = 11$ supergravity, which has the action (1.1). In searching for p-brane solutions to this action, there are two particular points to note. The first is that no scalar field is present in (1.1). This follows from the supermultiplet structure of the $D = 11$ theory, in which all fields are gauge fields. In lower dimensions, of course, scalars do appear; *e.g.* the dilaton in $D = 10$ type IIA supergravity emerges out of the $D = 11$ metric upon dimensional reduction from $D = 11$ to $D = 10$. The absence of the scalar that we had in our general discussion may be handled here simply by identifying the scalar coupling parameter a with zero, so that the scalar may be consistently truncated from our general action (2.1). Since $a^2 = \Delta - 2d\tilde{d}/(D-2)$, we identify $\Delta = 2\cdot 3\cdot 6/9 = 4$ for the $D = 11$ cases.

Now let us consider the consistency of dropping contributions arising from the FFA Chern-Simons term in (1.1). Note that for $n = 4$, the $F_{[4]}$ antisymmetric tensor field strength supports either an elementary/electric solution with $d = n-1 = 3$ (*i.e.* a $p = 2$ membrane) or a solitonic/magnetic solution with $\tilde{d} = 11-3-2 = 6$ (*i.e.* a $p = 5$ brane). In both these elementary and solitonic cases, the FFA term in the action (1.1) vanishes and hence this term does not make any non-vanishing contribution to the metric field equations for our ansätze. For the antisymmetric tensor field equation, a further check is necessary, since there one requires the *variation* of the FFA term to vanish in order to consistently ignore it. The field equation for $A_{[3]}$ is (1.2), which when written out explicitly becomes

$$\partial_M\left(\sqrt{-g}F^{MUVW}\right)+\frac{1}{2(4!)^2}\epsilon^{UVWx_1x_2x_3x_4y_1y_2y_3y_4}F_{x_1x_2x_3x_4}F_{y_1y_2y_3y_4}=0\ . \quad (3.1)$$

By direct inspection, one sees that the second term in this equation vanishes for both ansätze.

Next, we shall consider the elementary/electric and the solitonic/magnetic $D=11$ cases in detail. Subsequently, we shall explore how these particular solutions fit into wider, "black," families of p-branes.

3.1 $D=11$ Elementary/electric 2-brane

From our general discussion in Sec. 2, we have the elementary-ansatz solution [18]

$$\begin{aligned} ds^2 &= (1+\tfrac{k}{r^6})^{-2/3}d^\mu dx^\nu\eta_{\mu\nu}+(1+\tfrac{k}{r^6})^{1/3}dy^m dy^m \\ A_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda}(1+\tfrac{k}{r^6})^{-1}\ , \qquad \text{other components zero.} \\ &\qquad \underline{\text{electric 2-brane: isotropic coordinates}} \end{aligned} \quad (3.2)$$

At first glance, this solution looks like it might be singular at $r=0$. However, if one calculates the invariant components of the curvature tensor R_{MNPQ} and of the field strength $F_{m\mu_1\mu_2\mu_3}$, subsequently referred to an orthonormal frame by introducing vielbeins as in (2.8), one finds these invariants to be nonsingular. Moreover, although the proper distance to the surface $r=0$ along a $t=x^0=$ const. geodesic diverges, the surface $r=0$ can be reached along null geodesics in finite affine parameter.[19]

Thus, one may suspect that the metric as given in (3.2) does not in fact cover the entire spacetime, and so one should look for an analytic extension of it. Accordingly, one may consider a change to "Schwarzschild-type" coordinates by setting $r=(\tilde r^6-k)^{1/6}$. The solution then becomes:[19]

$$\begin{aligned} ds^2 &= (1+\tfrac{k}{r^6})^{2/3}(-dt^2+d\sigma^2+d\rho^2)+(1+\tfrac{k}{r^6})^{-2}d\tilde r^2+\tilde r^2 d\Omega_7^2 \\ A_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda}(1+\tfrac{k}{r^6})\ , \qquad \text{other components zero,} \\ &\qquad \underline{\text{electric 2-brane: Schwarzschild-type coordinates}} \end{aligned} \quad (3.3)$$

where we have supplied explicit worldvolume coordinates $x^\mu=(t,\sigma,\rho)$ and where $d\Omega_7^2$ is the line element on the unit 7-sphere, corresponding to the boundary $\partial\mathcal{M}_{8\mathrm{T}}$ of the $11-3=8$ dimensional transverse space.

The Schwarzschild-like coordinates make the surface $\tilde{r} = k^{1/6}$ (corresponding to $r = 0$) look like a horizon. One may indeed verify that the normal to this surface is a *null* vector, confirming that $\tilde{r} = k^{1/6}$ is in fact a horizon. This horizon is *degenerate*, however. Owing to the $2/3$ exponent in the g_{00} component, curves along the t axis for $\tilde{r} < k^{1/6}$ remain timelike, so that light cones do not "flip over" inside the horizon, unlike the situation for the classic Schwarzschild solution.

In order to see the structure of the membrane spacetime more clearly, let us change coordinates once again, setting $\tilde{r} = k^{1/6}(1 - R^3)^{-1/6}$. Overall, the transformation from the original isotropic coordinates to these new ones is effected by setting $\tilde{r} = k^{1/6}R^{1/2}/(1 - R^3)^{1/6}$. In these new coordinates, the solution becomes [19]

$$\begin{aligned} ds^2 &= \left\{R^2(-dt^2 + d\sigma^2 + d\rho^2) + 4k^{1/3}R^{-2}dR^2\right\} + k^{1/3}d\Omega_7^2 && \text{(a)} \\ &\quad + k^{1/3}[(1 - R^3)^{-1/3} - 1][4R^{-2}dR^2 + d\Omega_7^2] && \text{(b)} \\ A_{\mu\nu\lambda} &= R^3\epsilon_{\mu\nu\lambda}\ , && \text{other components zero.} \end{aligned}$$

$$\underline{\text{electric 2-brane: interpolating coordinates}} \tag{3.4}$$

This form of the solution makes it clearer that the light-cones do not "flip over" in the region inside the horizon (which is now at $R = 0$, with $R < 0$ being the interior). The main usefulness of the third form (3.4) of the membrane solution, however, is that it reveals how the solution *interpolates* between other "vacuum" solutions of $D = 11$ supergravity.[19] As $R \to 1$, the solution becomes flat, in the asymptotic exterior transverse region. As one approaches the horizon at $R = 0$, line (b) of the metric in (3.4) vanishes at least linearly in R. The residual metric, given in line (a), may then be recognized as a standard form of the metric on $(\mathrm{AdS})_4 \times \mathcal{S}^7$, generalizing the Robinson-Bertotti solution on $(\mathrm{AdS})_2 \times \mathcal{S}^2$ in $D = 4$. Thus, the membrane solution interpolates between flat space as $R \to 1$ and $(\mathrm{AdS})_4 \times \mathcal{S}^7$ as $R \to 0$ at the horizon.

Continuing on inside the horizon, one eventually encounters a true singularity at $\tilde{r} = 0$ ($R \to -\infty$). Unlike the singularity in the classic Schwarzschild solution, which is spacelike and hence unavoidable, the singularity in the membrane spacetime is *timelike.* Generically, geodesics do not intersect the singularity at a finite value of an affine parameter value. Radial null geodesics do intersect the singularity at finite affine parameter, however, so the spacetime is in fact genuinely singular. The timelike nature of this singularity, however, invites one to consider coupling a δ-function *source* to the solution at $\tilde{r} = 0$. Indeed, the $D = 11$ supermembrane action,[20] which generalizes the Nambu-Goto action for the string, is the unique "matter" system that can consistently couple to $D = 11$ supergravity.[20,22] Analysis of this coupling yields a relation between the parameter k in the solution (3.2) and the *tension* T of the supermembrane action:[18]

$$k = \frac{\kappa^2 T}{3\Omega_7} \ , \tag{3.5}$$

where $1/(2\kappa^2)$ is the coefficient of $\sqrt{-g}R$ in the Einstein-Hilbert Lagrangian and Ω_7 is the volume of the unit 7-sphere $\mathcal{S}^7$, *i.e.* the solid angle subtended by the boundary at transverse infinity.

The global structure of the membrane spacetime [19] is similar to the extreme Reissner-Nordstrom solution of General Relativity.[24] This global structure is summarized by a Carter-Penrose diagram as shown in Figure 1, in which the angular coordinates on $\mathcal{S}^7$ and also two ignorable worldsheet coordinates have been suppressed. As one can see, the region mapped by the isotropic coordinates does not cover the whole spacetime. This region, shaded in the diagram, is geodesically incomplete, since one may reach its boundaries $\mathcal{H}^+$, $\mathcal{H}^-$ along radial null geodesics at a finite affine-parameter value. These boundary surfaces are not singular, but, instead, constitute future and past horizons (one can see from the form (3.3) of the solution that the normals to these surfaces are null). The "throat" $\mathcal{P}$ in the diagram should be thought of as an exceptional point at infinity, and not as a part of the central singularity.

The region exterior to the horizon interpolates between flat regions $\mathcal{J}^\pm$ at future and past null infinities and a geometry that asymptotically tends to $(\mathrm{AdS})_4 \times \mathcal{S}^7$ on the horizon. This interpolating portion of the spacetime, corresponding to the shaded region of Figure 1 which is covered by the isotropic coordinates, may be sketched as shown in Figure 2.

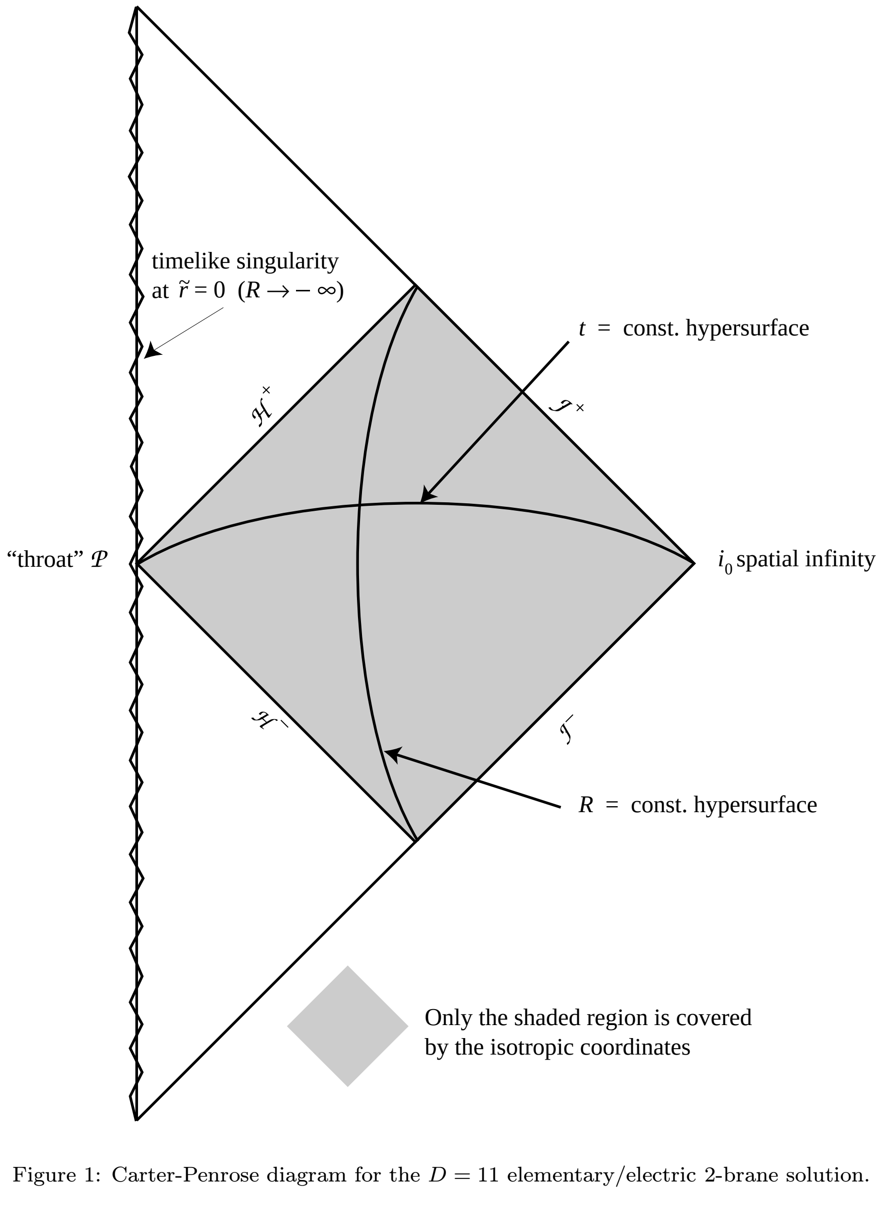

Figure 1: Carter-Penrose diagram for the $D = 11$ elementary/electric 2-brane solution.

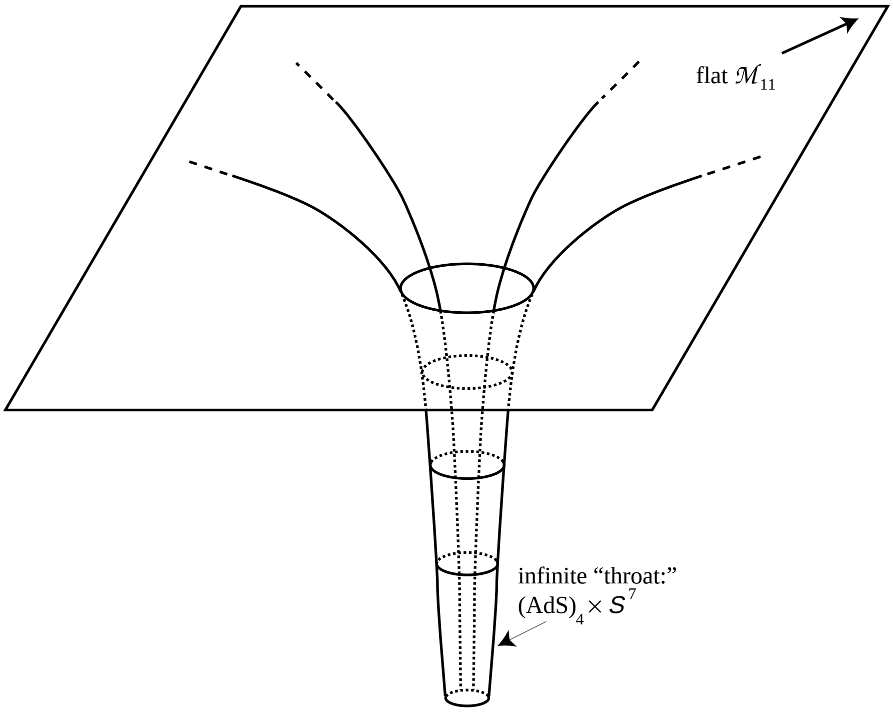

Figure 2: The $D = 11$ elementary/electric 2-brane solution interpolates between flat space at $\mathcal{J}^{\pm}$ and $(\mathrm{AdS})_4 \times \mathcal{S}^7$ at the horizon.

3.2 $D = 11$ Solitonic/magnetic 5-brane

Now consider the 5-brane solution to the $D = 11$ theory given by the solitonic ansatz for $F_{[4]}$. In isotropic coordinates, this solution is a magnetic 5-brane:[25]

$$\begin{aligned} ds^2 &= (1+\tfrac{k}{r^3})^{-\,1/3} dx^\mu dx^\nu \eta_{\mu\nu} + (1+\tfrac{k}{r^3})^{\,2/3} dy^m dy^m \qquad \mu,\nu = 0,\cdots,5 \\ F_{m_1\cdots m_4} &= 3k\epsilon_{m_1\cdots m_4 p}\frac{y^p}{r^5} \qquad \text{other components zero.} \end{aligned}$$

$$\underline{\text{magnetic 5-brane: isotropic coordinates}} \qquad (3.6)$$

As in the case of the elementary/electric membrane, this solution interpolates between two "vacua" of $D = 11$ supergravity. Now, however, these asymptotic geometries consist of the flat region encountered as $r \to \infty$ and of $(\mathrm{AdS})_7 \times \mathcal{S}^4$ as one approaches $r = 0$, which once again is a degenerate horizon. Combining two coordinate changes analogous to those of the elementary case,

$r = (\tilde{r}^3 - k)^{1/3}$ and $\tilde{r} = k^{1/3}(1 - R^6)^{-1/3}$, one has an overall transformation

$$r = \frac{k^{1/3} R^2}{(1 - R^6)^{1/3}} \ . \tag{3.7}$$

After these coordinate changes, the metric becomes

$$\underbrace{ds^2 \ = \ R^2 dx^\mu dx^\nu \eta_{\mu\nu} + k^{2/3} \left[\tfrac{4R^{-2}}{(1-R^6)^{8/3}} dR^2 + \tfrac{d\Omega_4^2}{(1-R^6)^{2/3}} \right]}_{\text{magnetic 5-brane: interpolating coordinates}} \ . \tag{3.8}$$

Once again, the surface $r = 0 \leftrightarrow R = 0$ may be seen from (3.8) to be a nonsingular degenerate horizon. In this case, however, not only do the light cones maintain their timelike orientation when crossing the horizon, as already happened in the electric case (3.4), but now the magnetic solution (3.8) is in fact fully *symmetric* [26] under a discrete isometry $R \to -R$.

Given this isometry $R \to -R$, one can *identify* the spacetime region $R \leq 0$ with the region $R \geq 0$. This identification is analogous[d] to the identification one naturally makes for flat space when written in polar coordinates, with the metric $ds^2_{\text{flat}} = -dt^2 + dr^2 + r^2 d^2$. Ostensibly, in these coordinates there appear to be separate regions of flat space with $r \gtrless 0$, but, owing to the existence of the isometry $r \to -r$, these regions may be identified. Accordingly, in the solitonic/magnetic 5-brane spacetime, we identify the region $-1 < R \leq 0$ with the region $0 \leq R < 1$. In the asymptotic limit where $R \to -1$, one finds an asymptotically flat geometry that is indistinguishable from the region where $R \to +1$, *i.e.* where $r \to \infty$. Thus, there is no singularity at all in the solitonic/magnetic 5-brane geometry. There is still an infinite "throat," however, at the horizon, and the region covered by the isotropic coordinates might again be sketched as in Figure 2, except now with the asymptotic geometry down the "throat" being $(\text{AdS})_7 \times \mathcal{S}^4$ instead of $(\text{AdS})_4 \times \mathcal{S}^7$ as for the elementary/electric solution. The Carter-Penrose diagram for the solitonic/magnetic 5-brane solution is given in Figure 3, where the full diagram extends indefinitely by "tiling" the section shown. Upon using the $R \to -R$ isometry to make discrete identifications, however, the whole of the spacetime may be considered to consist of just region **I**, which is the region covered by the isotropic coordinates (3.6).

[d] In considering this analogy, one should also take into account the possibility of conical singularities. In the case of flat space, a conical singularity with deficit angle $\theta = 2\pi$ arises at the origin $R = 0$ if one chooses not to make a discrete identification of the two regions $R \gtrless 0$. This is most easily seen by considering a combined pair of forward and backward cones with deficit angle $\theta < 2\pi$, then taking the limit $\theta \to 2\pi$. In this case, as in the case of the magnetic 5-brane geometry, the elimination of conical singularities actually *requires* making the discrete identification.

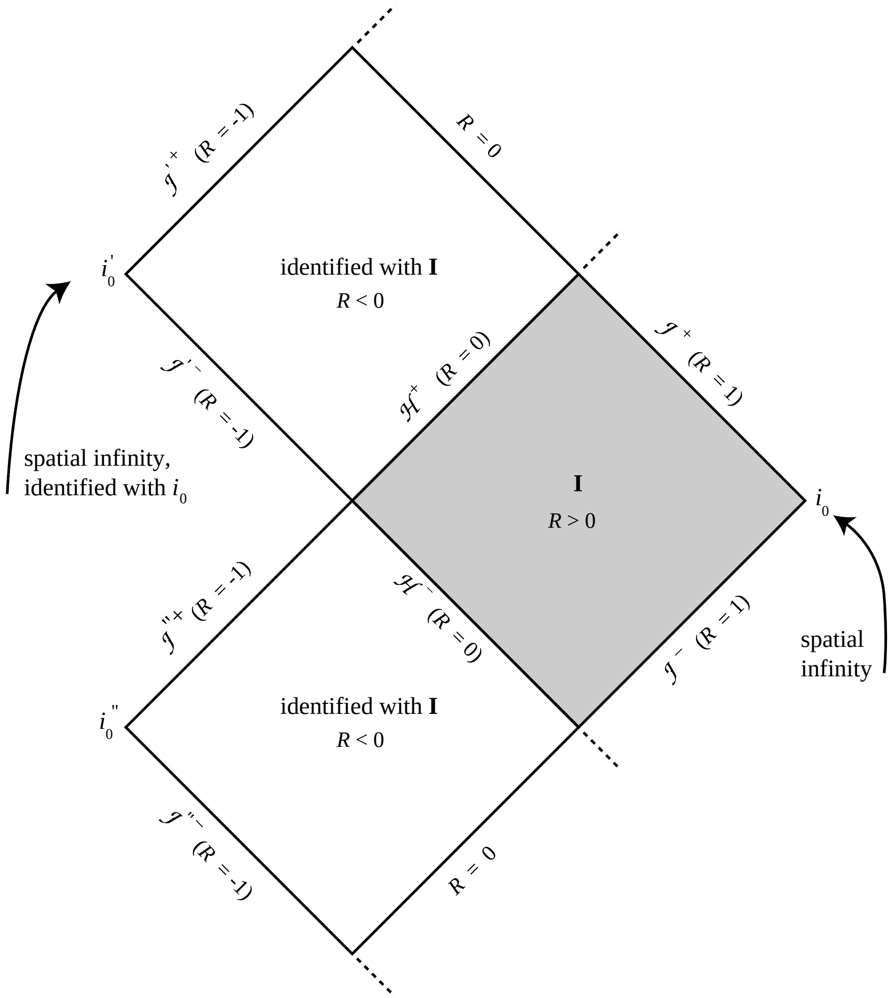

Figure 3: Carter-Penrose diagram for the solitonic/magnetic 5-brane solution.

After identification of the $R \gtrless 0$ regions, the 5-brane spacetime (3.6) is *geodesically complete.* Unlike the case of the elementary membrane solution (3.2, 3.4), one finds in the solitonic/magnetic case that the null geodesics passing through the horizon at $R = 0$ continue to evolve in their affine parameters without bound as $R \to -1$. Thus, the solitonic 5-brane solution is *completely non-singular.*

The electric and magnetic $D = 11$ solutions discussed here and in the previous subsection are special in that they do not involve a scalar field, since the bosonic sector of $D = 11$ supergravity (1.1) does not even contain a scalar field. Similar solutions occur in other situations where the parameter a (2.18) for a field strength supporting a p-brane solution vanishes, in which cases the scalar fields may consistently be set to zero; this happens for $(D, d) = (11, 3)$, (11,5), (10,4), (6,2), (5,1), (5,2) and (4,1). In these special cases, the solutions are nonsingular at the horizon and so one may analytically continue through to the other side of the horizon. When d is even for "scalarless" solutions of this type, there exists a discrete isometry analogous to the $R \to -R$ isometry of the $D = 11$ 5-brane solution (3.8), allowing the outer and inner regions to be identified.[26] When d is odd in such cases, the analytically-extended metric eventually reaches a timelike curvature singularity at $\tilde r = 0$.

When $a \neq 0$ and the scalar field associated to the field strength supporting a solution cannot be consistently set to zero, then the solution is singular at the horizon, as can be seen directly in the scalar solution (2.21) itself (where we recall that in isotropic coordinates, the horizon occurs at $r = 0$)

3.3 Black branes

In order to understand better the family of supergravity solutions that we have been discussing, let us now consider a generalization that lifts the degenerate nature of the horizon. Written in Schwarzschild-type coordinates, one finds the generalized "black brane" solution [27,28]

$$\begin{aligned} ds^2 &= -\frac{\Sigma_+}{\Sigma_-^{\left\{1-\frac{4\tilde d}{\Delta(D-2)}\right\}}}dt^2 + \Sigma_-^{\frac{4\tilde d}{\Delta(D-2)}}dx^i dx^i \\ &\quad + \frac{\Sigma_-^{\left\{\frac{2a^2}{\Delta\tilde d}-1\right\}}}{\Sigma_+}d\tilde r^2 + \tilde r^2 \Sigma_-^{\frac{2a^2}{\Delta\tilde d}}d\Omega^2_{D-d-1} \\ e^{\frac{\varsigma\Delta}{2a}\phi} &= \Sigma_-^{-1} \qquad\qquad \Sigma_\pm = 1 - \left(\tfrac{r_\pm}{\tilde r}\right)^{\tilde d}\ . \end{aligned} \tag{3.9}$$

black brane: Schwarzschild-type coordinates

The antisymmetric tensor field strength for this solution corresponds to a charge parameter $\lambda = 2\tilde d/\sqrt{\Delta}(r_= r_-)^{\tilde d/2}$, either electric or magnetic.

The characteristic feature of the above "blackened" p-branes is that they have a nondegenerate, nonsingular outer horizon at $\tilde{r} = r_+$, at which the light cones "flip over." At $\tilde{r} = r_-$, one encounters an inner horizon, which, however, coincides in general with a curvature singularity. The singular nature of the solution at $\tilde{r} = r_-$ is apparent in the scalar ϕ in (3.9). For solutions with $p \geq 1$, the singularity at the inner horizon persists even in cases where the scalar ϕ is absent.

The *extremal* limit of the black brane solution occurs for $r_+ = r_-$. When $a = 0$ and scalars may consistently be set to zero, the singularity at the horizon $r_+ = r_-$ disappears and then one may analytically continue through the horizon. In this case, the light cones do not "flip over" at the horizon because one is really crossing two coalesced horizons, and the coincident "flips" of the light cones cancel out.

The generally singular nature of the inner horizon of the non-extreme solution (3.9) shows that the "location" of the p-brane in spacetime should normally be thought to coincide with the inner horizon, or with the degenerate horizon in the extremal case.

4 Charges, Masses and Supersymmetry

The p-brane solutions that we have been studying are supported by antisymmetric tensor gauge field strengths that fall off at transverse infinity like $r^{-(\tilde{d}+1)}$, as one can see from $(2.5, 2.25, 2.6)$. This asymptotic falloff is slow enough to give a nonvanishing total charge density from a Gauss' law flux integral at transverse infinity, and we shall see that, for the "extremal" class of solutions that is our main focus, the mass density of the solution saturates a "Bogomol'ny bound" with respect to the charge density. In this Section, we shall first make more precise the relation between the geometry of the p-brane solutions, the p-form charges U_{AB} and V_{ABCDE} and the scalar charge magnitudes U and V $(1.3, 1.4)$; we shall then discuss the relations between these charges, the energy density and the preservation of unbroken supersymmetry.

4.1 p-form charges

Now let us consider the inclusion of sources into the supergravity equations. The harmonic function (2.21) has a singularity which has for simplicity been placed at the origin of the transverse coordinates y^m. As we have seen in Sections 3.1 and 3.2, whether or not this gives rise to a physical singularity in a solution depends on the global structure of that solution. In the electric 2-brane case, the solution does in the end have a singularity.[26] This singularity is unlike the Schwarzschild singularity, however, in that it is a *timelike* curve, and thus it may be considered to be the wordvolume of a δ-function source. The electric source that couples to $D = 11$ supergravity is the fundamental supermembrane action,[20] whose bosonic part is

$$I_{\rm source} = Q_{\rm e} \int_{\mathcal{W}_3} d^3\xi \Big[\sqrt{-\det(\partial_\mu x^M \partial_\nu x^N g_{MN}(x))} + \frac{1}{3!}\epsilon^{\mu\nu\rho}\partial_\mu x^M \partial_\nu x^N \partial_\rho x^R A_{MNR}(x) \Big] \ . \tag{4.1}$$

The source strength $Q_{\rm e}$ will shortly be found to be equal to the electric charge U upon solving the coupled equations of motion for the supergravity fields and a single source of this type. Varying the source action (4.1) with $\delta/\delta A_{[3]}$, one obtains the δ-function current

$$J^{MNR}(z) = Q_{\rm e} \int_{\mathcal{W}_3} \delta^3(z - x(\xi)) dx^M \wedge dx^N \wedge dx^R \ . \tag{4.2}$$

This current now stands on the RHS of the $A_{[3]}$ equation of motion:

$$d({}^*F_{[4]} + \tfrac{1}{2}A_{[3]} \wedge F_{[4]}) = {}^*J_{[3]} \ . \tag{4.3}$$

Thus, instead of the Gauss' law expression for the charge, one may instead rewrite the charge as a volume integral of the source,

$$U = \int_{\mathcal{M}_8} {}^*J_{[3]} = \frac{1}{3!}\int_{\mathcal{M}_8} J^{0MN} d^8 S_{MN} \ , \tag{4.4}$$

where $d^8 S_{MN}$ is the 8-volume element on $\mathcal{M}_8$, specified within a $D = 10$ spatial section of the supergravity spacetime by a 2-form. The charge derived in this way from a single 2-brane source is thus $U = Q_{\rm e}$ as expected.

Now consider the effect of making different choices of the $\mathcal{M}_8$ integration volume within the $D = 10$ spatial spacetime section, as shown in Figure 4. Let the difference between the surfaces $\mathcal{M}_8$ and $\mathcal{M}_8'$ be infinitesimal and be given by a vector field $v^N(x)$. The difference in the electric charges obtained is then given by

$$\delta U = \int_{\mathcal{M}_8} \mathcal{L}_v {}^*J_{[3]} = \frac{1}{3!}\int_{\partial\mathcal{M}_8} J^{0MN} v^R d^7 S_{MNR} \ , \tag{4.5}$$

where $\mathcal{L}_v$ is the Lie derivative along the vector field v. The second equality in (4.5) follows using Stokes' theorem and the conservation of the current $J_{[3]}$.

Now a topological nature of the charge integral (1.3) becomes apparent; similar considerations apply to the magnetic charge (1.4). As long as the current $J_{[3]}$ vanishes on the boundary $\partial\mathcal{M}_8$, the difference (4.5) between the charges calculated using the integration volumes $\mathcal{M}_8$ and $\mathcal{M}'_8$ will vanish. This divides the electric-charge integration volumes into two topological classes distinguishing those for which $\partial\mathcal{M}_8$ "captures" the p-brane current, as shown in Figure 4 and giving $U = Q_{\rm e}$, from those that do not capture the current, giving U=0.

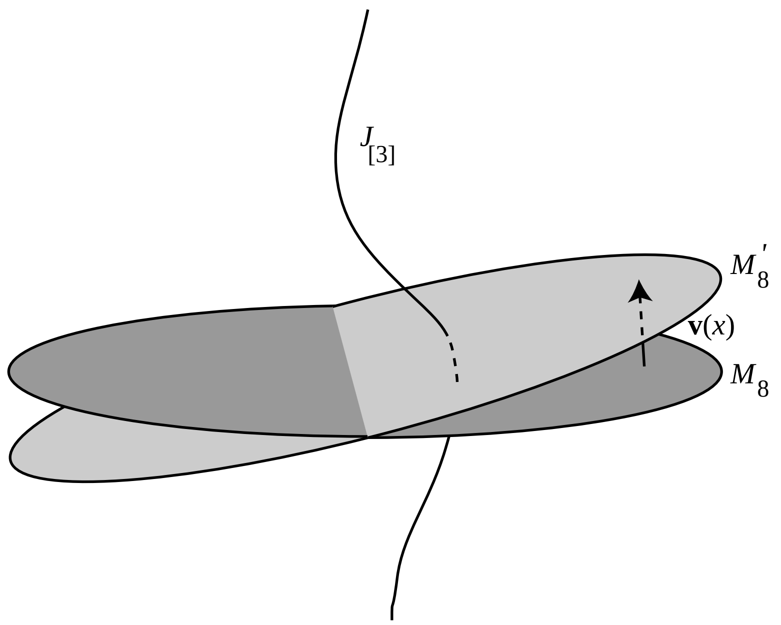

Figure 4: Different choices of charge integration volume "capturing" the current $J_{[3]}$.

The above discussion shows that the orientation-dependence of the U charges (1.3) is essentially topological. The topological classes for the charge integrals are naturally labeled by the asymptotic orientations of the p-brane spatial surfaces; an integration volume $\mathcal{M}_8$ extending out to infinity flips from the "capturing" class into the "non-capturing" class when $\partial\mathcal{M}_8$ crosses the δ-function surface defined by the current $J_{[3]}$. The charge thus naturally has a magnitude $|Q_{[p]}| = Q_{\rm e}$ and a unit p-form orientation $Q_{[p]}/|Q_{[p]}|$ that is proportional to the *asymptotic spatial volume form* of the p=brane. Both the magnitude and the orientation of this p-form charge are conserved using the supergravity equations of motion.

The necessity of considering asymptotic p-brane volume forms arises because the notion of a p-form charge is not limited to static, flat p-brane solutions such as (2.3, 2.5, 2.6). Such charges can also be defined for any solution whose energy differs from that of a flat, static one by a finite amount. The charges for such solutions will also appear in the supersymmetry algebra (1.5) for such backgrounds, but the corresponding energy densities will not in general saturate the BPS bounds. For a finite energy difference with respect to a flat, static p-brane, the asymptotic orientation of the p-brane volume form must tend to that of a static flat solution, which plays the rôle of a "BPS vacuum" in a given p-form charge sector of the theory.

In order to have a non-vanishing value for a charge (1.3) or (1.4) occurring in the supersymmetry algebra (1.5), the p-brane must be either *infinite* or *wrapped* around a compact spacetime dimension. The case of a finite p-brane is sketched in Figure 5. Since the boundary $\partial\mathcal{M}$ of the infinite integration volume $\mathcal{M}$ does not capture the locus where the p-brane current is non-vanishing, the current calculated using $\mathcal{M}$ will vanish as a result. Instead of an infinite p-brane, one may alternately have a p-brane wrapped around a compact dimension of spacetime, so that an integration-volume boundary $\partial\mathcal{M}_8$ is still capable of capturing the p-brane locus (if one considers this case as an infinite, but periodic, solution, this case may be considered simultaneously with that of the infinite p-branes). Only in such cases do the p-form charges occurring in the supersymmetry algebra (1.5) take non-vanishing values.[e]

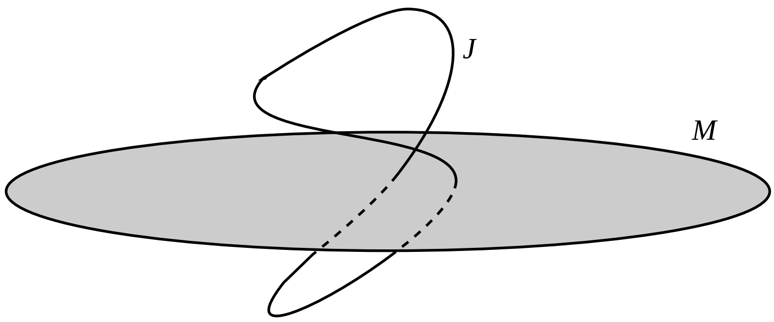

Figure 5: Finite p-brane not captured by $\partial\mathcal{M}$, giving zero charge.

[e]If one considers integration volumes that do not extend out to infinity, then one can construct integration surfaces that capture finite p-branes. Such charges do not occur in the supersymmetry algebra (1.5), but they are still of importance in determining the possible intersections of p-branes.[21]

4.2 p-brane mass densities

Now let us consider the mass density of a p-brane solution. Since the p-brane solutions have translational symmetry in their p spatial worldvolume directions, the total energy as measured by a surface integral at spatial infinity diverges, owing to the infinite extent. What is thus more appropriate to consider instead is the value of the density, energy/(unit p-volume). Since we are considering solutions in their rest frames, this will also give the value of mass/(unit p-volume), or *tension* of the solution. Instead of the standard spatial $d^{D-2}\Sigma^a$ surface integral, this will be a $d^{(D-d-1)}\Sigma^m$ surface integral over the boundary $\partial\mathcal{M}_{\rm T}$ of the transverse space.

The ADM formula for the energy density written as a Gauss'-law integral (see, *e.g.,* Ref.[16]) is, dropping the divergent spatial $d\Sigma^{\mu=i}$ integral,

$$\mathcal{E} = \int_{\partial\mathcal{M}_{\rm T}} d^{D-d-1}\Sigma^m(\partial^n h_{mn} - \partial_m h^b_b) \ , \tag{4.6}$$

written for $g_{MN} = \eta_{MN} + h_{MN}$ tending asymptotically to flat space in Cartesian coordinates, and with a, b *spatial* indices running over the values $\mu = i = 1, \ldots, d-1$; $m = d, \ldots, D-1$. For the general p-brane solution (2.24), one finds

$$h_{mn} = \frac{4kd}{\Delta(D-2)r^{\tilde{d}}}\delta_{mn} \ , \qquad\qquad h^b_b = \frac{8k(d+\frac{1}{2}\tilde{d})}{\Delta(D-2)r^{\tilde{d}}} \ , \tag{4.7}$$

and, since $d^{(D-d-1)}\Sigma^m = r^{\tilde{d}}y^m d\Omega^{(D-d-1)}$, one finds

$$\mathcal{E} = \frac{4k\tilde{d}\Omega_{D-d-1}}{\Delta} \ , \tag{4.8}$$

where Ω_{D-d-1} is the volume of the $\mathcal{S}^{D-d-1}$ unit sphere. Recalling that $k = \sqrt{\Delta}\lambda/(2\tilde{d})$, we consequently have a relation between the mass per unit p volume and the charge parameter of the solution

$$\mathcal{E} = \frac{2\lambda\Omega_{D-d-1}}{\sqrt{\Delta}} \ . \tag{4.9}$$

By contrast, the black brane solution (3.9) has $\mathcal{E} > 2\lambda\Omega_{D-d-1}/\sqrt{\Delta}$, so the extremal p-brane solution (2.24) is seen to *saturate* the inequality $\mathcal{E} \geq 2\lambda\Omega_{D-d-1}/\sqrt{\Delta}$.

4.3 p-brane charges

As one can see from (4.8, 4.9), the relation (2.26) between the integration constant k in the solution (2.24) and the charge parameter λ implies a deep link between the energy density and certain electric or magnetic charges. In the electric case, this charge is a quantity conserved by virtue of the equations of motion for the antisymmetric tensor gauge field $A_{[n-1]}$, and has generally become known as a "Page charge," after its first discussion in Ref.[2] To be specific, if we once again consider the bosonic sector of $D = 11$ supergravity theory (1.1), for which the antisymmetric tensor field equation was given in (3.1), one finds the Gauss'-law form conserved quantity [2] U (1.3).

For the p-brane solutions (2.24), the $\int A \wedge F$ term in (1.3) vanishes. The $\int {}^*F$ term does, however, give a contribution in the elementary/electric case, provided one picks $\mathcal{M}_8$ to coincide with the transverse space to the $d = 3$ membrane worldvolume, $\mathcal{M}_{8\mathrm{T}}$. The surface element for this transverse space is $d\Sigma^m_{(7)}$, so for the $p = 2$ elementary membrane solution (3.2), one finds

$$U = \int_{\partial M_{8\mathrm{T}}} d\Sigma^m_{(7)} F_{m012} = \lambda\Omega_7 \ . \tag{4.10}$$

Since the $D = 11$ $F_{[4]}$ field strength supporting this solution has $\Delta = 4$, the mass/charge relation is

$$\mathcal{E} = U = \lambda\Omega_7 \ . \tag{4.11}$$

Thus, like the classic extreme Reissner-Nordstrom black-hole solution to which it is strongly related (as can be seen from the Carter-Penrose diagram given in Figure 1), the $D = 11$ membrane solution has equal mass and charge densities, saturating the inequality $\mathcal{E} \geq U$.

Now let us consider the charge carried by the solitonic/magnetic 5-brane solution (3.6). The field strength in (3.6) is purely transverse, so no electric charge (1.3) is present. The magnetic charge (1.4) is carried by this solution, however. Once again, there is only one orientation of the subsurface $\widetilde{\mathcal{M}}_5$ that gives a nonvanishing contribution, *i.e.* that with $\widetilde{\mathcal{M}}_5 = \mathcal{M}_{5\mathrm{T}}$, the transverse space to the $d = 6$ worldvolume:

$$V = \int_{\partial \mathcal{M}_{5\mathrm{T}}} d\Sigma^m_{(4)} \epsilon_{mnpqr} F^{npqr} = \lambda\Omega_4 \ . \tag{4.12}$$

Thus, in the solitonic/magnetic 5-brane case as well, we have a saturation of the mass-charge inequality:

$$\mathcal{E} = V = \lambda\Omega_4 \ . \tag{4.13}$$

4.4 Preserved supersymmetry

Since the bosonic solutions that we have been considering are *consistent truncations* of $D = 11$ supergravity, they must also possess another conserved quantity, the *supercharge.* Admittedly, since the supercharge is a Grassmanian (anticommuting) quantity, its value will clearly be zero for the class of purely bosonic solutions that we have been discussing. However, the functional form of the supercharge is still important, as it determines the form of the asymptotic supersymmetry algebra. The Gauss'-law form of the supercharge is given as an integral over the boundary of the spatial hypersurface. For the $D = 11$ solutions, this surface of integration is the boundary at infinity $\partial\mathcal{M}_{10}$ of the $D = 10$ spatial hypersurface; the supercharge is then [1]

$$Q = \int_{\partial\mathcal{M}_{10}} \Gamma^{0bc}\psi_c d\Sigma_{(9)b} \ . \tag{4.14}$$

One can also rewrite this in fully Lorentz-covariant form, where $d\Sigma_{(9)b} = d\Sigma_{(9)0b} \to d\Sigma_{(9)AB}$:

$$Q = \int_{\partial\mathcal{M}_{10}} \Gamma^{ABC}\psi_C d\Sigma_{(9)AB} \ . \tag{4.15}$$

After appropriate definitions of Poisson brackets, the $D = 11$ supersymmetry algebra for the supercharge (4.14, 4.15) is found to be given [29] by (1.5) Thus, the supersymmetry algebra wraps together all of the conserved Gauss'-law type quantities that we have discussed.

The positivity of the Q^2 operator on the LHS of the algebra (1.5) is at the root of the *Bogomol'ny bounds* [30,26,32]

$$\begin{aligned} \mathcal{E} &\geq (2/\sqrt{\Delta})U \qquad\qquad \underline{\text{electric bound}} \qquad (4.16\text{a}) \\ \mathcal{E} &\geq (2/\sqrt{\Delta})V \qquad\qquad \underline{\text{magnetic bound}} \qquad (4.16\text{b}) \end{aligned}$$

that are saturated by the p-brane solutions.

The saturation of the Bogomol'ny inequalities by the p-brane solutions is an indication that they fit into special types of supermultiplets. All of these bound-saturating solutions share the important property that they *leave some portion of the supersymmetry unbroken.* Within the family of p-brane solutions that we have been discussing, it turns out [32] that the Δ values of such "supersymmetric" p-branes are of the form $\Delta = 4/N$, where N is the number of antisymmetric tensor field strengths participating in the solution (distinct, but of the same rank). The different charge contributions to the supersymmetry algebra occurring for different values of N (hence different Δ) affect the Bogomol'ny bounds as shown in (4.16).

In order to see how a purely bosonic solution may leave some portion of the supersymmetry unbroken, consider specifically once again the membrane solution of $D = 11$ supergravity.[18] This theory [1] has just one spinor field, the gravitino ψ_M. Checking for the consistency of setting $\psi_M = 0$ with the supposition of some residual supersymmetry with parameter $\epsilon(x)$ requires solving the equation

$$\delta\psi_{A|_{\psi=0}} = \tilde{D}_A\epsilon = 0 \ , \tag{4.17}$$

where $\psi_A = e_A{}^M\psi_M$ and

$$\begin{aligned}\tilde{D}_A\epsilon &= D_A\epsilon - \frac{1}{288}\left(\Gamma_A{}^{BCDE} - 8\delta_A{}^B\Gamma^{CDE}\right)F_{BCDE}\epsilon \\ D_A\epsilon &= (\partial_A + \tfrac{1}{4}\omega_A{}^{BC}\Gamma_{BC})\epsilon \ .\end{aligned} \tag{4.18}$$

Solving the equation $\tilde{D}_A\epsilon = 0$ amounts to finding a *Killing spinor* field in the presence of the bosonic background. Since the Killing spinor equation (4.17) is linear in $\epsilon(x)$, the Grassmanian (anticommuting) character of this parameter is irrelevant to the problem at hand, which thus reduces effectively to solving (4.17) for a commuting quantity.

In order to solve the Killing spinor equation (4.17) in a p-brane background, it is convenient to adopt an appropriate basis for the $D = 11$ Γ matrices. For the $d = 3$ membrane background, one would like to preserve $\mathrm{SO}(2,1)\times\mathrm{SO}(8)$ covariance. An appropriate basis that does this is

$$\Gamma_A = (\gamma_\mu \otimes \Sigma_9, \mathbb{1}_{(2)} \otimes \Sigma_m) \ , \tag{4.19}$$

where γ_μ and $\mathbb{1}_{(2)}$ are 2×2 $\mathrm{SO}(2,1)$ matrices; Σ_9 and Σ_m are 16×16 $\mathrm{SO}(8)$ matrices, with $\Sigma_9 = \Sigma_3\Sigma_4\ldots\Sigma_{10}$, so $\Sigma_9^2 = \mathbb{1}_{(16)}$. The most general spinor field consistent with $(\text{Poincaré})_3 \times \mathrm{SO}(8)$ invariance in this spinor basis is of the form

$$\epsilon(x,y) = \epsilon_2 \otimes \eta(r) \ , \tag{4.20}$$

where ϵ_2 is a *constant* $\mathrm{SO}(2,1)$ spinor and $\eta(r)$ is an $\mathrm{SO}(8)$ spinor depending only on the isotropic radial coordinate r; η may be further decomposed into Σ_9 eigenstates by the use of $\frac{1}{2}(\mathbb{1}\pm\Sigma_9)$ projectors.

Analysis of the the Killing spinor condition (4.17) in the above spinor basis leads to the following requirements [12,18] on the background and on the spinor field $\eta(r)$:

1) The background must satisfy the conditions $3A' + 6B' = 0$ and $C'e^C = 3A'e^{3A}$. The first of these conditions is, however, precisely the linearity-condition refinement (2.15) that we made in the p-brane ansatz; the second condition follows from the ansatz refinement (2.17) (considered as a condition on ϕ'/a) and from (2.22). Thus, what appeared previously to be simplifying specializations in the derivation given in Section 2 turn out in fact to be conditions *required* for supersymmetric solutions.

2) $\eta(r) = H^{-1/6}(y) = e^{C(r)/6}\eta_0$, where η_0 is a constant SO(8) spinor. Thus, the surviving local supersymmetry parameter $\epsilon(x,y)$ must take the form $\epsilon(x,y) = H^{-1/6}\epsilon_\infty$, where $\epsilon_\infty = \epsilon_2 \otimes \eta_0$. Note that, after imposing this requirement, at most a finite number of parameters can remain unfixed in the product spinor $\epsilon_2 \otimes \eta_0$; *i.e.* the *local* supersymmetry of the $D = 11$ theory is almost entirely broken by any particular solution. So far, the requirement (4.17) has cut down the amount of surviving supersymmetry from $D = 11$ local supersymmetry (*i.e.* effectively an infinite number of components) to the finite number of independent components present in $\epsilon_2 \otimes \eta_0$. The maximum number of such *rigid* unbroken supersymmetry components is achieved for $D = 11$ flat space, which has a full set of 32 constant components.

3) $(\mathbb{1} + \Sigma_9)\eta_0 = 0$, so the constant SO(8) spinor η_0 is also required to be *chiral.*[f] This cuts the number of surviving parameters in the product $\epsilon_\infty = \epsilon_2 \otimes \eta_0$ by *half:* the total number of surviving rigid supersymmetries in $\epsilon(x,y)$ is thus $2 \cdot 8 = 16$ (counting real spinor components). Since this is half of the maximum rigid number (*i.e.* half of the 32 for flat space), one says that the membrane solution "preserves half" of the supersymmetry.

In general, the procedure for checking how much supersymmetry is preserved by a given BPS solution follows steps analogous to points 1) – 3) above: first a check that the conditions required on the background fields are satisfied, then a determination of the functional form of the supersymmetry parameter in terms of some finite set of spinor components, and finally the imposition of projection conditions on that finite set. In a more telegraphic partial discussion, one may jump straight to the projection conditions 3). These must, of course, also emerge from a full analysis of equations like (4.17). But one can also see more directly what they will be simply by considering the supersymmetry algebra (1.5), specialised to the BPS background. Thus, for example, in the case of a $D = 11$ membrane solution oriented in the $\{012\}$ directions, one has, after normalising to a unit 2-volume,

$$\frac{1}{\text{2-vol}}\{Q_\alpha, Q_\beta\} = -(C\Gamma^0)_{\alpha\beta}\mathcal{E} + (C\Gamma^{12})_{\alpha\beta}U_{12} \ . \tag{4.21}$$

Since, as we have seen in (4.11), the membrane solution saturating the Bogomol'ny bound (4.16a) with $\mathcal{E} = U = U_{12}$, one may rewrite (4.21) as

$$\frac{1}{\text{2-vol}}\{Q_\alpha, Q_\beta\} = 2\mathcal{E}P_{012} \qquad\qquad P_{012} = \tfrac{1}{2}(\mathbb{1} + \Gamma^{012}) \ , \tag{4.22}$$

[f] The specific chirality indicated here is correlated with the sign choice made in the elementary/electric form ansatz (2.4); one may accordingly observe from (1.1) that a $D = 11$ parity transformation requires a sign flip of $A_{[3]}$.

where P_{012} is a projection operator (*i.e.* $P_{012}^2 = P_{012}$) whose trace is $\mathrm{tr}P_{012} = \frac{1}{2}\cdot 32$; thus, half of its eigenvalues are zero, and half are unity. Any surviving supersymmetry transformation must give zero when acting on the BPS background fields, and so the anticommutator $\{Q_\alpha, Q_\beta\}$ of the generators must give zero when contracted with a surviving supersymmetry parameter ϵ^α. From (4.22), this translates to

$$P_{012}\,\epsilon_\infty = 0\ , \tag{4.23}$$

which is equivalent to condition 3) above, $(\mathbb{1} + \Sigma_9)\eta_0 = 0$. Thus, we once again see that the $D = 11$ supermembrane solution (3.2) preserves half of the maximal rigid $D = 11$ supersymmetry. When we come to discuss the cases of "intersecting" p-branes in Section 6, it will be useful to have quick derivations like this for the projection conditions that must be satisfied by surviving supersymmetry parameters.

More generally, the positive semi-definiteness of the operator $\{Q_\alpha, Q_\beta\}$ is the underlying principle in the derivation [30,26,32] of the Bogomol'ny bounds (4.16). A consequence of this positive semi-definiteness is that zero eigenvalues correspond to solutions that saturate the Bogomol'ny inequalities (4.16), and these solutions preserve one component of unbroken supersymmetry for each such zero eigenvalue.

Similar consideration of the solitonic/magnetic 5-brane solution [25] (3.6) shows that it also preserves half the rigid $D = 11$ supersymmetry. In the 5-brane case, the analogue of condition 2) above is $\epsilon(x, y) = H^{-1/12}(y)\epsilon_\infty$, and the projection condition following from the algebra of preserved supersymmetry generators for a 5-brane oriented in the $\{012345\}$ directions is $P_{012345}\,\epsilon_\infty = 0$, where $P_{012345} = \frac{1}{2}(\mathbb{1} + \Gamma^{012345})$.

5 Kaluza-Klein dimensional reduction

Let us return now to the arena of purely bosonic field theories, and consider the relations between various bosonic-sector theories and the corresponding relations between p-brane solutions. It is well-known that supergravity theories are related by dimensional reduction from a small set of basic theories, the largest of which being $D = 11$ supergravity. The spinor sectors of the theories are equally well related by dimensional reduction, but in the following, we shall restrict our attention to the purely bosonic sector.

In order to set up the procedure, let us consider a theory in $(D + 1)$ dimensions, but break up the metric in D-dimensionally covariant pieces:

$$d\hat{s}^2 = e^{2\alpha\varphi}ds^2 + e^{2\beta\phi}(dz + \mathcal{A}_{M}dx^{M})^2 \tag{5.1}$$

where carets denote $(D+1)$-dimensional quantities corresponding to the $(D+1)$-dimensional coordinates $x^{\hat{M}} = (x^M, z)$; ds^2 is the line element in D dimensions and α and β are constants. The scalar φ in D dimensions emerges from the metric in $(D+1)$ dimensions as $(2\beta)^{-1}\ln g_{zz}$. Adjustment of the constants α and β is necessary to obtain desired structures in D dimensions. In particular, one should pick $\beta = -(D-2)\alpha$ in order to arrange for the Einstein-frame form of the gravitational action in $(D+1)$ dimensions to go over to the Einstein-frame form of the action in D dimensions.

The essential step in a Kaluza-Klein dimensional reduction is a *consistent truncation* of the field variables, generally made by choosing them to be independent of the reduction coordinate z. By a consistent truncation, we always understand a restriction on the variables that commutes with variation of the action to produce the field equations, *i.e.* a restriction such that solutions to the equations for the restricted variables are also solutions to the equations for the unrestricted variables. This ensures that the lower-dimensional solutions which we shall obtain are also particular solutions to higher-dimensional supergravity equations as well. Making the parameter choice $\beta = -(D-2)\alpha$ to preserve the Einstein-frame form of the action, one obtains

$$\sqrt{-\hat{g}}R(\hat{g}) = \sqrt{-g}\Big(R(g) - (D-1)(D-2)\alpha^2\nabla_M\varphi\nabla^M\varphi - \tfrac{1}{4}e^{-2(D-1)\alpha\varphi}\mathcal{F}_{MN}\mathcal{F}^{MN}\Big) \tag{5.2}$$

where $\mathcal{F} = d\mathcal{A}$. If one now chooses $\alpha^2 = [2(D-1)(D-2)]^{-1}$, the φ kinetic term becomes conventionally normalized.

Next, one needs to establish the reduction ansatz for the $(D+1)$-dimensional antisymmetric tensor gauge field $\hat{F}_{[n]} = d\hat{A}_{[n-1]}$. Clearly, among the $n-1$ antisymmetrised indices of $\hat{A}_{[n-1]}$ at most one can take the value z, so we have the decomposition

$$\hat{A}_{[n-1]} = B_{[n-1]} + B_{[n-2]} \wedge dz\ . \tag{5.3}$$

All of these reduced fields are to be taken to be functionally independent of z. For the corresponding field strengths, first define

$$G_{[n]} = dB_{[n-1]} \tag{5.4a}$$
$$G_{[n-1]} = dB_{[n-2]}\ . \tag{5.4b}$$

However, these are not exactly the most convenient quantities to work with, since a certain "Chern-Simons" structure appears upon dimensional reduction. The metric in $(D+1)$ dimensions couples to all fields, and, consequently, dimensional reduction will produce some terms with undifferentiated Kaluza-Klein vector fields $\mathcal{A}_M$ coupling to D-dimensional antisymmetric tensors. Accordingly, it is useful to introduce

$$G'_{[n]} = G_{[n]} - G_{[n-1]} \wedge \mathcal{A} \ , \tag{5.5}$$

where the second term in (5.5) may be viewed as a Chern-Simons correction from the reduced D-dimensional point of view.

At this stage, we are ready to perform the dimensional reduction of our general action (2.1). We find that

$$\hat{I} =: \int d^{D+1}x\sqrt{-\hat{g}}\Big[R(\hat{g}) - \tfrac{1}{2}\nabla_{\hat{M}}\phi\nabla^{\hat{M}}\phi - \frac{1}{2n!}e^{\hat{a}\phi}\hat{F}^2_{n]}\Big] \tag{5.6}$$

reduces to

$$\begin{aligned} I \quad = \quad & \int d^D x\sqrt{-g}\Big[R - \tfrac{1}{2}\nabla_M\phi\nabla^M\phi - \tfrac{1}{2}\nabla_M\varphi\nabla^M\varphi - \tfrac{1}{4}e^{-2(D-1)\alpha\varphi}\mathcal{F}^2_{[2]} \\ & - \frac{1}{2n!}e^{-2(n-1)\alpha\varphi+\hat{a}\phi}G'^2_{[n]} - \frac{1}{2(n-1)!}e^{2(D-n)\alpha\varphi+\hat{a}\phi}G^2_{[n-1]}\Big] \ . \end{aligned} \tag{5.7}$$

Although the dimensional reduction (5.7) has produced a somewhat complicated result, the important point to note is that each of the D-dimensional antisymmetric-tensor field strength terms $G'^2_{[n]}$ and $G^2_{[n-1]}$ has an exponential prefactor of the form $e^{a_r\tilde{\phi}_r}$, where the $\tilde{\phi}_r$, $r = (n, n-1)$ are $SO(2)$-rotated combinations of φ and ϕ. Now, keeping just one while setting to zero the other two of the three gauge fields $(\mathcal{A}_{[1]}, B_{[n-2]}, B_{n-1]})$, but retaining at the same time the scalar-field combination appearing in the corresponding exponential prefactor, is a *consistent truncation.* Thus, any one of the three field strengths $(\mathcal{F}_{[2]}, G_{[n-1]}, G'_{[n]})$, retained alone together with its corresponding scalar-field combination, can support p-brane solutions in D dimensions of the form that we have been discussing.

An important point to note here is that, in each of the $e^{a_r\tilde{\phi}}$ prefactors, the coefficient a_r satisfies

$$a_r^2 = \Delta - \frac{2d_r\tilde{d}_r}{(D-2)} = \Delta - \frac{2(r-1)(D-r-1)}{(D-2)} \tag{5.8}$$

with the *same* value of Δ as for the "parent" coupling parameter $\hat{a}$, satisfying

$$\hat{a}_r^2 = \Delta - \frac{2d_{(n)}\tilde{d}_{(n)}}{((D+1)-2)} = \Delta - \frac{2(n-1)(D-n)}{(D-1} \tag{5.9}$$

in $D+1$ dimensions. Thus, although the individual parameters a_r are both D- and r-dependent, the quantity Δ is *preserved* under Kaluza-Klein reduction for both of the "descendant" field-strength couplings (to $G'^2_{[n]}$ or to $G^2_{[n-1]}$) coming from the original term $e^{\hat{a}\phi}\hat{F}^2_{[n]}$. The 2-form field strength $\mathcal{F}_{[2]} = d\mathcal{A}$, on the other hand, emerges out of the gravitational action in $D+1$ dimensions; its coupling parameter corresponds to $\Delta = 4$.

If one retains in the reduced theory only one of the field strengths ($\mathcal{F}_{[2]}$, $G_{[n-1]}$, $G'_{[n]}$), together with its corresponding scalar-field combination, then one finds oneself back in the situation described by our general action (2.1), and then the p brane solutions obtained for the general case in Sec. 2 immediately become applicable. Moreover, since retaining only one field strength & scalar combination in this way effects a consistent truncation of the theory, solutions to this simple truncated system are *also* solutions to the untruncated theory, and indeed are also solutions to the original $(D+1)$-dimensional theory, since the Kaluza-klein dimensional reduction is also a consistent truncation.

5.1 Multiple field-strength solutions and the single-charge truncation

After repeated single steps of Kaluza-Klein dimensional reduction from $D = 11$ down to D dimensions, the metric takes the form [31,32]

$$ds^2_{11} \;=\; e^{-\frac{1}{3}\vec{a}\cdot\vec{\phi}}\, ds^2_{D} + \sum_i e^{(-\frac{4}{3}\vec{a}-\vec{a}_i)\cdot\vec{\phi}}\,(h^i)^2 \tag{5.10a}$$

$$h^i \;=\; dz^i + \mathcal{A}^i_{[1]} + \mathcal{A}^{ij}_{[0]}\, dz^j\ , \tag{5.10b}$$

where the $\mathcal{A}^i_{M}$ are a set of $(11-D)$ Kaluza-Klein vectors generalising the vector $\mathcal{A}_{M}$ in (5.1), emerging from the higher-dimensional metric upon dimensional reduction. Once such Kaluza-Klein vectors have appeared, subsequent dimensional reduction also gives rise to the zero-form gauge potentials $\mathcal{A}^{ij}_{[0]}$ appearing in (5.10b) as a consequence of the usual one-step reduction (5.3) of a 1-form gauge potential.

We shall also need the corresponding reduction of the $\hat{F}_{[4]}$ field strength[g] (where hatted quantities refer to the original, higher, dimension) and, for later reference, we shall also give the reduction of its Hodge dual $\hat{*}\hat{F}_{[4]}$:

$$\hat{F}_{[4]} \;=\; F_{[4]} + F^i_{[3]}\wedge h^i + \tfrac{1}{2}F^{ij}_{[2]}\wedge h^i\wedge h^j + \tfrac{1}{6}F^{ijk}_{[1]}\wedge h^i\wedge h^j\wedge h^k \tag{5.11a}$$

$$\hat{*}\hat{F}_{[4]} \;=\; e^{\vec{a}\cdot\vec{\phi}}\,{*F_{[4]}}\wedge v + e^{\vec{a}_i\cdot\vec{\phi}}\,{*F^i_{[3]}}\wedge v^i + \tfrac{1}{2}e^{\vec{a}_{ij}\cdot\vec{\phi}}\,{*F^{ij}_{[2]}}\wedge v^{ij} + \tfrac{1}{6}e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,{*F^{ijk}_{[1]}}\wedge v^{ijk}\ , \tag{5.11b}$$

[g]Note that the lower-dimensional field strengths $F_{[n]}$ include "Chern-Simons" corrections similar to those in (5.5).

(noting that, since the Hodge dual is a metric-dependent construction, exponentials of the dilatonic vectors $\vec{\phi}$ appear in the reduction of $\hat{*}\hat{F}_{[4]}$) where the forms v, v_i, v_{ij} and v_{ijk} appearing in (5.11b) are given by

$$\begin{aligned} v &= \tfrac{1}{(11-D)!}\epsilon_{i_1\cdots i_{11-D}}\, h^{i_1}\wedge\cdots\wedge h^{i_{11-D}} \\ v_i &= \tfrac{1}{(10-D)!}\epsilon_{ii_2\cdots i_{11-D}}\, h^{i_2}\wedge\cdots\wedge h^{i_{11-D}} \\ v_{ij} &= \tfrac{1}{(9-D)!}\epsilon_{iji_3\cdots i_{11-D}}\, h^{i_3}\wedge\cdots\wedge h^{i_{11-D}} \\ v_{ijk} &= \tfrac{1}{(8-D)!}\epsilon_{ijki_4\cdots i_{11-D}}\, h^{i_4}\wedge\cdots\wedge h^{i_{11-D}}\ . \end{aligned} \tag{5.12}$$

Using (5.10, 5.11a), the bosonic sector of maximal supergravity (1.1) now reduces to [31,32]

$$\begin{aligned} I_D = \int d^Dx\sqrt{-g}\Big[&R - \tfrac12(\partial\vec{\phi})^2 - \tfrac{1}{48}e^{\vec{a}\cdot\vec{\phi}}F_{[4]}^2 - \tfrac{1}{12}\sum_i e^{\vec{a}_i\cdot\vec{\phi}}(F^i_{[3]})^2 \\ &-\tfrac14\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}(F^{ij}_{[2]})^2 - \tfrac14\sum_i e^{\vec{b}_i\cdot\phi}(\mathcal{F}^i_{[2]})^2 \\ &-\tfrac12\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}(F^{ijk}_{[1]})^2 - \tfrac12\sum_{ij} e^{\vec{b}_{ij}\cdot\phi}(\mathcal{F}^{ij}_{[1]})^2\Big] + \mathcal{L}_{FFA}\ , \end{aligned} \tag{5.13}$$

where $i,j = 1,\ldots,11-D$, and field strengths with multiple i,j indices may be taken to be antisymmetric in those indices since these "internal" indices arise in the stepwise reduction procedure, and two equal index values never occur in a multi-index sum. From (5.11), one sees that the "straight-backed" field strengths $F_{[4]}$, $F^i_{[3]}$, $F^{ij}_{[2]}$ and $F^{ijk}_{[1]}$ are descendants from $F_{[4]}$ in $D=11$. The "calligraphic" field strengths $\mathcal{F}^i_{[2]}$, on the other hand, are the field strengths for the Kaluza-Klein vectors $\mathcal{A}^i_M$ appearing in (5.10b). Similarly, one also has a set of 1-form field strengths $\mathcal{F}^{ij}_{[1]}$ for the Kaluza-Klein zero-form gauge potentials $\mathcal{A}^{ij}_{[0]}$ appearing in (5.10b).

The nonlinearity of the original $D=11$ action (1.1) in the metric tensor produces a consequent nonlinearity in the $(11-D)$ dilatonic scalar fields $\vec{\phi}$ appearing in the exponential prefactors of the antisymmetric-tensor kinetic terms in (5.13). For each field-strength kinetic term in (5.13), there is a corresponding "dilaton vector" of coefficients determining the linear combination of the dilatonic scalars appearing in its exponential prefactor. For the 4-, 3-, 2- and 1-form "straight-backed" field strengths emerging from $F_{[4]}$ in $D=11$, these coefficients are denoted correspondingly $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$ and $\vec{a}_{ijk}$; for the "calligraphic" field strengths corresponding to Kaluza-Klein vectors and zero-form gauge potentials emerging out of the metric, these are denoted $\vec{b}_i$ and $\vec{b}_{ij}$. However, not all of these dilaton vectors are independent; in fact, they may all be expressed in terms of the 4-form and 3-form dilaton vectors $\vec{a}$ and $\vec{a}_{ij}$:[31,32]

$$\begin{aligned} \vec{a}_{ij} &= \vec{a}_i + \vec{a}_j - \vec{a} & \qquad b_i &= -\vec{a}_i + \vec{a} \\ \vec{a}_{ijk} &= \vec{a}_i + \vec{a}_j + \vec{a}_k - 2\vec{a} & \qquad \vec{b}_{ij} &= -\vec{a}_i + \vec{a}_j \ . \end{aligned} \tag{5.14}$$

Another important feature of the dilaton vectors is that they satisfy the following dot-product relations:

$$\begin{aligned} \vec{a}\cdot\vec{a} &= \frac{2(11-D)}{D-2} \\ \vec{a}\cdot\vec{a}_i &= \frac{2(8-D)}{D-2} \\ \vec{a}_i\cdot\vec{a}_j &= 2\delta_{ij} + \frac{2(6-D)}{D-2} \ . \end{aligned} \tag{5.15}$$

Throughout this discussion, we have emphasized *consistent truncations* in making simplifying restrictions of complicated systems of equations, so that the solutions of a simplified system are nonetheless perfectly valid solutions of the more complicated untruncated system. With the equations of motion following from (5.13) we face a complicated system that calls for analysis in simplified subsectors. Accordingly, we now seek a consistent truncation down to a simplified system of the form (2.1), retaining just one dilatonic scalar combination ϕ and one rank-n field strength combination $F_{[n]}$ constructed out of a certain number N of "retained" field strengths $F_{\alpha\,[n]}$, $\alpha = 1,\ldots,N$, (this could possibly be a straight-backed/calligraphic mixture) selected from those appearing in (5.13), with all the rest being set to zero.[32] Thus, we let

$$\vec{\phi} = \vec{n}\phi + \vec{\phi}_\perp \ , \tag{5.16}$$

where $\vec{n}\cdot\vec{\phi}_\perp = 0$; in the truncation we then seek to set consistently $\vec{\phi}_\perp = 0$.

We shall see that consistency for the retained field strengths $F_{\alpha\,[n]}$ requires them all to be proportional.[32] We shall let the dot product matrix for the dilaton vectors of the retained field strengths be denoted $M_{\alpha\beta} =: \vec{a}_\alpha\cdot\vec{a}_\beta$. Consistency of the truncation requires that the $\phi_\perp$ field equation be satisfied:

$$\Box\vec{\phi}_\perp - \sum_\alpha \Pi_\perp\cdot\vec{a}_\alpha (F_{\alpha\,[n]})^2 = 0 \ , \tag{5.17}$$

where $\Pi_\perp$ is the projector into the dilaton-vector subspace orthogonal to the retained dilaton direction $\vec{n}$. Setting $\vec{\phi}_\perp = 0$ in (5.17) and letting the retained

$F_{\alpha\,[n]}$ be proportional, one sees that achieving consistency is hopeless unless all the $e^{\vec{a}_\alpha\cdot\vec{\phi}}$ prefactors are the same, thus requiring

$$\vec{a}_\alpha\cdot\vec{n}=a \qquad \forall\alpha=1,\ldots,N\ , \tag{5.18}$$

where the constant a will play the role of the dilatonic scalar coefficient in the reduced system (2.1). Given a set of dilaton vectors for retained field strengths satisfying (5.18), consistency of (5.17) with the imposition of $\vec{\phi}_\perp=0$ requires

$$\Pi_\perp\cdot\sum_\alpha\vec{a}_\alpha(F_{\alpha\,[n]})^2=0\ . \tag{5.19}$$

This equation requires, for every point x^M in spacetime, that the combination $\sum_\alpha\vec{a}_\alpha(F_{\alpha\,[n]})^2$ be parallel to $\vec{n}$ in the dilaton-vector space. Combining this with the requirement (5.18), one has

$$\sum_\alpha\vec{a}_\alpha(F_{\alpha\,[n]})^2=a\vec{n}\sum_\alpha(F_{\alpha\,[n]})^2\ . \tag{5.20}$$

Taking then a dot product of this with $\vec{a}_\beta$, one has

$$\sum_\alpha M_{\beta\alpha}(F_{\alpha\,[n]})^2=a^2\sum_\alpha(F_{\alpha\,[n]})^2\ . \tag{5.21}$$

Detailed analysis [32] shows it to be sufficient to consider the cases where $M_{\alpha\beta}$ is invertible, so by applying $M^{-1}_{\alpha\beta}$ to (5.21), one finds

$$(F_{\alpha\,[n]})^2=a^2\sum_\beta M^{-1}_{\alpha\beta}\sum_\gamma(F_{\gamma\,[n]})^2\ , \tag{5.22}$$

and, indeed, we find that the $F_{\alpha\,[n]}$ must all be proportional. Summing on α, one has

$$a^2=(\sum_{\alpha,\beta}M^{-1}_{\alpha\beta})^{-1}\ ; \tag{5.23}$$

one then defines the retained field-strength combination $F_{[n]}$ so that

$$(F_{\alpha\,[n]})^2=a^2\sum_\beta M^{-1}_{\alpha\beta}(F_{[n]})^2\ . \tag{5.24}$$

The only remaining requirement for consistency of the truncation down to the simplified (g_{MN}, ϕ, $F_{[n]}$) system (2.1) arises from the necessity to ensure that the variation of the $\mathcal{L}_{FFA}$ term in (5.13) is not inconsistent with setting to zero the discarded dilatonic scalars and gauge potentials. In general, this imposes a somewhat complicated requirement. In the present review, however, we shall concentrate mainly on either purely-electric cases satisfying the elementary ansatz (2.4) or purely-magnetic cases satisfying the solitonic ansatz (2.6). As one can see by inspection, for pure electric or magnetic solutions of these sorts, the terms that are dangerous for consistency arising from the variation of $\mathcal{L}_{FFA}$ all vanish. Thus, for such solutions one may safely ignore the complications of the $\mathcal{L}_{FFA}$ term. This restriction to pure electric or magnetic solutions does, however, leave out the very interesting cases of dyonic solutions that exist in $D = 8$ and $D = 4$, upon which we shall comment later on in Section 7.

After truncating down to the system (2.1), the analysis proceeds as in Section 2. It turns out [32] that *supersymmetric* p-brane solutions arise when the matrix $M_{\alpha\beta}$ for the retained $F_{\alpha\,[n]}$ satisfies

$$M_{\alpha\beta} = 4\delta_{\alpha\beta} - \frac{2d\tilde{d}}{D-2} \ , \tag{5.25}$$

and the corresponding Δ value for $F_{[n]}$ is

$$\Delta = \frac{4}{N} \ , \tag{5.26}$$

where we recall that N is the number of retained field strengths. A generalization of this analysis leads to a classification of solutions with more than one independent retained scalar-field combination.[32] We shall see in Section 6 that the $N > 1$ solutions to single-charge truncated systems (2.1) may also be interpreted as special solutions of the full reduced action (5.13) containing N constituent $\Delta = 4$ brane components that just happen to have coincident charge centers. Consequently, one may consider only the $N = 1$, $\Delta = 4$ solutions to be fundamental.

5.2 Diagonal dimensional reduction of p-branes

The family of p-brane solutions is ideally suited to interpretation as solutions of Kaluza-Klein reduced theories, because they are naturally independent of the "worldvolume" x^μ coordinates. Accordingly, one may let the reduction coordinate z be one of the x^μ. Consequently, the only thing that needs to be done to such a solution in order to *reinterpret* it as a solution of a reduced system (5.7) is to perform a Weyl rescaling on it in order to be in accordance with the form of the metric chosen in the Kaluza-Klein ansatz (5.10), which was adjusted so as to maintain the Einstein-frame form of the gravitational term in the dimensionally reduced action.

Upon making such a reinterpretation, elementary/solitonic p-branes in $(D+1)$ dimensions give rise to elementary/solitonic $(p-1)$-branes in D dimensions, corresponding to the *same* value of Δ, as one can see from $(5.8, 5.9)$. Note that in this process, the quantity $\tilde{d}$ is conserved, since both D and d reduce by one. Reinterpretation of p brane solutions in this way, corresponding to standard Kaluza-Klein reduction on a worldvolume coordinate, proceeds diagonally on a D versus d plot, and hence is referred to as *diagonal* dimensional reduction. This procedure is the analogue, for supergravity field-theory solutions, of the procedure of *double dimensional reduction* [22] for p-brane worldvolume actions, which can be taken to constitute the δ-function *sources* for singular p-brane solutions, coupled in to resolve the singularities, as we discussed in subsection 4.1.

5.3 Multi-center solutions and vertical dimensional reduction

As we have seen, translational Killing symmetries of p-brane solutions allow a simultaneous interpretation of these field configurations as solutions belonging to several different supergravity theories, related one to another by Kaluza-Klein dimensional reduction. For the original single p-brane solutions (2.24), the only available translational Killing symmetries are those in the worldvolume directions, which we have exploited in describing diagonal dimensional reduction above. One may, however, generalize the basic solutions (2.24) by replacing the harmonic function $H(y)$ in (2.21) by a different solution of the Laplace equation (2.20). Thus, one can easily extend the family of p-brane solutions to *multi-center* p-brane solutions by taking the harmonic function to be

$$H(y) = 1 + \sum_\alpha \frac{k_\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}} \qquad k_\alpha > 0 \ . \tag{5.27}$$

Once again, the integration constant has been adjusted to make $H_{|\infty} = 1 \leftrightarrow \phi_{|\infty} = 0$. The generalized solution (5.27) corresponds to *parallel* and *similarly-oriented* p-branes, with all charge parameters $\lambda_\alpha = 2\tilde{d}k_\alpha/\sqrt{\Delta}$ required to be positive in order to avoid naked singularities. The "centers" of the individual "leaves" of this solution are at the points $y = y_\alpha$, where α ranges over any number of centers. The metric and the electric-case antisymmetric tensor gauge potential corresponding to (5.27) are given again in terms of $H(y)$ by (2.24a,2.25). In the solitonic case, the ansatz (2.6) needs to be modified so as to accommodate the multi-center form of the solution:

$$F_{m_1\ldots m_n} = -\tilde{d}^{-1}\epsilon_{m_1\ldots m_n p}\partial_p \sum_\alpha \frac{\lambda_\alpha}{|\vec{y}-\vec{y}_\alpha|^{\tilde{d}}} \ , \tag{5.28}$$

which ensures the validity of the Bianchi identity just as well as (2.6) does. The mass/(unit p-volume) density is now

$$\mathcal{E} = \frac{2\Omega_{D-d-1}}{\sqrt{\Delta}} \sum_\alpha \lambda_\alpha \ , \tag{5.29}$$

while the total electric or magnetic charge is given by $\Omega_{D-d-1}\sum\lambda_\alpha$, so the Bogomol'ny bounds (4.16) are saturated just as they are for the single-center solutions (2.24). Since the multi-center solutions given by (5.27) satisfy the same supersymmetry-preservation conditions on the metric and antisymmetric tensor as (2.24), the multi-center solutions leave the same amount of supersymmetry unbroken as the single-center solution.

From a mathematical point of view, the multi-center solutions (5.27) exist owing to the properties of the Laplace equation (2.20). From a physical point of view, however, these static solutions exist as a result of *cancellation* between *attractive* gravitational and scalar-field forces against *repulsive* antisymmetric-tensor forces for the similarly-oriented p-brane "leaves."

The multi-center solutions given by (5.27) can now be used to prepare solutions adapted to dimensional reduction in the *transverse* directions. This combination of a modification of the solution followed by dimensional reduction on a transverse coordinate is called *vertical* dimensional reduction [23] because it relates solutions vertically on a D versus d plot.[h] In order to do this, we need first to develop translation invariance in the transverse reduction coordinate. This can be done by "stacking" up identical p branes using (5.27) in a periodic array, *i.e.* by letting the integration constants k_α all be equal, and aligning the "centers" y_α along some axis, *e.g.* the z axis. Singling out one "stacking axis" in this way clearly destroys the overall isotropic symmetry of the solution, but, provided the centers are all in a line, the solution will nonetheless remain isotropic in the $D-d-1$ dimensions orthogonal to the stacking axis. Taking the limit of a densely-packed infinite stack of this sort, one has

$$\sum_\alpha \frac{k_\alpha}{|\vec{y}-\vec{y}_\alpha|^{\tilde{d}}} \quad\longrightarrow\quad \int_{-\infty}^{+\infty} \frac{k\,dz}{(\hat{r}^2+z^2)^{\tilde{d}/2}} = \frac{\tilde{k}}{\hat{r}^{\tilde{d}-1}} \tag{5.30a}$$

[h] Similar procedures have been considered in a number of articles in the literature; see, *e.g.* Refs.[33]

$$\hat{r}^2 = \sum_{m=d}^{D-2} y^m y^m \tag{5.30b}$$

$$\hat{k} = \frac{\sqrt{\pi} k \Gamma(\tilde{d} - \frac{1}{2})}{2\Gamma(\tilde{d})}\ , \tag{5.30c}$$

where $\hat{r}$ in (5.30b) is the radial coordinate for the $D - d - 1$ residual isotropic transverse coordinates. After a conformal rescaling in order to maintain the Einstein frame for the solution, one can finally reduce on the coordinate z along the stacking axis.

After stacking and reduction in this way, one obtains a p-brane solution with the *same* worldvolume dimension as the original higher-dimensional solution that was stacked up. Since the same antisymmetric tensors are used here to support both the stacked and the unstacked solutions, and since Δ is preserved under dimensional reduction, it follows that vertical dimensional reduction from D to $D - 1$ spacetime dimensions preserves the value of Δ just like the diagonal reduction discussed in the previous subsection. Note that under vertical reduction, the worldvolume dimension d is preserved, but $\tilde{d} = D - d - 2$ is reduced by one with each reduction step.

Combining the diagonal and vertical dimensional reduction trajectories of "descendant" solutions, one finds the general picture given in the plot of Figure 6. In this plot of spacetime dimension D versus worldvolume dimension d, reduction families emerge from certain basic solutions that cannot be "oxidized" back up to higher-dimensional isotropic p-brane solutions, and hence can be called "stainless" p-branes.[13] In Figure 6, these solutions are indicated by the large circles, with the corresponding Δ values shown adjacently. The indication of elementary or solitonic type relates to solutions of supergravity theories in versions with the lowest possible choice of rank ($n \leq D/2$) for the supporting field strength, obtainable by appropriate dualization. Of course, every solution to a theory obtained by dimensional reduction from $D = 11$ supergravity (1.1) may be oxidised back up to some solution in $D = 11$. We shall see in Section 6 that what one obtains upon oxidation of the "stainless" solutions in Figure 6 falls into the interesting class of "intersecting branes" built from four basic "elemental" solutions of $D = 11$ supergravity.

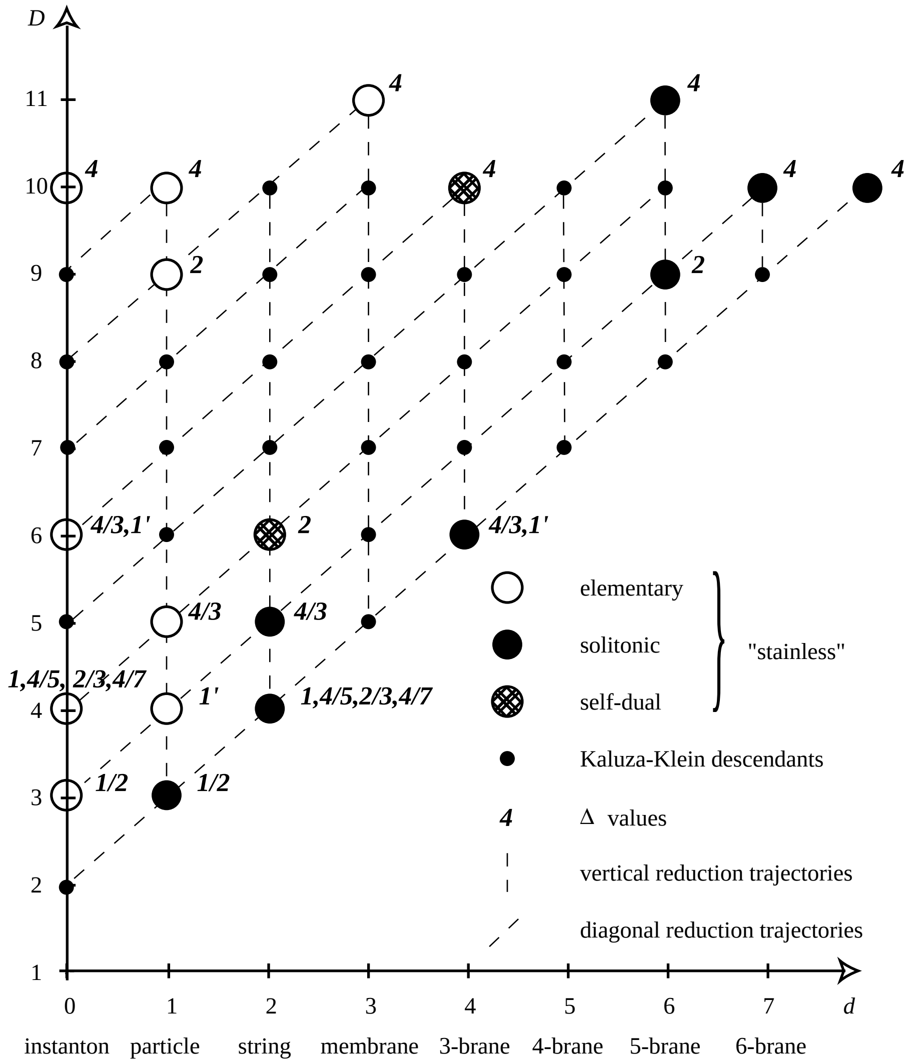

Figure 6: Brane-scan of supergravity p-brane solutions ($p \leq (D-3)$)

5.4 The geometry of $(D-3)$-branes

The process of vertical dimensional reduction described in the previous subsection proceeds uneventfully until one makes the reduction from a $(D, d = D-3)$ solution to a $(D-1, d = D-3)$ solution.[i] In this step, the integral (5.30) contains an additive divergence and needs to be renormalized. This is easily handled by putting finite limits $\pm L$ on the integral, which becomes $\int_{-L}^{L} d\tilde{z}(r^2 + \tilde{z}^2)^{-1/2}$, and then by subtracting a divergent term $2\ln L$ before taking the limit $L \to \infty$. Then the integral gives the expected $\ln \hat{r}$ harmonic function appropriate to two transverse dimensions.

Before proceeding any further with vertical dimensional reduction, let us consider some of the specific properties of $(D-3)$-branes that make the next vertical step down problematic. Firstly, the asymptotic metric of a $(D-3)$-brane is not a globally flat space, but only a locally flat space. This distinction means that there is in general a deficit solid angle at transverse infinity, which is related to the total mass density of the $(D-3)$-brane.[34] This means that any attempt to stack up $(D-3)$-branes within a standard supergravity theory will soon consume the entire solid angle at transverse infinity, thus destroying the asymptotic spacetime in the construction.

In order to understand the global structure of the $(D-3)$-branes in some more detail, consider the supersymmetric string in $D = 4$ dimensions.[12] In $D = 4$, one may dualize the 2-form $A_{\mu\nu}$ field to a pseudoscalar, or axion, field χ, so such strings are also solutions to dilaton-axion gravity. The p-brane ansatz gives a spacetime of the form $\mathcal{M}^4 = \mathcal{M}^2 \times \Sigma^2$, where $\mathcal{M}^2$ is $D = 2$ Minkowski space. Supporting this string solution, one has the 2-form gauge field $A_{\mu\nu}$ and the dilaton ϕ. These fields give rise to a field stress tensor of the form

$$\begin{aligned} T_{\mu\nu}(A,\phi) &= -\tfrac{1}{16}(a^2+4)\partial_m K \partial_m K \eta_{\mu\nu} \\ T_{mn}(A,\phi) &= \tfrac{1}{8}(a^2-4)(\partial_m K \partial_n K - \tfrac{1}{2}\delta_{mn}(\partial_p K \partial_p K)) \ , \end{aligned} \tag{5.31}$$

where a is as usual the dilaton coupling parameter and $e^{-K} = H = 1 - 8GT\ln(r)$, with $r = \sqrt{y^m y^m}$, $m = 2,\ 3$. If one now puts in an elementary string source action, with the string aligned along the $\mu, \nu = 0, 1$ subspace, so that $T_{mn\,(\text{source})} = 0$, then one has the source stress tensor

$$T_{\mu\nu\,(\text{source})} = \frac{-T}{\sqrt{-g}} \int d^2\xi \sqrt{-\gamma}\gamma^{ij}\partial_i X_\mu \partial_j X_\nu e^{-\frac{1}{2}a\phi}\delta(x - X) \ . \tag{5.32}$$

[i] Solutions with worldvolume dimension two less than the spacetime dimension will be referred to generally as $(D-3)$-branes, irrespective of whether the spacetime dimension is D or not.

By inspection of the field solution, one has $T_{mn}(A,\phi)=0$, while the contributions to $T_{\mu\nu}$ from the $A_{\mu\nu}$ and ϕ fields and also from the source (5.32) are both of the form $\mathrm{diag}(\rho,-\rho)$. Thus, the overall stress tensor is of the form $T_{MN}=\mathrm{diag}(\rho,-\rho,0,0)$.

Consequently, the Einstein equation in the transverse m, n indices becomes $R_{mn}-\frac{1}{2}g_{mn}R=0$, since the transverse stress tensor components vanish. This equation is naturally satisfied for a metric of the form of the p-brane ansatz, because this form causes the transverse components of the Ricci tensor to be proportional to the Ricci tensor of a $D=2$ spacetime, for which $R_{mn}-\frac{1}{2}g_{mn}R\equiv 0$ is an *identity,* corresponding to the fact that the usual Einstein action, $\sqrt{-g}R$, is a topological invariant in $D=2$. Accordingly, in the transverse directions, the equations are satisfied simply by by $0=0$.

In the world-sheet directions, the equations become

$$-\tfrac{1}{2}R\eta_{\mu\nu}=-8\pi G\rho\eta_{\mu\nu}\ , \tag{5.33}$$

or just

$$R=16\pi G\rho\ , \tag{5.34}$$

and as we have already noted, $R=R_{mm}$. Owing to the fact that the $D=2$ Weyl tensor vanishes, the transverse space Σ^2 is conformally flat; Eq. (5.34) gives its conformal factor. Thus, although there is no sensible Einstein action in the transverse $D=2$, space, a usual form of the Einstein equation nonetheless applies to that space as a result of the symmetries of the p-brane ansatz.

The above supersymmetric string solution may be compared to the cosmic strings arising in gauge theories with spontaneous symmetry breaking. There, the Higgs fields contributing to the energy density of the string are displaced from their usual vacuum values to unbroken-symmetry configurations at a stationary point of the Higgs potential, within a very small transverse-space region that may be considered to be the string "core." Approximating this by a delta function in the transverse space, the Ricci tensor and hence the full curvature *vanish* outside the string core, so that one obtains a *conical* spacetime, which is flat except at the location of the string core. The total energy is given by the *deficit angle* $8\pi GT$ of the conical spacetime. In contrast, the supersymmetric string has a field stress tensor $T_{\mu\nu}(A,\phi)$ which is not just concentrated at the string core but instead is smeared out over spacetime. The difference arises from the absence of a potential for the fields $A_{\mu\nu}$, ϕ supporting the solution in the supersymmetric case. Nonetheless, as one can see from the behavior of the stress tensor T_{mn} in Eq. (5.31), the transverse space Σ^2 is asymptotically locally flat (ALF), with a total energy density given by the overall deficit angle measured at infinity. For multiple-centered string solutions, one has

$$H = 1 - \sum_i 8GT_i \ln |\vec{y} - \vec{y}_i| \ . \tag{5.35}$$

Consequently, when considered within the original supergravity theory, the indefinite stacking of supersymmetric strings leads to a destruction of the transverse asymptotic space.

A second problem with any attempt to produce $(D-2)$-branes in ordinary supergravity theories is simply stated: starting from the p-brane ansatz (2.3, 2.6) and searching for $(D-2)$ branes in ordinary massless supergravity theories, one simply doesn't find any such solutions.

5.5 Beyond the $(D-3)$-brane barrier: Scherk-Schwarz reduction and domain walls

Faced with the above puzzles about what sort of $(D-2)$-brane could result by vertical reduction from a $(D-3)$-brane, one can simply decide to be brave, and to just proceed anyway with the established mathematical procedure of vertical dimensional reduction and see what one gets. In the next step of vertical dimensional reduction, one again encounters an additive divergence: the integral $\int_{-L}^{L} dz \ln(y^2+z^2)$ needs to be renormalized by subtracting a divergent term $4L(\ln L - 1)$. Upon subsequently performing the integral, the harmonic function $H(y)$ becomes linear in the one remaining transverse coordinate.

While the mathematical procedure of vertical dimensional reduction so as to produce some sort of $(D-2)$-brane proceeds apparently without serious complication, an analysis of the physics of the situation needs some care.[35] Consider the reduction from a $(D, d = D-2)$ solution (a $p = (D-3)$ brane) to a $(D-1, d = D-3)$ solution (a $p = (D-2)$ brane). Note that both the $(D-3)$ brane and its descendant $(D-2)$-brane have harmonic functions $H(y)$ that blow up at infinity. For the $(D-3)$-brane, however this is not in itself particularly remarkable, because, as one can see by inspection of (2.24) for this case, the metric asymptotically tends to a locally flat space as $r \to \infty$, and also in this limit the antisymmetric-tensor one-form field strength

$$F_m = -\epsilon_{mn}\partial_n H \tag{5.36}$$

tends asymptotically to zero, while the dilatonic scalar ϕ tends to its modulus value ϕ_∞ (set to zero for simplicity in (2.24)). The expression (5.36) for the field strength, however, shows that the next reduction step down to the $(D-1, d = D-2)$ solution has a significant new feature: upon stacking up $(D-3)$ branes prior to the vertical reduction, thus producing a linear harmonic function in the transverse coordinate y,

$$H(y) = \text{const.} + my \ , \tag{5.37}$$

the field strength (5.36) acquires a *constant* component along the stacking axis $\leftrightarrow$ reduction direction z,

$$F_z = -\epsilon_{zy}\partial_y H = m \ , \tag{5.38}$$

which implies an unavoidable dependence[j] of the corresponding zero-form gauge potential on the reduction coordinate:

$$A_{[0]}(x, y, z) = mz + \chi(x, y) \ . \tag{5.39}$$

From a Kaluza-Klein point of view, the unavoidable linear dependence of a gauge potential on the reduction coordinate given in (5.39) appears to be problematic. Throughout this review, we have dealt only with *consistent* Kaluza-Klein reductions, for which solutions of the reduced theory are also solutions of the unreduced theory. Generally, retaining any dependence on a reduction coordinate will lead to an *inconsistent truncation* of the theory: attempting to impose a z dependence of the form given in (5.39) prior to varying the Lagrangian will give a result different from that obtained by imposing this dependence in the field equations after variation.

The resolution of this difficulty is that in performing a Kaluza-Klein reduction with an ansatz like (5.39), one ends up outside the standard set of massless supergravity theories. In order to understand this, let us again focus on the problem of consistency of the Kaluza-Klein reduction. As we have seen, consistency of any restriction means that the restriction may either be imposed on the field variables in the original action prior to variation so as to derive the equations of motion, or instead may be imposed on the field variables in the equations of motion after variation, with an equal effect. In this case, solutions obeying the restriction will also be solutions of the general unrestricted equations of motion.

The most usual guarantee of consistency in Kaluza-Klein dimensional reduction is obtained by restricting the field variables to carry zero charge with respect to some conserved current, *e.g.* momentum in the reduction dimension. But this is not the only way in which consistency may be achieved. In

[j]Note that this vertical reduction from a $(D-3)$-brane to a $(D-2)$-brane is the first case in which one is *forced* to accept a dependence on the reduction coordinate z; in all higher-dimensional vertical reductions, such z dependence can be removed by a gauge transformation. The zero-form gauge potential in (5.39) does not have the needed gauge symmetry, however.

the present case, retaining a linear dependence on the reduction coordinate as in (5.39) would clearly produce an inconsistent truncation if the reduction coordinate were to appear explicitly in any of the field equations. But this does not imply that a truncation is necessarily inconsistent just because a gauge potential contains a term linear in the reduction coordinate. Inconsistency of a Kaluza-Klein truncation occurs when the original, unrestricted, field equations imply a condition that is inconsistent with the reduction ansatz. If a particular gauge potential appears in the action *only through its derivative, i.e.* through its field strength, then a consistent truncation may be achieved provided that the restriction on the gauge potential implies that the field strength is independent of the reduction coordinate. A zero-form gauge potential on which such a reduction may be carried out, occurring in the action only through its derivative, will be referred to as an *axion.*

Requiring axionic field strengths to be independent of the reduction coordinate amounts to extending the Kaluza-Klein reduction framework so as to allow for *linear* dependence of an axionic zero-form potential on the reduction coordinate, precisely of the form occurring in (5.39). So, provided $A_{[0]}$ is an axion, the reduction (5.39) turns out to be consistent after all. This extension of the Kaluza-Klein ansatz is in fact an instance of *Scherk-Schwarz* reduction.[36,37] The basic idea of Scherk-Schwarz reduction is to use an Abelian rigid symmetry of a system of equations in order to generalize the reduction ansatz by allowing a linear dependence on the reduction coordinate in the parameter of this Abelian symmetry. Consistency is guaranteed by cancellations orchestrated by the Abelian symmetry in field-equation terms where the parameter does not get differentiated. When it does get differentiated, it contributes only a term that is itself independent of the reduction coordinate. In the present case, the Abelian symmetry guaranteeing consistency of (5.39) is a simple shift symmetry $A_{[0]} \to A_{[0]} + \text{const}$.

Unlike the original implementation of the Scherk-Schwarz reduction idea,[36] which used an Abelian $U(1)$ phase symmetry acting on spinors, the Abelian shift symmetry used here commutes with supersymmetry, and hence the reduction does not spontaneously break supersymmetry. Instead, gauge symmetries for some of the antisymmetric tensors will be broken, with a corresponding appearance of mass terms. As with all examples of vertical dimensional reduction, the Δ value corresponding to a given field strength is also preserved. Thus, p-brane solutions related by vertical dimensional reduction, even in the enlarged Scherk-Schwarz sense, preserve the same amount of unbroken supersymmetry and have the same value of Δ.

It may be necessary to make several redefinitions and integrations by parts in order to reveal the axionic property of a given zero-form, and thus to prepare the theory for a reduction like (5.39). This is most easily explained by an example, so let us consider the first possible Scherk-Schwarz reduction[k] in the sequence of theories descending from (1.1), starting in $D = 9$ where the first axion field appears.[35] The Lagrangian for massless $D = 9$ maximal supergravity is obtained by specializing the general dimensionally-reduced action (5.13) given in Section 2 to this case:

$$\begin{aligned}\mathcal{L}_9 \;=\;& \sqrt{-g}\Big[R - \tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{2}e^{-\frac{3}{2}\phi_1+\frac{\sqrt{7}}{2}\phi_2}(\partial\chi)^2 - \tfrac{1}{48}e^{\vec{a}\cdot\vec{\phi}}(F_{[4]})^2\\ &- \tfrac{1}{2}e^{\vec{a}_1\cdot\vec{\phi}}(F^{(1)}_{[3]})^2 - \tfrac{1}{2}e^{\vec{a}_2\cdot\vec{\phi}}(F^{(2)}_{[3]})^2 - \tfrac{1}{4}e^{\vec{a}_{12}\cdot\vec{\phi}}(F^{(12)}_{[2]})^2 - \tfrac{1}{4}e^{\vec{b}_1\cdot\vec{\phi}}(\mathcal{F}^{(1)}_{[2]})^2\\ &- \tfrac{1}{4}e^{\vec{b}_2\cdot\vec{\phi}}(\mathcal{F}^{(2)}_{[2]})^2\Big] - \tfrac{1}{2}\tilde{F}_{[4]}\wedge\tilde{F}_{[4]}\wedge A^{(12)}_{[1]} - \tilde{F}^{(1)}_{[3]}\wedge\tilde{F}^{(2)}_{[3]}\wedge A_{[3]}\ , \end{aligned}\tag{5.40}$$

where $\chi = \mathcal{A}^{(12)}_{[0]}$ and $\vec{\phi} = (\phi_1, \phi_2)$.

Within the scalar sector $(\vec{\phi}, \chi)$ of (5.40), the dilaton coupling has been made explicit; in the rest of the Lagrangian, the dilaton vectors have the general structure given in (5.14, 5.16). The scalar sector of (5.40) forms a nonlinear σ-model for the manifold $\mathrm{GL}(2,\mathbb{R})/\mathrm{SO}(2)$. This already makes it appear that one may identify χ as an axion available for Scherk-Schwarz reduction. However, account must still be taken of the Chern-Simons structure lurking inside the field strengths in (5.11, 5.40). In detail, the field strengths are given by

$$\begin{aligned} F_{[4]} &= \tilde{F}_{[4]} - \tilde{F}^{(1)}_{[3]}\wedge\mathcal{A}^{(1)}_{[1]} - \tilde{F}^{(2)}_{[3]}\wedge\mathcal{A}^{(2)}_{[1]}\\ &\qquad + \chi\tilde{F}^{(1)}_{[3]}\wedge\mathcal{A}^{(2)}_{[1]} - \tilde{F}^{(12)}_{[2]}\wedge\mathcal{A}^{(1)}_{[1]}\wedge\mathcal{A}^{(2)}_{[1]} && (5.41\mathrm{a})\\ F^{(1)}_{[3]} &= \tilde{F}^{(1)}_{[3]} - \tilde{F}^{(12)}_{[2]}\wedge\mathcal{A}^{(2)}_{[1]} && (5.41\mathrm{b})\\ F^{(2)}_{[3]} &= \tilde{F}^{(2)}_{[3]} + \tilde{F}^{(12)}_{[2]}\wedge\mathcal{A}^{(1)}_{[1]} - \chi\tilde{F}^{(1)}_{[3]} && (5.41\mathrm{c})\\ F^{(12)}_{[2]} &= \tilde{F}^{(12)}_{[2]} \qquad \mathcal{F}^{(1)}_{[2]} = \tilde{\mathcal{F}}^{(1)}_{[2]} - d\chi\wedge\mathcal{A}^{(2)}_{[1]} && (5.41\mathrm{d})\\ \mathcal{F}^{(2)}_{[2]} &= \tilde{\mathcal{F}}^{(2)}_{[2]} \qquad \mathcal{F}^{(12)}_{[1]} = d\chi\ , && (5.41\mathrm{e}) \end{aligned}$$

where the field strengths carrying tildes are the naïve expressions without Chern-Simons corrections, *i.e.* $\tilde{F}_{n]} = dA_{[n-1]}$. Now the appearance of undifferentiated χ factors in (5.41a,c) makes it appear that a Scherk-Schwarz reduction would be inconsistent. However, one may eliminate these undifferentiated factors by making the field redefinition

$$A^{(2)}_{[2]} \longrightarrow A^{(2)}_{[2]} + \chi A^{(1)}_{[2]}\ , \tag{5.42}$$

[k] A higher-dimensional Scherk-Schwarz reduction is possible [37] starting from type IIB supergravity in $D = 10$, using the axion appearing in the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ scalar sector of that theory.

after which the field strengths (5.41a,c) become

$$\begin{aligned} F_{[4]} &= \tilde{F}_{[4]} - \tilde{F}^{(1)}_{[3]} \wedge \mathcal{A}^{(1)}_{[1]} - \tilde{F}^{(2)}_{[3]} \wedge \mathcal{A}^{(2)}_{[1]} \\ &\quad - d\chi \wedge A^{(1)}_{[2]} \wedge \mathcal{A}^{(2)}_{[1]} - \tilde{F}^{(12)}_{[2]} \wedge \mathcal{A}^{(1)}_{[1]} \wedge \mathcal{A}^{(2)}_{[1]} \qquad (5.43a) \\ F^{(2)}_{[3]} &= \tilde{F}^{(2)}_{[3]} + F^{(12)}_{[2]} \wedge \mathcal{A}^{(1)}_{[1]} + d\chi \wedge A^{(1)}_{[2]} \ , \qquad (5.43c) \end{aligned}$$

the rest of (5.41) remaining unchanged.

After the field redefinitions (5.42), the axion field $\chi = \mathcal{A}^{(12)}_{[0]}$ is now ready for application of the Scherk-Schwarz reduction ansatz (5.39). The coefficient of the term linear in the reduction coordinate z has been denoted m because it carries the dimensions of mass, and correspondingly its effect on the reduced action is to cause the appearance of mass terms. Applying (5.39) to the $D = 9$ Lagrangian, one obtains a $D = 8$ reduced Lagrangian[l]

$$\begin{aligned} \mathcal{L}_{8\,\mathrm{ss}} &= \\ &\sqrt{-g}\Big[R - \tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - \tfrac{1}{2}(\partial\phi_3)^2 \\ &-\tfrac{1}{2}e^{\vec{b}_{12}\cdot\vec{\phi}}(\partial\chi - m\mathcal{A}^{(3)}_{[1]})^2 - \tfrac{1}{2}e^{\vec{b}_{13}\cdot\vec{\phi}}(\partial\mathcal{A}^{(13)}_{[0]} - \partial\chi\mathcal{A}^{(23)}_{[0]} + m\mathcal{A}^{(2)}_{[1]})^2 \\ &-\tfrac{1}{2}e^{\vec{b}_{23}\cdot\vec{\phi}}(\partial\mathcal{A}^{(23)}_{[0]})^2 - \tfrac{1}{2}e^{\vec{a}_{123}\cdot\vec{\phi}}(\partial A^{(123)}_{[0]})^2 \\ &-\tfrac{1}{48}e^{\vec{a}\cdot\vec{\phi}}(F_{[4]} - mA^{(1)}_{[2]} \wedge \mathcal{A}^{(2)}_{[1]} \wedge \mathcal{A}^{(3)}_{[1]})^2 - \tfrac{1}{12}e^{\vec{a}_1\cdot\vec{\phi}}(F^{(1)}_{[3]})^2 \\ &-\tfrac{1}{12}e^{\vec{a}_2\cdot\vec{\phi}}(F^{(2)}_{[3]} - mA^{(1)}_{[2]} \wedge \mathcal{A}^{(3)}_{[1]})^2 - \tfrac{1}{12}e^{\vec{a}_3\cdot\vec{\phi}}(F^{(3)}_{[3]} + mA^{(1)}_{[2]} \wedge \mathcal{A}^{(2)}_{[1]})^2 \\ &-\tfrac{1}{4}e^{\vec{a}_{12}\cdot\vec{\phi}}(F^{(12)}_{[2]})^2 - \tfrac{1}{4}e^{\vec{a}_{13}\cdot\vec{\phi}}(F^{(13)}_{[2]})^2 - \tfrac{1}{4}e^{\vec{a}_{23}\cdot\vec{\phi}}(F^{(23)}_{[2]} + mA^{(1)}_{[2]})^2 \\ &-\tfrac{1}{4}e^{\vec{b}_1\cdot\vec{\phi}}(\mathcal{F}^{(1)}_{[2]} - m\mathcal{A}^{(2)}_{[1]} \wedge \mathcal{A}^{(3)}_{[1]})^2 - \tfrac{1}{4}e^{\vec{b}_2\cdot\vec{\phi}}(\mathcal{F}^{(2)}_{[2]})^2 - \tfrac{1}{4}e^{\vec{b}_3\cdot\vec{\phi}}(\mathcal{F}^{(3)}_{[2]})^2 \\ &-\tfrac{1}{2}m^2 e^{\vec{b}_{123}\cdot\vec{\phi}}\Big] + mF^{(1)}_{[3]} \wedge A^{(1)}_{[2]} \wedge A_{[3]} + \mathcal{L}_{FFA} \ , \qquad (5.44) \end{aligned}$$

where the dilaton vectors are now those appropriate for $D = 8$; the term $\mathcal{L}_{FFA}$ contains only m-independent terms.

It is apparent from (5.44) that the fields $\mathcal{A}^{(3)}_{[1]}$, $\mathcal{A}^{(2)}_{[1]}$ and $A^{(1)}_{[2]}$ have become massive. Moreover, there are field redefinitions under which the fields χ, $\mathcal{A}^{(13)}_{[0]}$ and $A^{(23)}_{[1]}$ may be absorbed. One way to see how this absorption happens is to notice that the action obtained from (5.44) has a set of three Stueckelberg-type

[l] I am grateful to Marcus Bremer for help in correcting some errors in the original expression of Eq. (5.44) given in Ref.[35]

gauge transformations under which $\mathcal{A}^{(3)}_{[1]}$, $\mathcal{A}^{(2)}_{[1]}$ and $A^{(1)}_{[2]}$ transform according to their standard gauge transformation laws. These three transformations are accompanied, however, by various compensating transformations necessitated by the Chern-Simons corrections present in (5.44) as well as by m-dependent shift transformations of χ, $\mathcal{A}^{(13)}_{[0]}$ and $A^{(23)}_{[1]}$, respectively. Owing to the presence of these local shift terms in the three Stuecelberg symmetries, the fields χ, $\mathcal{A}^{(13)}_{[0]}$ and $A^{(23)}_{[1]}$ may be gauged to zero. After gauging these three fields to zero, one has a clean set of mass terms in (5.44) for the fields $\mathcal{A}^{(3)}_{[1]}$, $\mathcal{A}^{(2)}_{[1]}$ and $A^{(1)}_{[2]}$.

As one descends through the available spacetime dimensions for supergravity theories, the number of axionic scalars available for a Scherk-Schwarz reduction step increases. The numbers of axions are given in the following Table:

Table 1: Supergravity axions versus spacetime dimension.

D	9	8	7	6	5	4
N_{axions}	1	4	10	20	36	63

Each of these axions gives rise to a *distinct* massive supergravity theory upon Scherk-Schwarz reduction,[35] and each of these reduced theories has its own pattern of mass generation. In addition, once a Scherk-Schwarz reduction step has been performed, the resulting theory can be further reduced using ordinary Kaluza-Klein reduction. Moreover, the Scherk-Schwarz and ordinary Kaluza-Klein processes do not commute, so the number of theories obtained by the various combinations of Scherk-Schwarz and ordinary dimensional reduction is cumulative. In addition, there are numerous possibilities of performing Scherk-Schwarz reduction simultaneously on a number of axions. This can be done either by arranging to cover a number of axions simultaneously with derivatives, or by further Scherk-Schwarz generalisations of the Kaluza-Klein reduction process.[38] For further details on the panoply of Scherk-Schwarz reduction possibilities, we refer the reader to Refs.[35,38]

The single-step procedure of Scherk-Schwarz dimensional reduction described above may be generalised to a procedure exploiting the various cohomology classes of a multi-dimensional compactification manifold.[39] The key to this link between the Scherk-Schwarz generalised dimensional reduction and the topology of the internal Kaluza-Klein manifold $\mathcal{K}$ is to recognise that the single-step reduction ansatz (5.39) may be generalised to

$$A_{[n-1]}(x,y,z) = \omega_{[n-1]} + A_{[n-1]}(x,y) \ , \tag{5.45}$$

where $\omega_{[n-1]}$ is an $(n-1)$ form defined locally on $\mathcal{K}$, whose exterior derivative $\Omega_{[n]} = d\omega_{[n-1]}$ is an element of the cohomology class $H^n(\mathcal{K}, \mathbb{R})$. For example, in the case of a single-step generalised reduction on a circle S^1, one has $\Omega_{[1]} = m dz \in H^1(S^1, \mathbb{R})$, reproducing our earlier single-step reduction (5.39).

As another example, consider a generalised reduction on a 4-torus T^4 starting in $D = 11$, setting $A_{[3]}(x, y, z) = \omega_{[3]} + A_{[3]}(x, y)$ with $\Omega_{[4]} = d\omega_{[3]} = m dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \in H^4(T^4, \mathbb{R})$. In this example, one may choose to write $\omega_{[3]}$ locally as $\omega_{[3]} = m z_1 dz_2 \wedge dz_3 \wedge dz_4$. All of the other fields are reduced using the standard Kaluza-Klein ansatz, with no dependence on any of the z_i coordinates. The theory resulting from this T^4 reduction is a $D = 7$ massive supergravity with a cosmological potential, analogous to the $D = 8$ theory (5.44). The same theory (up to field redefinitions) can also be obtained [35] by first making an ordinary Kaluza-Klein reduction from $D = 11$ down to $D = 8$ on a 3-torus T^3, then making an S^1 single-step generalised Scherk-Schwarz reduction (5.39) from $D = 8$ to $D = 7$. Although the T^4 reduction example simply reproduces a massive $D = 7$ theory that can also be obtained *via* the single-step ansatz (5.39), the recognition that one can use any of the $H^n(\mathcal{K}, \mathbb{R})$ cohomology classes of the compactification manifold $\mathcal{K}$ significantly extends the scope of the generalised reduction procedure. For example, it allows one to make generalised reductions on manifolds such as K3 or on Calabi-Yau manifolds.[39]

For our present purposes, the important feature of theories obtained by Scherk-Schwarz reduction is the appearance of cosmological potential terms such as the penultimate term in Eq. (5.44). Such terms may be considered within the context of our simplified action (2.1) by letting the rank n of the field strength take the value zero. Accordingly, by consistent truncation of (5.44) or of one of the many theories obtained by Scherk-Schwarz reduction in lower dimensions, one may arrive at the simple Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big[R - \tfrac{1}{2}\nabla_M\phi\nabla^M\phi - \tfrac{1}{2}m^2 e^{a\phi}\Big] . \tag{5.46}$$

Since the rank of the form here is $n = 0$, the elementary/electric type of solution would have worldvolume dimension $d = -1$, which is not very sensible, but the solitonic/magnetic solution has $\tilde{d} = D - 1$, corresponding to a $p = D - 2$ brane, or *domain wall,* as expected. Relating the parameter a in (5.46) to the reduction-invariant parameter Δ by the standard formula (2.18) gives $\Delta = a^2 - 2(D-1)/(D-2)$; taking the corresponding $p = D - 2$ brane solution from (2.24), one finds

$$ds^2 = H^{\frac{4}{\Delta(D-2)}}\, \eta_{\mu\nu}\, dx^\mu dx^\nu + H^{\frac{4(D-1)}{\Delta(D-2)}}\, dy^2 \tag{5.47a}$$

$$e^{\phi} = H^{2a/\Delta} \ , \tag{5.47b}$$

where the harmonic function $H(y)$ is now a linear function of the single transverse coordinate, in accordance with (5.37).[m] The curvature of the metric (5.47a) tends to zero at large values of $|y|$, but it diverges if H tends to zero. This latter singularity can be avoided by taking H to be

$$H = \text{const.} + M|y| \tag{5.48}$$

where $M = \frac{1}{2}m\sqrt{\Delta}$. With the choice (5.48), there is just a delta-function singularity at the location of the domain wall at $y = 0$, corresponding to the discontinuity in the gradient of H.

The domain-wall solution (5.47, 5.48) has the peculiarity of tending asymptotically to flat space as $|y| \to \infty$, within a theory that does not naturally admit flat space as a solution (by "naturally," we are excluding the case $a\phi \to -\infty$). Moreover, the theory (5.46) does not even admit a non-flat maximally-symmetric solution, owing to the complication of the cosmological potential. The domain-wall solution (5.47, 5.48), however, manages to "cancel" this potential at transverse infinity, allowing at least asymptotic flatness for this solution.

This brings us back to the other facets of the consistency problem for vertical dimensional reduction down to $(D-2)$-branes as discussed in subsection 5.4. There is no inconsistency between the existence of domain-wall solutions like (5.47, 5.48) and the inability to find such solutions in standard supergravity theories, or with the conical-spacetime character of $(D-3)$-branes, because these domain walls exist only in *massive* supergravity theories like (5.44), with a vacuum structure different from that of standard massless supergravities. Because the Scherk-Schwarz generalised dimensional reduction used to obtain them was a consistent truncation, such domain walls can be oxidised back to solutions of higher-dimensional massless supergravities, but in that case, they have the form of stacked solutions prepared for vertical reduction, with non-zero field strengths in the reduction directions, as in our example (5.38).

6 Intersecting branes, scattering branes

6.1 Multiple component solutions

Given the existence of solutions (5.24) with several active field strengths $F^{\alpha}_{[n]}$, but with coincident charge centers, it is natural to try to find solutions where

[m] Domain walls solutions such as (5.47) in supergravity theories were found for the $D = 4$ case in Ref.[40] and a review of them has been given in Ref.[41]

the charge centers for the different $F^{\alpha}_{[n]}$ are separated.[42] This will lead us to a better understanding of the $\Delta \neq 4$ solutions shown in Figure 6. Consider a number of field strengths that individually have $\Delta = 4$ couplings, but now look for a solution where ℓ of these field strengths are active, with centers $\vec{y}_\alpha$, $\alpha = 1, \ldots, \ell$. Let the charge parameter for F^α be λ^α. Thus, for example, in the magnetic case, one sets

$$F^{\alpha}_{m_1,\ldots,m_n} = \lambda^\alpha \epsilon_{m_1,\ldots,m_n p} \frac{y^p}{|\vec{y} - \vec{y}_\alpha|^{n+1}} \ . \tag{6.1}$$

In both the electric and the magnetic cases, the λ^α are related to the integration constants k^α appearing in the metric by $k^\alpha = \lambda^\alpha/\tilde{d}$. Letting $\varsigma = \pm 1$ in the electric/magnetic cases as before, the solution for the metric and the active dilatonic combinations $e^{\varsigma \vec{a}_\alpha \cdot \vec{\phi}}$ is given by

$$\begin{aligned} ds^2 &= \prod_{\alpha=1}^{\ell} H_\alpha^{\frac{-\tilde{d}}{D-2}} dx^\mu dx_\mu + \prod_{\alpha_1}^{\ell} H_\alpha^{\frac{d}{D-2}} dy^m dy^m \\ e^{\varsigma \vec{a}_\alpha \cdot \vec{\phi}} &= H_\alpha^2 \sum_{\beta_1}^{\ell} H_\beta^{\frac{-d\tilde{d}}{D-2}} \\ H_\alpha &= 1 + \frac{k^\alpha}{|\vec{y} - \vec{y}_\alpha|^{\tilde{d}}} \ . \end{aligned} \tag{6.2}$$

The non-trivial step in verifying the validity of this solution is the check that the non-linear terms still cancel in the Einstein equations, even with the multiple centers.[42]

Now consider a solution with two field strengths $(F^1_{[n]}, F^2_{[n]})$ in which the two charge parameters are taken to be the same, $\lambda_\alpha = \lambda$, while the charge centers are allowed to coalesce. When the charge centers have coalesced, the resulting solution may be viewed as a single-field-strength solution for a field strength rotated by $\pi/4$ in the space of field strengths $(F^1_{[n]}, F^2_{[n]})$. Since the charges add *vectorially,* the net charge parameter in this case will be $\lambda = \sqrt{2}\lambda$, and the net charge density will be $U = \sqrt{2}\lambda\Omega_{D-d-1}/4$. On the other hand, the total mass density will add as a *scalar* quantity, so $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = 2\lambda\Omega_{D-d-1}/4 = \sqrt{2}U$. Thus, the coalesced solution satisfies $\mathcal{E} = 2U/\sqrt{\Delta}$ with $\Delta = 2$. Direct comparison with our general p-brane solution (2.24) shows that the coalesced solution agrees precisely with the single-field-strength $\Delta = 2$ solution. Generalizing this construction to a case with N separate $\Delta = 4$ components, one finds in the coincident limit a $\Delta = 4/N$ supersymmetric solution from the single-field-strength analysis. In the next subsection, we shall see that as one adds new components, each one separately charged with respect to a different $\Delta = 4$ field strength, one progressively breaks more and more supersymmetry. For example, the above solution (6.2) leaves unbroken 1/4 of the original supersymmetry. Since the $\Delta = 4/N$ solutions may in this way be separated into $\Delta = 4$ components while still preserving some degree of unbroken supersymmetry, and without producing any relative forces to disturb their equilibrium, they may be considered to be "bound states at threshold."[42] We shall shortly see that the zero-force property of such multiple-component solutions is related to their managing still to preserve unbroken a certain portion of rigid supersymmetry, even though this portion is reduced with respect to the half-preservation characterising single-component $\Delta = 4$ solutions.

6.2 Intersecting branes and the four elements in $D = 11$

The multiple-charge-center solutions (6.2) to the dimensionally reduced theory (5.13) may automatically be interpreted as solutions of any one of the higher-dimensional theories descending from the $D = 11$ theory (1.1). This automatic "oxidation" is possible because we have insisted throughout on considering only consistent truncations. Although all lower-dimensional solutions may automatically be oxidised in this way into solutions of higher-dimensional supergravity theories, it is not guaranteed that these oxidised branes always fall into the class of isotropic p-brane solutions that we have mainly been discussing. For example, in $D = 9$, one has a two-black-hole solution of the form (6.2), supported by a 1-form gauge potential $A^{12}_{[1]}$ descending from the $D = 11$ gauge potential $A_{[3]}$ and also by another 1-form gauge potential, *e.g.* $\mathcal{A}^2_{[1]}$, emerging from the metric as a Kaluza-Klein vector field. Upon oxidising the two-black-hole solution back to $D = 11$, one finds the solution

$$
\begin{aligned}
ds^2_{11} &= H_1^{\frac{1}{3}}(y)\left[H_1^{-1}(y)\{-dt^2+d\rho^2+d\sigma^2+(H_2(y)-1)(dt+d\rho)^2\}+dy^m dy^m\right]\\
A_{[3]} &= H_1^{-1}(y)dt\wedge d\rho\wedge d\sigma\ , \qquad m = 3,\ldots,10, \qquad \underline{\text{wave}\|\text{2-brane}} \qquad (6.3)
\end{aligned}
$$

which depends on two independent harmonic functions $H_1(y)$ and $H_2(y)$, where the y^m are an 8-dimensional set of "overall transverse" coordinates.

Although the solution (6.3) clearly falls outside the class of p-brane or multiple p-brane solutions that we have considered so far, it nonetheless has two clearly recognisable elements, associated to the two harmonic functions $H_1(y)$ and $H_2(y)$. In order to identify these two elements, we may use the freedom to trivialise one or the other of these harmonic functions by setting it equal to unity. Thus, setting $H_2 = 1$, one recovers

$$\begin{aligned} ds^2_{11} &= H^{\frac{1}{3}}(y)\left[H^{-1}(y)\{-dt^2 + d\rho^2 + d\sigma^2 + dy^m dy^m\right] \\ A_{[3]} &= H^{-1}(y) dt \wedge d\rho \wedge d\sigma, \qquad m = 3, \ldots, 10, \qquad \underline{\text{2-brane}} \end{aligned} \tag{6.4}$$

which one may recognised as simply a certain style of organising the harmonic-function factors in the $D = 11$ membrane solution [18] (3.2), generalised to an arbitrary harmonic function $H(y) \leftrightarrow H_1(y)$ in the membrane's transverse space.

Setting $H_1 = 1$ in (6.3), on the other hand, produces a solution of $D = 11$ supergravity that is not a p-brane (*i.e.* it is not a Poincaré-invariant hyperplane solution). What one finds for $H_1 = 1$ is a classic solution of General Relativity found originally in 1923 by Brinkmann,[43] the *pp wave*:

$$\begin{aligned} ds^2_{11} &= \{-dt^2 + d\rho^2 + (H(y) - 1)(dt + d\rho)^2\} + dy^m dy^m \\ A_{[3]} &= 0\ , \qquad m = 2, \ldots, 10, \qquad \underline{\text{pp wave}} \end{aligned} \tag{6.5}$$

where for a general wave solution, $H(y)$ could be harmonic in the 9 dimensions y^m transverse to the two lightplane dimensions $\{t, \rho\}$ in which the wave propagates; for the specific case obtained by setting $H_1 = 1$ in (6.3), $H(y) \leftrightarrow H_2(y)$ is constant in one of these 9 directions, corresponding to the coordinate σ in (6.3).

The solution (6.3) thus may be viewed as a $D = 11$ pp wave superposed on a membrane. Owing to the fact that the harmonic function $H_2(y)$ depends only on the overall transverse coordinates y^m, $m = 3, \ldots, 10$, the wave is actually "delocalised" in the third membrane worldvolume direction, *i.e.* the solution (6.3) is independent of σ as well as of its own lightplane coordinates. Of course, this delocalisation of the wave in the σ direction is just what makes it possible to perform a dimensional reduction of (6.3) on the $\{\rho, \sigma\}$ coordinates down to a $D = 9$ configuration of two particles of the sort considered in (6.2), *i.e.* the wave in (6.3) has already been stacked up in the σ direction as is necessary in preparation for a vertical dimensional reduction. Another point to note about (6.3) is that the charge centers of the two harmonic functions H_1 and H_2 may be chosen completely independently in the overall transverse space. Thus, although this is an example of an "intersecting" brane configuration, it should be understood that the two components of (6.3) need not actually overlap on any specific subspace of spacetime. The term "intersecting" is generally taken to mean that there are shared worldvolume coordinates, in this case the $\{t, \rho\}$ overlap between the membrane worldvolume and the lightplane coordinates.[44]

A very striking feature of the family of multiple-component p-brane solutions is that their oxidations up to $D = 11$ involve combinations of only 4 basic "elemental" $D = 11$ solutions. Two of these we have just met in the oxidised solution (6.3): the membrane and the pp wave. The two others are the "duals" of these: the 5-brane [25] and a solution describing the oxidation to $D = 11$ of the "Kaluza-Klein monopole."[45] The 5-brane may be written in a style similar to that of the membrane (6.4):

$$\begin{aligned} ds_{11}^2 &= H^{\frac{2}{3}}(y)\left[H^{-1}(y)\{-dt^2 + dx_1^2 + \ldots + dx_5^2\} + dy^m dy^m\right] \\ F_{[4]} &= {}^*dH(y), \qquad m = 6, \ldots, 10, \qquad\qquad \underline{\text{5-brane}} \end{aligned} \tag{6.6}$$

where the $H(y)$ is a general harmonic function in the 5-dimensional transverse space.

The Kaluza-Klein monopole oxidised up to $D = 11$ is the solution

$$\begin{aligned} ds_{11}^2 &= -dt^2 + dx_1^2 + \ldots + dx_6^2 + ds_{\rm TN}^2(y) \\ A_{[3]} &= 0 \end{aligned} \tag{6.7a}$$

$$\begin{aligned} ds_{\rm TN}^2 &= H(y)dy^i dy^i + H^{-1}(y)(d\psi + V_i(y)dy^i)^2 \ , \qquad i = 1, 2, 3, \\ \vec{\nabla} \times \vec{V} &= \vec{\nabla} H \ , \qquad\qquad \underline{\text{Taub-NUT}} \end{aligned} \tag{6.7b}$$

where $ds_{\rm TN}^2$ is the Taub-NUT metric, a familiar four-dimensional Euclidean gravitational instanton. The harmonic function H in (6.7) is a function only of the 3 coordinates y^i, and not of the coordinate ψ, which plays a special rôle. Generally, the solution (6.7) has a conical singularity on the hyperplane $y^i = 0$, but this becomes a mere coordinate singularity, similar to that for flat space in polar coordinates, providing the coordinate ψ is periodically identified. For a single-center harmonic function $H(y) = 1 + k/(|y|)$, the appropriate identification period for ψ is $4\pi k$.

Thus, the Taub-NUT solution naturally invites interpretation as a compactified solution in one less dimension, after reduction on ψ. In the case of the original Kaluza-Klein monopole,[45] the starting solution had 4+1 dimensions, giving rise after compactification to a magnetically-charged particle in $D = 4$ dimensions. The solution (6.7) has an additional 6 spacelike worldvolume dimensions $x_1, \ldots, x_6$, so after reduction on the ψ coordinate one has a magnetically-charged 6-brane solution in $D = 10$.

The relation $\Delta\psi = 4\pi k$ between the compactification period of ψ and the charge-determining integration constant k in the harmonic function H of the solution (6.7) gives rise to a quantisation condition at the quantum level involving the magnetic charge of the dimensionally-reduced $D = 10$ 6-brane descending from (6.7) and the electric charge of the extreme black hole particle obtained by reducing the pp wave (6.5). This quantisation condition is nothing other than an ordinary quantisation of momentum for Fourier wave components on a compact space, in this case the compact ψ direction. In terms of the electric and magnetic charges U and V of the dimensionally reduced particle and 6-brane, one finds $UV = 2\pi\kappa_{10}^2 n$, with $n \in \mathbb{Z}$ (where κ_{10}^2 occurs because the charges U and V as defined in (1.3, 1.4) are not dimensionless). This is precisely of the form expected for a Dirac charge quantisation condition. In Section 7 we shall return to the subject of charge quantisation conditions more generally for the charges carried by p-branes.

Let us now return to the question of supersymmetry preservation and enquire whether intersecting branes like (6.3) can also preserve some portion of unbroken rigid supersymmetry. All four of the elemental $D = 11$ solutions (6.4 – 6.7) preserve half the $D = 11$ rigid supersymmetry. We have already seen this for the membrane solution in subsection 4.4. As another example, one may consider the supersymmetry preservation conditions for the pp wave solution (6.5). We shall skip over points 1) and 2) of the discussion analogous to that of subsection 4.4 and shall instead concentrate just on the projection conditions that must be satisfied by the surviving rigid supersymmetry parameter ϵ_∞. Analogously to our earlier abbreviated discussion using just the supersymmetry algebra, consider this algebra in the background of a pp wave solution (6.5) propagating in the $\{01\}$ directions of spacetime, with normalisation to unit length along the wave's propagation direction:

$$\frac{1}{\text{length}}\{Q_\alpha, Q_\beta\} = 2\mathcal{E}P_{01} \qquad\qquad P_{01} = \tfrac{1}{2}(\mathbb{1} + \Gamma^{01}) \ , \tag{6.8}$$

where P_{01} is again a projection operator with half of its eigenvalues zero, half unity. Consequently, the pp wave solution (6.5) preserves half of the $D = 11$ rigid supersymmetry.

Now let us apply the projection-operator analysis to the wave$\|$2-brane solution (6.3). Supersymmetry preservation in a membrane background oriented parallel to the $\{012\}$ hyperplane requires the projection condition $P_{012}\epsilon_\infty = 0$ (4.23), while supersymmetry preservation in a pp wave background with a $\{01\}$ lightplane requires $P_{01}\epsilon_\infty = 0$. Imposing these two conditions simultaneously is consistent because these projectors commute,

$$[P_{012}, P_{01}] = 0 \ . \tag{6.9}$$

Since $\text{tr}(P_{012}P_{01}) = \frac{1}{4}\cdot 32$, the imposition of both projection conditions on ϵ_∞ cuts the preserved portion of rigid $D = 11$ supersymmetry down to $\frac{1}{4}$.

Now, let's consider another example of an intersecting-brane solution, containing as elements a $D = 11$ membrane, 5-brane pair. The solution is

$$
\begin{aligned}
ds^2 &= H_1^{\frac{1}{3}}(y)H_2^{\frac{2}{3}}(y)\big[H_1^{-1}(y)H_2^{-1}(y)(-dt^2+dx_1^2) && \underline{2\perp 5(1)}\\
&\qquad +H_1^{-1}(y)(dx_2^2)+H_2^{-1}(y)(dx_3^2+\ldots+dx_6^2)\\
&\qquad\qquad +dy^m dy^m\big] \qquad\qquad m=7,\ldots,10 && (6.10\text{a})\\
&F_{m012}=\partial_m(H_1^{-1}) \qquad\qquad F_{2mnp}=-\epsilon_{mnpq}\partial_q H_2\ , && (6.10\text{b})
\end{aligned}
$$

where as in the wave-on-a-membrane solution (6.3), the harmonic functions $H_1(y)$ and $H_2(y)$ depend only on the overall transverse coordinates. By considering special cases where $H_2 = 1$ or $H_1 = 1$, one identifies the membrane and 5-brane elements of the solution (6.10); as before, these elements are delocalised in (*i.e.,* independent of) the "relative transverse" directions, by which one means the directions transverse to one element's worldvolume but belonging to the worldvolume of the other element, *i.e.* the directions $\{2;3,\ldots,6\}$ for the solution (6.10). Note that both the membrane and 5-brane elements share the worldvolume directions $\{01\}$; these are accordingly called "overall worldvolume" directions. Considering this "intersection" to be a string (but recall, however, that the overall-transverse charge centers of H_1 and H_2 need not coincide, so there is not necessarily a true string overlap), the solution (6.10) is denoted $2\perp 5(1)$.

The forms of the wave$\|$2-brane solution (6.3) and the $2\perp 5(1)$ solution (6.10) illustrate the general structure of intersecting-brane solutions. For a two-element solution, there are four sectors among the coordinates: overall worldvolume, two relative transverse sectors and the overall transverse sector. One may make a sketch of these relations for the $2\perp 5(1)$ solution (6.10):

0	1	2	3	4	5	6	7	8	9	10
x	x	x								
x	x		x	x	x	x				
W25		W2T5	W5T2				T25			

The character of each coordinate is indicated in this sketch: W2 and W5 indicating worldvolume coordinates with respect to each of the two elements and T2 and T5 indicating transverse coordinates with respect to each of the two elements. Thus, the overall worldvolume coordinates are the W25 coordinates and the overall transverse coordinates are the T25 coordinates. Having established this coordinate classification, the general structure of the intersecting brane metric is as follows. For each element, one puts an overall conformal factor $H_i^{d/(D-2)}(y)$ for the whole metric, and then in addition one puts a factor $H_i^{-1}(y)$ in front of each dx^2 term belonging to the worldvolume of the i^{th}

element. One may verify this pattern in the structure of (6.10). This pattern has been termed the *harmonic function rule*.[44]

This summary of the structure of intersecting brane solutions does not replace a full check that the supergravity equations of motion are solved, and in addition one needs to establish which combinations of the $D = 11$ elements may be present in a given solution. For a fuller review on this subject, we refer the reader to Ref.[46] For now, let us just check point 3) in the supersymmetry-preservation analysis for the $2 \perp 5(1)$ solution (6.10). For each of the two elements, one has a projection condition on the surviving rigid supersymmetry parameter ϵ_∞: $P_{012}\epsilon_\infty = 0$ for the membrane and $P_{01345}\epsilon_\infty = 0$ for the 5-brane. These may be consistently imposed at the same time, because $[P_{012}, P_{01345}] = 0$, similarly to our discussion of the wave$\|$2-brane solution. The amount of surviving supersymmetry in the $2 \perp 5(1)$ solution is $\frac{1}{4}$, because $\mathrm{tr}(P_{012}P_{01345}) = \frac{1}{4} \cdot 32$.

6.3 Brane probes, scattering branes and modulus σ-model geometry

The existence of static configurations such as the wave$\|$2-brane solution (6.3) or the $2 \perp 5(1)$ intersecting-brane solution (6.10) derives from the properties of the transverse-space Laplace equation (2.20) arising in the process of solving the supergravity equations subject to the p-brane ansätze (2.3–2.6). The Laplace equation has the well-known property of admitting multi-center solutions, which we have already encountered in Eq. (5.27). Physically, the existence of such multi-center solutions corresponds either to a cancellation of attractive gravitational and dilatonic forces against repulsive antisymmetric-tensor forces, or to the fact that one brane couples to the background supergravity fields with a conformal factor that wipes out the effects of the other brane. In order to see such cancellations more explicitly, one may use a source coupling analogous to the $D = 11$ bosonic supermembrane action (4.1) in order to treat the limiting problem of a light brane probe moving in the background of a heavy brane.[18,47] In this limit, one may ignore the deformation of the heavy-brane background caused by the light brane. The use of the brane-probe coupling is a simple way to approximately treat time-dependent brane configurations. For a p-brane probe of this sort coupled to a D-dimensional supergravity background, the probe action is

$$I_{\mathrm{probe}} = -T_\alpha \int d^{p+1}\xi \,(-\det(\partial_\mu x^m \partial_\nu x^n g_{mn}(x))^{\frac{1}{2}} e^{\frac{1}{2}\varsigma^{\mathrm{pr}}\vec{a}_\alpha\cdot\vec{\phi}} + Q_\alpha \int \tilde{A}^\alpha_{[p+1]} \tag{6.11a}$$

$$\tilde{A}^{\alpha}_{[p+1]} = (p+1)^{-1}\partial_{\mu_1}x^{m_1}\cdots\partial_{\mu_{p+1}}x^{m_{p+1}}A^{\alpha}_{m_1\cdots m_{p+1}}d\xi^{\mu_1}\wedge\cdots\wedge d\xi^{\mu_{p+1}} \ . \tag{6.11b}$$

The dilaton coupling in (6.11a) occurs because one needs to have the correct source for the D-dimensional Einstein frame, *i.e.* the conformal frame in which the D-dimensional Einstein-Hilbert action is free from dilatonic scalar factors. Requiring that the source match correctly to a p-brane (probe) solution demands the presence of the dilaton coupling $e^{\frac{1}{2}\varsigma^{\rm pr}\vec{a}_\alpha\cdot\vec{\phi}}$, where $\varsigma^{\rm pr}=\pm1$ according to whether the p-brane probe is of electric or magnetic type and where $\vec{a}_\alpha$ is the dilaton vector appearing in the kinetic-term dilaton coupling in (5.13) for the gauge potential $A^{\alpha}_{[p+1]}$, to which the p brane probe couples.

As a simple initial example of such a brane probe, one may take a light $D=11$ membrane probe in the background of a parallel and similarly-oriented heavy membrane.[18] In this case, the brane-probe action (6.11) becomes just the $D=11$ supermembrane action (4.1). If one takes the form of the heavy-membrane background from the electric ansatz (2.3, 2.4), and if one chooses the "static" worldvolume gauge $\xi^\mu = x^\mu$, $\mu=0,1,2$, then the bosonic probe action becomes

$$I_{\rm probe} = -T\int d^3\xi\left(\sqrt{-\det(e^{2A(y)}\eta_{\mu\nu}+e^{2B(y)}\partial_\mu y^m\partial_\nu y^m)}-e^{C(y)}\right) \ . \tag{6.12}$$

Expanding the square root in (6.12), one finds at order $(\partial y)^0$ the effective potential

$$V_{\rm probe} = T(e^{3A(y)}-e^{C}) \ . \tag{6.13}$$

Recalling the condition (2.22), which becomes just $e^{3A(y)}=e^{C(y)}$ for the membrane background, one has directly $V_{\rm probe}=0$, confirming the absence of static forces between the two membrane components.

Continuing on in the expansion of (6.12) to order $(\partial y)^2$ in the probe velocity, one has the effective probe σ-model

$$I^{(2)}_{\rm probe} = -\frac{T}{2}\int d^3\xi e^{3A(y)}e^{2(B(y)-A(y))}\partial_\mu y^m\partial_\nu y^m\eta_{\mu\nu} \ , \tag{6.14}$$

but, recalling the supersymmetry-preservation condition (2.17) characterising the heavy-membrane background, the probe σ-model metric in (6.14) reduces simply to

$$\gamma^{mn} = e^{A(y)+2B(y)}\delta^{mn} = \delta^{mn} \ , \tag{6.15}$$

i.e., the membrane-probe σ-model metric is *flat*.

The flatness of the membrane-probe σ-model metric (6.15) accords precisely with the degree of rigid supersymmetry that survives in the underlying supergravity solution with two parallel, similarly oriented $D = 11$ membranes, which we found in subsection 4.4 to have $\frac{1}{2} \cdot 32$ components, *i.e.* $d = 2 + 1$, $N = 8$ probe-worldvolume supersymmetry. This high degree of surviving supersymmetry is too restrictive in its constraints on the form of the σ-model to allow for anything other than a flat metric, precisely as one finds in (6.15). Continuing on with the expansion of (6.12), one first finds a nontrivial interaction between the probe and the heavy membrane background at order $(\partial y)^4$ (odd powers being ruled out by time-reversal invariance of the $D = 11$ supergravity equations).

Now consider a brane-probe configuration with less surviving supersymmetry, and with correspondingly weaker constraints on the probe worldvolume σ-model. Corresponding to the wave$\|$2-brane solution (6.3), one has, after dimensional reduction down to $D = 9$ dimensions, a system of two black holes supported by *different* $\Delta = 4$, $D = 9$ vector fields: one descending from the $D = 11$ 3-form gauge potential and one descending from the metric.

Now repeat the brane-probe analysis for the two-black-hole configuration, again choosing a static gauge on the probe worldvolume, which in the present case just becomes $\xi^0 = t$. Again expand the determinant of the induced metric in (6.11). At order $(\partial y)^0$, this now gives $V_{\rm probe} = e^A e^{-\frac{3}{2\sqrt{7}}\phi}$, but this potential turns out to be just a constant because the heavy-brane background satisfies $A = \frac{3}{2\sqrt{7}}\phi$. Thus, we confirm the expected static zero-force condition for the $\frac{1}{4}$ supersymmetric two-black-hole configuration descending from the wave$\|$2-brane solution (6.3). This zero-force condition arises not so much as a result of a cancellation between different forces but as a result of the probe's coupling to the background with a dilatonic factor in (6.11) that wipes out the conformal factor occurring in the heavy brane background metric.

Proceeding on to (velocity)^2 order, one now obtains a non-trivial probe σ-model, with metric

$$\gamma^{mn} = H_{\rm back}(y)\delta^{mn} \ , \tag{6.16}$$

where $H_{\rm back}$ is the harmonic function controlling the heavy brane's background fields; for the case of two black holes in $D = 9$, the harmonic function $H_{\rm back}$ has the structure $(1 + k/r^6)$.

The above test-brane analysis for two $D = 9$ black holes is confirmed by a more detailed study of the low-velocity scattering of supersymmetric black holes performed by Shiraishi.[48] The procedure is a standard one in soliton physics: one promotes the moduli of a static solution to time-dependent functions and then substitutes the resulting generalized field configuration back into the original field equations. This leads to a set of differential equations on the modulus variables which may be viewed as effective equations for the moduli. In the general case of multiple black hole scattering, the resulting system of differential equations may be quite complicated. The system of equations, however, simplifies dramatically in cases corresponding to the scattering of supersymmetric black holes, *e.g.* the above pair of $D = 9$ black holes, where the result turns out to involve only 2-body forces. These two-body forces may be derived from an effective action involving the position vectors of the two black holes. Separating the center-of-mass motion from the relative motion, one obtains the same modulus metric (6.16) as that found in the brane-probe analysis above, except for a rescaling which replaces the brane-probe mass by the reduced mass of the two-black-hole system.

Now we should resolve a puzzle of how this non-trivial $d = 1$ scattering modulus σ-model turns out to be consistent with the surviving supersymmetry.[49] The modulus variables of the two-black-hole system are fields in one dimension, *i.e.* time. The N-extended supersymmetry algebra in $d = 1$ is

$$\{Q^I, Q^J\} = 2\delta^{IJ}\hat{H} \qquad I = 1, \ldots, N \ , \tag{6.17}$$

where $\hat{H}$ is the Hamiltonian. A $d = 1$, $N = 1$ σ-model is specified by a triple $(\mathcal{M}, \gamma, A_{[3]})$, where $\mathcal{M}$ is the Riemannian σ-model manifold, γ is the metric on $\mathcal{M}$ and $A_{[3]}$ is a 3-form on $\mathcal{M}$ which plays the rôle of torsion in the derivative operator acting on fermions, $\nabla^{(+)}_t = \partial_t x^i \nabla^{(+)}_i$, where $\nabla^{(+)}_i \lambda^j = \nabla_i \lambda^j + \frac{1}{2} A^j{}_{ik} \lambda^k$. The σ-model action may be written using $N = 1$ superfields $x^i(t, \theta)$ (where $x^i(t) = x^i{}_{|\theta=0}$, $\lambda^i(t) = Dx^i{}_{|\theta=0}$) as

$$I = -\tfrac{1}{2} \int dt d\theta (\mathrm{i}\gamma_{ij} Dx^i \frac{d}{dt} x^j + \frac{1}{3!} A_{ijk} Dx^i Dx^j Dx^k) \ . \tag{6.18}$$

One may additionally [50] have a set of spinorial $N = 1$ superfields ψ^a, with Lagrangian $-\frac{1}{2} h_{ab} \psi^a \nabla_t \psi^b$, where h_{ab} is a fibre metric and ∇_t is constructed using an appropriate connection for the fibre corresponding to the ψ^a. However, in the present case we shall not include this extra superfield. In order to have extended supersymmetry in (6.18), one starts by positing a second set of supersymmetry transformations of the form $\delta x^i = \eta I^i{}_j Dx^j$, and then requires these transformations to close to form the $N = 2$ algebra (6.17); then one also requires that the action (6.18) be invariant. In this way, one obtains the equations

$$I^2 = -\mathbb{1} \tag{6.19a}$$

$$N^i_{jk} \equiv I^i_{[j,k]} = 0 \tag{6.19b}$$

$$\gamma_{kl} I^k{}_i I^l{}_j = \gamma_{ij} \tag{6.19c}$$

$$\nabla^{(+)}_{(i} I^k{}_{j)} = 0 \qquad (6.19d)$$

$$\partial_{[i}(I^m{}_j A_{|m|kl]}) - 2I^m{}_{[i}\partial_{[m} A_{jkl]]}) = 0 \, , \qquad (6.19e)$$

where (6.19a,b) follow from requiring the closure of the algebra (6.17) and (6.19c-e) follow from requiring invariance of the action (6.18). Conditions (6.19a,b) imply that $\mathcal{M}$ is a complex manifold, with $I^i{}_j$ as its complex structure.

The structure of the conditions (6.19) is more complicated than might have been expected. Experience with $d = 1+1$ extended supersymmetry [50] might have lead one to expect, by simple dimensional reduction, just the condition $\nabla^{(+)}_i I^j{}_k = 0$. Certainly, solutions of this condition also satisfy (6.19c–e), but the converse is not true, *i.e.* the $d = 1$ extended supersymmetry conditions are "weaker" than those obtained by dimensional reduction from $d = 1+1$, even though the $d = 1+1$ minimal spinors are, as in $d = 1$, just real single-component objects. Conversely, the $d = 1+1$ theory implies a "stronger" condition; the difference is explained by $d = 1+1$ Lorentz invariance: not all $d = 1$ theories can be "oxidized" up to Lorentz-invariant $d = 1+1$ theories. In the present case with two $D = 9$ black holes, this is reflected in the circumstance that after even one dimensional oxidation from $D = 9$ up to $D = 10$, the solution already contains a pp wave element (so that we have a $D = 10$ "wave-on-a-string" solution), with a lightplane metric that is not Poincaré invariant.

Note also that the $d = 1$ "torsion" $A_{[3]}$ is not required to be closed in (6.19). $d = 1$ supersymmetric theories satisfying (6.19) are *analogous* to (2,0) chiral supersymmetric theories in $d = 1+1$, but the weaker conditions (6.19) warrant a different notation for this wider class of models; one may call them 2b supersymmetric σ-models.[49] Such models are characterized by a Kähler geometry with torsion.

Continuing on to $N = 8$, $d = 1$ supersymmetry, one finds an 8b generalization [49] of the conditions (6.19), with 7 independent complex structures built using the octonionic structure constants[n] $\varphi_{ab}{}^c$: $\delta x^i = \eta^a I_a{}^i{}_j D x^j$, $a = 1, \ldots 7$, with $(I_a)^8{}_b = \delta_{ab}$, $(I_a)^b{}_8 = -\delta^b{}_a$, $(I_a)^b{}_c = \varphi_a{}^b{}_c$, where the octonion multiplication rule is $e_a e_b = -\delta_{ab} + \varphi_{ab}{}^c e_c$. Models satisfying such conditions have an "octonionic Kähler geometry with torsion," and are called OKT models.[49]

Now, are there any non-trivial solutions to these conditions? Evidently, from the brane-probe and Shiraishi analyses, there must be. For our two $D = 9$ black holes with a $D = 8$ transverse space, one may start from the ansatz $ds^2 = H(y) ds^2(\mathbb{E}^8)$, $A_{ijk} = \Omega_{ijk}{}^\ell \partial_\ell H$, where Ω is a 4-form on $\mathbb{E}^8$.

[n] We let the conventional octonionic "0" index be replaced by "8" here in order to avoid confusion with a timelike index; the $\mathbb{E}^8$ transverse space is of Euclidean signature.

Then, from the 8b generalization of condition (6.19d) one learns $\Omega_{8abc} = \varphi_{abc}$ and $\Omega_{abcd} = -{}^*\varphi_{abcd}$; from the 8b generalization of condition (6.19e) one learns $\delta^{ij}\partial_i\partial_j H = 0$. Thus we recover the familiar dependence of p-brane solutions on transverse-space harmonic functions, and we reobtain the brane-probe or Shiraishi structure of the black-hole modulus scattering metric with

$$H_{\rm relative} = 1 + \frac{k_{\rm red}}{|y_1 - y_2|^6} \ , \tag{6.20}$$

where $k_{\rm red}$ determines the reduced mass of the two black holes.

7 Duality symmetries and charge quantisation

As one can see from our discussion of Kaluza-Klein dimensional reduction in Section 5, progression down to lower dimensions D causes the number of dilatonic scalars $\vec{\phi}$ and also the number of zero-form potentials of 1-form field strengths to proliferate. When one reaches $D = 4$, for example, a total of 70 such spin-zero fields has accumulated. In $D = 4$, the maximal ($N = 8$) supergravity equations of motion have a linearly-realized $H = \mathrm{SU}(8)$ symmetry; this is also the automorphism symmetry of the $D = 4$, $N = 8$ supersymmetry algebra relevant to the (self-conjugate) supergravity multiplet. In formulating this symmetry, it is necessary to consider *complex* self-dual and anti-self-dual combinations of the 2-form field strengths, which are the highest-rank field strengths occurring in $D = 4$, higher ranks having been eliminated in the reduction or by dualization. Using two-component notation for the $D = 4$ spinors, these combinations transform as $F^{[ij]}_{\alpha\beta}$ and $\bar{F}_{\dot\alpha\dot\beta\,[ij]}$, $i, j = 1, \ldots, 8$, *i.e.* as a complex 28-dimensional dimensional representation of SU(8). Since this complex representation can be carried only by the complex field-strength combinations and not by the 1-form gauge potentials, it cannot be locally formulated at the level of the gauge potentials or of the action, where only an SO(8) symmetry is apparent.

Taking all the spin-zero fields together, one finds that they form a rather impressive nonlinear σ-model with a 70-dimensional manifold. Anticipating that this manifold must be a coset space with $H = \mathrm{SU}(8)$ as the linearly-realized denominator group, Cremmer and Julia [51] deduced that it had to be the manifold $\mathrm{E}_{7(+7)}/\mathrm{SU}(8)$; since the dimension of E_7 is 133 and that of SU(8) is 63, this gives a 70-dimensional manifold. Correspondingly, a nonlinearly-realized $\mathrm{E}_{7(+7)}$ symmetry also appears as an invariance of the $D = 4$, $N = 8$ maximal supergravity equations of motion. Such nonlinearly-realized symmetries of supergravity theories have always had a somewhat mysterious character. They arise in part out of general covariance in the higher dimensions, from which supergravities arise by dimensional reduction, but this is not enough: such symmetries act transitively on the σ-model manifolds, mixing both fields arising from the metric and also from the reduction of the $D = 11$ 3-form potential $A_{[3]}$ in (1.1).

In dimensions $4 \le D \le 9$, maximal supergravity has the sets of σ-model nonlinear G and linear H symmetries shown in Table 2. In all cases, the spin-zero fields take their values in "target" manifolds G/H. Just as the asymptotic value at infinity of the metric defines the reference, or "vacuum" spacetime with respect to which integrated charges and energy/momentum are defined, so do the asymptotic values of the spin-zero fields define the "scalar vacuum." These asymptotic values are referred to as the *moduli* of the solution. In string theory, these moduli acquire interpretations as the *coupling constants* and vacuum *θ-angles* of the theory. Once these are determined for a given "vacuum," the *classification symmetry* that organizes the distinct solutions of the theory into multiplets with the same energy must be a subgroup of the *little group*, or *isotropy group,* of the vacuum. In ordinary General Relativity with asymptotically flat spacetimes, the analogous group is the spacetime Poincaré group times the appropriate "internal" classifying symmetry, *e.g.* the group of rigid (*i.e.* constant-parameter) Yang-Mills gauge transformations.

Table 2: Supergravity σ-model symmetries.

D	G	H
9	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{SO}(2)$
8	$\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(3) \times \mathrm{SO}(2)$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$
6	$\mathrm{SO}(5,5)$	$\mathrm{SO}(5) \times \mathrm{SO}(5)$
5	$\mathrm{E}_{6(+6)}$	$\mathrm{USP}(8)$
4	$\mathrm{E}_{7(+7)}$	$\mathrm{SU}(8)$
3	$\mathrm{E}_{8(+8)}$	$\mathrm{SO}(16)$

The isotropy group of any point on a coset manifold G/H is just H, so this is the classical "internal" classifying symmetry for multiplets of supergravity solutions.

7.1 An example of duality symmetry: $D = 8$ supergravity

In maximal $D = 8$ supergravity, one sees from Table 2 that $G = \mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and the isotropy group is $H = \mathrm{SO}(3) \times \mathrm{SO}(2)$. We have an $(11 - 3 = 8)$ vector of dilatonic scalars as well as a singlet $F^{ijk}_{[1]}$ and a triplet $\mathcal{F}^{ij}_{[1]}$

($i,j,k = 1,2,3$) of 1-form field strengths for zero-form potentials. Taken all together, we have a manifold of dimension 7, which fits in precisely with the dimension of the $(\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}))/(\mathrm{SO}(3)\times\mathrm{SO}(2))$ coset-space manifold: $8+3-(3+1)=7$.

Owing to the direct-product structure, we may for the time being drop the 5-dimensional $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ sector and consider for simplicity just the 2-dimensional $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ sector. Here is the relevant part of the action:[52]

$$\begin{aligned} I_8^{\mathrm{SL}(2)} \quad = \quad & \int d^8x\sqrt{-g}\Big[R-\tfrac{1}{2}\nabla_M\sigma\nabla^M\sigma-\tfrac{1}{2}e^{-2\sigma}\nabla_M\chi\nabla^M\chi \\ & \qquad -\frac{1}{2\cdot 4!}e^{\sigma}(F_{[4]})^2-\frac{1}{2\cdot 4!}\chi F_{[4]}{}^*F_{[4]}\Big] \end{aligned} \tag{7.1}$$

where ${}^*F^{MNPQ} = 1/(4!\sqrt{-g})\epsilon^{MNPQx_1x_2x_3x_4}F_{x_1x_2x_3x_4}$ (the $\epsilon^{[8]}$ is a density, hence purely numerical).

On the scalar fields (σ,χ), the $\mathrm{SL}(2,\mathbb{R})$ symmetry acts as follows: let $\lambda=\chi+\mathrm{i}e^{\sigma}$; then

$$\Lambda=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \tag{7.2}$$

with $ab-cd=1$ is an element of $\mathrm{SL}(2,\mathbb{R})$ and acts on λ by the fractional-linear transformation

$$\lambda\longrightarrow\frac{a\lambda+b}{c\lambda+d}\ . \tag{7.3}$$

The action of the $\mathrm{SL}(2,\mathbb{R})$ symmetry on the 4-form field strength gives us an example of a symmetry of the equations of motion that is not a symmetry of the action. The field strength $F_{[4]}$ forms an $\mathrm{SL}(2,\mathbb{R})$ *doublet* together with

$$G_{[4]}=e^{\sigma}{}^*F_{[4]}-\chi F_{[4]}\ , \tag{7.4}$$

i.e.,

$$\begin{pmatrix} F_{[4]} \\ G_{[4]}\end{pmatrix}\longrightarrow(\Lambda^{\mathrm{T}})^{-1}\begin{pmatrix} F_{[4]} \\ G_{[4]}\end{pmatrix}\ . \tag{7.5}$$

One may check that these transform the $F_{[4]}$ field equation

$$\nabla_M(e^{\sigma}F^{MNPQ}+\chi{}^*F^{MNPQ})=0 \tag{7.6}$$

into the corresponding Bianchi identity,

$$\nabla_M{}^*F^{MNPQ}=0\ . \tag{7.7}$$

Since the field equations may be expressed purely in terms of $F_{[4]}$, we have a genuine symmetry of the field equations in the transformation (7.5), but since this transformation cannot be expressed locally in terms of the gauge potential $A_{[3]}$, this is not a local symmetry of the action. The transformation (7.3, 7.5) is a $D = 8$ analogue of ordinary Maxwell duality transformation in the presence of scalar fields. Accordingly, we shall refer generally to the supergravity σ-model symmetries as duality symmetries.

The $F_{[4]}$ field strength of the $D = 8$ theory supports elementary/electric p-brane solutions with $p = 4 - 2 = 2$, *i.e.* membranes, which have a $d = 3$ dimensional worldvolume. The corresponding solitonic/magnetic solutions in $D = 8$ have worldvolume dimension $\tilde{d} = 8 - 3 - 2 = 3$ also. So in this case, $F_{[4]}$ supports *both* electric and magnetic membranes. It is also possible in this case to have solutions generalizing the purely electric or magnetic solutions considered so far to solutions that carry both types of charge, *i.e. dyons*.[52] This possibility is also reflected in the combined Bogomol'ny bound[o] for this situation, which generalizes the single-charge bounds (4.16):

$$\mathcal{E}^2 \geq e^{-\sigma_\infty}(U + \chi_\infty V)^2 + e^{\sigma_\infty} V^2 \ , \tag{7.8}$$

where U and V are the electric and magnetic charges and σ_∞ and χ_∞ are the moduli, *i.e.* the constant asymptotic values of the scalar fields $\sigma(x)$ and $\chi(x)$. The bound (7.8) is itself $\mathrm{SL}(2, \mathbb{R})$ invariant, provided that one transforms both the moduli $(\sigma_\infty, \chi_\infty)$ (according to (7.3)) and also the charges (U, V). For the simple case with $\sigma_\infty = \chi_\infty = 0$ that we have mainly chosen in order to simplify the writing of explicit solutions, the bound (7.8) reduces to $\mathcal{E}^2 \geq U^2 + V^2$, which is invariant under an obvious isotropy group $H = \mathrm{SO}(2)$.

7.2 p-form charge quantisation conditions

So far, we have discussed the structure of p-brane solutions at a purely classical level. At the classical level, a given supergravity theory can have a continuous spectrum of electrically and magnetically charged solutions with respect to any one of the n-form field strengths that can support the solution. At the quantum level, however, an important restriction on this spectrum of solutions enters into force: the Dirac-Schwinger-Zwanziger (DSZ) quantisation conditions for particles with electric or magnetic or dyonic charges.[53,54] As we have seen, however, the electric and magnetic charges carried by branes and appearing in the supersymmetry algebra (1.5) are *forms,* and the study of their charge-quantisation properties involves some special features not seen in the $D = 4$ Maxwell case.[55]

[o]In comparing (7.8) to the single-charge bounds (4.16), one should take note that for $F_{[4]}$ in (7.1) we have $\Delta = 4$, so $2/\sqrt{\Delta} = 1$.

We shall first review a Wu-Yang style of argument,[53] (for a Dirac-string argument, see Ref.[54,56]) considering a closed sequence $\mathcal{W}$ of deformations of one p-brane, say an electric one, in the background fields set up by a dual, magnetic, $\hat{p} = D-p-4$ brane. After such a sequence of deformations, one sees from the supermembrane action (4.1) that the electric p-brane wavefunction picks up a phase factor

$$\exp\left(\frac{\mathrm{i}Q_\mathrm{e}}{(p+1)!}\oint_{\mathcal{W}} A_{M_1\ldots M_{p+1}} dx_1^M \wedge \ldots \wedge dx^{M_{p+1}}\right) \ , \tag{7.9}$$

where $A_{[p+1]}$ is the gauge potential set up (locally) by the magnetic $\hat{p}$-brane background.

A number of differences arise in this problem with respect to the ordinary Dirac quantisation condition for $D = 4$ particles. One of these is that, as we have seen in subsection 4.1, objects carrying p-form charges appearing in the supersymmetry algebra (1.5) are necessarily either infinite or are wrapped around compact spacetime dimensions. For infinite p-branes, some deformation sequences $\mathcal{W}$ will lead to a divergent integral in the exponent in (7.9); such deformations would also require an infinite amount of energy, and so should be excluded from consideration. In particular, this excludes deformations that involve rigid rotations of an entire infinite brane. Thus, at least the *asymptotic* orientation of the electric brane must be preserved throughout the sequence of deformations. Another way of viewing this restriction on the deformations is to note that the asymptotic orientation of a brane is encoded into the electric p-form charge, and so one should not consider changing this p-form in the course of the deformation any more than one should consider changing the magnitude of the electric charge in the ordinary $D = 4$ Maxwell case.

We shall see shortly that another difference with respect to the ordinary $D = 4$ Dirac quantisation of particles in Maxwell theory will be the existence of “Dirac-insensitive” configurations, for which the phase in (7.9) vanishes.

Restricting attention to deformations that give non-divergent phases, one may use Stoke’s theorem to rewrite the integral in (7.9):

$$\begin{aligned}\frac{Q_\mathrm{e}}{(p+1)!}\oint_{\mathcal{W}} A_{M_1\ldots M_{p+1}} dx_1^M \wedge \ldots \wedge dx^{M_{p+1}} = \\ \frac{Q_\mathrm{e}}{(p+2)!}\int_{\mathcal{M}_\mathcal{W}} F_{M_1\ldots M_{p+2}} dx^{M_1} \wedge \ldots \wedge dx^{M_{p+2}} = Q_\mathrm{e}\Phi_{\mathcal{M}_\mathcal{W}} \ ,\end{aligned} \tag{7.10}$$

where $\mathcal{M}_\mathcal{W}$ is any surface “capping” the closed surface $\mathcal{W}$, *i.e.* a surface such that $\partial\mathcal{M}_\mathcal{W} = \mathcal{W}$; $\Phi_{\mathcal{M}_\mathcal{W}}$ is then the flux through the cap $\mathcal{M}_\mathcal{W}$. Choosing the capping surface in two different ways, one can find a flux discrepancy $\Phi_{\mathcal{M}_1} -$

$\Phi_{\mathcal{M}_2} = \Phi_{\mathcal{M}_1 \cap \mathcal{M}_2} = \Phi_{\mathcal{M}_{\rm total}}$ (taking into account the orientation sensitivity of the flux integral). Then if $\mathcal{M}_{\rm total} = \mathcal{M}_1 \cap \mathcal{M}_2$ "captures" the magnetic $\hat{p}$-brane, the flux $\Phi_{\mathcal{M}_{\rm total}}$ will equal the magnetic charge $Q_{\rm m}$ of the $\hat{p}$-brane; thus the discrepancy in the phase factor (7.9) will be simply $\exp({\rm i}Q_{\rm e}Q_{\rm m})$. Requiring this to equal unity gives,[53] in strict analogy to the ordinary case of electric and magnetic particles in $D = 4$, the Dirac quantisation condition

$$Q_{\rm e}Q_{\rm m} = 2\pi n \ , \qquad n \in \mathbb{Z} \ . \tag{7.11}$$

The charge quantisation condition (7.11) is almost, but not quite, the full story. In deriving (7.11), we have not taken into account the p-form character of the charges. Taking this into account shows that the phase in (7.9) vanishes for a measure-zero set of configurations of the electric and magnetic branes.[55] This is easiest to explain in a simplified case where the electric and magnetic branes are kept in static flat configurations, with the electric p-brane oriented along the directions $\{x^{M_1} \dots x^{M_p}\}$. The phase factor (7.9) then becomes $\exp({\rm i}Q_{\rm e} \oint_{\mathcal{W}} A_{M_1 \dots M_p R}\, \partial x^R/\partial\sigma)$, where σ is an ordering parameter for the closed sequence of deformations $\mathcal{W}$. In making this deformation sequence, we recall from the above discussion that one should restrict the deformations to preserve the asymptotic orientation of the deformed p-brane. For simplicity, one may simply consider moving the electric p-brane by parallel transport around the magnetic $\hat{p}$-brane in a closed loop. The accrued phase factor is invariant under gauge transformations of the potential $A_{[p+1]}$. This makes it possible to simplify the discussion by making use of a specially chosen gauge. Note that magnetic $\hat{p}$-branes have purely transverse field strengths like (2.27b); there is accordingly a gauge in which the gauge potential $A_{[p+1]}$ is also purely transverse, *i.e.* it vanishes whenever any of its indices point along a worldvolume direction of the magnetic $\hat{p}$-brane. Consideration of more general deformation sequences yields the same result.[55]

Now one can see how the Dirac-insensitive configurations arise: the phase in (7.9) vanishes whenever there is even a partial alignment between the electric and the magnetic branes, *i.e.* when there are shared worldvolume directions between the two branes. This measure-zero set of Dirac-insensitive configurations may be simply characterised in terms of the p and $\hat{p}$ charges themselves by the condition $Q^{\rm el}_{[p]} \wedge Q^{\rm mag}_{[\hat{p}]} = 0$. For such configurations, one obtains no Dirac quantisation condition. To summarise, one may incorporate this orientation restriction into the Dirac quantisation condition (7.11) by writing a $(p+\hat{p})$-form quantisation condition

$$Q^{\rm el}_{[p]} \wedge Q^{\rm mag}_{[\hat{p}]} = 2\pi n \frac{Q^{\rm el}_{[p]} \wedge Q^{\rm mag}_{[\hat{p}]}}{|Q^{\rm el}_{[p]}|\,|Q^{\rm mag}_{[\hat{p}]}|} \ , \qquad n \in \mathbb{Z} \ , \tag{7.12}$$

which reduces to (7.11) for all except the Dirac-insensitive set of configurations.

7.3 Charge quantisation conditions and dimensional reduction

The existence of Dirac-insensitive configurations may seem to be of only peripheral importance, given that they constitute only a measure-zero subset of the total set of asymptotic brane configurations. However, their relevance becomes more clear when one considers the relations existing between the p-form charges under dimensional reduction. Let us recall the relations (5.11) between the field strengths in different dimensions. Now, the electric and magnetic charges carried by branes in D dimensions take the forms

$$Q_{\rm e} = \int \left(e^{\vec{c}\cdot\vec{\phi}}\, {}^*F + \kappa(A) \right) \tag{7.13a}$$

$$Q_{\rm m} = \int \tilde{F} \ , \tag{7.13b}$$

where $\tilde{F} = dA$, $F = \tilde{F} +$ (Chern-Simons modifications) (*i.e.* modifications involving lower-order forms arising in the dimensional reduction similar to those in the $D = 9$ case (5.41)) and $\vec{c}$ is the dilaton vector corresponding to F in the dimensionally-reduced action (5.13). The term $\kappa(A)$ in (7.13a) is the analogue of the term $\frac{1}{2}A_{[3]} \wedge F_{[4]}$ in (1.3). From the expressions (5.11) for the reduced field strengths and their duals, one obtains the following relations between the original charges in $D = 11$ and those in the reduced theory:

Table 3: Relations between Q^{11} and Q^D

	$F_{[4]}$	$F^i_{[3]}$	$F^{ij}_{[2]}$	$F^{ijk}_{[1]}$
Electric $Q^{11}_{\rm e} =$	$Q^D_{\rm e}\, V$	$Q^D_{\rm e}\, \frac{V}{L_i}$	$Q^D_{\rm e}\, \frac{V}{L_i\, L_j}$	$Q^D_{\rm e}\, \frac{V}{L_i\, L_j\, L_k}$
Magnetic $Q^{11}_{\rm m} =$	$Q^D_{\rm m}$	$Q^D_{\rm m}\, L_i$	$Q^D_{\rm m}\, L_i\, L_j$	$Q^D_{\rm m}\, L_i\, L_j\, L_k$

where $L_i = \int dz^i$ is the compactification period of the reduction coordinate z^i and $V = \int d^{11-D}z = \prod_{i=1}^{11-D} L_i$ is the total compactification volume. Note that the factors of L_i cancel out in the various products of electric and magnetic charges only for charges belonging to the same field strength in the reduced dimension D.

Now consider the quantisation conditions obtained between the various dimensionally reduced charges shown in Table 3. We need to consider the various schemes possible for dimensional reduction of dual pairs of (p,$\hat{p}$) branes. We have seen that for single-element brane solutions, there are two basic schemes, as explained in Section 5: diagonal, which involves reduction on a worldvolume coordinate, and vertical, which involves reduction on a transverse coordinate after preparation by "stacking up" single-center solutions so as to generate a transverse-space translation invariance needed for the dimensional reduction.

For the dimensional reduction of a solution containing two elements, there are then four possible schemes, depending on whether the reduction coordinate z belongs to the worldvolume or to the transverse space of each brane. For an electric/magnetic pair, we have the following four reduction possibilities: diagonal/diagonal, diagonal/vertical, vertical/diagonal and vertical/vertical. Only the mixed cases will turn out to preserve Dirac sensitivity in the lower dimension after reduction.

This is most easily illustrated by considering the diagonal/diagonal case, for which z belongs to the worldvolumes of both branes. With such a shared worldvolume direction, one has clearly fallen into the measure-zero set of Dirac-insensitive configurations with $Q^{\rm el}_{[p]}\wedge Q^{\rm mag}_{[\hat p]}=0$ in the higher dimension D. Correspondingly, in $(D-1)$ dimensions one finds that the diagonally reduced electric $(p-1)$ brane is supported by an $n=p+1$ form field strength, but the diagonally reduced magnetic $(\hat p-1)$ brane is supported by an $n=p+2$ form; since only branes supported by the same field strength can have a Dirac quantisation condition, this diagonal/diagonal reduction properly corresponds to a Dirac-insensitive configuration.

Now consider the mixed reductions, *e.g.* diagonal/vertical. In performing a vertical reduction of a magnetic $\hat p$-brane by stacking up an infinite deck of single-center branes in order to create the $\mathbb{R}$ translational invariance necessary for the reduction, the total magnetic charge will clearly diverge. Thus, in a vertical reduction it is necessary to reinterpret the magnetic charge $Q_{\rm m}$ as a charge density per unit z compactification length. Before obtaining the Dirac quantisation condition in the lower dimension, it is necessary to restore a gravitational-constant factor of κ^2 that should properly have appeared in the quantisation conditions (7.11, 7.12). As one may verify, the electric and magnetic charges as defined in (1.3, 1.4) are not dimensionless. Thus, (7.11) in $D=11$ should properly have been written $Q_{\rm e}Q_{\rm m}=2\pi\kappa_{11}^2 n$. If one lets the compactification length be denoted by L in the D-dimensional theory prior to dimensional reduction, then one obtains a Dirac phase $\exp({\rm i}\kappa_{D-1}^{-2}Q_{\rm e}Q_{\rm m}L)$. This fits precisely, however, with another aspect of dimensional reduction: the gravitational constants in dimensions D and $D-1$ are related by $\kappa_D^2=L\kappa_{D-1}^2$. Thus, in dimension $D-1$ one obtains the expected quantisation condition $Q_{\rm e}Q_{\rm m}=2\pi\kappa_{D-1}^2 n$. Note, correspondingly, that upon making a mixed diagonal/vertical reduction, the electric and magnetic branes remain dual to each other in the lower dimension, supported by the same $n = p-1+2 = p+1$ form field strength. The opposite mixed vertical/diagonal reduction case goes similarly, except that the dual branes are then supported by the same $n = p+2$ form field strength.

In the final case of vertical/vertical reduction, Dirac sensitivity is lost in the reduction, not owing to the orientation of the branes, but because in this case both the electric and the magnetic charges need to be interpreted as densities per unit compactification length, and so one obtains a phase $\exp(\mathrm{i}\kappa_D^{-2}Q_{\mathrm{e}}Q_{\mathrm{m}}L^2)$. Only one factor of L is absorbed into κ_{D-1}^2, and one has $\lim_{L\to 0}L^2/\kappa_D^2 = 0$. Correspondingly, the two dimensionally reduced branes in the lower dimension are supported by different field strengths: an $n = p+2$ form for the electric brane and an $n = p+1$ form for the magnetic brane.

Thus, there is a perfect accord between the structure of the Dirac quantisation conditions for p-form charges in the various supergravity theories related by dimensional reduction. The existence of Dirac-insensitive configurations plays a central rôle in establishing this accord, even though they represent only a subset of measure zero from the point of view of the higher-dimensional theory.

Another indication of the relevance of the Dirac-insensitive configurations is the observation [55] that all the intersecting-brane solutions with some degree of preserved supersymmetry, as considered in Section 6, correspond to Dirac-insensitive configurations. This may immediately be seen in such solutions as the $2 \perp 5(1)$ solution (6.10), but it is also true for solutions involving pp wave and Taub-NUT elements.

7.4 Counting p-branes

As we have seen at the classical level, the classifying symmetry for solutions in a given scalar vacuum, specified by the values of the scalar moduli, is the linearly-realized isotropy symmetry H given in Table 2. When one takes into account the Dirac quantisation condition, this classifying symmetry becomes restricted to a discrete group, which clearly must be a subgroup of the corresponding $G(\mathbb{Z})$ duality group, so in general one seeks to identify the group $G(\mathbb{Z})\cap H$. The value of this intersection is modulus-dependent, showing that the homogeneity of the G/H coset space is broken at the quantum level by the quantisation condition. Classically, of course, the particular point on the vacuum manifold G/H corresponding to the scalar moduli can be changed by application of a transitively-acting G transformation, for example with a group element g. Correspondingly, the isotropy subgroup H moves by conjugation with g,

$$H \longrightarrow gHg^{-1} \ . \tag{7.14}$$

The discretized duality group $G(\mathbb{Z})$, on the other hand, does not depend upon the moduli. This is because the modulus dependence cancels out in the "canonical" charges that we have defined in Eq. (7.13). One way to see this is to use the relations between charges in different dimensions given in Table 3, noting that there are no scalar moduli in $D = 11$, so the modulus-independent relations of Table 3 imply that the lower-dimensional charges (7.13) do not depend on the moduli.[p]

Another way to understand this is by comparison with ordinary Maxwell electrodynamics, where an analogous charge would be that derived from the action $I_{\rm Max} = -{}^1\!/_{(4e^2)}\int F^{\rm can}_{\mu\nu}F^{{\rm can}\,\mu\nu}$, corresponding to a covariant derivative $D_\mu = \partial_\mu + {\rm i}A^{\rm can}_\mu$. This is analogous to our dimensionally reduced action (5.13) from which the charges (7.13) are derived, because the modulus factors $e^{\vec{c}\cdot\vec{\phi}_\infty}$ appearing in (5.13) (together with the rest of the $\vec{\phi}$ dilatonic scalar dependence) play the rôles of coupling constant factors like e^{-2}. If one wants to compare this to the "conventional" charges defined with respect to a conventional gauge potential $A^{\rm conv}_\mu = e^{-1}A^{\rm can}_\mu$, for which the action is $-{}^1\!/_4\int F^{\rm conv}_{\mu\nu}F^{{\rm conv}\,\mu\nu}$, then the canonical and conventional charges obtained *via* Gauss's law surface integrals are related by

$$Q_{\rm can} = \frac{1}{2e^2}\int d^2\Sigma^{ij}\epsilon_{ijk}F^{{\rm can}\,0k} = \frac{1}{2e}\int d^2\Sigma^{ij}\epsilon_{ijk}F^{{\rm conv}\,0k} = \frac{1}{e}Q_{\rm conv}\ . \qquad (7.15)$$

Thus, in the Maxwell electrodynamics case, the dependence on the electric charge unit e drops out in $Q_{\rm can}$, although the conventional charge $Q_{\rm conv}$ scales proportionally to e. The modulus independence of the charges (7.13) works in a similar fashion. Then, given that the discretised quantum duality group $G(\mathbb{Z})$ is *defined* by the requirement that it map the set of Dirac-allowed charges onto itself, it is evident that the group $G(\mathbb{Z})$, referred to the canonical charges (7.13), does not depend on the moduli.

As a consequence of the different modulus dependences of H and of $G(\mathbb{Z})$, it follows that the size of the intersection group $G(\mathbb{Z}) \cap H$ is dependent on the moduli. The analogous feature in ordinary Maxwell theory is that a true duality symmetry of the theory only arises when the electric charge takes the value $e = 1$ (in appropriate units), since the duality transformation maps $e \to e^{-1}$. Thus, the value $e = 1$ is a distinguished value.

[p]Note that the compactification periods L_i appearing in Table 3 have values that may be adjusted by convention. These should not be thought of as determining the geometry of the compactifying internal manifold, which is determined instead by the scalar moduli. Thus, the relations of Table 3 imply the independence of the canonically-defined charges from the physically relevant moduli.

The distinguished point on the scalar vacuum manifold for general supergravity theories is the one where all the scalar moduli vanish. This is the point where $G(\mathbb{Z}) \cap H$ is maximal. Let us return to our $D = 8$ example to help identify what this group is. In that case, for the scalars (σ, χ), we may write out the transformation in detail using (7.3):

$$\begin{aligned} e^{-\sigma} &\longrightarrow (d + c\chi)^2 e^{-\sigma} + c^2 e^{\sigma} \\ \chi e^{-\sigma} &\longrightarrow (d + c\chi)(b + a\chi)e^{-\sigma} + ace^{\sigma} \ . \end{aligned} \tag{7.16}$$

Requiring $a, b, c, d \in \mathbb{Z}$ and also that the modulus point $\sigma_\infty = \chi_\infty = 0$ be left invariant, we find only two transformations: the identity and the transformation $a = d = 0, b = -1, c = 1$, which maps σ and χ according to

$$\begin{aligned} e^{-\sigma} &\longrightarrow e^{\sigma} + \chi^2 e^{-\sigma} \\ \chi e^{-\sigma} &\longrightarrow -\chi e^{-\sigma} \ . \end{aligned} \tag{7.17}$$

Thus, for our truncated system, we find just an S_2 discrete symmetry as the quantum isotropy subgroup of $\mathrm{SL}(2, \mathbb{Z})$ at the distinguished point on the scalar vacuum manifold. This S_2 is the natural analogue of the S_2 symmetry that appears in Maxwell theory when $e = 1$.

In order to aid in identifying the pattern behind this $D = 8$ example, suppose that the zero-form gauge potential χ is small, and consider the S_2 transformation to lowest order in χ. To this order, the transformation just flips the signs of σ and χ. Acting on the field strengths $(F_{[4]}, G_{[4]})$, one finds

$$(F_{[4]}, G_{[4]}) \longrightarrow (-G_{[4]}, F_{[4]}) \ . \tag{7.18}$$

One may again check (in fact to all orders, not just to lowest order in χ) that (7.18) maps the field equation for $F_{[4]}$ into the corresponding Bianchi identity:

$$\nabla_M (e^{\sigma} F^{MNPQ} + \chi {}^*F^{MNPQ}) \longrightarrow -\nabla_M {}^*F^{MNPQ} \ . \tag{7.19}$$

Considering this S_2 transformation to lowest order in the zero-form χ has the advantage that the sign-flip of ϕ may be "impressed" upon the $\vec{a}$ dilaton vector for $F_{[4]}$: $\vec{a} \to -\vec{a}$. The general structure of such $G(\mathbb{Z}) \cap H$ transformations will be found by considering the impressed action of this group on the dilaton vectors.

Now consider the $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3)$ sector of the $D = 8$ scalar manifold, again with the moduli set to the distinguished point on the scalar manifold. To lowest order in zero-form gauge potentials, the action of $\mathrm{SL}(3, \mathbb{Z}) \cap H$ may similarly by impressed upon the 3-form dilaton vectors, causing in this case a *permutation* of the $\vec{a}_i$, generating for the $D = 8$ case overall the discrete group

$S_3 \times S_2$. Now that we have a bit more structure to contemplate, we can notice that the $G(\mathbb{Z}) \cap H$ transformations *leave the* $(\vec{a}, \vec{a}_i)$ *dot products invariant*.[57]

The invariance of the dilaton vectors' dot products prompts one to return to the algebra (5.16) of these dot products and see what else we may recognize in it. Noting that the duality groups given in Table 2 for the higher dimensions D involve $\mathrm{SL}(N, \mathbb{R})$ groups, we recall that the *weight vectors* $\vec{h}_i$ of the fundamental representation of $\mathrm{SL}(N, \mathbb{R})$ satisfy

$$\vec{h}_i \cdot \vec{h}_j = \delta_{ij} - \frac{1}{N} \ , \qquad \qquad \sum_{i=1}^{N} \vec{h}_i = 0 \ . \tag{7.20}$$

These relations are precisely those satisfied by $\frac{\pm 1}{\sqrt{2}}\vec{a}$ and $\frac{1}{\sqrt{2}}\vec{a}_i$, corresponding to the cases $N = 2$ and $N = 3$. This suggests that the action of the maximal $G(\mathbb{Z}) \cap H$ group (*i.e.* for scalar moduli set to the distinguished point on the scalar manifold) may be identified in general with the symmetry group of the set of fundamental weights for the corresponding supergravity duality group G as given in Table 2. The symmetry group of the fundamental weights is the *Weyl group* [57] of G, so the action of the maximal $G(\mathbb{Z}) \cap H$ p-brane classifying symmetry is identified with that of the Weyl group of G.

As one proceeds down through the lower-dimensional cases, where the supergravity symmetry groups shown in Table 2 grow in complexity, the above pattern persists:[57] in all cases, the action of the maximal classifying symmetry $G(\mathbb{Z}) \cap H$ may be identified with the Weyl group of G. This is then the group that *counts* the distinct p-brane solutions[q] of a given type (4.6), subject to the Dirac quantisation condition and referred to the distinguished point on the scalar modulus manifold. For example, in $D = 7$, where from Table 2 one sees that $G = \mathrm{SL}(5, \mathbb{R})$ and $H = \mathrm{SO}(5)$, one finds that the action of $G(\mathbb{Z}) \cap H$ is equivalent to that of the discrete group S_5, which is the Weyl group of $\mathrm{SL}(5, \mathbb{R})$. In the lower-dimensional cases shown in Table 2, the discrete group $G(\mathbb{Z}) \cap H$ becomes less familiar, and is most simply described as the Weyl group of G.

From the analysis of the Weyl-group duality multiplets, one may tabulate [57] the multiplicities of p-branes residing at each point of the plot given in Figure 6. For supersymmetric p-branes arising from a set of N participating field strengths $F_{[n]}$, corresponding to $\Delta = 4/N$ for the dilatonic scalar coupling, one finds the multiplicities given in Table 4. By combining these duality multiplets together with the diagonal and vertical dimensional reduction fam-

[q] Of course, these solutions must also fall into supermultiplets with respect to the unbroken supersymmetry; the corresponding supermultiplet structures have been discussed in Ref.[60]

ilies discussed in Sections 5 and 5.3, the full set of $p \leq (D-3)$ branes shown in Figure 6 becomes "welded" together into one overall symmetrical structure.

Table 4: Examples of p-brane Weyl-group multiplicities

		D						
$F_{[n]}$	Δ	10	9	8	7	6	5	4
$F_{[4]}$	4	1	1	2				
$F_{[3]}$	4	1	2	3	5	10		
$F_{[2]}$	4	1	1+2	6	10	16	27	56
	2		2	6	15	40	135	756
	$4/3$						45	2520
$F_{[1]}$	4		2	8	20	40	72	126
	2			12	60	280	1080	3780
	$4/3$					480	4320	30240+2520

7.5 The charge lattice

For the electric and magnetic BPS brane solutions supported by a given field strength, we have seen above that the Dirac charge quantisation condition (7.12) implies that, given a certain minimum "electric" charge (7.13a), the allowed set of magnetic charges is determined. Then, taking the minimum magnetic charge from this set, the argument may be turned around to show that the set of allowed electric charges is given by integer multiples of the minimum electric charge. This argument does not directly establish, however, what the minimum electric charge is, *i.e.* the value of the charge unit. This cannot be established by use of the Dirac quantisation condition alone.

There are other tools, however, that one can use to fix the charge lattice completely. To do so, we shall need to exploit the existence of certain special "unit-setting" brane types, and also to exploit fully the consequences of the assumption that the $G(\mathbb{Z})$ duality symmetry remains exactly valid at the quantum level. We have already encountered one example of a "unit-setting" brane in subsection 6.2, where we encountered the pp wave/Taub-NUT pair of $D = 11$ solutions. We saw there that the Taub-NUT solution (6.7) is nonsingular provided that the coordinate ψ is periodically identified with period $L = 4\pi k$, where k is the charge-determining parameter in the 3-dimensional harmonic function $H(y) = 1 + k/(|y|)$. Upon dimensional reduction down to $D = 10$, one obtains a magnetic 6-brane solution, with a charge *classically* discretised to take a value in the set

$$Q_{\rm m} = rL \ , \qquad r \in \mathbb{Z} \ . \tag{7.21}$$

Given these values for the magnetic charge, the $D = 10$ Dirac quantisation condition

$$Q_{\rm e} Q_{\rm m} = 2\pi\kappa_{10}^2 n \ , \qquad n \in \mathbb{Z} \ , \tag{7.22}$$

or, equivalently, as we saw in subsection 6.2, the quantisation of $D = 11$ pp wave momentum in the compact ψ direction, gives an allowed set of electric charges

$$Q_{\rm e} = \frac{2\pi\kappa_{10}^2}{L} n \ , \qquad n \in \mathbb{Z} \ . \tag{7.23}$$

Thus, the requirement that magnetic $D = 10$ 6-branes oxidise up to non-singular Taub-NUT solutions in $D = 11$ fully determines the 6-brane electric and magnetic charge units and not just the product of them which occurs in the Dirac quantisation condition

If one assumes that the $G(\mathbb{Z})$ duality symmetries remain strictly unbroken at the quantum level, then one may relate the 6-brane charge units to those of other BPS brane types.[r] In doing so, one must exploit the fact that brane solutions with Poincaré worldvolume symmetries may be dimensionally reduced down to lower dimensions, where the duality groups shown in Table 2 grow larger. In a given dimension D, the $G(\mathbb{Z})$ duality symmetries only rotate between p-branes of the same worldvolume dimension, supported by the same kind of field strength, as we have seen from our discussion of the Weyl-group action on p-branes given in subsection 7.4. Upon reduction down to dimensions $D_{\rm red} < D$, however, the solutions descending from an original p-brane in D dimensions are subject to a larger $G(\mathbb{Z})$ duality symmetry, and this can be used to rotate a descendant brane into descendants of p'-branes for various values of p'. Dimensional oxidation back up to D dimensions then completes the link, establishing relations *via* the duality symmetries between various BPS brane types which can be supported by different field strengths, including field strengths of different rank.[61] This link may be used to establish relations between the charge units for the various p-form charges of differing rank, even though the corresponding solutions are Dirac-insensitive to each each other.

Another charge-unit-setting BPS brane species occurs in the $D = 10$ type IIB theory. This theory has a well-known difficulty with the formulation of a satisfactory action, although its field equations are perfectly well-defined. The difficulty in formulating an action arise from the presence of a self-dual 5-form

[r] For details of the duality relations between charge units for different p-branes, see Ref.[55]

field strength, $H_{[5]} = {}^*H_{[5]}$. The corresponding electrically and magnetically charged BPS solutions are 3-branes, and, owing to the self-duality condition, these solutions are actually *dyons*, with a charge vector at 45° to the electric axis. We shall consider the type IIB theory in some more detail in Section 8; for now, it will be sufficient for us to note that the dyonic 3-branes of $D = 10$ type IIB theory are also a unit-setting brane species.[55] The unit-setting property arises because of a characteristic property of the Dirac-Schwinger-Zwanziger quantisation condition for dyons in dimensions $D = 4r + 2$: for dyons $(Q_{\rm e}^{(1)}, Q_{\rm m}^{(1)})$, $(Q_{\rm e}^{(2)}, Q_{\rm m}^{(2)})$, this condition is *symmetric*:[55]

$$Q_{\rm e}^{(1)}Q_{\rm m}^{(2)} + Q_{\rm e}^{(2)}Q_{\rm m}^{(1)} = 2\pi\kappa_{4r+2}^2 n\ , \qquad n \in \mathbb{Z}\ , \tag{7.24}$$

unlike the more familiar antisymmetric DSZ condition that is obtained in dimensions $D = 4r$. The symmetric nature of (7.24) means that dyons may be Dirac-sensitive to others of their own type,[s] quite differently from the antisymmetric cases in $D = 4r$ dimensions. For the 45° dyonic 3-branes, one thus obtains the quantisation condition

$$|Q_{[3]}| = n\sqrt{\pi}\kappa_{\rm IIB}\ , \qquad n \in \mathbb{Z} \tag{7.25}$$

where $\kappa_{\rm IIB}$ is the gravitational constant for the type IIB theory. Then, using duality symmetries, one may relate the $\sqrt{\pi}\kappa_{\rm IIB}$ charge unit to those of other supergravity R–R charges.

Thus, using duality symmetries together with the pp wave/Taub-NUT and self-dual 3-brane charge scales, one may determine the charge-lattice units for all BPS brane types.[61,55]. It is easiest to express the units of the resulting overall charge lattice by making a specific choice for the compactification periods. If one lets all the compactification periods L_i be equal,

$$L_i = L_{\rm IIB} = L = (2\pi\kappa_{11}^2)^{\frac{1}{9}}\ , \tag{7.26}$$

then the electric and magnetic charge-lattice units for rank-n field strengths in dimension D are determined to be [55]

$$\Delta Q_{\rm e} = L^{D-n-1}\ , \qquad \Delta Q_{\rm m} = L^{n-1}\ . \tag{7.27}$$

8 Local versus active dualities

The proper interpretation of the discretised Cremmer-Julia $G(\mathbb{Z})$ duality symmetry at the level of supergravity theory is subject to a certain amount of

[s] They will be Dirac-sensitive provided one of them is slightly rotated so as to avoid having any common worldvolume directions with the other, in order to avoid having a Dirac-insensitive configuration as discussed in subsection 7.3.

debate, but at the level of string theory the situation becomes more clear. In any dimension D, there is a subgroup of $G(\mathbb{Z})$ that corresponds to *T duality,* which is a perturbative symmetry holding order-by-order in the string loop expansion. T duality [62] consists of transformations that invert the radii of a toroidal compactification, under which quantised string oscillator modes and string winding modes become interchanged. Aside from such a relabeling, however, the overall string spectrum remains unchanged. Hence, T duality needs to be viewed as a *local* symmetry in string theory, *i.e.* string configurations on compact manifolds related by T duality are *identified.* Depending on whether one considers $(D-3)$ branes to be an unavoidable component of the spectrum, the same has also been argued to be the case at the level of the supergravity effective field theory.[63] The well-founded basis, in string theory at least, for a local interpretation of the T duality subgroup of $G(\mathbb{Z})$ has led subsequently to the hypothesis [58,59] that the full duality group $G(\mathbb{Z})$ should be given a local interpretation: sets of string solutions and moduli related by $G(\mathbb{Z})$ transformations are to be treated as equivalent descriptions of a single state. This local interpretation of the $G(\mathbb{Z})$ duality transformations is similar to that adopted for general coordinate transformations viewed *passively,* according to which, *e.g.,* flat space in Cartesian or in Rindler coordinates is viewed as one and the same solution.

As with general coordinate transformations, however, duality symmetries may occur in several different guises that are not always clearly distinguished. As one can see from the charge lattice discussed in subsection 7.5, there is also a $G(\mathbb{Z})$ covariance of the set of charge vectors for *physically inequivalent* BPS brane solutions. In the discussion of subsection 7.5, we did not consider in detail the action of $G(\mathbb{Z})$ on the moduli, because, as we saw in subsection 7.4, the canonically-defined charges (7.13) are in fact modulus-independent.

Since the dilatonic and axionic scalar moduli determine the coupling constants and vacuum θ-angles of the theory, these quantities should be fixed when quantising about a given vacuum state of the theory. This is similar to the treatment of asymptotically flat spacetime in gravity, where the choice of a particular asymptotic geometry is necessary in order to establish the "vacuum" with respect to which quantised fluctuations can be considered.

Thus, in considering physically-inequivalent solutions, one should compare solutions with the same asymptotic values of the scalar fields. When this is done, one finds that solutions carrying charges (7.13) related by $G(\mathbb{Z})$ transformations generally have differing mass densities. Since the standard Cremmer-Julia duality transformations, such as those of our $D=8$ example in subsection 7.1, commute with P^0 time translations and so necessarily preserve mass densities, it is clear that the BPS spectrum at fixed scalar moduli cannot form a multiplet under the standard Cremmer-Julia $G(\mathbb{Z})$ duality symmetry. This conclusion is in any case unavoidable, given the local interpretation adopted for the standard duality transformations as discussed above: once one has identified solution/modulus sets under the standard $G(\mathbb{Z})$ duality transformations, one cannot then turn around and use the same $G(\mathbb{Z})$ transformations to generate inequivalent solutions.

Thus, the question arises: is there any spectrum-generating symmetry lying behind the apparently $G(\mathbb{Z})$ invariant charge lattices of inequivalent solutions that we saw in subsection 7.5? At least at the classical level, and for single-charge (*i.e.* $\Delta = 4$) solutions, the answer [64] turns out to be 'yes.' We shall illustrate the point using type IIB supergravity as an example.[t]

8.1 The symmetries of type IIB supergravity

Aside from the difficulties arising from the self-duality condition for the 5-form field strength $H_{[5]}$, the equations of motion of the bosonic fields of the IIB theory may be derived from the action

$$\begin{aligned} I_{10}^{\rm IIB} \;=\; \int d^{10}x[eR+\tfrac{1}{4}e\,{\rm tr}(\nabla_\mu\mathcal{M}^{-1}\,\nabla^\mu\mathcal{M})-\tfrac{1}{12}e\,H_{[3]}^T\,\mathcal{M}\,H_{[3]}-\tfrac{1}{240}e\,H_{[5]}^2 \\ -\tfrac{1}{2\sqrt{2}}\epsilon_{ij}\,{}^*(B_{[4]}\wedge dA_{[2]}^{(i)}\wedge dA_{[2]}^{(j)})]\ . \end{aligned} \tag{8.1}$$

The 5-form self-duality condition

$$H_{[5]} = {}^*H_{[5]} \tag{8.2}$$

may be handled in the fashion of Ref.[66], being imposed by hand as an extra constraint on the field equations obtained by varying (8.1). This somewhat hybrid procedure will be sufficient for our present purposes.

The matrix $\mathcal{M}$ in (8.1) contains two scalar fields: a dilatonic scalar ϕ which occurs nonlinearly through its exponential, and an axionic scalar χ, which may also be considered to be a zero-form gauge potential; explicitly, one has

$$\mathcal{M} = \begin{pmatrix} e^{-\phi}+\chi^2\,e^{\phi} & \chi\,e^{\phi} \\ \chi\,e^{\phi} & e^{\phi} \end{pmatrix}\ . \tag{8.3}$$

The doublet $H_{[3]}$ contains the field strengths of the 2-form gauge potentials $A_{[2]}$:

$$H_{[3]} = \begin{pmatrix} dA_{[2]}^{(1)} \\ dA_{[2]}^{(2)} \end{pmatrix}\ . \tag{8.4}$$

[t]For a detailed discussion of SL(2, $\mathbb{R}$) duality in type IIB supergravity, see Ref.[65]

The action (8.1) is invariant under the SL(2, IR) transformations

$$H_{[3]} \longrightarrow (\Lambda^T)^{-1} H_{[3]} \ , \qquad \mathcal{M} \longrightarrow \Lambda \, \mathcal{M} \, \Lambda^T \ , \tag{8.5}$$

where the SL(2, IR) parameter matrix is

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \tag{8.6}$$

and the SL(2, IR) constraint is $ad - bc = 1$. If one defines the complex scalar field $\tau = \chi + i\, e^{-\phi}$, then the transformation on $\mathcal{M}$ can be rewritten as the fractional linear transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \ . \tag{8.7}$$

Note that since $H_{[5]}$ is a singlet under SL(2, IR), the self-duality constraint (8.2), which is imposed by hand, also preserves the SL(2, IR) symmetry. Since this SL(2, IR) transformation rotates the doublet $A_{2]}$ of electric 2-form potentials amongst themselves, this is an "electric-electric" duality, as opposed to the "electric-magnetic" duality discussed in the $D = 8$ example of subsection 7.1. Nonetheless, similar issues concerning duality multiplets for a fixed scalar vacuum arise in both cases.

There is one more symmetry of the equations of motion following from the action (8.1). This is a rather humble symmetry that is not often remarked upon, but which will play an important role in constructing active SL(2, IR) duality transformations for the physically distinct BPS string and 5-brane multiplets of the theory. As for pure source-free Einstein theory, the action (8.1) transforms homogeneously as λ^3 under the following scaling transformations:

$$g_{\mu\nu} \longrightarrow \lambda^2 \, g_{\mu\nu} \ , \qquad A^{(i)}_{[2]} \longrightarrow \lambda^2 \, A^{(i)}_{[2]} \ , \qquad H_{[5]} \longrightarrow \lambda^4 \, H_{[5]} \ ; \tag{8.8}$$

note that the power of λ in each field's transformation is equal to the number of indices it carries, and, accordingly, the scalars ϕ and χ are not transformed. Although the transformation (8.8) does not leave the action (8.1) invariant, the λ^3 homogeneity of this scaling for all terms in the action is sufficient to produce a symmetry of the IIB equations of motion. It should be noted that the SL(2, IR) electric-magnetic duality of the $D = 8$ example given in subsection 7.1 shares with the transformation (8.8) the feature of being a symmetry only of the equations of motion, and not of the action.

The $SL(2, \mathrm{IR})$ transformations map solutions of (8.1) into other solutions. We shall need to consider in particular the action of these transformations on the charges carried by solutions. From the equations of motion of the 3-form field strength $H_{[3]}$ in (8.1),

$$d\,{}^*(MH_{[3]}) = -\frac{1}{\sqrt{2}}H_{[5]} \wedge \Omega H_{[3]} \ , \tag{8.9}$$

where Ω is the $SL(2,\mathbb{R})$-invariant tensor

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{8.10}$$

one finds that the following two-component quantity is conserved:

$$Q_{\rm e} = \int \left({}^*(MH_{[3]}) + \frac{1}{3\sqrt{2}}\Omega(2B_{[4]} \wedge H_{[3]} - H_{[5]} \wedge A_{[2]}) \right) \ . \tag{8.11}$$

Under an $SL(2,\mathbb{R})$ transformation, $Q_{\rm e}$ transforms covariantly as a doublet: $Q_{\rm e} \to \Lambda Q_{\rm e}$.

By virtue of the Bianchi identities for the 3-form field strength, one has in addition a topologically-conserved magnetic charge doublet,

$$Q_{\rm m} = \int H_{[3]} \ , \tag{8.12}$$

which transforms under $SL(2,\mathbb{R})$ as $Q_{\rm m} \to (\Lambda^{\rm T})^{-1}Q_{\rm m}$, *i.e.* contragrediently to $Q_{\rm e}$. The transformation properties of the electric and magnetic charge doublets are just such as to ensure that the Dirac quantization condition $Q_{\rm m}^{\rm T}Q_{\rm e} \in 2\pi\kappa_{\rm IIB}^2\mathbb{Z}$ is $SL(2,\mathbb{R})$ invariant.

The overall effect of this standard $\mathrm{SL}(2,\mathbb{R})$ symmetry on type IIB supergravity solutions may be expressed in terms of its action on the solutions' charges and on the scalar moduli. This group action may be viewed as an automorphism of a vector bundle, with the scalar fields' $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ target manifold as the base space, and the charge vector space as the fiber.

We have seen in our general discussion of charge lattices in subsection 7.5 that the continuous classical Cremmer-Julia symmetries G break down to discretised $G(\mathbb{Z})$ symmetries that map between states on the quantum charge lattice. In the present type IIB case, the classical $\mathrm{SL}(2,\mathbb{R})$ symmetry breaks down to $\mathrm{SL}(2,\mathbb{Z})$ at the quantum level. Taking the basis states of the IIB charge lattice to be

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad\qquad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \tag{8.13}$$

the surviving $\mathrm{SL}(2,\mathbb{Z})$ group will be represented by $\mathrm{SL}(2,\mathbb{R})$ matrices with integral entries.

As we have discussed above, the discretised duality symmetries $G(\mathbb{Z})$ are given a local interpretation in string theory. In the case of the type IIB theory, this is a hypothesis rather than a demonstrated result, because the $\mathrm{SL}(2,\mathbb{Z})$ transformations map between NS–NS and R–R states, and this is a distinctly non-perturbative transformation. Adopting this hypothesis nonetheless, an orbit of the standard $\mathrm{SL}(2,\mathbb{Z})$ transformation reduces to a single point; after making the corresponding identifications, the scalar modulus space becomes the double coset space $\mathrm{SL}(2,\mathbb{Z})\backslash{}^{\mathrm{SL}(2,\mathbb{R})}/\mathrm{SO}(2)$.

8.2 Active duality symmetries

Now let us see how duality multiplets of the physically inequivalent BPS states can occur, even though they will contain states with different mass densities. This latter fact alone tells us that we must include some transformation that acts on the metric. We shall continue with our exploration of the continuous classical $\mathrm{SL}(2,\mathbb{R})$ symmetry of the type IIB theory. Finding the surviving quantum-level $\mathrm{SL}(2,\mathbb{Z})$ later on will be a straightforward matter of restricting the transformations to a subgroup. The procedure starts with a standard $\mathrm{SL}(2,\mathbb{R})$ transformation, which transforms the doublet charges (8.11) in a straightforwardly linear fashion, but which also transforms in an unwanted way the scalar moduli. Subsequent compensating transformations will then have the task of eliminating the unwanted transformation of the scalar moduli, but without changing the "already final" values of the charges. Let us suppose that this initial transformation, with parameter Λ, maps the charge vector and complex scalar modulus (Q,τ_∞) to new values (Q',τ'_∞).

After this initial Λ transformation, one wishes to return the complex scalar modulus τ'_∞ to its original value τ_∞, in order to obtain an overall transformation that does not in the end disturb the complex modulus. To do this, notice that within $\mathrm{SL}(2,\mathbb{R})$ there is a subgroup that leaves a doublet charge vector Q' invariant up to an overall rescaling. This projective stability group of Q' is isomorphic to the **Borel** subgroup of $\mathrm{SL}(2,\mathbb{R})$:

$$\mathbf{Borel} = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\Big|\, a,b\in\mathbb{R} \right\} \ . \qquad (8.14)$$

This standard representation of the $\mathrm{SL}(2,\mathbb{R})$ Borel subgroup clearly leaves the basis charge vector $\boldsymbol{e}_1$ of Eq. (8.13) invariant up to scaling by a. For a general charge vector Q', there will exist a corresponding projective stability subgroup which is isomorphic to (8.14), but obtained by conjugation of (8.14) with an element of $H \cong \mathrm{SO}(2)$. The importance of the Borel subgroup for our present purposes is that it acts *transitively* on the ${}^{G}\!/\!_{H} = {}^{\mathrm{SL}(2,\mathbb{R})}\!/\!_{\mathrm{SO}(2)}$ coset space in which the scalar fields take their values, so this transformation may be used to return the scalar moduli to the original values they had before the Λ transformation.

The next step in the construction is to correct for the unwanted scaling $Q' \to aQ'$ which occurs as a result of the Borel compensating transformation, by use of a further compensating scaling of the form (8.8), $aQ' \to \lambda^2 aQ'$, in which one picks the rigid parameter λ such that $\lambda^2 a = 1$. This almost completes the construction of the active $\mathrm{SL}(2,\mathbb{R})$. For the final step, note that the transformation (8.8) also scales the metric, $g_{\mu\nu} \to \lambda^2 g_{\mu\nu} = a^{-1} g_{\mu\nu}$. Since one does not want to alter the asymptotic metric at infinity, one needs to compensate for this scaling by a final general coordinate transformation, $x^\mu \to x'^\mu = a^{-1/2} x^\mu$.

The overall active $\mathrm{SL}(2,\mathbb{R})$ duality package constructed in this way transforms the charges in a linear fashion, $Q \to \lambda Q'$, in exactly the same way as the standard supergravity Cremmer-Julia $\mathrm{SL}(2,\mathbb{R})$ duality, but now leaving the complex scalar modulus τ_∞ unchanged. This is achieved by a net construction that acts upon the field variables of the theory in a quite nonlinear fashion. This net transformation may be explicitly written by noting that for $\mathrm{SL}(2,\mathbb{R})$ there is an Iwasawa decomposition

$$\Lambda = \tilde{b} h \ , \tag{8.15}$$

where $\tilde{b} \in \mathbf{Borel}_{Q'}$ is an element of the projective stability group of the final charge vector Q' and where $h \in \mathrm{H}_{\tau_\infty}$ is an element of the stability subgroup of τ_∞. Clearly, the Borel transformation that is needed in this construction is just $b = (\tilde{b})^{-1}$, leaving thus a transformation $h \in \mathrm{H}_{\tau_\infty}$ which does not change the complex modulus τ_∞. The compensating scaling transformation t of the form (8.8) and the associated general coordinate transformation also leave the scalar moduli unchanged. The net active $\mathrm{SL}(2,\mathbb{R})$ transformation thus is just $bt\Lambda = th$. Specifically, for $\tau_\infty = \chi_\infty + \mathrm{i}e^{-\phi_\infty}$ and a transformation Λ mapping $Q_{\mathrm{i}} = \begin{pmatrix} p_{\mathrm{i}} \\ q_{\mathrm{i}} \end{pmatrix}$ to $Q_{\mathrm{f}} = \begin{pmatrix} p_{\mathrm{f}} \\ q_{\mathrm{f}} \end{pmatrix} = \Lambda Q_{\mathrm{i}}$, the $h \in H_{\tau_\infty}$ group element is

$$\begin{aligned} h_{\mathrm{fi}} &= V_\infty \begin{pmatrix} \cos\tilde{\theta}_{\mathrm{fi}} & \sin\tilde{\theta}_{\mathrm{fi}} \\ -\sin\tilde{\theta}_{\mathrm{fi}} & \cos\tilde{\theta}_{\mathrm{fi}} \end{pmatrix} V_\infty^{-1} \ , \qquad && \tilde{\theta}_{\mathrm{fi}} = \tilde{\theta}_{\mathrm{f}} - \tilde{\theta}_{\mathrm{i}} \\ \tan\tilde{\theta}_{\mathrm{i}} &= e^{\phi_\infty}(\tan\theta_{\mathrm{i}} - \chi_\infty) \ , && \tan\theta_{\mathrm{i}} = p_{\mathrm{i}}/q_{\mathrm{i}} \ , \end{aligned} \tag{8.16}$$

where the matrix V_∞ is an element of **Borel** that has the effect of moving the scalar modulus from the point $\tau = \mathrm{i}$ to the point τ_∞:

$$V_\infty = e^{-\phi_\infty/2}\begin{pmatrix} 1 & e^{\phi_\infty}\chi_\infty \\ 0 & e^{\phi_\infty} \end{pmatrix} \ . \tag{8.17}$$

The matrix V_∞ appearing here is also the asymptotic limit of a matrix $V(\phi, \chi)$ that serves to factorize the matrix $\mathcal{M}$ given in (8.3), $M = VV^{\mathrm{T}}$. This factorization makes plain the transitive action of the Borel subgroup on the $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset space in which the scalar fields take their values. Note that the matrix $\mathcal{M}$ determines both the scalar kinetic terms and also their interactions with the various antisymmetric-tensor gauge fields appearing in the action (8.1).

The scaling-transformation part of the net active $\mathrm{SL}(2,\mathbb{R})$ construction is simply expressed as a ratio of mass densities,

$$t_{\mathrm{fi}} = \frac{m_{\mathrm{f}}}{m_{\mathrm{i}}} \ , \qquad m_{\mathrm{i}}^2 = Q_{\mathrm{i}}^{\mathrm{T}}\mathcal{M}_\infty^{-1}Q_{\mathrm{i}} \ . \tag{8.18}$$

This expression reflects the fact that the scaling symmetry (8.8) acts on the metric and thus enables the active $\mathrm{SL}(2,\mathbb{R})$ transformation to relate solutions at different mass-density levels $m_{\mathrm{i,f}}$. Since, by contrast, the mass-density levels are invariant under the action of the standard $\mathrm{SL}(2,\mathbb{R})$, it is clear that the two realizations of this group are distinctly different. Mapping between different mass levels, referred to a given scalar vacuum determined by the complex modulus τ_∞, can only be achieved by including the scaling transformation (8.18).

The group composition property of the active $\mathrm{SL}(2,\mathbb{R})$ symmetry needs to be checked in the same fashion as for nonlinear realizations generally, *i.e.* one needs to check that a group operation $\mathcal{O}(\Lambda, Q) = th$ acting on an initial state characterized by a charge doublet Q combines with a second group operation according to the rule

$$\mathcal{O}(\Lambda_2, \Lambda_1 Q)\mathcal{O}(\Lambda_1, Q) = \mathcal{O}(\Lambda_2\Lambda_1, Q) \ . \tag{8.19}$$

One may verify directly that the nonlinear realization given by (8.16, 8.18) does in fact satisfy this composition law, when acting on any of the fields of the type IIB theory.

At the quantum level, the Dirac quantization condition restricts the allowed states of the theory to a discrete charge lattice, as we have seen. The standard $\mathrm{SL}(2,\mathbb{R})$ symmetry thus becomes restricted to a discrete $\mathrm{SL}(2,\mathbb{Z})$ subgroup in order to respect this charge lattice, and the active $\mathrm{SL}(2,\mathbb{R})$ constructed above likewise becomes restricted to an $\mathrm{SL}(2,\mathbb{Z})$ subgroup. This quantum-level discretised group of active transformations is obtained simply by restricting the matrix parameters Λ for a classical active $\mathrm{SL}(2,\mathbb{R})$ transformation so as to lie in $\mathrm{SL}(2,\mathbb{Z})$.

In lower-dimensional spacetime, the supergravity duality groups G shown in Table 2 grow in rank and the structure of the charge orbits becomes progressively more and more complicated, but the above story is basically repeated for an important class of p-brane solutions. This is the class of single-charge solutions, for which the charges Q fall into *highest-weight* representations of G. The duality groups shown in Table 2 are all maximally noncompact, and possess an Iwasawa decomposition generalizing the $\mathrm{SL}(2,\mathbb{R})$ case (8.15):

$$\Lambda = \tilde{b}h \qquad \tilde{b} \in \mathbf{Borel}_Q \ , \ h \in H_{\mathrm{moduli}} \ , \tag{8.20}$$

where $\mathbf{Borel}_Q$ is isomorphic to the Borel subgroup of G. Once again, this subgroup acts transitively on the coset space G/H in which the scalar fields take their values, so this is the correct subgroup to use for a compensating transformation to restore the moduli to their original values in a given scalar vacuum. As in the $\mathrm{SL}(2,\mathbb{R})$ example of the type IIB theory, one may see that this group action is transitive by noting that the matrix $\mathcal{M}$ (8.3) which governs the scalar kinetic terms and interactions can be parameterized in the form $\mathcal{M} = VV^{\#}$, where V is an element of the Borel subgroup. The operation $\#$ here depends on the groups G and H in question; in spacetime dimensions $D \geq 4$ we have

$$V^{\#} = \begin{cases} V^{\mathrm{T}}, & \text{for } H \text{ orthogonal} \\ V^{\dagger}, & \text{for } H \text{ unitary} \\ \Omega V^{\dagger}, & \text{for } H \text{ a USp group.} \end{cases} \tag{8.21}$$

(The $D = 3$ case in which $G = E_{8(+8)}$ and $H = \mathrm{SO}(16)$ needs to be treated as a special case.[67])

Given the above group-theoretical structure, the construction of active G symmetry transformations that preserve the scalar moduli proceeds in strict analogy with the type IIB $\mathrm{SL}(2,\mathbb{R})$ example that we have presented. This construction depends upon the existence of a projective stability group [64,67] of the charge Q that is isomorphic to the Borel subgroup of G. This is the case whenever Q transforms according to a highest-weight representation of G. The BPS brane solutions with this property are the single-charge solutions with $\Delta = 4$. As we have seen in Section 6, BPS brane solutions with $\Delta = 4/N$ can be interpreted as coincident-charge-center cases of intersecting-brane solutions with N elements, each of which would separately be a $\Delta = 4$ solution on its own. The construction of active duality symmetries for such multiple-charge solutions remains an open problem, for they have a larger class of integration constants, representing relative positions and phases of the charge components. Only the asymptotic scalar moduli can be moved transitively by the Borel subgroup of G and, correspondingly, the representations carried by the charges in such multi-charge cases are not of highest-weight type.

The active $G(\mathbb{Z})$ duality constructions work straightforwardly enough at the classical level, but their dependence on symmetries of field equations that are not symmetries of the corresponding actions gives a reason for caution about their quantum durability. This may be a subject where string theory needs to intervene with its famed "miracles." Some of these miracles can be seen in supergravity-level analyses of the persistence of BPS solutions with arbitrary mass scales, despite the presence of apparently threatening quantum corrections,[64] but a systematic way to understand the remarkable identities making this possible is not known. Thus, there still remain some areas where string theory appears to be more clever than supergravity.

9 Non-compact σ-models, null geodesics, and harmonic maps

A complementary approach [68,69,70] to the analysis of brane solutions in terms of the four $D = 11$ elemental solutions presented in Section 6 is to make a dimensional reduction until only overall-transverse dimensions remain, and then to consider the resulting nonlinear σ-model supporting the solution. In such a reduction, all of the worldvolume and relative-transverse coordinates are eliminated, *including the time coordinate,* which is possible because the BPS solutions are all time independent. The two complementary approaches to the analysis of BPS brane solutions may thus be characterised as oxidation up to the top of Figure 6, or reduction down to the left edge Figure 6, *i.e.* reduction down to BPS "instantons," or $p = -1$ branes, with worldvolume dimension $d = 0$.

The $d = 0$ instanton solutions are supported by 1-form field strengths, *i.e.* the derivatives of axionic scalars, $F_{[1]} = d\chi$. Taken together with the dilatonic scalars accumulated in the process of dimensional reduction, these form a *noncompact nonlinear σ-model* with a target manifold G/H', where G is the usual supergravity symmetry group shown in Table 2 for the corresponding (reduced) dimension D but H' is a noncompact form of the modulus little group H shown in Table 2. The difference between the groups H' and H arises because dimensional reduction on the time coordinate introduces extra minus signs, with respect to the usual spatial-coordinate Kaluza-Klein reduction, in "kinetic" terms for scalars descending from vector fields in the $(D + 1)$ dimensional theory including the time dimension. Scalars descending from scalars or from the metric in $(D + 1)$ dimensions do not acquire extra minus signs. The change to the little group H' is also needed for the transformation of field strengths of higher rank, but these need not be considered for our discussion of the BPS instantons. The relevant groups for the noncompact σ-models in dimensions $9 \geq D \geq 3$ are given in Table 5. These should be compared to the standard Cremmer-Julia groups given in Table 2.

Table 5: Symmetries for BPS instanton σ-models.

D	G	H'
9	$\mathrm{GL}(2,\mathbb{R})$	$\mathrm{SO}(1,1)$
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$\mathrm{SO}(2,1)\times\mathrm{SO}(1,1)$
7	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(3,2)$
6	$\mathrm{SO}(5,5)$	$\mathrm{SO}(3,2)\times\mathrm{SO}(5)$
5	$\mathrm{E}_{6(+6)}$	$\mathrm{USP}(4,4)$
4	$\mathrm{E}_{7(+7)}$	$\mathrm{SU}^*(8)$
3	$\mathrm{E}_{8(+8)}$	$\mathrm{SO}^*(16)$

The sector of dimensionally-reduced supergravity that is relevant for the instanton solutions consists just of the transverse-space Euclidean-signature metric and the G/H' σ-model, with an action

$$I_\sigma = \int d^D y\sqrt{g}\left(R - \tfrac{1}{2}G_{AB}(\phi)\partial_i\phi^A\partial_j\phi^B g^{ij}\right) \ , \tag{9.1}$$

where the ϕ^A are σ-model fields taking values in the G/H' target space, G_{AB} is the target-space metric and $g^{ij}(y)$ is the Euclidean-signature metric for the σ-model domain space. The equations of motion following from (9.1) are

$$\tfrac{1}{\sqrt{g}}\nabla_i(\sqrt{g}g^{ij}G_{AB}(\phi)\partial_j\phi^B) = 0 \tag{9.2a}$$

$$R_{ij} = \tfrac{1}{2}G_{AB}(\phi)\partial_i\phi^A\partial_j\phi^B \ , \tag{9.2b}$$

where ∇_i is a covariant derivative; when acting on a target-space vector V_A, it is given by

$$\nabla_i V_A = \partial_i V_A - \Gamma^D_{AE}(G)\partial_i\phi^E V_D \ , \tag{9.3}$$

in which $\Gamma^A_{BC}(G)$ is the Christoffel connection for the target-space metric G_{AB}. The action (9.1) and the field equations (9.2) are covariant with respect to general-coordinate transformations on the σ-model target manifold G/H'. The action (9.1) and the field equations (9.2) are also covariant with respect to general $y^i \rightarrow y'^i$ coordinate transformations of the domain space. These two types of general coordinate transformations are quite different, however, in that the domain-space transformations constitute a true gauge symmetry of the dynamical system (9.1), while the σ-model target-space transformations generally change the metric $G_{AB}(\phi^A)$ and so correspond to an actual symmetry of (9.1) only for the finite-parameter group G of target-space *isometries*.

As in our original search for p-brane solutions given in Section 2, it is appropriate to adopt an *ansatz* in order to focus the search for solutions. In the search for instanton solutions, the metric ansatz can take a particularly simple form:

$$g^{ij} = \delta^{ij} \ , \tag{9.4}$$

in which the domain-space metric is assumed to be *flat*. The σ-model equations and domain-space gravity equations for the flat metric (9.4) then become

$$\nabla^i(G_{AB}(\phi)\partial_i\phi^B) = 0 \tag{9.5a}$$
$$R_{ij} = \tfrac{1}{2}G_{AB}(\phi)\partial_i\phi^A\partial_j\phi^B = 0 \quad . \tag{9.5b}$$

Now comes the key step [68] in finding instanton solutions to the specialised equations (9.5): for single-charge solutions, one supposes that the σ-model fields ϕ^A depend on the domain-space coordinates y^i only trough some intermediate scalar functions $\sigma(y)$, *i.e.*

$$\phi^A(y) = \phi^A(\sigma(y)) \ . \tag{9.6}$$

After making this assumption, the σ-model ϕ^A equations (9.5a) become

$$\nabla^2\sigma\frac{d\phi^A}{d\sigma} + (\partial_i\sigma)(\partial_i\sigma)\left[\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(G)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma}\right] = 0 \ , \tag{9.7}$$

while the gravitational equation (9.5b) becomes the *constraint*

$$G_{AB}(\phi)\frac{d\phi^A}{d\sigma}\frac{d\phi^B}{d\sigma} = 0 \ . \tag{9.8}$$

An important class of solutions to (9.7) is obtained by taking

$$\nabla^2\sigma = 0 \tag{9.9a}$$
$$\frac{d^2\phi^A}{d\sigma^2} + \Gamma^A_{BC}(G)\frac{d\phi^B}{d\sigma}\frac{d\phi^C}{d\sigma} = 0 \ . \tag{9.9b}$$

At this point, one can give a picture of the σ-model maps involved in the system of equations (9.8, 9.9), noting that (9.9a) is just Laplace's equation and that (9.9b) is the geodesic equation on G/H', while the constraint (9.8) requires the tangent vector to a geodesic to be a null vector. The intermediate function $\sigma(y)$ is required by (9.9a) to be a *harmonic* function mapping from the flat (9.4) Euclidean domain space onto a *null geodesic* on the target space ${}^{G}\!/_{H'}$. Clearly, the harmonic map $\sigma(y)$ should be identified with the harmonic function $H(y)$ that controls the single-charge brane solutions (2.24). On the geodesic in ${}^{G}\!/_{H'}$, on the other hand, σ plays the role of an affine parameter. The importance of the noncompact structure of the target space manifold ${}^{G}\!/_{H'}$, for the groups G and H' given in Table 5, now becomes clear: only on such a noncompact manifold does one have nontrivial null geodesics as required by the gravitational constraint (9.8). The σ-model solution (9.6) oxidises back up to one of the single-charge brane solutions shown in Figure 6, and, conversely, any solution shown in Figure 6 may be reduced down to a corresponding noncompact σ-model solution of this type. This sequence of σ-model maps is sketched in Figure 7.

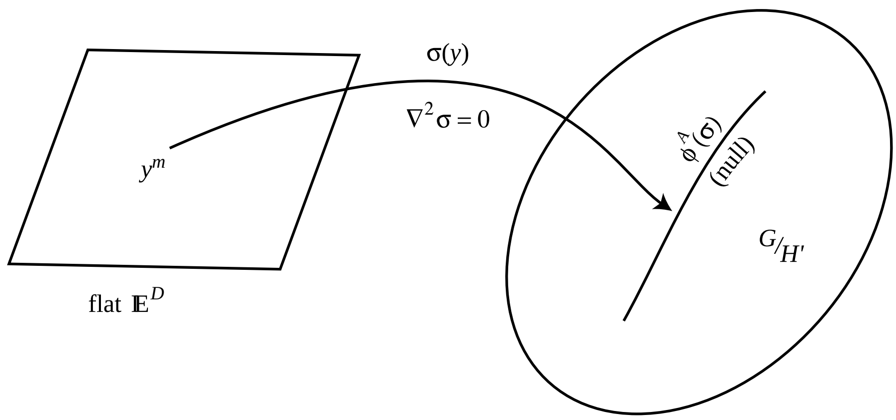

Figure 7: Harmonic map from $\mathbb{E}^D$ to a null geodesic in ${}^{G}\!/_{H'}$.

An extension [69,70] of this σ-model picture allows for solutions involving multiple harmonic maps $\sigma_a(y)$. In that case, one deals not with a single geodesic, but with a totally geodesic submanifold of ${}^{G}\!/_{H'}$, and, moreover, the geodesics generated by any curve in the intermediate σ_a parameter space must be null. This is the σ-model construction that generates multi-charge solutions, giving rise to intersecting-brane solutions of the types discussed in Section 6. As with the intersecting-brane solutions, however, there are important compatibility conditions that must be satisfied in order for such multi-charge solutions to exist. We saw in subsection 6.2 that, in order for some portion of the rigid supersymmetry to remain unbroken, the projectors constraining the surviving supersymmetry parameter need to be consistent. In the σ-model picture, a required condition is expressed in terms of the velocity vectors for the null geodesics. If one adopts a matrix representation $\boldsymbol{M}$ for points in the coset manifold G/H', the σ-model equations for the matrix fields $\boldsymbol{M}(y^m)$ are simply written

$$\nabla^i(\boldsymbol{M}^{-1}\partial_i\boldsymbol{M}) = 0 \ . \tag{9.10}$$

Points on the geodesic submanifold with affine parameters σ_a may be written

$$\boldsymbol{M} = \boldsymbol{A}\exp(\sum_a \boldsymbol{B}_a\sigma_a) \ , \tag{9.11}$$

where the constant matrices $\boldsymbol{B}_a$ give the velocities for the various geodesics parametrised by the σ_a, while an initial point on these geodesics is specified by the constant matrix $\boldsymbol{A}$. The compatibility condition between these velocities is given by the double-commutator condition [70]

$$[[\boldsymbol{B}_a, \boldsymbol{B}_b], \boldsymbol{B}_c] = 0 \ . \tag{9.12}$$

This condition allows one to rewrite (9.11) as

$$\boldsymbol{M} = \boldsymbol{A}\exp\left(-\tfrac{1}{2}\sum_{c>b}\sum_b [\boldsymbol{B}_b, \boldsymbol{B}_c]\sigma_b\sigma_c\right)\prod_a \exp(\boldsymbol{B}_a\sigma_a) \ , \tag{9.13}$$

where the first factor commutes with the $\boldsymbol{B}_a$ as a result of (9.12). The matrix current then becomes

$$\boldsymbol{M}^{-1}\partial_i\boldsymbol{M} = \sum_a \boldsymbol{B}_a\partial_i\sigma_a - \tfrac{1}{2}\sum_{c>b}\sum_b [\boldsymbol{B}_b, \boldsymbol{B}_c](\sigma_b\partial_i\sigma_c - \sigma_c\partial_i\sigma_b) \ , \tag{9.14}$$

and this is then seen to be conserved provided the σ_a satisfy $\nabla^2\sigma_a(y) = 0$, *i.e.* they are harmonic maps from the Euclidean overall-transverse space of the y^m into the geodesic submanifold (9.11). The constraint imposed by the gravitational equation is

$$R_{ij} = \tfrac{1}{4}\sum_{a,b} \mathrm{tr}(\boldsymbol{B}_a\boldsymbol{B}_b)\partial_i\sigma_a\partial_j\sigma_b = 0 \ , \tag{9.15}$$

which is satisfied provided the geodesics parametrised by the σ_a are null and orthogonal, *i.e.*

$$\mathrm{tr}(\boldsymbol{B}_a\boldsymbol{B}_b) = 0 \ . \tag{9.16}$$

The general set of stationary multi-charge brane solutions is thus obtained in the σ-model construction by identifying the set of totally null, totally geodesic submanifolds of G/H' such that the velocity vectors satisfy the compatibility condition (9.12).

Aside from the elegance of the above σ-model picture of the equations governing BPS brane solutions, these constructions make quite clear the places where assumptions have been made that are more stringent than are really necessary. One example of this is the assumption that the transverse-space geometry is flat, Eq. (9.4). This is clearly more restrictive than is really necessary; one could just as well have a more general Ricci-flat domain-space geometry, with a correspondingly covariantised constraint for the null geodesics on the noncompact manifold G/H'. The use of more general Ricci-flat transverse geometries is at the basis of "generalised p-brane" solutions that have been considered in Refs.[73,74]

10 Concluding remarks

In this review, we have discussed principally the structure of classical p-brane solutions to supergravity theories. Some topics that deserve a fuller treatment have only been touched upon here. For example, the worldvolume symmetries of p-brane sources, and in particular the important subject of κ symmetry, which bridges the gap between the full target-space supersymmetry of the ambient supergravity theory and the fractional supersymmetry surviving in the BPS brane background, have only been touched upon. For a fuller treatment, the reader is referred to Refs [4,7], or to the more recent discussions of κ-symmetric actions for cases involving R–R sector antisymmetric-tensor fields.[75]

Another aspect of the p-brane story, which we have only briefly presented here in Section 6, is the large family of intersecting branes. These now include [74] intersections at angles other than 90°, and can involve fractions of preserved supersymmetry other than inverse powers of 2. For a fuller treatment of some of these subjects, the reader is referred to Ref.[44,46], and for the implications of charge conservation in determining the allowed intersections to Refs. [21]

Yet another aspect of this subject that we have not dwelt upon here is the intrinsically string-theoretic side, in which some of the BPS supergravity solutions that we have discussed appear as Dirichlet surfaces on which open strings can end; for this, we refer the reader to Ref.[9]

Of course, the real fascination of this subject lies in its connection to the emerging picture in string theory/quantum gravity, and in particular to the rôles that BPS supergravity solutions play as states stable against the effects of quantum corrections. In this emerging picture, the duality symmetries that we have discussed in Section 7 play an essential part, uniting the underlying type IIA, IIB, $E_8 \times E_8$ and SO(32) heterotic, and also the type I string theories into one overall theory, which then also has a phase with $D = 11$ supergravity as its field-theory limit. The usefulness of classical supergravity considerations in probing the structure of this emerging "M theory" is one of the major surprises of the subject.

Acknowledgments

The author would like to acknowledge helpful conversations with Marcus Bremer, Bernard de Wit, Mike Duff, François Englert, Gary Gibbons, Hong Lü, George Papadopoulos, Chris Pope, and Paul Townsend. The author would like to thank The Abdus Salam International Center for Theoretical Physics for the invitations to give lectures at two successive summer schools, on which this review is based, and also to CERN, SISSA, the Ecole Normale Supérieure, UCLA and the University of Pennsylvania for hospitality at different periods during the writing. This work was supported in part by the Commission of the European Communities under contract ERBFMRX-CT96-0045.

References

1. E. Cremmer, B. Julia and J. Scherk, *Phys. Lett.* B **76**, 409 (1978).
2. D.N. Page, *Phys. Rev.* D **28**, 2976 (1983).
3. J.W. van Holten and A. van Proeyen, "$N = 1$ Supersymmetry Algebras in $D = 2$, $D = 3$, $D = 4$ mod-8," *J. Phys.* A**15**, 3763 (1982).
4. K.S. Stelle and P.K. Townsend, "Are 2-branes better than 1?" in *Proc. CAP Summer Institute, Edmonton, Alberta, July 1987,* KEK library accession number `8801076`.
5. P.S. Howe and K.S. Stelle, "The Ultraviolet Properties of Supersymmetric Field Theories," *Int. J. Mod. Phys.* A **4**, 1871 (1989).
6. P.K. Townsend, "Three lectures on supersymmetry and extended objects," in *Integrable Systems, Quantum Groups and Quantum Field Theories (23rd GIFT Seminar on Theoretical Physics, Salamanca, June, 1992),* eds L.A. Ibort and M.A. Rodriguez (Kluwer, 1993).
7. M.J. Duff, R.R. Khuri and J.X. Lu, "String solitons," *Physics Reports* **259**, 213 (1995), `hep-th/9412184;`
8. M.J. Duff, "Supermembranes," `hep-th/9611203.`
9. J. Polchinski, "Tasi Lectures on D-branes," `hep-th/9611050.`

10. E.S. Fradkin and A.A. Tseytlin, *Phys. Lett.* B **158**, 316 (1985); *Nucl. Phys.* B **261**, 1 (1985).
11. C. Callan, D. Friedan, E. Martinec and M. Perry, *Nucl. Phys.* B **262**, 593 (1985).
12. A. Dabholkar, G. Gibbons, J.A. Harvey and F. Ruiz Ruiz, "Superstrings and Solitons," *Nucl. Phys.* B **340**, 33 (1990).
13. H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, "Stainless Super p-branes," *Nucl. Phys.* B **456**, 669 (1996), `hep-th/9508042.`
14. I.C. Campbell and P.C. West, *Nucl. Phys.* B **243**, 112 (1984); F. Giani and M. Pernici, *Phys. Rev.* D **30**, 325 (1984); M. Huq and M.A. Namazie, *Class. Quantum Grav.* **2**, 293 (1985); *ibid.* **2**, 597 (1985).
15. M.B. Green and J.H. Schwarz, *Phys. Lett.* B **122**, 143 (1983); J.H. Schwarz and P.C. West, *Phys. Lett.* B **126**, 301 (1983); J.H. Schwarz, *Nucl. Phys.* B **226**, 269 (1983); P.S. Howe and P.C. West, *Nucl. Phys.* B **238**, 181 (1984).
16. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (W.H. Freeman and Co., San Francisco, 1973), Box 14.5.
17. H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, "Dilatonic p-brane solitons," *Nucl. Phys.* B **371**, 46 (1996), `hep-th/9511203.`
18. M.J. Duff and K.S. Stelle, "Multi-membrane solutions of $D = 11$ Supergravity," *Phys. Lett.* B **253**, 113 (1991).
19. G.W. Gibbons and P.K. Townsend, *Phys. Rev. Lett.* **71**, 3754 (1993), `hep-th/9302049;` M.J. Duff, G.W. Gibbons and P.K. Townsend, *Phys. Lett.* B **332**, 321 (1994), `hep-th/9405124.`
20. E. Bergshoeff, E. Sezgin and P.K. Townsend, *Phys. Lett.* B **189**, 75 (1987).
21. E. Witten, "Bound states of strings and p-branes," *Nucl. Phys.* B **460**, 335 (1996), `hep-th/9510135;` M.R. Douglas, "Branes within branes," `hep-th/9512077;` P.K. Townsend, "Brane Surgery," in *Proc. European Res. Conf. on Advanced Quantum Field Theory, La Londe-les-Maures, Sept. 1996,* `hep-th/9609217.`
22. M.J. Duff, P.S. Howe, T. Inami and K.S. Stelle, "Superstrings in $D = 10$ from Supermembranes in $D = 11$," *Phys. Lett.* B **191**, 70 (1987).
23. H. Lü, C.N. Pope and K.S. Stelle, "Vertical Versus Diagonal Dimensional Reduction for p-branes," *Nucl. Phys.* B **481**, 313 (1996), `hep-th/9605082.`

24. S.W. Hawking and G.F.R. Ellis, *The Large-Scale Structure of Space-Time,* (Cambridge University Press, 1973).
25. R. Güven, "Black p-brane solutions of $D = 11$ supergravity theory," *Phys. Lett.* B **276**, 49 (1992).
26. G.W. Gibbons, G.T. Horowitz and P.K. Townsend, *Class. Quantum Grav.* **12**, 297 (1995), `hep-th/9410073.`
27. G. Horowitz and A. Strominger, *Nucl. Phys.* B **360**, 197 (1991); M.J. Duff and J.X. Lu, *Nucl. Phys.* B **416**, 301 (1994), `hep-th/9306052.`
28. M.J. Duff, H. Lü and C.N. Pope, "The Black Branes of M-theory," *Phys. Lett.* B **382**, 73 (1996), `hep-th 9604052.`
29. J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo and P.K. Townsend, *Phys. Rev. Lett.* **63**, 2443 (1989).
30. G.W. Gibbons and C.M. Hull, *Phys. Lett.* B **109**, 190 (1982).
31. H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities, Nucl. Phys.* B **465**, 127 (1996), hep-th/9512012.
32. H. Lü and C.N. Pope, "An approach to the classification of p-brane solitons," `hep-th/9601089.`
33. R. Khuri, *Nucl. Phys.* B **387**, 315 (1992), `hep-th/9205081;` J.P. Gauntlett, J.A. Harvey and J.T. Liu, *Nucl. Phys.* B **409**, 363 (1993), `hep-th/9211056.`
34. B.R. Greene, A. Shapere, C. Vafa and S-T. Yau, *Nucl. Phys.* B **337**, 1 (1990);
G.W. Gibbons, M.B. Green and M.J. Perry, *Phys. Lett.* B **370**, 37 (1996), `hep-th/9511080.`
35. P.M. Cowdall, H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, "Domain Walls in Massive Supergravities," *Nucl. Phys.* B **486**, 49 (1997), `hep-th/9608173.`
36. J. Scherk and J.H. Schwarz, *Phys. Lett.* B **82**, 60 (1979).
37. E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, "Duality of Type II 7-branes and 8-branes," *Nucl. Phys.* B **470**, 113 (1996), `hep-th/9601150.`
38. N. Kaloper, R.R. Khuri and R.C. Myers, "On generalised axion reductions," `hep-th/9803066.`
39. H. Lü and C.N. Pope, "Domain walls from M-branes," *Mod. Phys. Lett.* A **12**, 1087 (1997), `hep-th/9611079;`
I.V. Lavrinenko, H. Lü and C.N. Pope, "From topology to generalised dimensional reduction," *Nucl. Phys.* B **492**, 278 (1997), `hep-th/9611134.`

40. M. Cvetic, “Extreme domain wall — black hole complementarity in $N = 1$ supergravity with a general dilaton coupling,” *Phys. Lett.* B **341**, 160 (1994).
41. M. Cvetic and H.H. Soleng, “Supergravity domain walls,” *Physics Reports* **282**, 159 (1997), `hep-th/9604090.`
42. J. Rahmfeld, *Phys. Lett.* B **372**, 198 (1996), `hep-th/9512089;`
N. Khviengia, Z. Khviengia, H. Lü and C.N. Pope, “Intersecting M-branes and Bound States,” `hep-th/9605077;`
M.J. Duff and J. Rahmfeld, “Bound States of Black Holes and Other p-branes,” *Nucl. Phys.* B **481**, 332 (1996), `hep-th/9605085.`
43. H.W. Brinkmann, *Proc. Nat. Acad. Sci.* **9**, 1 (1923).
44. G. Papadopoulos and P.K. Townsend, “Intersecting M-branes,” *Phys. Lett.* B **380**, 273 (1996), `hep-th/9603087;`
A. Tseytlin, “Harmonic superpositions of M-branes,” *Nucl. Phys.* B **475**, 149 (1996);
I.R. Klebanov and A.A. Tseytlin, “Intersecting M-branes as four-dimensional black holes,” *Nucl. Phys.* B **475**, 179 (1996);
K. Berndt, E. Bergshoeff and B. Janssen, “Intersecting D-branes in ten dimensions and six dimensions,” *Phys. Rev.* D **55**, 3785 (1997), `hep-th/9604168;`
J. Gauntlett, D. Kastor and J. Traschen, “Overlapping branes in M-theory,” *Nucl. Phys.* B **478**, 544 (1996).
45. R. Sorkin, *Phys. Rev. Lett.* **51**, 87 (1983);
D.J. Gross and M.J. Perry, *Nucl. Phys.* B **226**, 29 (1983).
46. J.P. Gauntlett, “Intersecting Branes,” `hep-th/9705011.`
47. A.A. Tseytlin, “‘No force’ condition and BPS combinations of p-branes in eleven dimensions and ten dimensions,” *Nucl. Phys.* B **487**, 141 (1997), `hep-th/9609212.`
48. K. Shiraishi, *Nucl. Phys.* B **402**, 399 (1993).
49. G.W. Gibbons, G. Papadopoulos and K.S. Stelle, “HKT and OKT geometries on soliton black hole moduli spaces,” *Nucl. Phys.* B **508**, 623 (1997), `hep-th/9706207.`
50. R. Coles and G. Papadopoulos, “The geometry of one-dimensional supersymmetric non-linear sigma models,” *Class. Quantum Grav.* **7**, 427 (1990).
51. E. Cremmer and B. Julia, *Nucl. Phys.* B **159**, 141 (1979).
52. J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend, “Dyonic Membranes,” *Nucl. Phys.* B **460**, 560 (1996), `hep-th/9508177.`

53. R. Nepomechie, “Magnetic monopoles from antisymmetric tensor gauge fields,” *Phys. Rev.* D **31**, 1921 (1985).
54. C. Teitelboim, *Phys. Lett.* B **67**, 63, 69 (1986).
55. M. Bremer, H. Lü, C.N. Pope and K.S. Stelle, “Dirac quantisation conditions and Kaluza-Klein reduction,” `hep-th/9710244`.
56. S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, “Duality, self-duality, sources and charge quantisation in Abelian N-form theories,” `hep-th/9702184`.
57. H. Lü, C.N. Pope and K.S. Stelle, “Weyl Group Invariance and p-brane Multiplets,” *Nucl. Phys.* B **476**, 89 (1996), `hep-th/9602140`.
58. C.M. Hull and P.K. Townsend, *Nucl. Phys.* B **438**, 109 (1995), `hep-th/9410167`.
59. E. Witten, “String theory dynamics in various dimensions,” *Nucl. Phys.* B **443**, 85 (1995).
60. M.J. Duff and J. Rahmfeld, “Bound States of Black Holes and Other p-branes,” *Nucl. Phys.* B **481**, 332 (1996), `hep-th/9605085`.
61. J.H. Schwarz, “The power of M-theory,” *Phys. Lett.* B **367**, 97 (1996), `hep-th/9510086`.
62. T.H. Buscher, “A symmetry of the string background field equations,” *Phys. Lett.* B **194**, 69 (1987);
A. Giveon, M. Porrati and E. Rabinovici, “Target space duality in string theory,” *Physics Reports* **244**, 77 (1994), `hep-th/9401139`.
63. A. Sen, “Electric and magnetic duality in string theory,” *Nucl. Phys.* B **404**, 109 (1993), `hep-th/9207053`;
“$SL(2,\mathbb{Z})$ duality and magnetically charged strings,” *Int. J. Mod. Phys.* A **8**, 5079 (1993), `hep-th/9302038`.
64. E. Cremmer, H. Lü, C.N. Pope and K.S. Stelle, “Spectrum-generating symmetries for BPS Solitons,” `hep-th/9707207`.
65. J.H. Schwarz, “An $SL(2,\mathbb{Z})$ multiplet of type IIB superstrings,” *Phys. Lett.* B **360**, 13 (1995); Erratum *ibid.* **364**, 252 (1995), `hep-th/9508143`.
66. E. Bergshoeff, C.M. Hull and T. Ortin, “Duality in the type II superstring effective action,” *Nucl. Phys.* B **451**, 547 (1995), `hep-th/9504081`.
67. E. Cremmer, B. Julia, H. Lü and C.N. Pope, “Dualisation of dualities, I,” `hep-th/9710119`.
68. G. Neugebaur and D. Kramer, *Ann. der Physik (Leipzig)* **24**, 62 (1969).
69. P. Breitenlohner, D. Maison and G. Gibbons, *Comm. Math. Phys.* **120**, 253 (1988).

70. G. Clement and D. Gal'tsov, "Stationary BPS solutions to dilaton-axion gravity," *Phys. Rev.* D **54**, 6136 (1996), `hep-th/9607043`;
D.V. Gal'tsov and O.A. Rytchkov, "Generating branes via sigma-models," `hep-th/9801160`.
71. H. Lü and C.N. Pope, "Multi-scalar p-brane solitons," *Int. J. Mod. Phys.* A **12**, 437 (1997), `hep-th/9512153`.
72. N. Khviengia, Z. Khviengia, H. Lü and C.N. Pope, "Intersecting M-branes and bound states," *Phys. Lett.* B **388**, 21 (1996), `hep-th/9605077`.
73. K. Becker and M. Becker, "M-theory on eight-manifolds," *Nucl. Phys.* B **477**, 155 (1996), `hep-th/9605053`.
74. M. Berkooz, M.R. Douglas and R.G. Leigh, "Branes intersecting at angles," *Nucl. Phys.* B **480**, 265 (1996), `hep-th/9606139`;
J.P. Gauntlett, G.W. Gibbons, G. Papadopoulos and P.K. Townsend, "HyperKahler manifolds and multiply intersecting branes," *Nucl. Phys.* B **500**, 133 (1997), `hep-th/9702202`.
75. P.S. Howe and E. Sezgin, "Superbranes," *Phys. Lett.* B **390**, 133 (1997), `hep-th/9607227`;
P.S. Howe and E. Sezgin, "$D = 11,\ p = 5$," *Phys. Lett.* B **394**, 62 (1997), `hep-th/9611008`;
M. Cederwall, A. von Gussich, B.E.W. Nilsson and A. Westerberg, "The Dirichlet Super Three Brane in Ten-Dimensional Type IIB Supergravity," *Nucl. Phys.* B **490**, 163 (1997), `hep-th/9610148`;
M. Aganagic, C. Popescu and J.H. Schwarz, "D-brane actions with local kappa-symmetry," *Phys. Lett.* B **393**, 311 (1997), `hep-th/9610249`;
M. Cederwall, A. von Gussich, B.E.W. Nilsson, P. Sundell and A. Westerberg, "The dirichlet super p-branes in ten-dimensional Type IIA and IIB supergravity," *Nucl. Phys.* B **490**, 179 (1997), `hep-th/9611159`;
E. Bergshoeff and P.K. Townsend, "Super D-branes," `hep-th/9611173`.
I. Bandos, D. Sorokin and M. Tonin, "Generalized Action Principle and Superfield Equations of Motion for $d = 10$ D-p-branes," *Nucl. Phys.* B **497**, 275 (1997), `hep-th/9701127`.