Poisson–Lie T-duality beyond the classical level and the renormalization group

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Abstract

In order to study quantum aspects of σ-models related by Poisson–Lie T-duality, we construct three- and two-dimensional models that correspond, in one of the dual faces, to deformations of S^3 and S^2. Their classical canonical equivalence is demonstrated by means of a generating functional, which we explicitly compute. We examine how they behave under the renormalization group and show that dually related models have the same 1-loop beta functions for the coupling and deformation parameters. We find non-trivial fixed points in the ultraviolet, where the theories do not become asymptotically free. This suggests that the limit of Poisson–Lie T-duality to the usual Abelian and non-Abelian T-dualities does not exist quantum mechanically, although it does so classically.

1   Introduction

A generalization of Abelian [1] and non-Abelian [2] target space duality (T-duality) is the so-called Poisson–Lie T-duality [3]. Its notable features for our purposes are that it does not rely on the existence of isometries but rather on a rigid group-theoretical structure [3] and also that it can be explicitly formulated as a canonical transformation between phase-space variables [4, 5], similarly to ordinary T-duality cases [6, 7].[1]

Up to now all considerations concerning Poisson–Lie T-duality were, almost exclusively, classical. It is the main objective of the present work to go beyond that level. The main obstacle in doing so is the (almost complete) lack of explicit examples in the literature of σ-models related by genuine Poisson–Lie T-duality. Namely, models such that they cannot be obtained in the context of the usual Abelian or non-Abelian T-duality and also such that their target spaces have the same dimension. We construct explicit examples in three and two target space dimensions; they represent a two- and a one-parameter deformations of S^3 and S^2 and of their non-Abelian duals, respectively, and they reduce to them in an appropriate limit. We also give the explicit expression for the generating functional of the canonical transformation relating them. We then study renormalization of σ-models with two-dimensional target spaces by treating them as ordinary 2-dimensional field theories. Renormalizability in this context means that the counter-terms arising at every order in perturbation theory can be absorbed into renormalizations of the coupling and the parameters, up to field redefinitions or, equivalently, diffeomorphisms in the target space. We show that the 1-loop beta functions for the coupling and the parameters of the dually related models are equivalent, which strongly hints towards their equivalence beyond the classical level.[2] Moreover, we find that their behaviour in the ultraviolet is completely different from the corresponding one for S^2 and its non-Abelian dual. The latter models are asymptotically free whereas our models have a non-trivial behaviour, i.e. the geometry is still curved. This suggests that Poisson–Lie T-duality is really distinct from the usual Abelian and non-Abelian T-dualities and that it does not correspond to a "soft breaking" of these. We have also considered renormalization at the 2-loop level. We have found that the models are not renormalizable in the strict field-theoretical sense, namely, the corresponding counter-term cannot be absorbed into coupling, parameter and field renormalization. This presumably implies that the Poisson–Lie T-duality transformation rules should be modified at this level.

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[1]Poisson–Lie T-duality is by construction a canonical transformation and an abstract expression for the generating functional was already given in [8]. Also, Poisson–Lie T-duality does not cover all known T-duality transformations. It does not cover, in particular, non-Abelian duality transformations with respect to vector subgroups, i.e. in WZW models [9], as well as the vector-axial T-duality [10]. For this type of dualities a new unified framework is required, as explained in [5].

[2]In this respect we note that the existence of a canonical transformation relating two different σ-models seems to be necessary for their equivalence at the quantum level. We note the case of the Principal Chiral model and the Pseudodual Chiral model [11] whose classical solutions are in one-to-one correspondence but which are not canonically equivalent [7]. It is well known that their quantum behaviours are drastically different [12].

In the rest of this section we will briefly review, by following the conventions of [4, 5], some facts about Poisson–Lie T-duality, which are necessary for our constructions. The form of 2-dimensional σ-model actions related by Poisson–Lie T-duality is [3][3]

$$S = \frac{1}{2\lambda} \int E_{ab} L_\mu^a L_\nu^b \partial_+ X^\mu \partial_- X^\nu \ , \qquad E = (E_0^{-1} + \Pi)^{-1} \ , \tag{1}$$

and

$$\tilde{S} = \frac{1}{2\lambda} \int \tilde{E}^{ab} \tilde{L}_{a\mu} \tilde{L}_{b\nu} \partial_+ \tilde{X}^\mu \partial_- \tilde{X}^\nu \ , \qquad \tilde{E} = (E_0 + \tilde{\Pi})^{-1} \ . \tag{2}$$

The field variables in (1) are X^μ, $\mu = 1, 2, \ldots, \dim(G)$ and parametrize an element g of a group G. We also introduce representation matrices $\{T_a\}$, with $a = 1, 2, \ldots, \dim(G)$ and the components of the left-invariant Maurer–Cartan forms L_μ^a. The light-cone coordinates on the world-sheet are $\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)$ and λ denotes the overall coupling constant. Similarly, for (2) the field variables are $\tilde{X}^\mu$, where $\tilde{X}^\mu$, $\mu = 1, 2, \ldots, \dim(G)$, parametrize a different group $\tilde{G}$, whose dimension is, however, equal to that of G. Accordingly, we introduce a different set of representation matrices $\{\tilde{T}^a\}$, with $a = 1, 2, \ldots, \dim(G)$, and the corresponding components of the left-invariant Maurer–Cartan forms $\tilde{L}_{a\mu}$. In (1) and (2), E_0 is a constant $\dim(G) \times \dim(G)$ matrix, whereas Π and $\tilde{\Pi}$ are antisymmetric matrices with the same dimension as E_0, but they depend on the variables X^μ and $\tilde{X}^\mu$ via the corresponding group elements g and $\tilde{g}$. Hence, we do not require any isometry associated with the groups G and $\tilde{G}$.

It is crucial [3] for Poisson–Lie T-duality that the algebras generated by $\{T_a\}$ and $\{\tilde{T}^a\}$ form a pair of maximally isotropic subalgebras into which the Lie algebra of a Lie group known as the Drinfeld double can be decomposed. This implies the existence of non-trivial mixed commutators between the T_a's and the $\tilde{T}^a$'s, of the type we will encounter in section 2. It is important that there exist also a bilinear invariant $\langle \cdot | \cdot \rangle$ with the various generators obeying

$$\langle T_a | T_b \rangle = \langle \tilde{T}^a | \tilde{T}^b \rangle = 0 \ , \qquad \langle T_a | \tilde{T}^b \rangle = \delta_a{}^b \ . \tag{3}$$

The matrices Π, $\tilde{\Pi}$ in (1) and (2) are defined as

$$\Pi^{ab} = b^{ca} a_c{}^b \ , \qquad \tilde{\Pi}_{ab} = \tilde{b}_{ca} \tilde{a}^c{}_b \ , \tag{4}$$

where the matrices $a(g)$, $b(g)$ are constructed using

$$g^{-1} T_a g = a_a{}^b T_b \ , \qquad g^{-1} \tilde{T}^a g = b^{ab} T_b + (a^{-1})_b{}^a \tilde{T}^b \ , \tag{5}$$

and similarly for $\tilde{a}(\tilde{g})$ and $\tilde{b}(\tilde{g})$. Consistency restricts them to obey

$$a(g^{-1}) = a^{-1}(g) \ , \qquad b^T(g) = b(g^{-1}) \ , \qquad \Pi^T(g) = -\Pi(g) \ , \tag{6}$$

and similarly for the tilded ones.

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[3]The most general such actions that include spectator extra fields can be found in [3, 13].

2 The models

2.1 Generalities: algebraic and group-theoretical structure

We start this chapter by first constructing an explicit Drinfeld double and a bilinear invariant for it, and then computing the matrices a, b, Π as well as their tilded counterparts.

Consider the 3-dimensional algebras $su(2)$ and e_3, with generators labelled $\{T_a\}$ and $\{\tilde{T}_a\}$, respectively ($a = 1, 2, 3$). For convenience we also split the index $a = (i, 3)$, $i = 1, 2$. The commutation relations are given by[4]

$$[T_a, T_b] = i\epsilon_{abc}T_c \ ,$$
$$[\tilde{T}_3, \tilde{T}_i] = \tilde{T}_i \ , \quad [\tilde{T}_i, \tilde{T}_j] = 0 \ . \tag{7}$$

The above algebras can be thought of as a pair of maximally isotropic subalgebras of a 6-dimensional algebra corresponding to a Drinfeld double Lie group. The complete set of commutation relations is given by (7) and the "mixed" ones

$$[T_i, \tilde{T}_j] = i\epsilon_{ij}\tilde{T}_3 - \delta_{ij}T_3 \ , \quad [T_3, \tilde{T}_i] = i\epsilon_{ij}\tilde{T}_j \ ,$$
$$[\tilde{T}_3, T_i] = i\epsilon_{ij}\tilde{T}_j - T_i \ . \tag{8}$$

It can be shown that the combined set of commutation relations in (7) and (8) obey, the Jacobi identities. Next we construct an invariant product for our 6-dimensional Drinfeld double algebra, with respect to which the various generators should obey (3). Consider two elements in the Lie algebra of the form (x_i, y_i), with $i = 1, 2$, where $x_i \in su(2)$ and $y_i \in e_3$. Then, the invariant inner product is defined as

$$\langle (x_1, y_1) | (x_2, y_2) \rangle = \{x_1, x_2\} - \{y_1, y_2\} \ , \tag{9}$$

where $\{\cdot, \cdot\}$ is the invariant product for the algebras $su(2)$ and e_3. A representation for the various generators is given by

$$T_a = (\frac{\sigma_a}{2}, \frac{\sigma_a}{2}) \ , \quad a = 1, 2, 3 \ ,$$
$$\tilde{T}_1 = (\sigma_+, -\sigma_-) \ , \quad \tilde{T}_2 = -i(\sigma_+, \sigma_-) \ , \quad \tilde{T}_3 = \frac{1}{2}(\sigma_3, -\sigma_3) \ , \tag{10}$$

where the σ_a's denote the three Pauli matrices and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. Then, the product $\{\cdot, \cdot\}$ of any two matrices is equal to their trace. We note that our double is similar to the $O(2, 2)$ non-compact double used in [14].

Next, we parametrize the $SU(2)$ group element in terms of the three Euler angles ϕ, ψ and θ. It is represented by the 4×4 block-diagonal matrix

$$g_{SU(2)} = \text{diag}(g, g) \ , \tag{11}$$

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[4]We note that e_3 is not the algebra for the Euclidean group in three dimensions. Also, in the rest of the paper we will not distinguish between upper and lower Lie-algebra indices.

where

$$g = e^{\frac{i}{2}\phi\sigma_3} e^{\frac{i}{2}\theta\sigma_2} e^{\frac{i}{2}\psi\sigma_3} = \begin{pmatrix} \cos\frac{\theta}{2} e^{\frac{i}{2}(\phi+\psi)} & \sin\frac{\theta}{2} e^{\frac{i}{2}(\phi-\psi)} \\ -\sin\frac{\theta}{2} e^{-\frac{i}{2}(\phi-\psi)} & \cos\frac{\theta}{2} e^{-\frac{i}{2}(\phi+\psi)} \end{pmatrix} . \tag{12}$$

Also the group element of E_3 is parametrized in terms of three variables y_1, y_2 and χ and represented by the following 4×4 block-diagonal matrix

$$\tilde{g}_{E_3} = \mathrm{diag}(\tilde{g}_+, \tilde{g}_-) , \tag{13}$$

where

$$\tilde{g}_+ = \begin{pmatrix} e^{+\frac{\chi}{2}} & \chi_+ \\ 0 & e^{-\frac{\chi}{2}} \end{pmatrix} , \qquad \tilde{g}_- = \begin{pmatrix} e^{-\frac{\chi}{2}} & 0 \\ \chi_- & e^{+\frac{\chi}{2}} \end{pmatrix} ,$$

$$\chi_\pm = \pm e^{-\frac{\chi}{2}}(y_1 \mp iy_2) . \tag{14}$$

The Maurer–Cartan forms are defined as $L_a = -i\langle g^{-1}dg|\tilde{T}_a\rangle$ and $\tilde{L}_a = \langle \tilde{g}^{-1}d\tilde{g}|T_a\rangle$. Using the parametrization of the $SU(2)$ group element in (12) we have explicitly

$$\begin{aligned} L_1 &= \cos\psi \sin\theta d\phi - \sin\psi d\theta , \\ L_2 &= \sin\psi \sin\theta d\phi + \cos\psi d\theta , \\ L_3 &= d\psi + \cos\theta d\phi . \end{aligned} \tag{15}$$

Similarly, using the parametrization (13) for the E_3 group element we find

$$\tilde{L}_1 = e^{-\chi}dy_1 , \qquad \tilde{L}_2 = e^{-\chi}dy_2 , \qquad \tilde{L}_3 = d\chi . \tag{16}$$

Also the matrices

$$(a_{ab}) = \begin{pmatrix} \cos\phi\cos\psi\cos\theta - \sin\phi\sin\psi & \cos\phi\sin\psi\cos\theta + \sin\phi\cos\psi & -\cos\phi\sin\theta \\ -\sin\phi\cos\psi\cos\theta - \cos\phi\sin\psi & \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & \sin\phi\sin\theta \\ \cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta \end{pmatrix} , \tag{17}$$

and

$$(b_{ab}) = \begin{pmatrix} a_{12} + a_{21} & a_{22} - a_{11} & a_{23} \\ a_{22} - a_{11} & -a_{12} - a_{21} & -a_{13} \\ a_{32} & -a_{31} & 0 \end{pmatrix} , \tag{18}$$

as well as

$$(\tilde{a}_{ab}) = \begin{pmatrix} e^{-\chi} & 0 & 0 \\ 0 & e^{-\chi} & 0 \\ y_1 e^{-\chi} & y_2 e^{-\chi} & 1 \end{pmatrix} , \tag{19}$$

and

$$(\tilde{b}_{ab}) = \begin{pmatrix} -y_1 y_2 e^{-\chi} & \sinh\chi + \frac{1}{2}(y_1^2 - y_2^2)e^{-\chi} & -y_2 \\ -\sinh\chi + \frac{1}{2}(y_1^2 - y_2^2)e^{-\chi} & y_1 y_2 e^{-\chi} & y_1 \\ y_2 e^{-\chi} & -y_1 e^{-\chi} & 0 \end{pmatrix} . \tag{20}$$

Then the matrices $\Pi = b^T a$ and $\tilde{\Pi} = \tilde{b}^T \tilde{a}$ can be computed. Since a 3×3 antisymmetric tensor has as many independent components as a three-vector, we may represent them in terms of two vectors $\vec{A}$ and $\vec{\tilde{A}}$ using

$$\Pi_{ab} = -\epsilon_{abc} A_c \,, \qquad \tilde{\Pi}_{ab} = -\epsilon_{abc} \tilde{A}_c \,. \tag{21}$$

In the case at hand we compute

$$\begin{aligned}
\vec{A} &= (\cos\psi\sin\theta, \sin\psi\sin\theta, \cos\theta - 1) \,, \\
\vec{\tilde{A}} &= \left(y_1 e^{-\chi}, y_2 e^{-\chi}, \sinh\chi e^{-\chi} - \frac{1}{2}(y_1^2 + y_2^2)e^{-2\chi} \right) \,.
\end{aligned} \tag{22}$$

2.2   The classical canonical transformation

Given the underlying algebraic and group theoretical structure, there exists a classical canonical transformation relating the corresponding σ-model actions [4, 5]. This is universal and does not depend on the particular choice of the matrix E_0 in (1), (2). Moreover it always has the same form, even when we include spectator fields. For σ-models related by Poisson–Lie T-duality corresponding to three-dimensional groups it can be shown that the general canonical transformation, given in [4, 5], takes the form

$$\begin{aligned}
\vec{P} &= \frac{1}{1 - \vec{A}\cdot\vec{\tilde{A}}} \left(\vec{\tilde{A}} \times \vec{L}_\sigma + \vec{\tilde{L}}_\sigma - (\vec{\tilde{A}}\cdot\vec{\tilde{L}}_\sigma)\vec{A} \right) \,, \\
\vec{\tilde{P}} &= \frac{1}{1 - \vec{A}\cdot\vec{\tilde{A}}} \left(\vec{A} \times \vec{\tilde{L}}_\sigma + \vec{L}_\sigma - (\vec{A}\cdot\vec{L}_\sigma)\vec{\tilde{A}} \right) \,,
\end{aligned} \tag{23}$$

where the three-vectors $\vec{P}$ and $\vec{\tilde{P}}$ have components given by $P_a = P_\mu L_{\mu a}^{-1}$ and $\tilde{P}_a = \tilde{P}_\mu \tilde{L}_{\mu a}^{-1}$, with P_μ, $\tilde{P}_\mu$ being the conjugate momenta to X^μ and $\tilde{X}^\mu$, respectively. These transformations can be derived from a generating functional of the form

$$F = \oint d\sigma (B_\mu \partial_\sigma X^\mu + \tilde{B}_\mu \partial_\sigma \tilde{X}^\mu) \,, \tag{24}$$

for some functions B_μ, $\tilde{B}_\mu$ of the target space variables, which are determined by solving a set of differential equations [5]. For three-dimensional cases these can be cast into the form

$$\begin{aligned}
\partial_{[\mu} B_{\nu]} &= -\left(1 - \vec{A}\cdot\vec{\tilde{A}}\right)^{-1} \vec{\tilde{A}}\cdot\vec{L}_\mu \times \vec{L}_\nu \,, \\
\tilde{\partial}_{[\mu} \tilde{B}_{\nu]} &= \left(1 - \vec{A}\cdot\vec{\tilde{A}}\right)^{-1} \vec{A}\cdot\vec{\tilde{L}}_\mu \times \vec{\tilde{L}}_\nu \,, \\
\partial_\mu \tilde{B}_\nu - \tilde{\partial}_\nu B_\mu &= \left(1 - \vec{A}\cdot\vec{\tilde{A}}\right)^{-1} \left(\vec{L}_\mu\cdot\vec{\tilde{L}}_\nu - (\vec{A}\cdot\vec{L}_\mu)(\vec{\tilde{A}}\cdot\vec{\tilde{L}}_\mu) \right) \,.
\end{aligned} \tag{25}$$

For the specific models at hand it is possible to find the explicit solution to them

$$B_\phi = -\ln\left(e^{2\chi}\cos^2\frac{\theta}{2} + e^\chi \sin\theta (y_1\cos\psi + y_2\sin\psi) + (1 + y_1^2 + y_2^2)\sin^2\frac{\theta}{2} \right) \,,$$

$$B_\psi = -\chi + 2\frac{y_1\cos\psi + y_2\sin\psi}{\sqrt{1+(y_1\sin\psi - y_2\cos\psi)^2}}\cot^{-1}\left(\frac{\sqrt{1+(y_1\sin\psi - y_2\cos\psi)^2}\,\tan\frac{\theta}{2}}{e^\chi + y_1\cos\psi + y_2\sin\psi}\right),$$

$$B_\theta = 0,$$

$$\tilde{B}_{y_1} = +2\frac{\sin\psi}{\sqrt{1+(y_1\sin\psi - y_2\cos\psi)^2}}\tan^{-1}\left(\frac{y_1\cos\psi + y_2\sin\psi + e^\chi\cot\frac{\theta}{2}}{\sqrt{1+(y_1\sin\psi - y_2\cos\psi)^2}}\right),$$

$$\tilde{B}_{y_2} = -2\frac{\cos\psi}{\sqrt{1+(y_1\sin\psi - y_2\cos\psi)^2}}\tan^{-1}\left(\frac{y_1\cos\psi + y_2\sin\psi + e^\chi\cot\frac{\theta}{2}}{\sqrt{1+(y_1\sin\psi - y_2\cos\psi)^2}}\right),$$

$$\tilde{B}_\chi = -\phi. \tag{26}$$

The above generating functional exhibits a generic, for Poisson–Lie T-duality, behaviour [5]. Namely, it is highly non-linear in the group variables of $SU(2)$ and E_3 and therefore it will receive quantum corrections when the canonical transformation is implemented in the full Hilbert space along the lines of [15]. Nevertheless, its existence seems to guarantee the quantum equivalence of the corresponding dually related σ-models, as noted in footnote 2 and as we will see in section 3.

2.3   Explicit three- and two-dimensional models

Let us consider three-dimensional dual models with no spectators at all and choose the constant matrix $E_0^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Then the metric and antisymmetric tensor corresponding to the action (1) can be written as

$$ds^2 = \frac{1}{V}\left(A_a A_b + \frac{\lambda_1\lambda_2\lambda_3}{\lambda_a}\delta_{ab}\right)L_a L_b,$$

$$B = \frac{1}{V}\epsilon_{abc}\lambda_c A_c L_a \wedge L_b, \tag{27}$$

$$V \equiv \lambda_1\lambda_2\lambda_3 + \lambda_a A_a^2.$$

We note that since there is no explicit dependence of L_a and A_a on the Euler angle ϕ the background (27) has the corresponding isometry. For the dual model the expressions for the background metric and antisymmetric tensor are identical to those in (27), with tilded symbols replacing the untilded ones (also $\tilde{\lambda}_a = 1/\lambda_a$). A greater simplification occurs if two of the constant coefficients are equal, i.e. if $\lambda_1 = \lambda_2$. It is then easy to see that there is a second commuting isometry corresponding to the vector field $\frac{\partial}{\partial\psi}$. Let us reparametrize the constants $\lambda_1 = \lambda_2$ and λ_3 in terms of two other constants κ and g as $\lambda_1 = \lambda_2 = \kappa(1+g)^{1/2}$ and $\lambda_3 = \kappa(1+g)^{-1/2}$. Then the metric and the antisymmetric tensors are given by

$$ds^2 = \kappa^{-1}(1+g)^{-1/2}V^{-1}\left(\kappa^2(L_a L_a + gL_3 L_3) + 4\sin^4\frac{\theta}{2}(d\phi - d\psi)^2\right),$$

$$B = 2\sin\theta V^{-1}\, d\theta \wedge \left((1 - 2g(1+g)^{-1}\sin^2\frac{\theta}{2})d\phi + d\psi \right) , \tag{28}$$

where

$$V \equiv \kappa^2 + \sin^2\theta + 4(1+g)^{-1}\sin^4\frac{\theta}{2} , \tag{29}$$

and

$$L_a L_a + g L_3 L_3 = (1+g)d\psi^2 + (1 + g\cos^2\theta)d\phi^2 + d\theta^2 + 2(1+g)\cos\theta d\psi d\phi \tag{30}$$

is the metric for S^3 deformed by the bilinear in the current L_3. Notice that the antisymmetric tensor is a pure gauge, i.e. $dB = 0$. Also the dual metric and the dual antisymmetric tensors are given by

$$d\tilde{s}^2 = \kappa^3(1+g)^{1/2}\tilde{V}^{-1}\left\{ e^{-2\chi}\left(e^{-\chi}(y_1 dy_1 + y_2 dy_2) + \left(\sinh\chi - \frac{1}{2}(y_1^2 + y_2^2)e^{-\chi} \right) d\chi \right)^2 \right.$$

$$\left. + \kappa^{-2}(1+g)^{-1}d\chi^2 + \kappa^{-2}e^{-2\chi}(dy_1^2 + dy_2^2) \right\} ,$$

$$\tilde{B} = 2\kappa^2 e^{-2\chi}\tilde{V}^{-1}\left\{ (1+g)e^{-\chi}\left(\sinh\chi - \frac{1}{2}(y_1^2 + y_2^2)e^{-\chi} \right) dy_1 \wedge dy_2 \right.$$

$$\left. + (y_1 dy_2 - y_2 dy_1) \wedge d\chi \right\} , \tag{31}$$

where

$$\tilde{V} \equiv 1 + \kappa^2 e^{-2\chi}\left(y_1^2 + y_2^2 + (1+g)\left(\sinh\chi - \frac{1}{2}(y_1^2 + y_2^2)e^{-\chi} \right)^2 \right) . \tag{32}$$

Notice that if we take the limit $\kappa \to \infty$ and also rescale the overall coupling constant as $\lambda \to \lambda\kappa^{-1}(1+g)^{-1/2}$, the metric in (28) reduces to the deformed S^3 metric in (30). In addition, we rescale $(y_1, y_2, \chi) \to \frac{1}{\kappa}(x_1, x_2, x_3)$. This contracts the group E_3 into an Abelian one. Then, (31) reduces to the expression for the usual non-Abelian dual of (30) with respect to the left action of $SU(2)$. If on the other hand we rescale as $\theta \to \kappa\rho$, $\phi \to \frac{1}{2}(\kappa x_3 + 2\alpha)$, $\psi \to \frac{1}{2}(\kappa x_3 - 2\alpha)$ and $\lambda \to \kappa\lambda(1+g)^{1/2}$, and if we let then $\kappa \to 0$ we obtain the non-Abelian dual of E_3 (effectively this contracts also $SU(2)$ into an Abelian group). In addition, for both of the limits we just briefly mentioned, the generating functional (26) reduces to the appropriate one for non-Abelian duality [7, 16].

We may also consider a different limit. Namely we reparametrize $\kappa = 2a^{-1}(1+g)^{-1/2}$ and then let $g \to -1$. This implies that $\lambda_1 = \lambda_2 = 2/a$ are finite in that limit, whereas $\lambda_3 = 2a^{-1}(1+g)^{-1}$ is sent to infinity. We also rescale the overall coupling constant as $\lambda \to \lambda a/2$. The resulting models have two-dimensional target spaces as the third dimension becomes suppressed in that limit. The metric corresponding to (28) becomes

$$ds^2 = \frac{1}{1 + a^2\sin^4\frac{\theta}{2}}\left(d\theta^2 + \sin^2\theta d\phi^2 \right) . \tag{33}$$

It represents a deformed 2-sphere, as can also be seen by explicitly verifying that the Euler characteristic $\chi = \frac{1}{4\pi}\int\sqrt{G}R$ equals 2. The metric corresponding to (31) is given

by

$$d\tilde{s}^2 = \frac{1/2}{r(1+az)} \left(dz^2 + \left(dr + \frac{z - ar/2}{1+az} dz \right)^2 \right) , \tag{34}$$

where we have changed variables as $y_1^2 + y_2^2 = \frac{1}{2}ra^2$ and $e^{2\chi} = 1 + az$. This is a non-compact manifold. In the limit $a \to 0$, (33) and (34) reduce to the metric for S^2 and its non-Abelian dual with respect to $SU(2)$ (see third article in ref. [2]).

In order to perform the 1-loop quantum analysis in a unified manner it will be useful to cast the metrics (33) and (34) into the conformally flat form

$$ds^2 = e^{-2\Phi}(dx^2 + dy^2) , \tag{35}$$

where the appropriate change of variables and the inverse of the conformal factor are given for (33) by

$$x = \cos \phi \cot \frac{\theta}{2} , \quad y = \sin \phi \cot \frac{\theta}{2} , \tag{36}$$

and

$$e^{2\Phi} = \frac{1}{4} \left((1 + x^2 + y^2)^2 + a^2 \right) , \tag{37}$$

and for (34) by

$$x = \frac{1}{a^2} \left((1 + az)^{1/2} - 2 \right) + \left(\frac{1}{a^2} + \frac{r}{2} \right) (1 + az)^{-1/2} ,$$

$$y = \frac{1}{a} \left((1 + az)^{1/2} - 1 \right) , \tag{38}$$

and

$$e^{2\Phi} = x(1 + ay) - y^2 . \tag{39}$$

The two-dimensional models (33) and (34) correspond to some analytic continuations of "dressed coset" models in [17].[5]

3 Renormalization

In this section we investigate the behaviour under the renormalization group of our 2-dimensional dually related models (35) with conformal factors given by (37) and (39). For the cases of the usual Abelian and non-Abelian dualities, a similar investigation was performed for a one-parameter family of deformations of the Principal Chiral model for $SU(2)$, with metric given in (30), and its non-Abelian dual [7, 18] in [18].

Consider a two-dimensional σ-model with Lagrangian density $\mathcal{L} = \frac{1}{2\lambda} Q^+_{\mu\nu} \partial_+ X^\mu \partial_- X^\nu$, where $Q^+_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$. It will be renormalizable if the corresponding counter-terms, at a given order in a loop expansion, can be absorbed into a renormalization of the coupling

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[5]A comparison of (34) with the corresponding metric (after analytic continuation) in [17] is facilitated when the latter is also cast into the form (35). The conformal factor is exactly the same as that in (39), but the necessary coordinate transformation to bring the metric into that form is different than (38).

constant λ and (or) of some parameters labelled collectively a. In addition, we allow for general field redefinitions, which are coordinate reparametrizations in the target space. This definition of renormalizability of σ-models is quiet strict and similar to that for ordinary field theories. A natural extension of this is to allow for the manifold to vary with the mass scale and the renormalization group to act in the infinite dimensional space of all metrics and torsions [19]. Further discussion of this generalized renormalizability will not be needed for our puproses. Perturbatively in powers of λ we express the bare quantities, denoted by a zero as a subscript, as

$$\lambda_0 = \mu^\epsilon \lambda \left(1 + \frac{J_1(a)}{\pi\epsilon}\lambda + \frac{J_2(a)}{8\pi^2\epsilon}\lambda^2 \cdots \right) \equiv \mu^\epsilon \lambda \left(1 + \frac{y_\lambda}{\epsilon} + \cdots \right) \ ,$$

$$a_0 = a + \frac{a_1(a)}{\pi\epsilon}\lambda + \frac{a_2(a)}{8\pi^2\epsilon}\lambda^2 + \cdots \equiv a \left(1 + \frac{y_a}{\epsilon} + \cdots \right) \ , \qquad (40)$$

$$X_0^\mu = X^\mu + \frac{X_1^\mu(X,a)}{\pi\epsilon}\lambda + \frac{X_2^\mu(X,a)}{8\pi^2\epsilon}\lambda^2 + \cdots \ .$$

The ellipses stand for higher-order loop- and pole-terms in λ and ϵ respectively. Then, the beta-functions up to two loops are given by $\beta_\lambda = \lambda^2 \frac{\partial y_\lambda}{\partial \lambda} = \frac{\lambda^2}{\pi}\left(J_1 + J_2\lambda/(4\pi)\right)$ and $\beta_a = \lambda a \frac{\partial y_a}{\partial \lambda} = \frac{\lambda}{\pi}\left(a_1 + a_2\lambda/(4\pi)\right)$. The equations to be satisfied by appropriately choosing J_i, a_i and X_i^μ, with $i = 1, 2$, are given by

$$T_{\mu\nu}^{(i)} = -J_i Q_{\mu\nu}^+ + \partial_a Q_{\mu\nu}^+ a_i + \partial_\lambda Q_{\mu\nu}^+ X_i^\lambda + Q_{\lambda\nu}^+ \partial_\mu X_i^\lambda + Q_{\mu\lambda}^+ \partial_\nu X_i^\lambda \ , \qquad i = 1, 2 \ , \qquad (41)$$

where the corresponding counter-terms computed in the dimensional regularization scheme are (see, for instance, [20])

$$T_{\mu\nu}^{(1)} = \frac{1}{2} R_{\mu\nu}^- \ , \qquad T_{\mu\nu}^{(2)} = \frac{1}{4} R_{\mu\lambda\rho\sigma}^- Y^{\rho\sigma\lambda}{}_\nu \ ,$$

$$Y_{\rho\sigma\lambda\nu} \equiv -2R_{\rho\sigma\lambda\nu}^- + 3R_{[\lambda\rho\sigma]\nu}^- + \frac{1}{2}(H_{\lambda\rho}^2 G_{\sigma\nu} - H_{\lambda\sigma}^2 G_{\rho\nu}) \ , \qquad (42)$$

with $R_{\mu\nu\rho\lambda}^-$ and $R_{\mu\nu}^-$ being the "generalized" curvature and Ricci tensors constructed with connections that include the torsion. In principle we may add a term on the right-hand side corresponding to a redefinition (gauge transformation) of the antisymmetric tensor as $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\Lambda_{\nu]}$. Such a gauge transformation will affect only the antisymmetric part of (41).

Next we specialize to the case of two-dimensional metrics of the form (35) with ($X^1 = z = x + iy$, $X^2 = \bar{z} = x - iy$). If we work out (41) for $\mu = \nu = 1$ and $\mu = \nu = 2$, we obtain the conditions $\partial_z \bar{z}_i = 0$ and $\partial_{\bar{z}} z_i = 0$ (for $i = 1, 2$) respectively. Setting $\mu = 1$, $\nu = 2$ we obtain instead

$$e^{2\phi}\partial_z\partial_{\bar{z}}\Phi = -\frac{J_1}{2} - \partial_a\Phi a_1 + \left(\frac{1}{2}\partial_z - \partial_z\Phi\right)z_1 + \left(\frac{1}{2}\partial_{\bar{z}} - \partial_{\bar{z}}\Phi\right)\bar{z}_1 \ , \qquad (43)$$

$$8e^{4\phi}(\partial_z\partial_{\bar{z}}\Phi)^2 = -\frac{J_2}{2} - \partial_a\Phi a_2 + \left(\frac{1}{2}\partial_z - \partial_z\Phi\right)z_2 + \left(\frac{1}{2}\partial_{\bar{z}} - \partial_{\bar{z}}\Phi\right)\bar{z}_2 \ . \qquad (44)$$

For two-dimensional models the antisymmetric tensor is a pure gauge. It has no effect on (41) since the latter has only a symmetric part. At the 1-loop level, there remains to compute the constants J_1, ϵ_1 and the holomorphic function $z_1(z)$. The function $\bar{z}_1(\bar{z})$ is the complex conjugate of $z_1(z)$. For the model (33) we found, using (43), that

$$J_1 = -\frac{1}{2}(1 + a^2) , \qquad a_1 = \frac{1}{2}a(1 + a^2) ,$$

$$x_1 = \frac{1}{4}a^2 x , \qquad y_1 = \frac{1}{4}a^2 y , \tag{45}$$

where $z_1 = x_1 + iy_1$. The corresponding computation for the model (39) gives

$$J_1 = -\frac{1}{2}(1 + a^2) , \qquad a_1 = \frac{1}{2}a(1 + a^2) ,$$

$$x_1 = \frac{1}{4} - \left(1 + \frac{3}{4}a^2\right)x - \frac{a}{4}y , \qquad y_1 = -\frac{a}{4} + \frac{a}{4}x - \left(1 + \frac{3}{4}a^2\right)y . \tag{46}$$

Since J_1, a_1 are the same for both models, we conclude that they have the same 1-loop beta-functions, which are explicitly given by[6]

$$\beta_\lambda = -\frac{\lambda^2}{2\pi}(1 + a^2) , \qquad \beta_a = \frac{\lambda}{2\pi}a(1 + a^2) . \tag{47}$$

Notice that according to this system of equations a runs to infinity and the product λa remains constant. Hence, under renormalization flow, λ approaches zero. However, in the ultraviolet the fixed point is not a trivial one. This is easily seen by performing the scaling of variables $x \to xa^{1/2}$ and $y \to ya^{1/2}$ in (37) and (39), and then taking the limit $a \to \infty$. We also have to rescale the coupling $\lambda \to \lambda/a$, so that the product λa is kept fixed. In the limit $a \to \infty$ the resulting metrics are well defined and again of the form (35), with the conformal factors given by $e^{2\Phi\infty} = 1/4(1 + (x^2 + y^2)^2)$ (also with the topology of the 2-sphere) and $\epsilon^{2\Phi\infty} = xy$ respectively. Since these do not correspond to flat metrics, the theories do not become asymptotically free. This is to be contrasted with the behaviour of β_λ for the $O(3)$-chiral model and its non-Abelian dual that is obtained from β_λ in (47) for $a = 0$. For these theories the coupling runs to zero in the ultraviolet and therefore they are asymptotically free. Hence, at the quantum level, a limit under which Poisson–Lie T-duality reduces to the usual non-Abelian duality does not exist. We also note that the σ-models corresponding to the geometries with the Φ_∞'s above are no longer renormalizable. This is natural as they correspond to the fixed point at

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[6]We have also checked the renormalizability of the models at the 2-loop level and found that (44) admits no solution with the exception of the model (33), which becomes renormalizable when $a = 0$ (with $J_2 = -1$, in agreement with [18]). Hence, the corresponding counter-term cannot be absorbed into a renormalization of the couplings and of the fields. This presumably implies that the corresponding Poisson–Lie T-duality transformation rules need to be corrected order by order in perturbation theory as it is the case for ordinary T-dualities [21, 18]. Such a conclusion is also supported by the expectation that the non-linear generating functional (26) will receive quantum corrections [15]. This implies that the induced canonical transformation, hence the Poisson–Lie T-duality rules, will be accordingly corrected.

the ultraviolet of the geometries corresponding to the Φ's, but it can also be checked independently by seeing that there are no J_1, a_1 and z_1 that solve (43).

We have mentioned that the Drinfeld double we have considered, in the limit of $SU(2)$ contracting into an Abelian group, will be appropriate for σ-models related by non-Abelian duality with respect to the E_3 group. As was shown in [22], such models are problematic as far as conformal invariance is concerned, since the algebra structure constants are not traceless. This is not in conflict with our result that for a particular "non-conformal" example the 1-loop beta-functions are equivalent. As we have seen, taking singular limits does not always produce quantum mechanically the expected result from classical considerations.[7] Contraction of $SU(2)$ into an Abelian group is certainly such a limit. This issue does deserve further investigation.

There are several directions in which the present work can be extended. One should further investigate the conditions (41) in full generality and classify all σ-models related by Poisson–Lie T-duality that have equivalent beta-functions, as in the example we have considered. An important question is whether or not the limit of Poisson–Lie T-duality to non-Abelian duality, i.e. when one of the groups becomes Abelian, exists quantum mechanically. Our present investigation suggests that this is not always the case, but one would like to know to what extent this is a general statement and not a model-dependent one. Finally, the construction of conformal σ-models related by genuine Poisson–Lie T-duality is an important open problem since it is of direct relevance to string theory. An important step towards this will be the complete classification of all Drinfeld doubles.

Note added

1. The comments of this paragraph follow a discussion with J. Iliopoulos.

The fact that the renormalization group flow drives the σ-model (33) (and also (34)) away from the point $a = 0$ is quite similar to phenomena discussed some years ago in the context of usual four dimensional field theories, in searching for the origin of symmetries in Nature (see [25] and refs. therein). In some of these examples, enhanced symmetry points in the space of couplings and parameters acted as infrared repulsors due to the fact that the theories corresponding to the enhanced symmetry points were asymptotically

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[7]Nevertheless, this was the case with the singular limits considered in [23, 9]. In these cases one starts with exact string solutions and, after the limit is taken, one obtains plane wave (among other) exact solutions to string theory as well. Hence, in these examples conformal invariance was already an input before any limits were taken.

free. In our case the enhanced symmetry point, with an $SU(2)$ symmetry group, is at $a = 0$ and the corresponding σ-model for the S^2 metric is indeed asymptotically free.

2. The referee suggested that the reason that the σ-models (33) and (34) are not 2-loop renormalizable, in the strict field theoretical sense, might be due to the fact that we have not allowed for the most general renormalization of constants possible. Indeed, there is a more general class of 2-dimensional σ-models that are related by Poisson–Lie T-duality (see also [17]). They can be obtained by taking a non-diagonal matrix E_0 in (1) and (2). For our purposes it is enough to concentrate on the generalization of (33), which reads

$$ds^2 = \frac{1}{1 + a^2(b - \sin^2 \frac{\theta}{2})^2} \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \ ,$$

where a, b are constants. It is easy to check that this is the most general form preserving the isometry corresponding to shifts of ϕ and also keeping $e^{2\Phi}$ a quadratic in $z\bar{z}$ polynomial, when the metric is written in the form (35) after using (36). It is a lengthly, but straightforward, calculation to verify that the condition for a 2-loop renormalizability (44) is not satisfied for any choice of a_2, b_2, J_2 and $z_2, \bar{z}_2$.

Hence, the only reasonable conclusion is that the Poisson–Lie T-duality transformations have to be modified to this order. This is in agreement with the general arguments presented in footnote 6.

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