RF Empty Bucket Channelling Combined with a Betatron Core to Improve Slow Extraction in Medical Synchrotrons

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ABSTRACT

The uniformity of a slow-extracted beam from a synchrotron is degraded by ripples from the power converters of the magnetic elements. This effect can be reduced by making the beam particles cross more quickly from the stable to the unstable region. Among the various methods that have been proposed for this purpose, RF bucket channelling seems to be a good candidate for compensating low frequency ripples in spills of the order of one second. The method is based on the technique of RF phase displacement acceleration. In the configuration studied, a coasting beam is accelerated slowly into a third-order resonance by a betatron core. The acceleration rate set by the betatron core determines the spill length. Empty buckets are then created at the resonance frequency and adjusted with a phase angle that would decelerate any trapped beam by an equal and opposite amount. The main RF system can be used for this purpose.

The empty buckets cause an obstruction in phase space and the beam particles are forced to channel around the buckets. The particle speed is thus increased as the particle crosses into the resonance region, making the extraction less sensitive to ripple. The energy at which the particles enter in the unstable region is not fixed, but depends on their momentum and betatron amplitude. An improvement factor is obtained for all momenta and betatron amplitudes, provided that the bucket is properly positioned and that its half height is greater than the energy spread engaged in the resonance. The existing theory is extended to show that particles entering at different energies get different improvement factors, and how the improvement in the spill depends on the frequency and amplitude of the ripple.

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1. INTRODUCTION

Resonant slow extraction was first proposed by H. Hereward [1] and is often used to obtain long spills. The basic scheme for extraction is described with the help of the Steinbach diagram in Figure 1. Sextupoles create a region of instability in the phase plane. A particle that enters this region is trapped in the resonance, rapidly increases its betatron amplitude, until it passes through an electrostatic septum (ES), where it receives a kick and is ejected.
During the spill, the borders of the resonance region are not well defined, due to ripples in the power supplies of the magnetic elements. The resonance line move as well as the position of the waiting stack, and this results in a modulation of the spill.

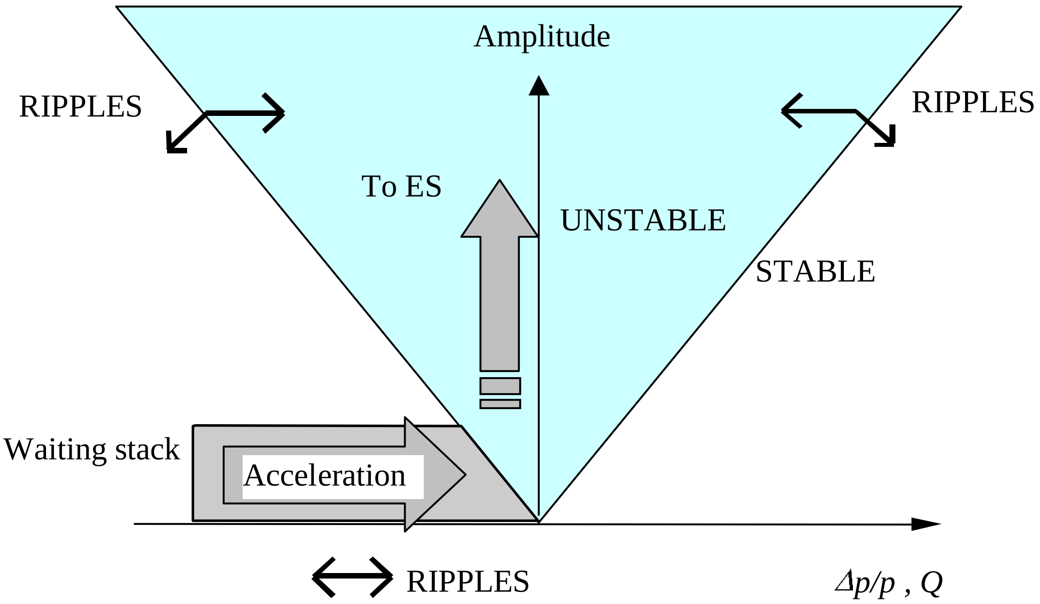


Figure 1. The Steinbach diagram for the extraction scheme (the black arrows show the relative movements of the resonance and the stack).

The spill *S(t)* can be written as:

$$S(t) = \frac{dN}{dt} = \frac{dN}{dQ} \cdot \frac{dQ}{dt} \qquad (1)$$

where N is the number of particles and Q the horizontal tune. The tune change can be expressed as the sum of two components, $\dot{Q}_0$ the constant component and $\dot{Q}_r$ the component due to unwanted ripple, so that:

$$\frac{dQ}{dt} = \dot{Q}_0 + \dot{Q}_r = \dot{Q}_0 \cdot (1 + \frac{\dot{Q}_r}{\dot{Q}_0}) \qquad (2)$$

Extraction can be performed by either accelerating the stack into the resonance, or by displacing the resonance so as to progressively ‘eat’ the stack. RF bucket channelling can be used in both cases, but in the following, the first possibility will be considered, since this is the solution adopted for the Proton Ion Medical Machine Study (PIMMS)[2] at CERN. An induction accelerator, a betatron core, is used to provide the acceleration. This high inductance device is well suited to deliver a very smooth spill, and, since the energy stored is high, it has the feature to respond slowly to transients that could give unintentional beam spikes to the patient (see reference [2]).

For a uniform spill *S(t)*, the product of *dN/dQ* and *dQ/dt* must be kept constant. The form of the stack determines *dN/dQ*, but this can be controlled by feedback on the acceleration rate from the betatron core.

Unfortunately, *dQ/dt* is affected strongly at all frequencies by $\dot{Q}_r$. The purpose of the empty bucket channelling is to reduce the effect of this contribution.
In general, if the spill time is fixed, the best is to have the largest possible momentum spread, since this reduces the time needed to cross the resonance boundary, which is the point at which the ripple disturbs the spill quality.

The contribution of $\dot{Q}_r$ can be reduced, if $\dot{Q}_0$ is increased. $\dot{Q}_0$ can not be changed for the whole stack since it is fixed by the spill time, but it can be increased in particular phase space regions, if the density of the particles *dN/dQ* in those regions is decreased accordingly, so as to keep a constant *S(t)* (see (1)). The scheme shown in Figure 2 has a region of high speed, low density created close to the resonance.

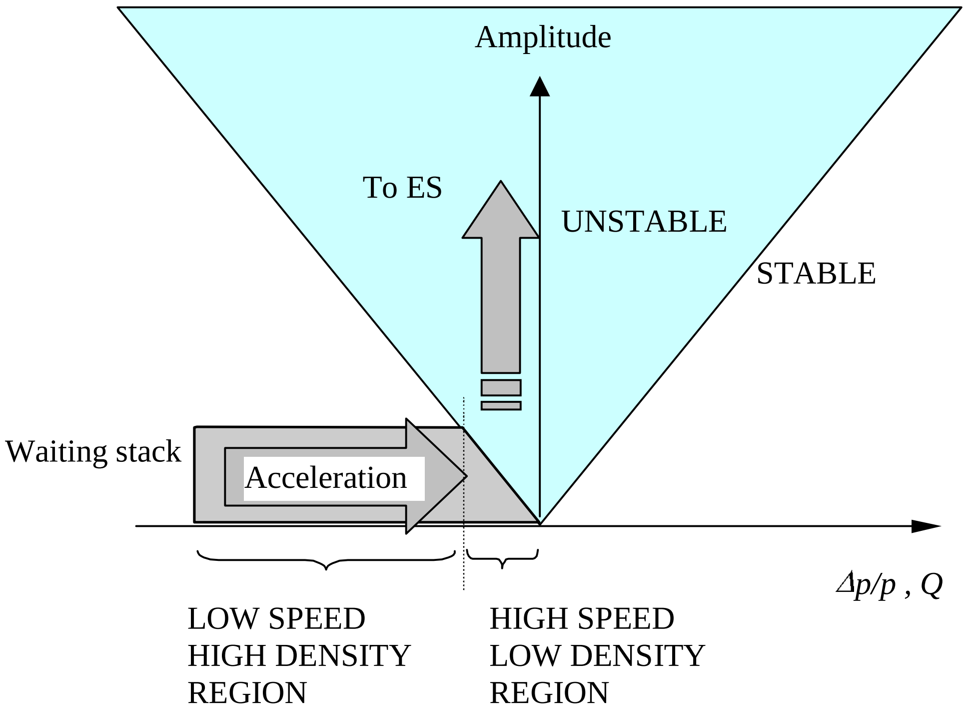


Figure 2. Scheme with the creation of a high speed low density region close to the resonance.

[2] PIMMS is the Proton Ion Medical Machine Study. This study is being hosted by the CERN PS Division and it is within the framework of this study that the present work has been carried out.

The parameter that indicates the quality of the extracted spill is the duty factor given by:

$$F = \frac{\left(\int_{T_{spill}} S(t)dt \right)^2}{T_{spill} \int_{T_{spill}} S^2(t)dt}$$

and after the substitution of (2):

$$F = \frac{1}{1 + \frac{1}{2}\left(\frac{\dot{Q}_r}{\dot{Q}_0} \right)^2} \qquad \text{for} \qquad \frac{\dot{Q}_r}{\dot{Q}_0} \leq 1.$$

The improvement in duty factor is given by an increase in $\dot{Q}_0$ by a factor *K*. Since the increase in $\dot{Q}_0$ is given by an increase in the local acceleration, the ratio $\frac{\dot{Q}_0}{\dot{Q}_r}$ can be re-expressed to give ($\dot{p}_r$ is the "acceleration" due to ripple):

$$F = \frac{1}{1 + \frac{1}{2}\left(\frac{\dot{p}_r}{K \cdot \dot{p}_0} \right)^2} \qquad .(3)$$

2. THE METHOD

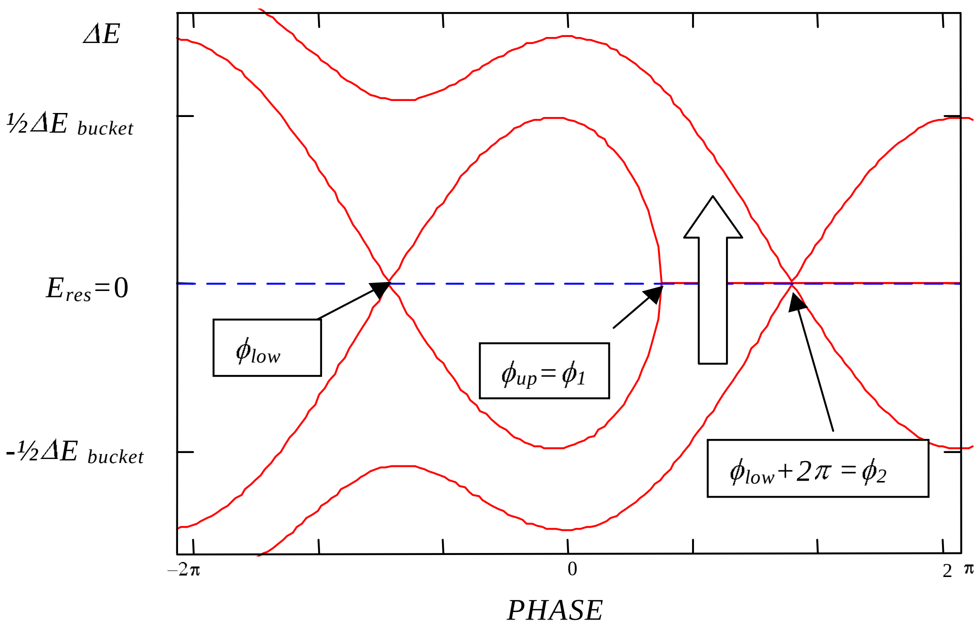


Figure 3. *Particle channelling between buckets (arrow), case of a decelerating bucket below transition, or accelerating bucket above transition (ϕ_s<0). E is the total energy, E_{res}=0 is the energy at which the empty bucket is positioned, and $^1/_2\Delta E_{bucket}$ is the bucket half height. In the case E = E_{res} (i.e. ΔE = 0), the bucket limits in phase are ϕ_{low} and ϕ_{up} and the channel width is $\Delta\phi = \phi_2 - \phi_1$).*

An empty bucket is created at the resonance frequency corresponding to the resonance energy E_{res}. The hardware should be set in order to keep the bucket frequency 'fixed' at the resonance frequency for the whole spill time. At the starting time the beam is out of the resonance. The flux of the extracted particles is set by the acceleration impressed by the betatron core.

In the longitudinal phase plane, the particles turn around the bucket, ideally without entering it. This is similar to what happens when accelerating a stack by the RF phase displacement method [3], except that the stack moves and the bucket remains fixed. At resonance energy, the particles are swept between the buckets in the small phase interval $\Delta\phi$ limited by the bucket separatrices. In this channel, dN/dQ is reduced, and dQ/dt increased. In other words, the empty bucket creates a 'bottle neck' in the phase space, through which the particles are swept with increased velocity $K\dot{Q}_0$ see Figure 3.

3. CALCULATION OF THE AVERAGE MULTIPLYING FACTOR *K*

Since dQ/dt is proportional to dp/dt, and dp/dt is proportional to dE/dt, it is sufficient to calculate the improvement of dE/dt when particles cross the resonance energy.

Using the Hamiltonian formalism (see reference [4]), the motion in the longitudinal phase plane can be described in terms of two first order differential equations in the conjugate variables ($\Delta E/h\omega_0$, $\Delta\phi=\phi-\phi_s$):

$$\frac{d\left(\dfrac{\Delta E}{h\omega_0}\right)}{dt} = \frac{qV}{2\pi h}(\sin\phi - \sin\phi_s) \tag{4.1}$$

$$\frac{d(\phi-\phi_s)}{dt} = -\omega_0^2 h^2 \frac{\eta}{\beta^2 E}\left(\frac{\Delta E}{h\omega_0}\right) \tag{4.2}$$

where q is the charge, ω_0 is the revolution frequency, V is the RF voltage, $\beta=v/c$ is the particle normalised velocity, h is the harmonic number, $\eta=1/\gamma^2-1/\gamma_t^2$ is the phase slip factor, γ is the relativistic mass factor, γ_t the γ at transition, E is the total energy, ϕ is the phase of the arbitrary particle and corresponds to the phase of the RF voltage, and 's' refers to the synchronous particle.

The corresponding Hamiltonian H is:

$$H = \frac{1}{2}\frac{\eta}{\beta^2 E}(\Delta E)^2 + \frac{qV}{2\pi h}(\sin\phi_s(\phi_s-\phi) + \cos\phi_s - \cos\phi) \quad . \tag{5}$$

In the hypothesis that the acceleration is smooth and continuous over one turn, we have (the bucket is empty, but the outside particles are still affected):

$$\frac{dE}{dt} = \frac{qV}{2\pi}\cdot\omega_0\cdot(\sin\phi - \sin\phi_s)\,. \tag{6}$$

A particle that crosses the resonance when $\phi = 0$ is not affected by the RF voltage. Its energy will vary by:

$$\left(\frac{dE}{dt}\right)_0 = -\frac{q\ V}{2\pi}\cdot\omega_0\cdot\sin\phi_s, \qquad (7)$$

which is exactly equal to the *dE/dt* given by the betatron core. Other particles will be affected in different ways depending on their arbitrary phase and energy. On average:

$$\left(\frac{dE}{dt}\right)_{AV} = \frac{1}{\phi_2-\phi_1}\int_{\phi_1}^{\phi_2}\frac{q\ V}{2\pi}\cdot\omega_0\cdot(\sin\phi-\sin\phi_s)d\phi$$

ϕ_1 and ϕ_2 are the minimum and maximum phases of the channel (see Figure 3 in the case $\Delta E = 0$). The average multiplying factor is given by:

$$K = \frac{\left(\frac{dE}{dt}\right)_{AV}}{\left(\frac{dE}{dt}\right)_0} = -\frac{1}{(\phi_2-\phi_1)\sin\phi_s}\int_{\phi_1}^{\phi_2}(\sin\phi-\sin\phi_s)d\phi \qquad (8)$$

which gives:

$$K = \frac{\Gamma\Delta\phi+\cos\phi_2-\cos\phi_1}{\Gamma\Delta\phi} = 1+\frac{\cos\phi_2-\cos\phi_1}{\Gamma\Delta\phi} \qquad (9)$$

where $\Gamma=sin\phi_s$ and $\Delta\phi=\phi_2-\phi_1$. Intuitively, it can be seen that the more the bucket obstructs the available phase space, the faster the particles must move. Thus, the closer the bucket becomes to a stationary bucket the greater the particle velocity enhancement. This is the reason to study, in the following, quasi-stationary buckets.

4. AVERAGE MULTIPLYING FACTOR IN THE CASE $\Delta E = E\text{-}E_{res} = 0$

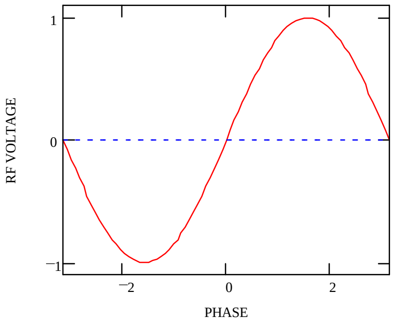
In the case $\Delta E = 0$, *K* can be calculated for all the different cases sketched in Figure 4. Table 1 gives the needed values for all the variables involved.

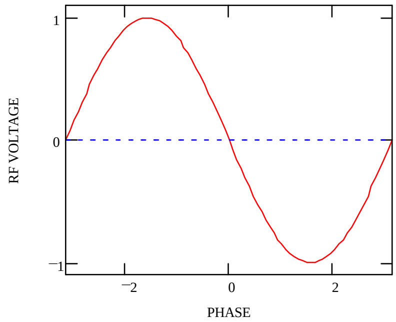
ϕ_s~0 ($\Gamma=sin\phi_s$)	*Below transition* Accelerating bucket $\phi_{s>0}$	Decelerating bucket $\phi_{s<0}$	*Above transition* Accelerating bucket $\phi_{s<0}$	Decelerating bucket $\phi_{s>0}$
$\Delta\phi = \phi_2\text{-}\phi_1$	$\phi_l\text{-}(\phi_u\text{-}2\pi)$	$(\phi_l+2\pi)\text{-}\phi_u$	$\phi_l\text{-}(\phi_u\text{-}2\pi)$	$(\phi_l+2\pi)\text{-}\phi_u$
$cos\phi_s$	$1-\Gamma^2/2$	$1-\Gamma^2/2$	$1-\Gamma^2/2$	$1-\Gamma^2/2$
ϕ_l	$\sim -\pi$	$-\pi\text{-}\phi_s$	$-\pi\text{-}\phi_s$	$\sim -\pi$
ϕ_u	$\pi\text{-}\phi_s$	$\sim \pi$	$\sim \pi$	$\pi\text{-}\phi_s$
$cos\phi_l$	$-1+\frac{(\phi_l+\pi)^2}{2}$	$-cos\phi_s$	$-cos\phi_s$	$-1+\frac{(\phi_l+\pi)^2}{2}$
$cos\phi_u$	$-cos\phi_s$	$-1+\frac{(\phi_u-\pi)^2}{2}$	$-1+\frac{(\phi_u-\pi)^2}{2}$	$-cos\phi_s$

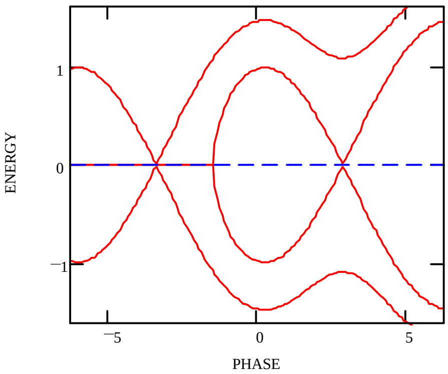
Table 1. Some useful bucket parameters. $Cos\phi_s$, $cos\phi_l$, and $cos\phi_u$ are given by the Taylor series expansions.

BELOW TRANSITION
particle rotation is counter clockwise

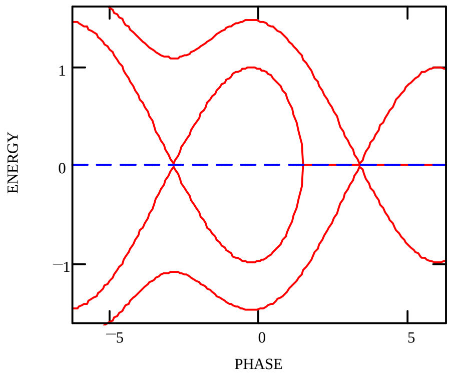
ABOVE TRANSITION
particle rotation is clockwise



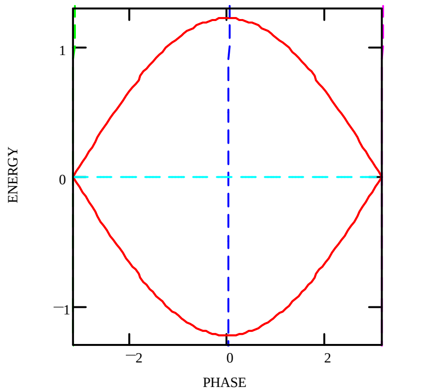




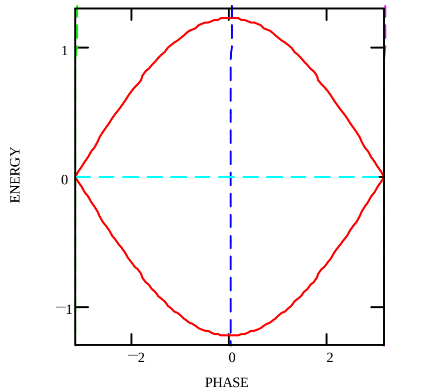
Accelerating bucket. $\phi_s>0$, $\Delta\phi=\phi_l-(\phi_u-2\pi)$



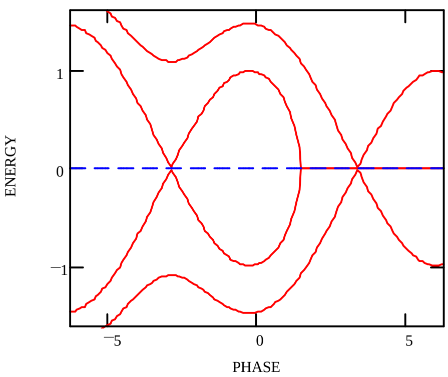
Accelerating bucket. $\phi_s<0$, $\Delta\phi=(\phi_l+2\pi)-\phi_u$



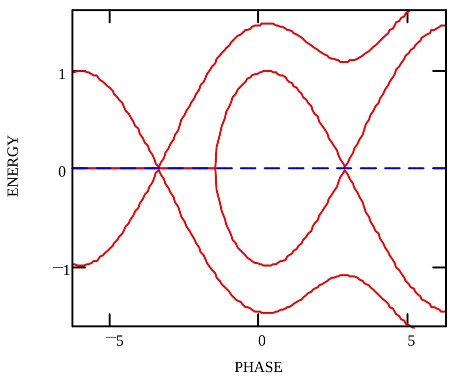
Stationary bucket. $\phi_s=0$, $\Delta\phi=0$ (no channel)



Stationary bucket. $\phi_s=0$, $\Delta\phi=0$ (no channel)



Decelerating bucket. $\phi_s<0$, $\Delta\phi=(\phi_l+2\pi)-\phi_u$



Decelerating bucket. $\phi_s>0$, $\Delta\phi=\phi_l-(\phi_u-2\pi)$

Figure 4. Possible cases for RF bucket channelling. ϕ_s is the synchronous phase and $\Delta\phi$ is the width of the channel for $\Delta E=0$.

Making use of the bucket relation [5]:

$$\phi_l \sin\phi_s + \cos\phi_l = \phi_u \sin\phi_s + \cos\phi_u \qquad (10)$$

and finding from Table 1 the values corresponding to the different cases, from equation (8) in any case we find the result:

$$\left(\frac{dE}{dt}\right)_{AV} = -\frac{q \cdot V}{2\pi} \cdot \omega_0 \cdot \frac{2\pi \cdot \sin\phi_s}{\Delta\phi}$$

and for the average multiplying factor:

$$K = \frac{\left(\frac{dE}{dt}\right)_{AV}}{\left(\frac{dE}{dt}\right)_0} = \frac{2\pi}{\Delta\phi} = \frac{2\pi}{2\pi - bucketwidth} \qquad (\Delta E = 0).$$

The formula has an easy geometrical interpretation: in the case $\Delta E = 0$, K is given by the ratio between the whole phase segment $0\text{-}2\pi$ and segment containing the allowed phases. In the hypothesis (usually true) that $\phi_s << 2\pi$, using (10) and Table 1 the channel width becomes:

$$\Delta\phi = \phi_l - \phi_u + 2\pi \approx 2\sqrt{\pi \cdot \Gamma}$$

and the multiplying factor:

$$K = \frac{2\pi}{\Delta\phi} = \sqrt{\frac{\pi}{\Gamma}} \qquad (\Delta E = 0). \qquad (11)$$

This agrees with the particular case given in reference [6] for the CERN/PS synchrotron, where an empty bucket is created above transition.

5. LINK BETWEEN THE RF VOLTAGE AND THE STABLE PHASE

The parameters of the RF bucket are determined by considering a fictitious particle trapped in the bucket at the synchronous phase. The RF frequency will be the revolution frequency (or harmonics) in order to keep the particle energy constant at the synchronous phase (in this case $dB/dt = 0$).

The energy losses or gains in the bucket will be those needed to compensate the changes that take place in the machine (in this case the energy gain in the betatron), i.e.:

$$\left(\frac{dE}{dt}\right)_{\text{cavity}} = -\left(\frac{dE}{dt}\right)_{\text{betatron core}}.$$

which can be written as [7]:

$$V \cdot |\Gamma| = 2\pi \cdot R \cdot \frac{d(B\rho)}{dt} \tag{12}$$

where R is the mean radius of the orbit, $B\rho$ is the beam rigidity, and all other parameters as before. The link with the beam momentum p is given by [8]:

$$p = \frac{Zq}{A} \cdot \frac{c}{10^9} \cdot B\rho \tag{13}$$

where Zq is the charge of the particle, A its atomic mass number, p is the beam momentum and c is the velocity of the light. The combination of (12) with (13) gives:

$$|\Gamma| = \frac{A}{Zq} \cdot \frac{2\pi \cdot 10^9}{c} \cdot \frac{R}{V} \cdot \frac{dp}{dt} \tag{14}$$

thus the RF voltage is proportional to the energy change in the core and inversely proportional to $\Gamma = \sin\phi_s$. In order to have a high multiplying factor, one should keep the RF voltage high in order to have a low Γ, since this shrinks the width of the channel. The limit will be fixed by the maximum RF voltage achievable.

6. EFFECT DUE TO DIFFERENT BETATRON AMPLITUDES

It can be seen directly from Figure 1 that the resonance energy of the particles depends on their betatron amplitude, the higher the betatron amplitude the lower the resonance energy, and vice versa. Qualitatively we can state that in order to obtain a high multiplying factor for all the particles, two conditions have to be fulfilled. One on the bucket height and the other on the bucket position.

The first condition says that the beam energy spread engaged into the resonance has to be smaller than the bucket half height, i.e.:

$$\Delta E_{\text{spread of resonance}} < \frac{1}{2} \cdot \Delta E_{\text{bucket}} \tag{15}$$

Since the height of the RF bucket is given by:

$$\frac{1}{2}\Delta E_{\text{bucket}} = \beta \cdot Y(\phi_s) \cdot \sqrt{\frac{qE \cdot V}{h\pi|\eta|}}$$

where $Y(\phi_s) = \sqrt{\sin\phi_s(2\phi_s \pm \pi) + 2\cos\phi_s}$, where the '+' sign is valid when $\phi_s > 0$.

This condition imposes a constraint on the RF voltage:

$$V > \frac{1}{\beta^2} \cdot \frac{1}{[Y(\phi_s)]^2} \cdot \frac{h\pi|\eta|}{qE} \cdot (\Delta E_{\text{spread of resonance}})^2 \qquad (16)$$

The second condition relates the position of the bucket to the resonance energy. In order to have a positive improvement for all the betatron amplitudes, one should position the bucket as shown in Figure 5. This is the case of a stack starting from energies lower than the resonance energy.

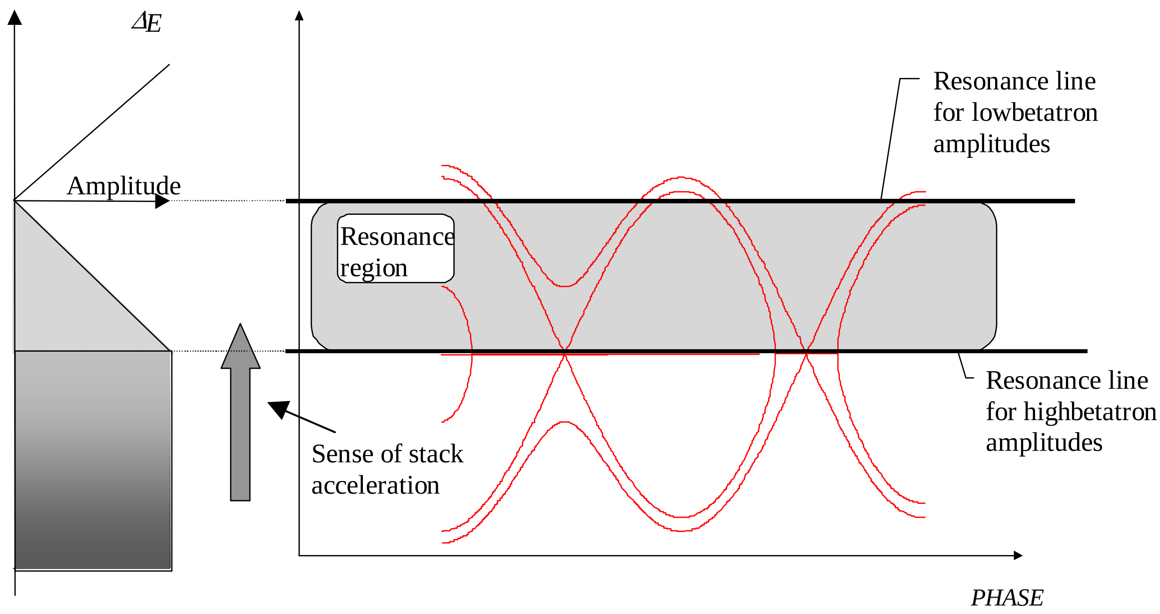


Figure 5. Position of the bucket with respect to the resonance region (in the case of PIMMS).

7. AVERAGE MULTIPLYING FACTOR IN THE GENERAL CASE

For the general case it is necessary to solve equation (9), with values of ϕ_1 and ϕ_2 depending on the value of ΔE (see Figure 5). Only $0 < \Delta E < \Delta E_{\text{bucket}}/2$, will be considered here, where the conditions of chapter 5 are fulfilled.

It is useful to calculate the Hamiltonians H_1 and H_2 corresponding to ϕ_1 and ϕ_2, the phase limits of the channel. The Hamiltonian H_1 corresponds to the 'internal' separatrix, and the Hamiltonian H_2 to the 'external' separatrix. From equation (5), and equation (10) (that links ϕ_1 and ϕ_2, in the case $\Delta E=0$), H_1 and H_2 for the accelerating bucket are found to be:

$$H_1 = \frac{qV}{2\pi h} \cdot (2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s) \qquad H_2 = \frac{qV}{2\pi h} \cdot (2\cos\phi_s + (2\phi_s + \pi)\sin\phi_s)$$

For the decelerating bucket H_1 becomes H_2 and vice versa.

ϕ_1 (ϕ_2) is the root of the equation:

$$H_{1,2} - \frac{1}{2}\frac{\eta}{\beta^2 E}\Delta E^2 - \frac{ZqV}{2\pi h}\cdot(\sin\phi_s(\phi_s - \phi) + \cos\phi_s - \cos\phi) = 0 \qquad (17)$$

for $$0 < \Delta E < \frac{\Delta E_{bucket}}{2}.$$

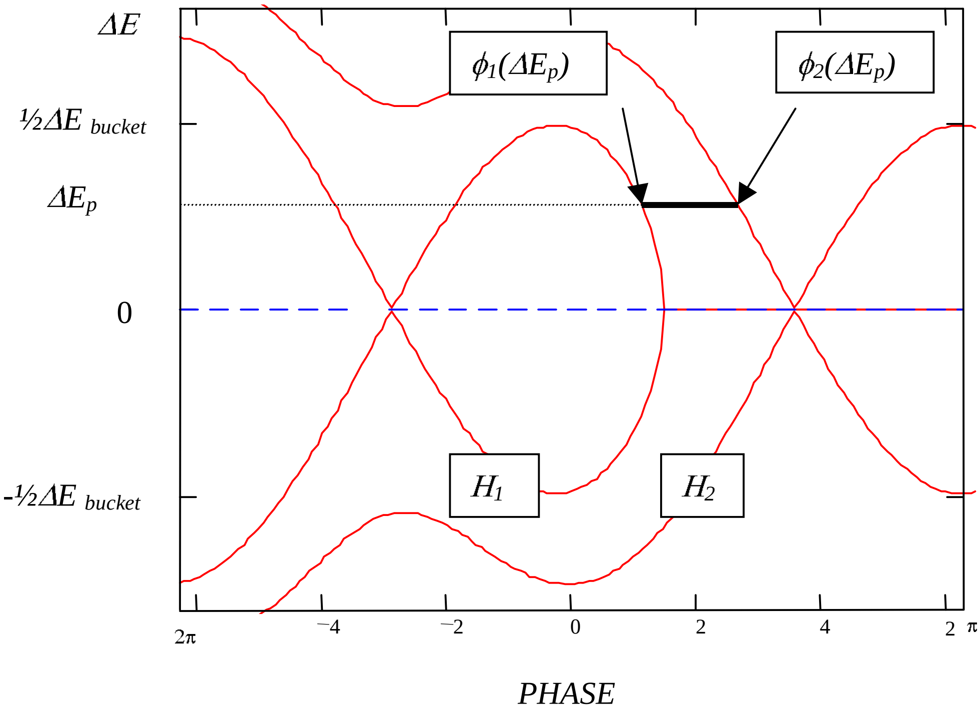


Figure 6. Position and parameters of the channel for a general $\Delta E = \Delta E_p \neq 0$. H_1 is the Hamiltonian corresponding to the 'internal' separatrix, H_2 is the Hamiltonian corresponding to the 'external' separatrix.

The solution of equation(s) (17) can be found numerically to give the value of the general multiplying factor *K* in function of ΔE.

$$K(\Delta E) = 1 + \frac{\cos[\phi_2(\Delta E)] - \cos[\phi_1(\Delta E)]}{\Gamma\cdot[\phi_2(\Delta E) - \phi_1(\Delta E)]}$$

8. DEPENDENCE OF THE AVERAGE MULTIPLYING FACTOR ON THE RIPPLE

In the hypothesis of a smooth and continuous momentum increase during extraction, we have:

$$\dot{p}_0 = \frac{\frac{\Delta p}{p}p_0}{T_{\text{spill}}} \qquad (18)$$

where p_0 is the momentum of the particle (without ripple), $\Delta p/p$ is the momentum spread at extraction energy, T_{spill} is the spill time in seconds.
The ripple contribution to the momentum at the frequency ω can be written as:

$$p_r(\omega) = p_{r0} + p_{r\omega}\sin\omega t \qquad (19)$$

where p_{r0} is the constant component, and $p_{r\omega}$ is the modulated component of the amplitude of the ripple at the particular frequency considered.
By defining:

$$\alpha_r = \frac{p_{r\omega}}{p_0 + p_{r0}} \approx \frac{p_{r\omega}}{p_0} \qquad (p_{r0} << p_0)$$

$\dot{p}_r$ becomes:

$$\dot{p}_r(\omega) = \omega p_{r\omega}\cos\omega t = \omega\alpha_r p_0 \cos\omega t\ .$$

The total momentum derivative during extraction is therefore:

$$\dot{p} = \dot{p}_0 + \dot{p}_{\text{r}}(\omega) = \frac{\frac{\Delta p}{p}\cdot p_0}{T_{spill}} + \omega\alpha_r p_0\cos\omega t\ . \qquad (20)$$

In the general case ($0 < \Delta E < \Delta E_{\text{bucket}}/2$) the average multiplying factor is given by (9). Inserting in (9) the results of (14) and (20) gives the new multiplying factor (for simplicity $\dot{p}_r(\omega) = \dot{p}_r$):

$$K(\Delta E,\phi) = 1 + \frac{\cos\phi_2 - \cos\phi_1}{\Delta\phi\cdot\left|\frac{A}{Zq}\cdot\frac{2\pi\cdot 10^9}{c}\cdot\frac{R}{V}\cdot p_0\cdot(v_0 + \omega\alpha_r\cos\omega t)\right|}$$

where: $v_0 = \frac{\Delta p}{p}\frac{1}{T_{spill}}$.

The duty factor is given by (3), inserting the new value for K:

$$F(\Delta E,\omega) = \frac{1}{1 + \frac{1}{2\cdot K^2(\Delta E,\omega)}\cdot\left(\frac{\dot{p}_r}{\dot{p}_0}\right)^2}\ .$$

In the case $\Delta E = 0$, we find a new expression for K: let K_0 be the average improving factor coming from (11). Inserting in (11) the results of (14) and (20) gives:

$$K(0,\omega) = \sqrt{\frac{Zq}{A} \cdot \frac{c}{2 \cdot 10^9} \cdot \frac{V}{R} \cdot \frac{1}{\dot{p}_0}} \cdot \sqrt{\frac{\dot{p}_0}{\dot{p}_r + \dot{p}_0}} = K_0 \cdot \sqrt{1 - \frac{\dot{p}_r}{\dot{p}_0 + \dot{p}_r}} \quad . \quad (21)$$

The duty factor can be written as:

$$F(0,\omega) = \frac{1}{1 + \frac{1}{2} \cdot \left(1 + \frac{\dot{p}_r}{\dot{p}_0}\right) \cdot \left(\frac{\dot{p}_r}{K_0 \dot{p}_0}\right)^2}$$

It is also interesting to know the minimum *K* and hence the poorest value to be expected for *F*. The minimum *K* and *F* are calculated for $\Delta E = 0$ and for $\dot{p}_r$ maximum, and are given by:

$$K_{\min}(0,\omega) = K_0 \sqrt{1 - \frac{\alpha_r \omega}{\frac{\Delta p}{p} \frac{1}{T_{\text{spill}}} + \alpha_r \omega}} \qquad F_{\min}(0,\omega) = \frac{1}{1 + \frac{1}{2} \cdot \left(1 + \frac{\omega \alpha_r}{v_0}\right) \cdot \left(\frac{\omega \alpha_r}{K_0 v_0}\right)^2} \ .$$

One sees that the multiplying factor decreases if the frequency of the ripple increases, or if the spill time increases. As an example, the multiplying factor for a 100% modulation (i.e. $\dot{p}_r = \dot{p}_0$) from (21) leads to:

$$K_{\min}(0,\omega) = \frac{K_0}{\sqrt{2}}$$

9. EFFECT DUE TO THE TRANSIT TIME INTO THE RESONANCE

Once in the resonance, the particles stay inside the machine for a certain time before they are extracted. This time is called transit time in the resonance and depends on the initial betatron amplitude of the particle. It is on average higher for particles with low betatron amplitudes [9]. During this time, the RF varies the energy of all the particles following hamiltonian trajectories and they are extracted at an energy different from the one at which they entered into the resonance. If this time is too long, some of the particles can cross back out of the resonance region and remain in the machine.
To analyse this behaviour, it is necessary to consider the particle motion outside the separatrix, when the motion is no longer oscillatory. From equation (4.1):

$$d\phi = -\omega_0 h \frac{\eta}{\beta^2 E} \Delta E \cdot dt \qquad (23)$$

where ΔE is found from equation (5):

$$\Delta E = \sqrt{\frac{2\beta^2 E}{\eta}} \cdot \sqrt{H - \frac{qV}{2\pi h} U(\phi)} \qquad (24)$$

with: $0 < \Delta E < \frac{1}{2}\Delta E_{bucket}$ and $U(\phi) = \sin\phi_s(\phi_s - \phi) + \cos\phi_s - \cos\phi$

After substitution (23):

$$dt = -\frac{\beta}{\omega_0 h}\cdot\sqrt{\frac{E}{2\eta}}\cdot\frac{1}{\sqrt{H - \frac{qV}{2\pi h}U(\phi)}}\cdot d\phi$$

and integration:

$$\tau = -\frac{\beta}{\omega_{rev}\cdot h}\cdot\sqrt{\frac{E}{2\eta}}\cdot\int\frac{1}{\sqrt{H - \frac{qV}{2\pi h}\cdot\left(\sin\phi_s\cdot(\phi - \phi_s) + \cos\phi_s - \cos\phi\right)}}\cdot d\phi \qquad (25)$$

where H is the Hamiltonian associated with the particle and E is the total energy of the particle. The integration has to start at the phase at which the particle crosses the resonance line, and ends at the phase of the re-crossing.

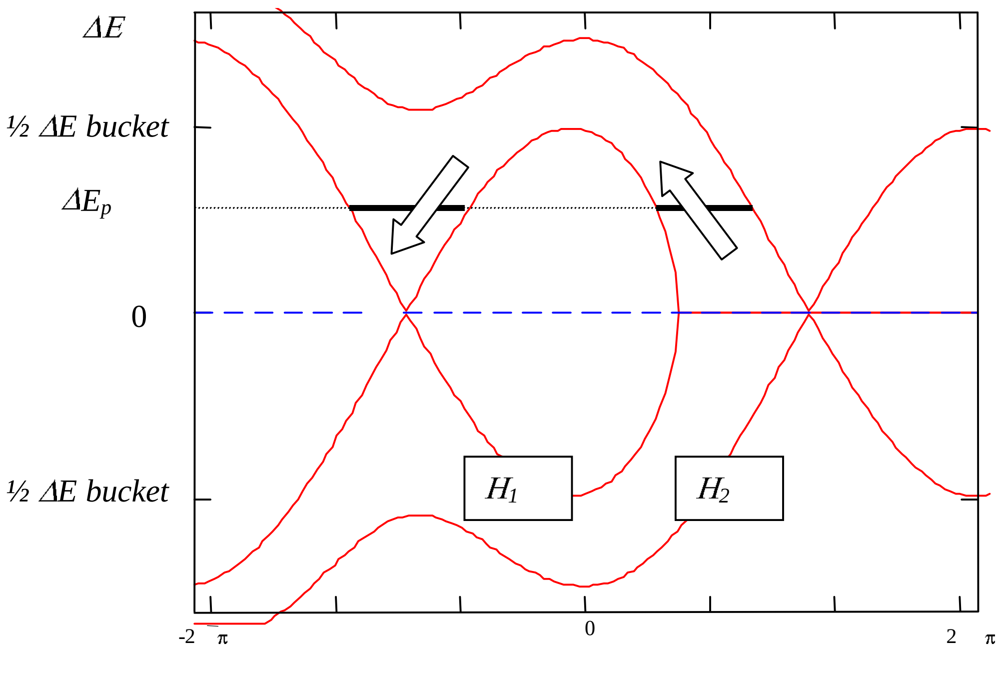


Figure 7. High betatron amplitude particles crossing into and then back out of the resonance (errors).

These integration limits are found solving equation (24) numerically for $\Delta E = \Delta E_p$, and $H = H_p$ where ΔE_p is the height of the resonance line for an arbitrary particle (see Figures 5 and 7), and H_p its Hamiltonian, which has two solutions in the interval $-\pi < \phi < \pi$. From the value of τ one can estimate the number of turns needed for the particle to re-cross (for the first time) the resonance.

In practice, this method gives an absolute minimum number of turns for the particle to cross back out of the resonance (the case of quasi-zero resonance excitation). While the particle is in the resonance its betatron amplitude grows and the resonance region expands downwards in Figure 5. The finite transit time into the resonance has no effect on the ripple, but can give a variation of the momentum spread of the extracted beam.

10. TYPICAL PARAMETERS FOR MEDICAL SLOW EXTRACTION

In the studied extraction scheme, the beam is prepared in a coasting stack outside the resonance region, with a suited momentum spread and then slowly accelerated into the resonance. The horizontal tune (i.e. the energy of the resonance) is kept constant and the stack is accelerated by a betatron core. In this way, the momentum spread of the beam delivered is small, because it is limited to the momentum spread engaged into the resonance and not influenced by the momentum spread in the stack. All the parameters of the lattice elements are also kept constant.

The list given follows the parameters chosen in PIMMS study, but the maximum energies considered are just indicative, since they can change depending on medical requests. The RF voltage satisfies the need to have a bucket half height bigger than the energy spread engaged.

Parameters of the machine:		
Type of particle	protons	C_{12}^{6+}
Energy [MeV/u]	300	425
γ_t	1.89	1.89
γ	1.32	1.456
β	0.653	0.727
η	0.294	0.192
Circumference [m]	75	75
Revolution time [μs]	0.3831	0.3439
Parameters of the stack:		
Beam rigidity $B\rho$ [Tm]	2.695	6.578
Flux variation [Weber]	0.809	1.974
$\Delta p/p$ momentum spread	4‰	4‰
$\Delta p/p$ momentum spread engaged into the resonance	1‰	1‰
ΔT energy spread engaged into the resonance [MeV]	0.5273	8.602
Parameters of the empty bucket (decelerating bucket below transition):		
RF Voltage [kV]	4	4
Harmonic number h	1	1
Spill time [s]	0.5	1
Number of turns needed for extraction	1.305E6	2.908E6
$\Gamma = sin\phi_s$	-4.043E-4	-4.933E-4
Bucket half height [MeV]	2.135	26.18
Hamiltonian of internal separatrix H_1	1272	7634
Hamiltonian of external separatrix H_2	1274	7645
Multiplying factor K_0 ($\Delta E=0$, no ripple)	88.131	79.77
Minimum number of turns to cross-back the resonance	1100	>5000

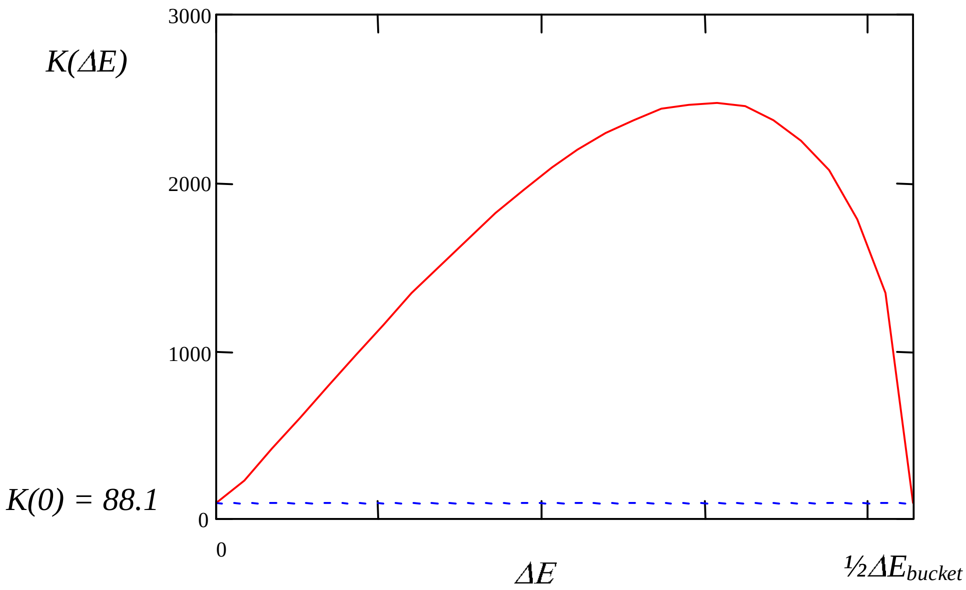


Figure 8. PROTONS 300MeV, Multiplying factor K versus ΔE for ω = 0 (PIMMS values).

The horizontal line corresponds to the value of K for ΔE = 0. Dependence on the ripple frequency and amplitude is not taken into account (ω = 0). The curve has a maximum inside the ΔE range. At the extremes, (i.e. for ΔE = 0 and $\Delta E = \Delta E_{bucket}/2$), the value given by equation (11) is still obtained.

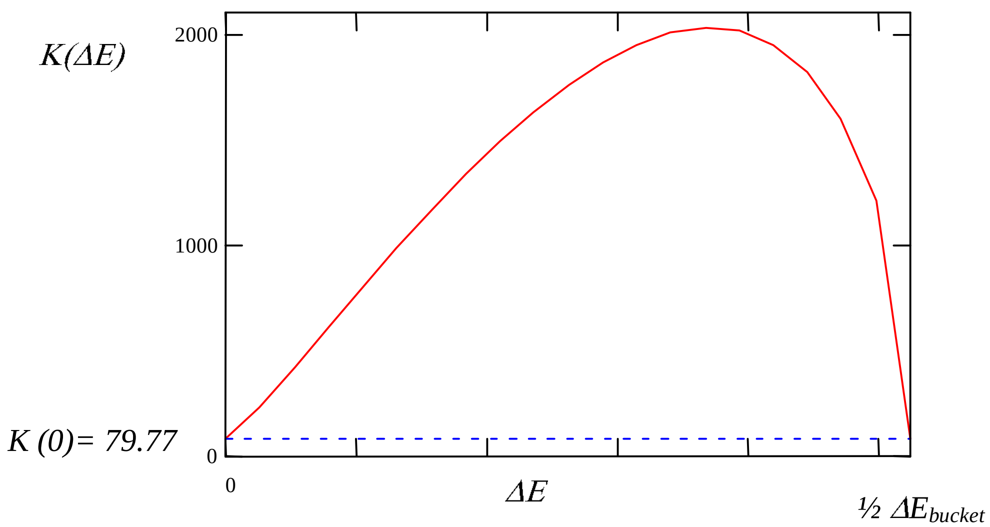
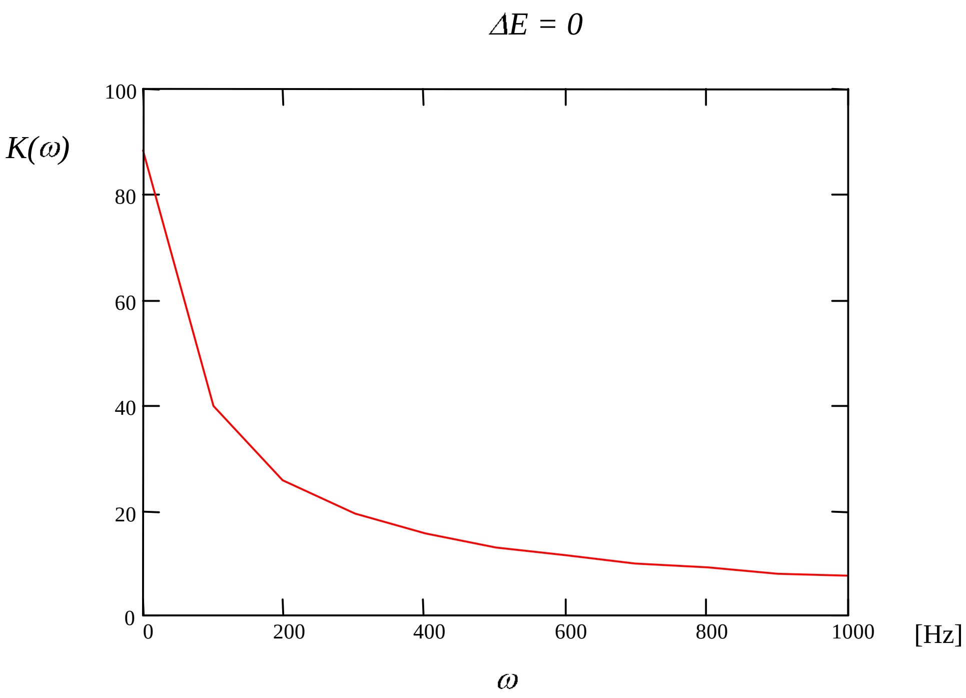


Figure 9. 425 MeV/u CARBON IONS C_{12}^{6+}, Multiplying factor K versus ΔE for ω = 0 (PIMMS values).

The horizontal line corresponds to the value of K for ΔE = 0. Dependence from the ripple frequency and amplitude is not taken into account (ω = 0). The curve has a maximum inside the ΔE range. At the extremes, (i.e. for ΔE = 0 and $\Delta E = \Delta E_{bucket}/2$), the value given by equation (11) is still obtained.



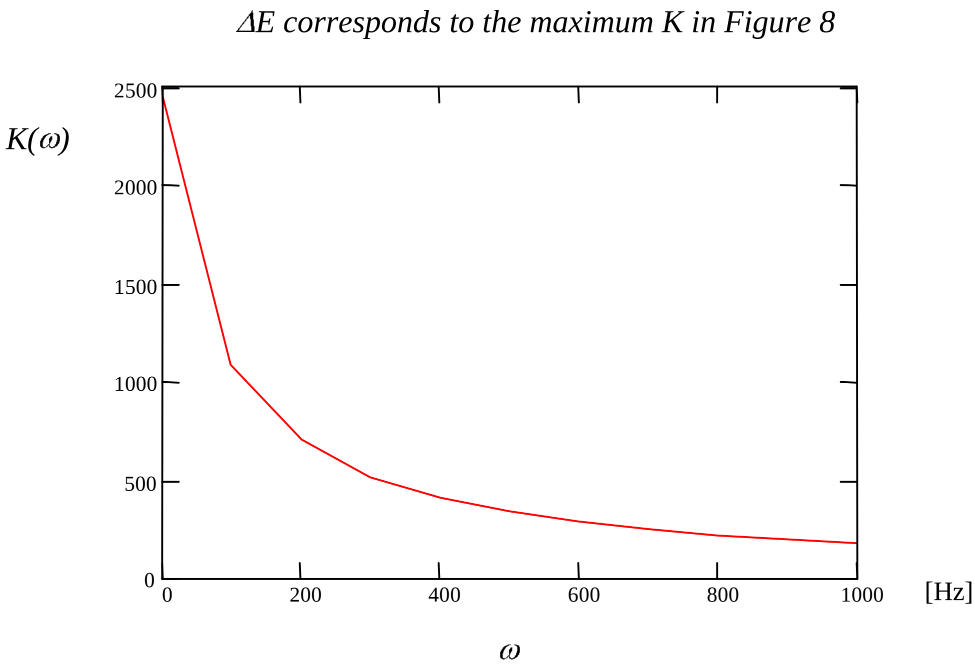
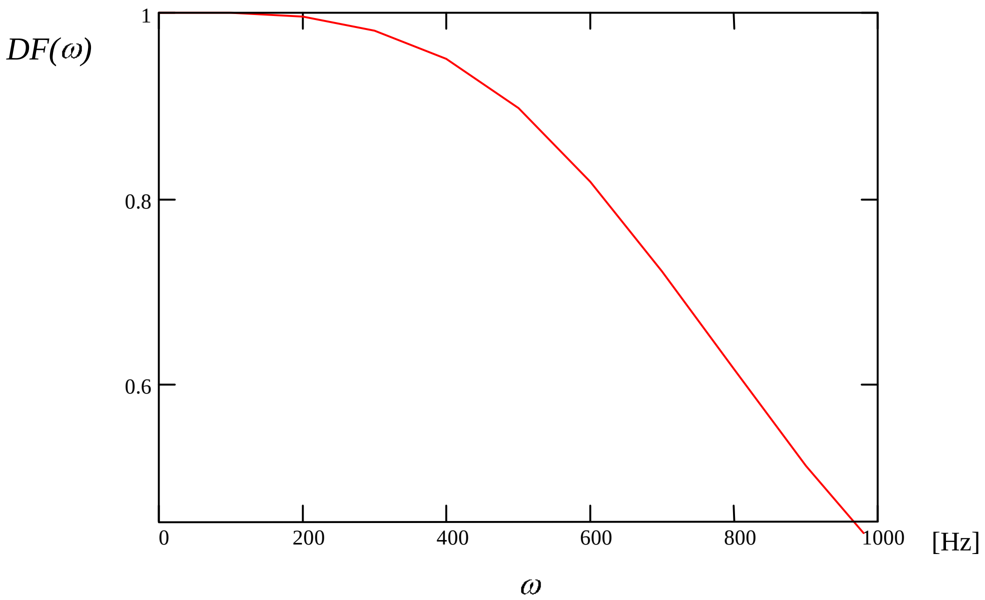


Figure 10. PROTONS 300 MeV: K versus ω, the ripple frequency.
(PIMMS values, with $\alpha_r(\omega) = p_r / p_0 = 10^{-4}$ taken arbitrarily).

ΔE=0



ΔE corresponds to the K maximum in Figure 8

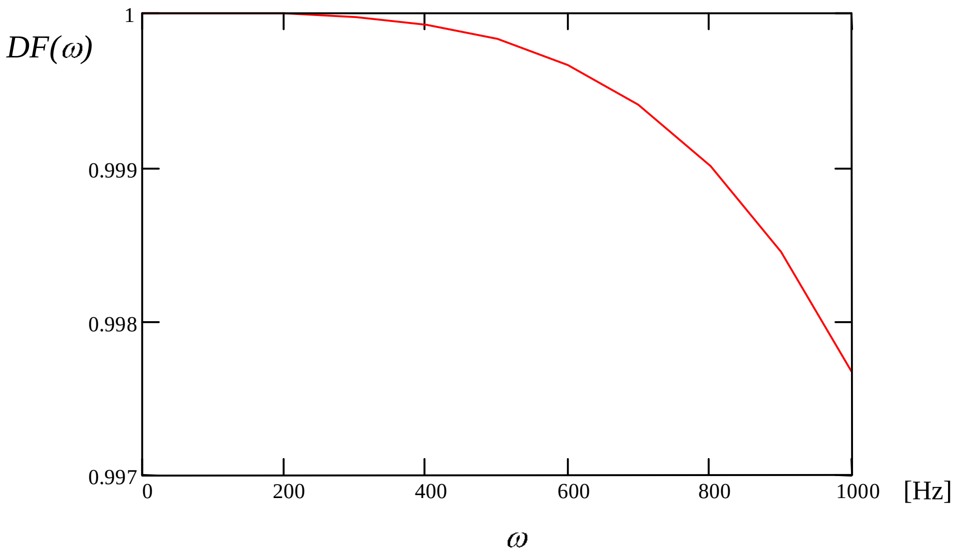
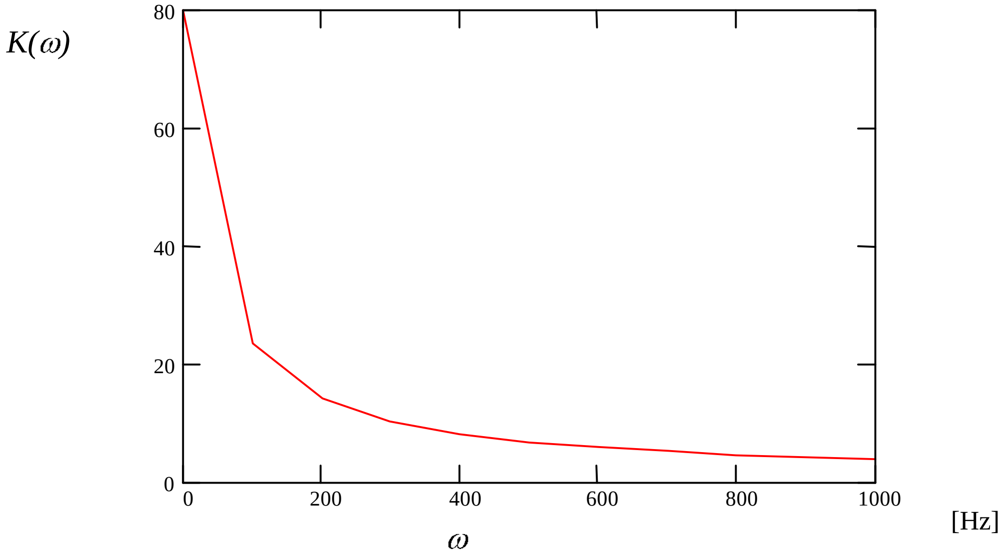


Figure 11. PROTONS 300MeV: improved duty factor versus ω, the ripple frequency.
(PIMMS values, with $\alpha_r(\omega)=p_r/p_0=10^{-4}$ taken arbitrarily).

ΔE=0



ΔE corresponds to the K maximum in Figure 9

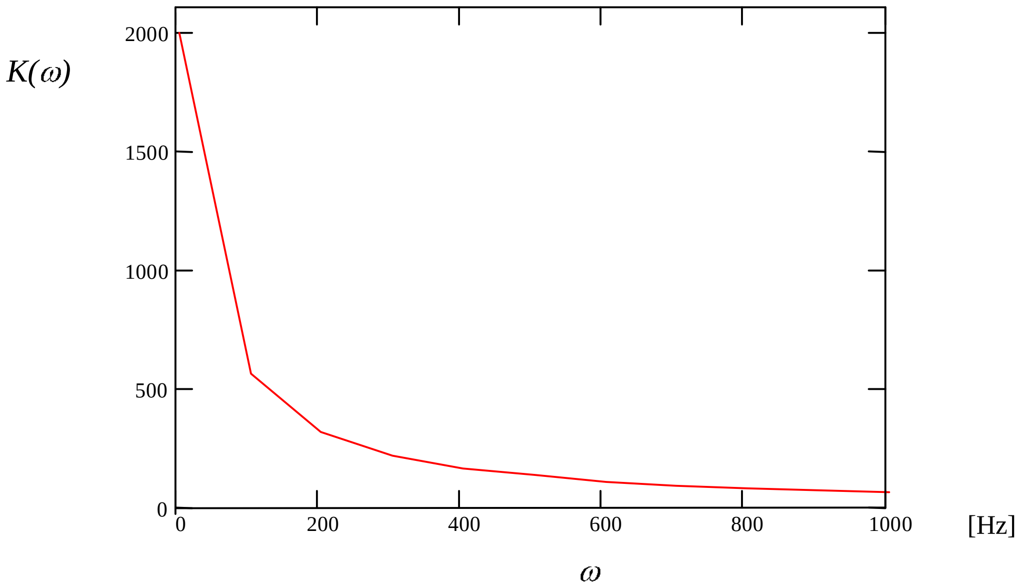
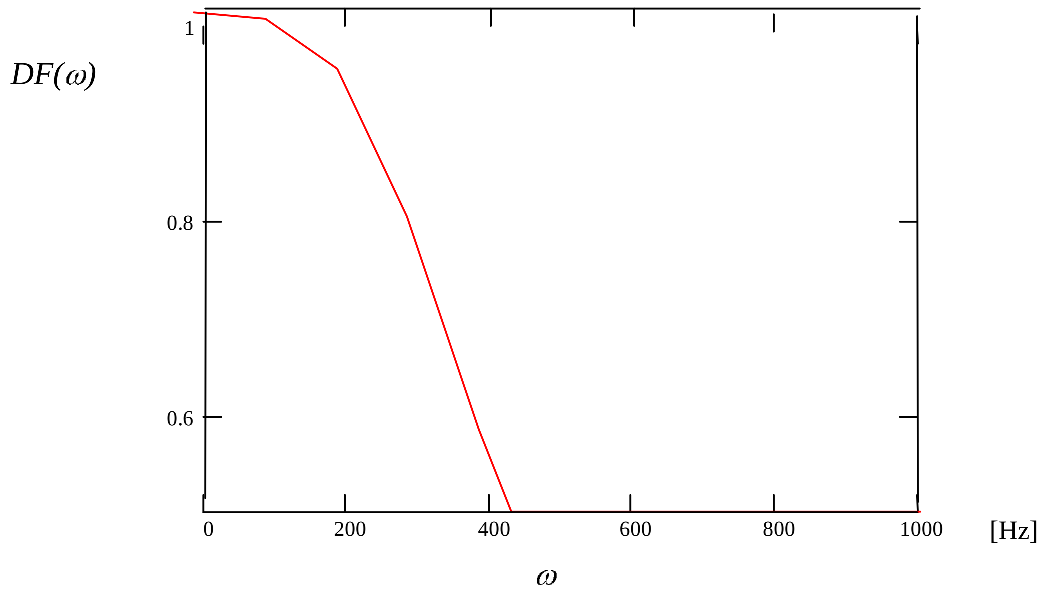


Figure 12. CARBON IONS 425MeV/u: multiplying factor K as function of ω, the ripple frequency.
(PIMMS values, with $\alpha_r(\omega) = p_r/p_0 = 10^{-4}$ taken arbitrarily).

ΔE=0



ΔE corresponds to the K maximum in Figure 10

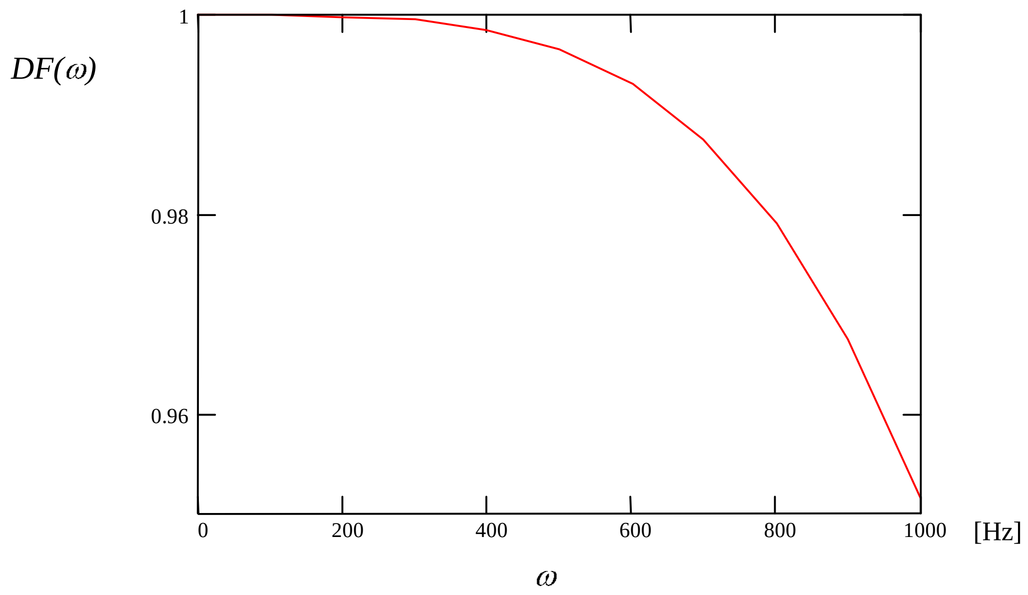


Figure 13. CARBON IONS 425MeV/u: improved duty factor versus ω, the ripple frequency.
(PIMMS values, with $\alpha_r(\omega) = p_r / p_0 = 10^{-4}$ taken arbitrarily).

CONCLUSION

RF empty bucket channelling increases the spill quality during slow extraction by increasing the *dp/dt* at the resonance crossing.
The improvement is not constant for all the particles, but depends on their betatron amplitude. Furthermore, it depends on the amplitude and the frequency of the ripple. The method becomes less effective as the frequency increases.
The bucket half height should be higher than the energy spread engaged in the resonance, which sets a minimum value needed for the RF voltage. The bucket should be properly positioned with respect to the resonance region. The RF voltage value will depend on the machine, on the hardware at disposition, and on beam parameters. The empty bucket must always give the same and opposite acceleration rate as that given by the betatron core.
Another characteristic of the method is due to the fact that the particles are extracted in a small interval of 0-2π. This results in a modulation of the spill at the harmonics of the RF frequency (few MHz). In the case of medical machines, the degradation of the spill quality at frequencies above 10 kHz is of no concern, because modulation coming from higher frequencies is averaged by the slow extraction process, the physical spot size and the integration time in the on-line dosimetry system [10].

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